



HAL
open science

The Poisson boundary of triangular matrices in a number field

Bruno Schapira

► **To cite this version:**

Bruno Schapira. The Poisson boundary of triangular matrices in a number field. 2007. hal-00119525v2

HAL Id: hal-00119525

<https://hal.science/hal-00119525v2>

Preprint submitted on 20 Feb 2007 (v2), last revised 19 Sep 2008 (v3)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

THE POISSON BOUNDARY OF TRIANGULAR MATRICES IN A NUMBER FIELD

BRUNO SCHAPIRA

ABSTRACT. The aim of this note is to describe the Poisson boundary of the group of invertible triangular matrices with coefficients in a number field. It generalizes to any dimension and to any number field a result of Brofferio [2] concerning the Poisson boundary of random rational affinities.

Key words: Random walks, Poisson boundary, triangular matrices, number field, Bruhat decomposition.

A.M.S. classification: 22D40; 28D05; 28D20; 60B15; 60J10; 60J50.

1. INTRODUCTION

The Poisson boundary is a measure space which describes the asymptotic behavior of random walks on groups. In the same time it gives information on the geometry of the group and provides a representation of bounded harmonic functions (we refer for instance to [7] [8] [6] or [10] for more details). Our aim in this paper is to explore the case of a group of matrices with coefficients in a number field. More precisely we study the group of upper diagonal matrices with non zero diagonal coefficients.

This example was treated previously by Brofferio [2] for matrices of size 2 and rational coefficients, which corresponds to the case of rational affinities. She proved that the Poisson boundary is the product over all prime numbers p (including $p = \infty$) of measure spaces (B_p, ν_p) . Moreover each factor B_p is either a p -adic line, or a point. This is determined explicitly in function of the random walk. In each case one can see B_p as a subspace of the p -adic projective line, which is known to be the Poisson boundary of the group of p -adic affinities (cf [5] [3]). The goal of this paper is to generalize this result in higher dimension $d \geq 2$. In other words we will prove that the Poisson boundary is a product of factors (B_p, ν_p) , where for each p , B_p is a subspace of the Furstenberg boundary, which is also the space of flags on \mathbb{Q}_p^d . It is well known that this space, or a quotient, is the Poisson boundary of a large class of random walks on groups of real matrices (see e.g. [7] [13] or [15]). In fact there is a well known decomposition of the Furstenberg boundary called Bruhat decomposition, which coincides for $d = 2$ to the decomposition of the projective line into a line and a point. So we will prove that each B_p is a component, also called a Bruhat cell, of this decomposition, that we determine in function of the random walk.

Such factor decompositions of the Poisson boundary were already observed so far. For instance Bader and Shalom [1] proved recently a general factor theorem in an adelic setting. It is in fact rather likely that our result should extend to a more general class of groups, such as $SL_d(\mathbb{Q})$ and its parabolic subgroups. But for this

our proof "with hands" should probably be replaced by more powerful tools, such as the Oseledec' theorem (see for instance its use by Ledrappier [13] for the study of discrete subgroups of semisimple groups).

Let us describe now the organization of the paper. In the next section we detail our notations and recall some preliminary background on μ -boundaries, the Poisson boundary and Bruhat decomposition. In section 3 we give a precise statement of our result in the case of rational coefficients, which is maybe easier in a first reading. The other sections (except the last one) are devoted to its proof. First in section 4, we find a μ -boundary which is a good candidate to be the Poisson boundary. Then we prove it in sections 5 to 7 (plus appendix). The general strategy is essentially the same than in [2], but with some additional technical points (see notably Proposition 6.1). The last section is devoted to the case of number fields. There is some adjustments to make in the proof that we explain.

Acknowledgments: I would like to thank Philippe Bougerol, Sara Brofferio, Yves Guivarc'h and Vadim Kaimanovich for enlightening discussions. I am also particularly indebted to Uri Bader for his comments on a previous version of this paper. In particular he draw my attention to the nice description of the Furstenberg boundary in terms of Bruhat cells. Apart from its own interest this helped me to clarify a few points, so that hopefully the present paper should be now a little bit more readable.

2. PRELIMINARIES

Let $d \geq 1$ be an integer. We denote by $A(\mathbb{Q})$ the set of upper triangular matrices of size d with coefficients in \mathbb{Q} and non zero diagonal coefficients. So if $a = (a_{i,j})_{i,j} \in A(\mathbb{Q})$, we have $a_{i,j} = 0$, if $i > j$, and $a_{i,i} \neq 0$ for $1 \leq i \leq d$. We denote by \mathcal{P} the set of prime numbers (including ∞), and by \mathcal{P}^* the set $\mathcal{P} - \{\infty\}$. Given $p \in \mathcal{P}$, we denote by \mathbb{Q}_p the set of p -adic numbers (\mathbb{Q}_∞ denotes by convention the field of reals \mathbb{R}), and by $|\cdot|_p$ the usual p -adic norm on \mathbb{Q}_p . For $n \geq 1$ and $z = (z_1, \dots, z_n) \in \mathbb{Q}_p^n$, we set $|z|_p = \max_{i=1, \dots, n} |z_i|_p$.

Random walk and Poisson boundary: Let μ be a measure on $A(\mathbb{Q})$. We consider a sequence $(g_n)_{n \geq 1}$ of i.i.d. random variables of law μ on $A(\mathbb{Q})$. The random walk $(x_n)_{n \geq 0}$ of law μ on $A(\mathbb{Q})$ is defined by

$$x_n := g_1 \cdots g_n.$$

We denote by \mathbb{P} the law of $(x_n)_{n \geq 1}$ on the path space $A(\mathbb{Q})^{\mathbb{N}}$. We say that two sequences $(x_n)_{n \geq 1}$ and $(y_n)_{n \geq 1}$ are equivalent, if there exists $n_1, n_2 \geq 1$ such that for all $n \geq 1$, $x_{n+n_1} = y_{n+n_2}$. For any element $\mathbf{x} \in A(\mathbb{Q})^{\mathbb{N}}$ we denote by $\mathbf{bnd} \mathbf{x}$ its equivalence class for this relation. The Poisson boundary is by definition the quotient of the measure space $(A(\mathbb{Q})^{\mathbb{N}}, \mathbb{P})$ by this relation. As proved by Kaimanovich [10], it is also the maximal μ -boundary. But let us give more details. Assume that \mathbf{B} is a locally compact space, endowed with a measure ν and an action of $A(\mathbb{Q})$. We say that ν is μ -invariant, if

$$\int_{A(\mathbb{Q})} (g\nu) d\mu(g) = \nu,$$

where for all $g \in A(\mathbb{Q})$, $g\nu$ is defined by

$$g\nu(f) = \int_{\mathbf{B}} f(gz) d\nu(z)$$

for all continuous function f . In this case, according to Furstenberg [7], we say that (\mathbf{B}, ν) is a μ -boundary if, \mathbb{P} -almost surely $x_n\nu$ converges vaguely to a Dirac measure. The result of Kaimanovich [10] states that each μ -boundary is a quotient of the Poisson boundary by some partition of measurable sets of equivalence classes of the path space.

Let $\text{sgr}(\mu)$ be the semi-group generated by the support of μ , i.e. $\text{sgr}(\mu) = \cup_n \text{supp}(\mu^{*n})$. We say that a function f on $\text{sgr}(\mu)$ is μ -harmonic if

$$\int_{A(\mathbb{Q})} f(gg') d\mu(g') = f(g),$$

for all $g \in \text{sgr}(\mu)$. It was proved in [10] that, there is an isometry between the space $\mathcal{H}^\infty(A(\mathbb{Q}), \mu)$ of bounded μ -harmonic functions f on $\text{sgr}(\mu)$ and the space $L^\infty(\mathbf{B}, \nu)$ of bounded functions F on \mathbf{B} . The isometry is given by the formulas

$$F(\mathbf{bnd} \mathbf{x}) = \lim_{n \rightarrow \infty} f(x_n), \quad f(g) = \int_{\mathbf{B}} F(gz) d\nu(z).$$

The second formula is the so called Poisson integral representation formula of bounded harmonic functions.

Some structure and the Bruhat decomposition: Let $G = GL_d$. We denote by Δ the set of diagonal matrices with non zero diagonal coefficients. We denote by $\delta = \text{diag}(\delta_1, \dots, \delta_d)$ the diagonal matrix with entries $\delta_{i,i} = \delta_i$, $i = 1, \dots, d$. Let U be the group of upper triangular matrices with one's in the diagonal (unipotent matrices). The notation $U(\mathcal{R})$, where \mathcal{R} is some ring, means that the coefficients strictly upper the diagonal are in \mathcal{R} . Let \bar{U} be the group of lower triangular matrices with one's in the diagonal. We set $A = \Delta U$ and $\bar{A} = \Delta \bar{U}$. We denote by W the Weyl group which will be identified with a subset of G . Its action by conjugation on Δ permutes the coordinates of the diagonal. In this way W will also be identified with the group of permutations of the set $\{1, \dots, d\}$. For $w \in W$, we set $U_w = w\bar{U}w^{-1} \cap U$ and $U^w = wUw^{-1} \cap U$. We have $U = U^w U_w$ and $U^w \cap U_w = \{\text{Id}\}$. If $u \in U$, and if there is no possible confusion on the w , we denote by \bar{u} its component in U^w according to the previous decomposition of U . An element $u \in U$ lies in U^w if, and only if, $u_{i,j} = 0$ whenever $i < j$ and $w(i) > w(j)$.

The *Bruhat decomposition* (see e.g. [9] or [18]) says that G can be decomposed in the following disjoint union:

$$G = \coprod_{w \in W} Aw\bar{A} = \coprod_{w \in W} U^w w\bar{A}.$$

The components $Aw\bar{A}$ are called the Bruhat cells. In the quotient space G/\bar{A} they are identified with the groups U^w .

3. STATEMENT OF THE RESULT

For $p \in \mathcal{P}$ and $i \in [1, \dots, d]$, we set

$$\phi_p(i) := \int_{A(\mathbb{Q})} \ln |a_{i,i}|_p d\mu(a).$$

Let w be the unique element in W such that $w(i) > w(j)$ if $i < j$ and $\phi_p(i) \geq \phi_p(j)$. We define now B_p as the Bruhat cell $U^w(\mathbb{Q}_p)$. The action of $a \in A(\mathbb{Q})$ on $b \in B_p$ is defined as follows. Assume that $a = u\delta$ with $u \in U$ and $\delta \in \Delta$. Then

$$a \cdot b := \overline{abd^{-1}}.$$

In other words $a \cdot b$ is the unique element of U^w representing ab in G/\overline{A} . The main result of this paper is

Theorem 3.1. *Let μ be a probability measure on $A(\mathbb{Q})$ such that*

$$\int_{A(\mathbb{Q})} \sum_{i \leq j} \left(\sum_{p \in \mathcal{P}} |\ln |a_{i,j}|_p| \right) d\mu(a) < +\infty.$$

Then there exists a unique μ -invariant probability measure ν on the space

$$\mathbf{B} := \prod_{p \in \mathcal{P}} B_p,$$

such that the measure space (\mathbf{B}, ν) is the Poisson boundary of the random walk of law μ on $A(\mathbb{Q})$.

In the rest of the paper we will always assume that the measure μ satisfies the hypothesis of the theorem.

Remark 3.1. Let

$$n_p := \text{Card} \left\{ (i, j) \mid i < j \text{ and } \phi_p(i) < \phi_p(j) \right\}.$$

Observe that B_p is isomorphic to $\mathbb{Q}_p^{n_p}$ and may naturally be endowed with the topology inherited from $|\cdot|_p$. In this way the space \mathbf{B} , endowed with the product topology, is not locally compact. Nevertheless, as explained by Brofferio [2], there exists a locally compact subspace \mathbf{B}^* of \mathbf{B} containing the support of ν . For instance, by identifying B_p with $\mathbb{Q}_p^{n_p}$, we may take

$$\mathbf{B}^* = \left\{ z = (z^p)_{p \in \mathcal{P}} \in \mathbf{B} \mid |z^p|_p \leq r_p, \text{ for all } p \text{ but a finite number} \right\},$$

for some well chosen r_p 's. Then (\mathbf{B}^*, ν) is well the maximal μ -boundary.

4. SOME μ -BOUNDARIES

Assume that $p \in \mathcal{P}$. Let (e_1, \dots, e_d) be the canonical basis of \mathbb{Q}_p^d . Let

$$J_d = \{i \leq d \mid \phi_p(i) \geq \phi_p(d)\},$$

and let r be the cardinal of J_d . Assume that $j_1 < \dots < j_r = d$ are the elements of J_d . We denote by Λ_{sub}^r the subspace of Λ^r generated by the elements $e_{i_1} \wedge \dots \wedge e_{i_r}$ such that $i_s \leq j_s$ for all $s \in [1, \dots, r]$. We denote by \mathcal{B}_r the basis of Λ_{sub}^r made up of these elements ranged in the lexicographic order. Let also m be the dimension of this subspace. Each $a \in A(\mathbb{Q})$ defines an endomorphism of Λ^r , by setting $a(v_1 \wedge \dots \wedge v_r) := av_1 \wedge \dots \wedge av_r$. We denote by $a^{(r)}$ the restriction of this endomorphism to Λ_{sub}^r . Observe that it has a triangular matrix representation in the basis \mathcal{B}_r . This provides a representation of $A(\mathbb{Q})$ on the subspace $P\mathbb{Q}_p^m$ of \mathbb{Q}_p^m whose vectors have last coordinate equal to 1. Indeed for $u \in P\mathbb{Q}_p^m$ and $a \in A(\mathbb{Q})$, we define the (projective) action $a \cdot u$ of a on u by

$$a \cdot u := \frac{1}{\prod_{j \in J} a_{j,j}} a^{(r)} u.$$

Lemma 4.1. *For any $u \in P\mathbb{Q}_p^m$, the sequence $(x_n \cdot u)_{n \geq 1}$ converges \mathbb{P} -a.s. Moreover the limit, that we denote by $(Z_k^p(d))_{k \leq m}$, does not depend on the choice of u .*

Proof: We assume that $m \geq 2$, otherwise there is nothing to prove. Let $a \in A(\mathbb{Q})$. We have observed that $a^{(r)}$ has a triangular matrix representation in the basis \mathcal{B}_r . We put $a' = \frac{1}{\prod_{j \in J} a_{j,j}} a^{(r)}$. Then $a'_{m,m} = 1$, and for $k < m$, there exists a subset $K \subset \{1, \dots, d\}$ of cardinal r , such that $a'_{k,k} = \frac{\prod_{j \in K} a_{j,j}}{\prod_{j \in J} a_{j,j}}$. Therefore

$$(1) \quad \begin{aligned} \phi'_p(k) &:= \mathbb{E}[\ln |a'_{k,k}|_p] = \sum_{j \in K} \mathbb{E}[\ln |a_{j,j}|_p] - \sum_{j \in J} \mathbb{E}[\ln |a_{j,j}|_p] \\ &= \sum_{j \in K \cap J^c} \underbrace{(\phi_p(j) - \phi_p(d))}_{<0} - \sum_{j \in J \cap K^c} \underbrace{(\phi_p(j) - \phi_p(d))}_{\geq 0} < 0. \end{aligned}$$

For $n \geq 1$, let

$$x'_n = g'_1 \cdots g'_n.$$

If $k < m$, by the law of large number, a.s.

$$(2) \quad \frac{\ln |(x'_n)_{k,k}|_p}{n} = \frac{1}{n} \sum_{l=1}^n \ln |(g'_l)_{k,k}|_p \rightarrow \phi'_p(k), \text{ when } n \rightarrow \infty.$$

first step: Let k be fixed. Let us prove by induction on $k' \in [k, \dots, m-1]$, that for all $k \leq k' \leq m-1$, there exists a.s. $\alpha > 0$ such that $\frac{1}{n} \ln |(x'_n)_{k,k'}|_p \leq -\alpha$ for n large enough. If $k' = k$ the result is immediate with (2). We assume now the result for $l \leq k' - 1$ (with $k' > k$), and we prove it for k' . By the induction hypothesis there exists a.s. $\alpha > 0$ and $N_1 \geq 1$ such that for all $l \in [k, \dots, k' - 1]$,

$$n \geq N_1 \Rightarrow |(x'_n)_{k,l}|_p \leq e^{-n\alpha}.$$

Let $\epsilon > 0$ be such that $\alpha - 4\epsilon > 0$, and $\phi'_p(k') + \epsilon < 0$. By (2) there exists a.s. $N_2 \geq N_1$ such that

$$(3) \quad n \geq N_2 \Rightarrow e^{n(\phi'_p(k) - \epsilon)} \leq |(x'_n)_{k',k'}|_p \leq e^{n(\phi'_p(k) + \epsilon)}.$$

We set $c_n = \max_{i,j} |(g'_{n+1})_{i,j}|_p$. We have a.s. $\frac{\ln c_n}{n} \rightarrow 0$. Thus there is a.s. some integer $N_3 \geq N_2$ such that $n \geq N_3 \Rightarrow c_n \leq e^{n\epsilon}$. Finally we set $u_n := \frac{(x'_n)_{k,k'}}{(x'_n)_{k',k'}}$. We have for all $n \geq 1$,

$$(4) \quad u_{n+1} = u_n + \underbrace{\frac{1}{(x'_{n+1})_{k',k'}} \sum_{l=k}^{k'-1} (x'_n)_{k,l} (g'_{n+1})_{l,k'}}_{:=r_n}.$$

With the previous notations we have a.s. for $n \geq N_3$,

$$(5) \quad |r_n|_p \leq d e^{-n\phi'_p(k') - n(\alpha - 3\epsilon)}.$$

We set $C = \max_{n \leq N_3} |r_n|_p$. Hence by (3), (4) and (5), we have a.s. for any $N \geq N_3$,

$$|(x'_N)_{k,k'}|_p \leq |(x'_N)_{k',k'}|_p \sum_{n=1}^N |r_n|_p \leq Nd(e^{-N(\alpha - 4\epsilon)} + C e^{N(\phi'_p(k') + \epsilon)}),$$

and the result for k' follows.

second step: We have for $n \geq 1$, and any $k < m$,

$$(x'_{n+1})_{k,m} = (x'_n)_{k,m} + \sum_{l=k}^{m-1} (x'_n)_{k,l} (g'_{n+1})_{l,m}.$$

As a consequence $(x'_n)_{k,m}$ is the partial sum of a series whose general term converges a.s. to 0 exponentially fast (by the first step). Thus it is almost surely convergent, and the lemma follows. \blacksquare

This lemma implies in fact that $P\mathbb{Q}_p^m$ equipped with the law of $(Z_k^p(d))_{k \leq m}$ is a μ -boundary. But we want to show that B_p is a μ -boundary. One way to see it is as follows. First for all $d' \leq d$, we define the minor of size d' of an element $a \in A(\mathbb{Q})$ as the matrix $d' \times d'$ in the upper left corner of a . These matrices act in the same way on $\Lambda_{\text{sub}}^{r(d')} \mathbb{Q}_p^{d'}$, where $r(d')$ is the cardinal of $J_{d'}$. Hence Lemma 4.1 holds as well in this setting. This provides new μ -boundaries and new vectors $(Z_k^p(d'))_{k \leq m(d')}$, where $m(d')$ is the dimension of $\Lambda_{\text{sub}}^{r(d')} \mathbb{Q}_p^{d'}$. We claim that the set of vectors $(Z^p(2), \dots, Z^p(d))$ is associated to an element $(Z_{i,j}^p)_{1 \leq i,j \leq d}$ of B_p . More precisely we claim that we can define the columns Z_j^p of Z^p , for $j = 1, \dots, d$, recursively by

$$Z_{i_1}^p \wedge \dots \wedge Z_j^p = Z^p(j),$$

where i_1, \dots, j are the elements of J_j . Indeed the set, let say S , of vectors $(V(2), \dots, V(d))$ which are associated to an element of B_p by this way is stable under the action of $A(\mathbb{Q})$. But since the limit $(Z^p(2), \dots, Z^p(d))$ is independent of the starting point, which can be chosen in S , it must be also in S . Thus if ν_p is the law of the associated Z^p , we get that (B_p, ν_p) is a μ -boundary. Hence (\mathbf{B}, ν) , where $\mathbf{B} = \prod_p B_p$ and ν is the law of $(Z^p)_p$, is also a μ -boundary.

5. GAUGES ON $A(\mathbb{Q})$

We denote by \mathbb{A} the adèle ring of \mathbb{Q} , i.e. the restricted product $\prod'_{p \in \mathcal{P}} \mathbb{Q}_p$ (cf e.g. [19]). The notation \prod' means that if $(z^p)_{p \in \mathcal{P}} \in \mathbb{A}$, then for all p but a finite number, $|z^p|_p \leq 1$. Let H be the group of upper triangular matrices with non zero rational diagonal coefficients and strictly upper diagonal coefficients in \mathbb{A} . In other words

$$H := U(\mathbb{A}) \Delta(\mathbb{Q}).$$

We have a natural injection $i_{\mathbb{A}}$ from \mathbb{Q} into \mathbb{A} and therefore also an injection i_H from $A(\mathbb{Q})$ into H . Via i_H we will sometimes identify elements in $A(\mathbb{Q})$ with their image in H . For $q \in \mathbb{Q}^*$, we set

$$\langle q \rangle := \sum_{p \in \mathcal{P}^*} |\ln |q|_p|.$$

In particular for every irreducible fraction $q = \pm r/s$ of integers, one has $\langle q \rangle = \ln r + \ln s$. If $\delta = \text{diag}(\delta_1, \dots, \delta_d) \in \Delta(\mathbb{Q})$, we set

$$\langle \delta \rangle := \sum_{i=1}^d \langle \delta_i \rangle.$$

For $b = (b^p)_{p \in \mathcal{P}} \in \mathbb{A}$ and $p \in \mathcal{P}$, we set

$$\langle b \rangle_p^+ := \ln^+ |b^p|_p,$$

where \ln^+ denotes the positive part of the function \ln and

$$\langle b \rangle^+ := \sum_{p \in \mathcal{P}} \langle b \rangle_p^+$$

If $u \in U(\mathbb{A})$ and $p \in \mathcal{P}$, we set

$$\langle u \rangle_p^+ := \sum_{i < j} \langle u_{i,j} \rangle_p^+ \quad \text{and} \quad \langle u \rangle^+ := \sum_{i < j} \langle u_{i,j} \rangle^+.$$

Let $h \in H$ and let $h = u\delta$ be its decomposition in $U(\mathbb{A})\Delta(\mathbb{Q})$. We define the adelic length of h by

$$\|h\| := \langle u \rangle^+ + \langle \delta \rangle.$$

The adelic length is not sub-additive, but as in dimension two [2], it is almost the case. Indeed for any $q, q' \in \mathbb{Q}^*$,

$$\langle qq' \rangle \leq \langle q \rangle + \langle q' \rangle,$$

and for any $b, b', b'' \in \mathbb{A}$, and $q \in \mathbb{Q}^*$,

$$\langle b + qb'b'' \rangle^+ \leq \ln 2 + \langle b \rangle^+ + \langle b' \rangle^+ + \langle b'' \rangle^+ + \langle q \rangle.$$

Using these relations we can find constants $K > 0$ and $K' > 0$ such that for all $h, h' \in H$,

$$\|hh'\| \leq K + K'(\|h\| + \|h'\|).$$

Now we consider the family of gauges $(\mathcal{G}_k^h)_{k \in \mathbb{N}}$ on $A(\mathbb{Q})$ defined for $k \geq 0$ and $h \in H$, by

$$\mathcal{G}_k^h := \{a \in A(\mathbb{Q}) \mid \|a^{-1}h\| \leq k\}.$$

Lemma 5.1. *The family of gauges $\{\mathcal{G}_k^h\}_{h \in H}$ has uniform exponential growth, i.e. there exists $C' > 0$ such that $\text{Card}\{\mathcal{G}_k^h\} \leq e^{C'k}$ for all $h \in H$ and all $k \in \mathbb{N} - \{0\}$.*

Proof: First, since the inverse map is a bijection of $A(\mathbb{Q})$, we can always replace a^{-1} by a in the definition of the gauges. Now let h_0 be the unit element of H , and let $a \in A(\mathbb{Q})$ be such that $\|ah_0\| \leq k$. In this case $\langle a_{i,i} \rangle \leq k$ for any $i \leq d$, and $\langle a_{i,j}/a_{j,j} \rangle^+ \leq k$ for any $i < j$. But the number of rational $q \neq 0$ such that $\langle q \rangle \leq k$ is lower than $2e^{2k}$. Moreover, for any rational q , $\langle q \rangle^+ \geq \langle q \rangle / 2$. Thus

$$\langle \frac{a_{i,j}}{a_{j,j}} \rangle^+ \geq \frac{1}{2} \langle \frac{a_{i,j}}{a_{j,j}} \rangle \geq \frac{1}{2} (\langle a_{i,j} \rangle - \langle a_{j,j} \rangle),$$

which implies

$$\text{Card}(\mathcal{G}_k^{h_0}) \leq (2e^{6k})^{d^2}.$$

Now let $h \in H$. Since the multiplication by any element is a bijection of $A(\mathbb{Q})$, we do not change the cardinality of the \mathcal{G}_k^h if we multiply to the left h by an element in $A(\mathbb{Q})$. Hence, multiplying them if necessary by $\text{diag}(h_1^{-1}, \dots, h_d^{-1})$ we can always suppose that $h_1 = \dots = h_d = 1$. Then as in [2] we can find $b \in A(\mathbb{Q})$ such that $\|h^{-1}b\| = 0$. Hence for any $a \in A(\mathbb{Q})$,

$$\|ab\| = \|ahh^{-1}b\| \leq K + K'(\|ah\| + \|h^{-1}b\|).$$

Thus $\mathcal{G}_k^h \subseteq \mathcal{G}_{K+K'k}^b$, which has the same cardinal as $\mathcal{G}_{K+K'k}^{h_0}$, since $b \in A(\mathbb{Q})$. This concludes the proof of the lemma. \blacksquare

6. SOME ESTIMATES

Let us begin with the lemma

Lemma 6.1. *For $i \leq d$, and $n \geq 1$, let $q_n^i = \prod_{p \in \mathcal{P}^*} p^{-\lceil n \frac{\phi_p(i)}{\ln p} \rceil}$, where for $x \in \mathbb{R}$, $\lceil x \rceil$ denotes the integer part of x if $x \geq 0$, and the opposite of the integer part of $-x$ otherwise. Then*

$$\frac{\langle (x_n)_{i,i}^{-1} q_n^i \rangle}{n} \rightarrow 0, \text{ in } L^1.$$

Proof: It is the same proof as in [2]. Let us recall it. For $p \in \mathcal{P}$, the ergodic theorem implies that

$$\frac{\ln |(x_n)_{i,i}^{-1} q_n^i|_p}{n} = \frac{-\sum_{k=1}^n \ln |(gk)_{i,i}|_p + \ln p \left\lceil n \frac{\phi_p}{\ln p} \right\rceil}{n} \rightarrow 0$$

in L^1 . Thus by the dominated convergence theorem, the sequence

$$\mathbb{E} \left[\frac{\langle (x_n)_{i,i}^{-1} q_n^i \rangle}{n} \right] = \sum_{p \in \mathcal{P}^*} \frac{\mathbb{E} [|\ln |(x_n)_{i,i}^{-1} q_n^i|_p|]}{n}$$

converges to zero because each term of the infinite sum converges to zero and is dominated by $\mathbb{E} [|\ln |a_{i,i}|_p|] + |\phi_p|$ whose series is convergent by hypothesis. \blacksquare

Let now P be a finite subset of \mathcal{P} and let $q_n = \text{diag}(q_n^1, \dots, q_n^d)$. We set

$$\begin{aligned} \pi_n^P : \prod_{p \in P} B_p &\longrightarrow H \\ (z^p)_{p \in P} &\longmapsto \mathbf{z} q_n, \end{aligned}$$

where for $i < j$, $\mathbf{z}_{i,j} \in \mathbb{A}$ is defined by

$$\mathbf{z}_{i,j}^P = \begin{cases} z_{i,j}^p & \text{if } p \in P, \\ 0 & \text{otherwise.} \end{cases}$$

We set also $Z^P := (Z^p)_{p \in P} \in \prod_{p \in P} B_p$. The main result of this section is the following proposition.

Proposition 6.1. *Let P be a finite subset of \mathcal{P} , such that $\infty \subset P$. For $p \in \mathcal{P}$, let $K_p = \sum_{r \leq s} \int_{A(\mathbb{Q})} |\ln |a_{r,s}|_p| d\mu(a)$. Then there exists a constant $C > 0$, such that*

$$\mathbb{P} \left[\frac{\|x_n^{-1} \pi_n^P(Z^P)\|}{n} \leq C \sum_{p \notin P} K_p \right] \rightarrow 1.$$

Proof: Assume that

$$x_n = u_n \delta_n \quad \forall n \geq 1,$$

with $u_n \in U$ and $\delta_n \in \Delta$. We have

$$\|x_n^{-1} \pi_n^P(Z^P)\| = \langle \delta_n^{-1} q_n \rangle + \langle x_n^{-1} \pi_n^P(Z^P) q_n^{-1} \delta_n \rangle^+.$$

First we know by Lemma 6.1 that $\langle \delta_n^{-1} q_n \rangle / n$ converges to 0 in L^1 . So it converges also to 0 in probability. Next

$$\langle x_n^{-1} \pi_n^P(Z^P) q_n^{-1} \delta_n \rangle^+ = \sum_{p \in P} \langle x_n^{-1} Z^p \delta_n \rangle_p^+ + \sum_{p \notin P} \langle x_n^{-1} \delta_n \rangle_p^+.$$

first step: the sum over $p \notin P$.

For $i \leq j$ and $N \geq 1$ we have

$$\frac{(x_{N+1}^{-1})_{i,j}}{(x_{N+1}^{-1})_{i,i}} = \frac{(x_N^{-1})_{i,j}}{(x_N^{-1})_{i,i}} + \underbrace{\sum_{k=i+1}^j \frac{(x_N^{-1})_{k,j}}{(x_{N+1}^{-1})_{i,i}} (g_{N+1}^{-1})_{i,k}}_{:=r_N}.$$

By the ultra-metric property we get

$$(6) \quad \ln^+ |(x_{N+1}^{-1})_{i,j}|_p \leq |\ln |(x_{N+1}^{-1})_{i,i}|_p| + \max_{1 \leq n \leq N} \ln^+ |r_n|_p.$$

For $n \geq 1$, we set $c_n = \max_{r,s} |(g_{n+1}^{-1})_{r,s}|_p$. Observe that for some constant $C_1 > 0$, $E[|\ln c_n|] \leq C_1 K_p$. By (6), we have

$$(7) \quad \max_{1 \leq n \leq N+1} \ln^+ |(x_n^{-1})_{i,j}|_p \leq 2 \sum_{n=1}^N (|\ln |(g_{n+1}^{-1})_{i,i}|_p| + |\ln c_n|) + \max_{i+1 \leq k \leq j} \max_{1 \leq n \leq N} \ln^+ |(x_n^{-1})_{k,j}|_p.$$

Now by an elementary induction on $(j-k) \in [0, \dots, j-i]$ (with j fixed), we get from (7)

$$(8) \quad \forall N \geq 1 \quad \frac{1}{N} \mathbb{E} \left[\max_{1 \leq n \leq N+1} \ln^+ |(x_n^{-1})_{i,j}|_p \right] \leq C K_p,$$

for some constant $C > 0$. Let now

$$\alpha_n^p := \frac{1}{n} \ln^+ |(x_n^{-1})_{i,j}|_p.$$

Again from (7) we get by induction on $j-k$ that a.s. for all $p \notin P$, $(\alpha_n^p - C K_p)^+$ tends to 0, when $n \rightarrow +\infty$. By (8) and Lebesgue theorem, we have even that $\sum_p (\alpha_n^p - C K_p)^+$ converges to 0 in L^1 . So for some constant $C' > 0$,

$$(9) \quad \mathbb{P} \left[\frac{1}{n} \sum_{p \notin P} \langle x_n^{-1} \delta_n \rangle_p^+ \leq C' \sum_{p \notin P} K_p \right] \rightarrow 1.$$

Second step: the sum over $p \in P$.

We will show now that for all $i \leq j$ and all $p \in P$,

$$\left| \frac{(x_n^{-1} Z^p)_{i,j}}{(x_n^{-1})_{j,j}} \right|_p \leq e^{n\epsilon}$$

for n large enough. Together with (9) this will conclude the proof of the proposition. Without loss of generality we can always suppose that $i = 1$ and $j = d$. For $n \geq 1$, and $l \geq 1$, we have

$$\frac{(x_n^{-1} Z^p)_{l,d}}{(x_n^{-1})_{d,d}} = \sum_{k=l}^d \frac{(x_n^{-1})_{l,k}}{(x_n^{-1})_{d,d}} Z_{k,d}^p := u_n^l$$

Using that $x_n x_n^{-1} = \text{Id}$, we get by an elementary induction on l that, for $l \geq 1$,

$$(10) \quad u_n^l = \frac{(x_n)_{d,d}}{(x_n)_{l,l}} Z_{l,d}^p - \sum_{k=l+1}^d \frac{(x_n)_{l,k}}{(x_n)_{l,l}} u_n^k.$$

Now for $l < k$ let $A(l, k) := ((x_n)_{i,j})_{l \leq i < k, l < j \leq k}$. Next we will need the elementary formula

$$(11) \quad \frac{\det A(l, k)}{\prod_{l'=l+1}^k (x_n)_{l', l'}} = \sum_{l'=l+1}^k (-1)^{l'-l+1} \frac{(x_n)_{l, l'}}{(x_n)_{l', l'}} \frac{\det A(l', k)}{\prod_{l''=l'+1}^k (x_n)_{l'', l''}},$$

where by convention $A(k, k) = (1)$. We denote also for $l < k_1 < k$, by $A(l, \widehat{k_1}, k)$ the matrix $A(l, k)$, where the k_1^{th} line and the $(k_1 - 1)^{\text{th}}$ column are omitted. With evident notation we define analogously $A(l, \widehat{k_1}, \dots, \widehat{k_r}, k)$ for $l < k_1 < \dots < k_r < k$. For any $l < d$ we set $I_d^l := \{l\} \cup (J - \{d\})$, $J_d^l := \{j \in J / j \neq l\}$ and

$$S_n^l := \frac{\epsilon_l^p}{\prod_{j \in J} (x_n)_{j, j}} \det \left(((x_n)_{i, j})_{(i, j) \in I_d^l \times J_d^l} \right),$$

where $\epsilon_l^p = (-1)^{\text{Card}\{l < i < d | \phi_p(i) \geq \phi_p(d)\}}$. By convention we set also $S_n^d = -1$. We will need the

Lemma 6.2. *When $l \notin J$,*

$$Z_{l, d}^p = \lim_{n \rightarrow \infty} S_n^l.$$

We postpone the proof of this lemma to the appendix. Let $\{i_1, \dots, i_s\} = J^c$ and for $l \geq 1$, let $k_l = \min\{k \leq s \mid i_k \geq l\}$. First we prove by induction on $d - l \geq 0$, that

$$(12) \quad \begin{aligned} u_n^l &= -\mathbf{1}_{(l \in J)} \frac{(x_n)_{d, d}}{(x_n)_{l, l}} S_n^l \\ &+ \sum_{k=k_l}^s (-1)^{i_k - l} \frac{(x_n)_{d, d}^{i_k - l + 1} \det A(l, i_k)}{(x_n)_{l, l} \cdots (x_n)_{i_k, i_k}} (Z_{i_k, d}^p - S_n^{i_k}), \end{aligned}$$

where we recall our convention $A(l, i_{k_l}) = (1)$ if $i_{k_l} = l$ (i.e. if $l \notin J$). In fact the result is trivial for $l = d$. Now we suppose that it is true for l strictly greater than some l_0 . Then Formula (12) for l_0 is a direct consequence of (10) and (11), which proves the induction step. Next we prove also by induction on $d - l \geq 0$, that

$$(13) \quad \begin{aligned} u_n^l &= \frac{(x_n)_{d, d}}{(x_n)_{l, l}} (Z_{l, d}^p - S_n^l) \\ &+ \sum_{k=k_l}^s \frac{(-1)^{i_k - l} \det A(l, \widehat{i_{k_l}}, \dots, \widehat{i_{k-1}}, i_k)}{(x_n)_{l, l} \cdots (\widehat{x_n})_{i_{k_l}, i_{k_l}} \cdots (\widehat{x_n})_{i_{k-1}, i_{k-1}} \cdots (x_n)_{i_{k-1}, i_{k-1}}} u_n^{i_k}, \end{aligned}$$

where the notation \widehat{x} means that x is omitted in the list. Formula (13) is true for $l = d$. So we suppose that it is true for l strictly greater than some l_0 . Then observe that for any $l < k' < k$,

$$\det A(l, k) = \det A(l, k') \det A(k', k) + (-1)^{k' - l} (x_n)_{k', k'} \det A(l, \widehat{k'}, k).$$

Injecting this in (12) and using the induction hypothesis we get (13) for l_0 , and we can conclude by the induction principle. Eventually we prove again by induction on $d - l \geq 0$, that $|u_n^l|_p \leq e^{n\epsilon}$ for n large enough. We suppose that it is true for l strictly greater than some l_0 . For any l and any $k > k_l$, $\det A(l, \dots, \widehat{i_{k-1}}, i_k)$ is equal to the component on $e_l \cdots \wedge \widehat{e_{i_{k-1}}} \cdots \wedge e_{i_{k-1}}$ of $(ae_2 \cdots \wedge \widehat{ae_{i_{k-1}}} \cdots \wedge ae_{i_k})$.

Therefore as in the proof of Lemma 4.1, we see that

$$\left| \frac{\det A(l, \widehat{i_{k_l}}, \dots, \widehat{i_{k-1}}, i_k)}{(x_n)_{l,l} \cdots (\widehat{x_n})_{i_{k_l}, i_{k_l}} \cdots (\widehat{x_n})_{i_{k-1}, i_{k-1}} \cdots (x_n)_{i_{k-1}, i_{k-1}}} \right|_p \leq e^{n\epsilon},$$

for n large enough. Moreover if $l_0 \in J$, in which case $Z_{l_0, d}^p = 0$, we have also by the same argument $\left| \frac{(x_n)_{d,d}}{(x_n)_{l_0, l_0}} S_n^{l_0} \right|_p \leq e^{n\epsilon}$ for n large enough. Then we immediately prove the result for l_0 , by using the induction hypothesis and Formula (13). This finishes the proof of the proposition. \blacksquare

7. THE POISSON BOUNDARY

We have already proved that \mathbf{B} can be endowed with a probability measure ν , for which it is a μ -boundary. We will now prove that it is in fact the Poisson boundary by using a criteria of Kaimanovich [10] on the entropy of the conditional expectation. Suppose that μ has a finite entropy on the group $A(\mathbb{Q})$ (which is the case if it satisfies the hypothesis of Theorem 3.1, see [2] or [4]). Consider the family $(\mathbb{P}^z)_{z \in \mathbf{B}}$ of probability measures obtained by conditioning the measure \mathbb{P} with respect to the events $\text{bnd}_{\mathbf{B}}(\mathbf{x}) = z$ (where $\text{bnd}_{\mathbf{B}}$ is the projection from $A(\mathbb{Q})^{\mathbb{N}}$ to \mathbf{B}). Let \mathbb{P}_n^z be the corresponding measures on $A(\mathbb{Q})$ obtained by the projection $\mathbf{x} \mapsto x_n$. Then Theorem 4.6 in [10] says that (\mathbf{B}, ν) is the Poisson boundary if, and only if, for ν -almost all $z \in \mathbf{B}$

$$-\frac{1}{n} \ln \mathbb{P}_n^z(x_n) \rightarrow 0, \quad \mathbb{P}^z(d\mathbf{x}) - \text{a.s.}$$

Now the end of the proof of Theorem 3.1 is the same as in [2]. We include it here for the convenience of the reader:

Proof of Theorem 3.1: Let P be a finite subset of \mathcal{P} containing ∞ . For $z \in \prod_{p \in P} \mathbb{Q}_p^{n_p}$ let z^P be its projection on $\prod_{p \in P} \mathbb{Q}_p^{n_p}$. Let $K = C \sum_{p \notin P} K_p$, where C is the constant appearing in Proposition 6.1. Then by Proposition 6.1

$$(14) \quad \mathbb{P} \left[x_n \in \mathcal{G}_{nK}^{\pi_n^P(Z_\infty^P)} \right] = \int_{\mathbf{B}} \mathbb{P}_n^z \left[\mathcal{G}_{nK}^{\pi_n^P(z^P)} \right] d\nu(z) \rightarrow 1.$$

Now let h be the \mathbb{P}^z -almost sure limit of $-\ln \mathbb{P}_n^z(x_n)/n$, which exists for ν -almost all z according to [10]. Next consider the set

$$A_n = \{g \in A(\mathbb{Q}) / -h - \epsilon < \ln \mathbb{P}_n^z(g)/n < -h + \epsilon\}.$$

Then

$$\mathbb{P}_n^z(A_n \cap \mathcal{G}_{nK}^{\pi_n^P(z^P)}) \leq e^{n(\epsilon-h)} \text{Card} \left(\mathcal{G}_{nK}^{\pi_n^P(z^P)} \right) \leq e^{n(\epsilon-h)} e^{C'nK},$$

where C' is the parameter of the exponential growth of the gauges $(\mathcal{G}^g)_{g \in H}$. Thus we must have $C'K - h + \epsilon \geq 0$. Otherwise this would contradict (14). Since ϵ was arbitrarily chosen,

$$h \leq C'K - \epsilon.$$

Letting now P grow to \mathcal{P} , we obtain $h = 0$, which concludes the proof of the theorem. \blacksquare

8. THE CASE OF A NUMBER FIELD

In this section \mathbb{K} denotes a number field, i.e. a finite extension of \mathbb{Q} . We refer to [16] [17] [19] for the general theory. Let \mathcal{O} be the ring of integers of \mathbb{K} . The main difference with the rational case is that except for \mathbb{Q} or imaginary quadratic extensions of \mathbb{Q} , the set \mathcal{O}^* of units (the invertible elements of \mathcal{O}) of \mathbb{K} is infinite. So we have to be careful when defining the gauges, to keep them with uniform exponential growth. Namely we have to define $\langle k \rangle$ for $k \in \mathbb{K}$, in a suitable way. More precisely, let \mathcal{P} be the set of prime ideals of \mathcal{O} , and for $\mathfrak{p} \in \mathcal{P}$ let $v_{\mathfrak{p}}$ be the associated discrete valuation. Let $N_{\mathfrak{p}} = \text{Card}(\mathcal{O}/\mathfrak{p})$. Following a usual convention (see e.g. [12]), we define the norm associated to \mathfrak{p} by

$$\|k\|_{\mathfrak{p}} := N_{\mathfrak{p}}^{-v_{\mathfrak{p}}(k)},$$

for all $k \in \mathbb{K}^*$. Let N be the norm function on \mathcal{O} . If $k = x^{-1}y$ with $x, y \in \mathcal{O}$ such that $(x) \wedge (y) = 1$, then

$$\sum_{\mathfrak{p} \in \mathcal{P}} |\ln \|k\|_{\mathfrak{p}}| = \ln |N(x)| + \ln |N(y)|.$$

Remember that the norm of any unit is equal to ± 1 . Thus with the previous notation we can not define $\langle k \rangle$ as the sum $\ln |N(x)| + \ln |N(y)|$ like in the rational case. Otherwise the associated gauges would have an infinite cardinal. In fact we will add in the sum a term corresponding to archimedean norms. Let us give more details. Let V_{∞} be the set of norms extending the usual absolute value $|\cdot|$ on \mathbb{Q} . If $v \in V_{\infty}$ we will write (with a slight abuse of notation) $v = |\cdot|_v$ and we define the norm $\|\cdot\|_v := |\cdot|_v^{\epsilon_v}$ on \mathbb{K} , where $\epsilon_v = 1$ if $\mathbb{K}_v = \mathbb{R}$ whereas $\epsilon_v = 2$ if $\mathbb{K}_v = \mathbb{C}$ (as usually \mathbb{K}_v denotes the completion of \mathbb{K} endowed with $|\cdot|_v$). Then we have the product formula (cf [12])

$$\prod_{\mathfrak{p} \in \mathcal{P}} \|k\|_{\mathfrak{p}} \times \prod_{v \in V_{\infty}} \|k\|_v = 1,$$

for all $k \in \mathbb{K}^*$, which implies by the way the identity

$$(15) \quad \sum_{\mathfrak{p} \in \mathcal{P}} \phi_{\mathfrak{p}} + \sum_{v \in V_{\infty}} \phi_v = 0,$$

with evident notation. Now we fix some archimedean norm $\|\cdot\|_{v_0}$ and we define

$$\langle k \rangle := \sum_{\mathfrak{p} \in \mathcal{P}} |\ln \|k\|_{\mathfrak{p}}| + \sum_{v \neq v_0} |\ln \|k\|_v|.$$

In this way the set of $k \in \mathbb{K}^*$ such that $\langle k \rangle \leq C$ has a cardinal bounded by $\text{const} \cdot e^{\text{const} \cdot C}$, where the constants are independent of C . Then we can define the height function on the adèle ring and the associated gauges, in the same way as in the rational case. Now the only other change in the proof is the definition of the q_n^i (see section 6). Remember that the set of units is isomorphic to $\mathbb{Z}^{r_1+r_2-1} \times G$, where G is cyclic, r_1 is the number of embedding of \mathbb{K} in \mathbb{R} and $2r_2$ the number of embedding in \mathbb{C} . We set

$$q_n^i = \prod_{\mathfrak{p} \in \mathcal{P}} p^{-[\frac{n\phi_{\mathfrak{p}}}{1n_{\mathfrak{p}}}]} \prod_{v \neq v_0} u_v^{-[n\alpha_v]},$$

where in the first product, for each \mathfrak{p} the prime p is such that $v_{\mathfrak{p}}$ extends v_p on \mathbb{Q}^* , and in the second product the $u_v \in \mathcal{O}^*$ and the $\alpha_v \in \mathbb{R}$ are chosen as follows. For

$(u_v)_{v \neq v_0}$ take any basis of $\mathbb{Z}^{r_1+r_2-1}$ (seen as a subset of \mathcal{O}^*). Then the matrix $(\ln \|u_v\|_w)_{v,w \neq v_0}$ is invertible (see the proof of Theorem 1 p.72 in [16]). So one can choose $(\alpha_v)_{v \neq v_0}$ such that

$$\sum_{v \neq v_0} \alpha_v \ln \|u_v\|_w = \sum_{v \neq w} \phi_v,$$

for all $w \neq v_0$. Thus with (15) one can check that the analogue of Lemma 6.1 holds. The other parts of the proof are unchanged. We leave the details to the reader. Thus one get (with analogous notation as in the rational case):

Theorem 8.1. *Let μ be a probability measure on $A(\mathbb{K})$ such that*

$$\int_{A(\mathbb{K})} \sum_{i \leq j} \left(\sum_{\mathfrak{p} \in \mathcal{P}} |\ln \|a_{i,j}\|_{\mathfrak{p}}| + \sum_{v \in V_{\infty}} |\ln \|a_{i,j}\|_v| \right) d\mu(a) < +\infty.$$

Then there exists a unique μ -invariant probability measure ν on the space

$$\mathbf{B} := \prod_{\mathfrak{p} \in \mathcal{P}} B_{\mathfrak{p}} \prod_{v \in V_{\infty}} B_v,$$

such that the measure space (\mathbf{B}, ν) is the Poisson boundary of the random walk of law μ on $A(\mathbb{K})$.

9. APPENDIX

Proof of Lemma 6.2: Let $l \notin J$. Let $i_1 < \dots < i_r$ be the elements of I_d^l . Assume that $e_{i_1} \wedge \dots \wedge e_{i_r}$ is the k^{th} element of the basis \mathcal{B}_r (with the notation of section 4). First by definition of Z^p , we can see that $Z_{i,d}^p = \epsilon_i^p Z_k^p(d)$. Next we have seen in the proof of Lemma 4.1 that $Z_k^p(d) = \lim_{n \rightarrow \infty} (x_n^l)_{k,m}$. We will show in fact directly that for any $a \in A(\mathbb{Q})$,

$$a_{k,m}^{(r)} = \det \left(\underbrace{(a_{i,j})_{(i,j) \in I_d^l \times J}}_{:=M(l)} \right).$$

Naturally it will imply the lemma. We prove the result by induction on $h = d - l$. If $h = 1$, i.e. $l = d - 1$, then $I_d^l = \{j_1, \dots, j_{r-1}, d - 1\}$, and $a_{k,m}^{(r)} = a_{j_1, j_1} \dots a_{j_{r-1}, j_{r-1}} a_{d-1, d}$. Then the result is immediate. We prove now the induction step from h to $h + 1$. We suppose that $j_{s-1} < l < j_s$ for some s (if $s = 1$ we have just $l < j_s$). The coefficient $a_{k,m}^{(r)}$ is equal to the component on $e_{j_1} \dots \wedge e_l \dots \wedge e_{j_{r-1}}$ of $(ae_{j_1} \wedge \dots \wedge ae_{j_r})$. This component is equal to the sum over $k \in [s, \dots, r]$, of the components of $(\prod_{j < l} a_{j,j})(e_{j_1} \dots \wedge ae_{j_s} \dots \wedge a_{l, j_k} e_l \dots \wedge \widehat{ae_{j_k}} \dots \wedge ae_{j_r})$ on $e_{j_1} \dots \wedge e_{j_k} \dots \wedge e_{j_{r-1}}$. But by the induction hypothesis, for each $k \in [s, \dots, r]$, the corresponding component is equal to a_{l, j_k} times the cofactor of a_{l, j_k} in the matrix $M(l)$. This gives exactly the formula of the determinant of $M(l)$. Therefore the proof of the lemma is finished. \blacksquare

REFERENCES

- [1] **Bader U., Shalom Y.:** *Factor and normal subgroup theorems for lattices in products of groups*, Invent. Math. 163, (2006), 415–454.
- [2] **Brofferio S.:** *The Poisson Boundary of random rational affinities*, Ann. Inst. Fourier 56, (2006), 499–515.

- [3] **Cartwright D. I., Kaimanovich V. A., Woess W.:** *Random walks on the affine group of local fields and of homogeneous trees*, Ann. Inst. Fourier 44 (1994), 1243–1288.
- [4] **Derriennic Y.:** *Entropie, théorèmes limite et marches aléatoires*, in Probability measures on groups VIII (Oberwolfach, 1985), LNM 1210, pp. 241–284, Springer, Berlin, (1986).
- [5] **Elie L.:** *Noyaux potentiels associés aux marches aléatoires sur les espaces homogènes. Quelques exemples clefs dont le groupe affine*, in Théorie du potentiel (Orsay, 1983), volume 1096 of Lectures Notes in Math., 223–260, Springer, Berlin, 1984.
- [6] **Furman A.:** *Random walks on groups and random transformations*, Handbook of dynamical systems, vol. 1A, pp. 931–1014, Amsterdam: North-Holland (2002).
- [7] **Furstenberg H.:** *A Poisson formula for semi-simple Lie groups*, Ann. of Math. 77 (1963), 335–386.
- [8] **Furstenberg H.:** *Boundary theory and stochastic processes on homogeneous spaces*, in Harmonic analysis on homogeneous spaces (Proc. Sympos. Pure Math., Vol. XXVI, Williams Coll., Williamstown, Mass., 1972), p. 193–229, Amer. Math. Soc., Providence, R.I., (1973).
- [9] **Guivarc’h Y., Raugi A.:** *Frontière de Furstenberg, propriétés de contraction et théorèmes de convergence*, (French) Z. Wahrsch. Verw. Gebiete 69, (1985), 187–242.
- [10] **Kaimanovich V. A.:** *Poisson boundary of discrete groups*, a survey, unpublished manuscript.
- [11] **Kaimanovich V. A., Vershik A. M.:** *Random walks on discrete groups: boundary and entropy*, Ann. Probab. 11, (1983), 457–490.
- [12] **Lang S.:** *Introduction to diophantine approximations*, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., (1966).
- [13] **Ledrappier F.:** *Poisson boundaries of discrete groups of matrices*, Israel J. Math. 50, (1985), 319–336.
- [14] **Margulis G. A.:** *Discrete subgroups of semisimple Lie groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete 17. Springer-Verlag, Berlin, (1991), x+388 pp.
- [15] **Raugi A.:** *Fonctions harmoniques sur les groupes localement compacts à base dénombrable*, Bull. Soc. Math. France Mém. No. 54, (1977), 5–118.
- [16] **Samuel P.:** *Théorie algébrique des nombres (French)*, Hermann, Paris (1967), 130 pp.
- [17] **Serre J.P.:** *Corps locaux (French)*, Sec. edition, Publications of University Nancago, No. VIII. Hermann, Paris, (1968), 245 pp.
- [18] **Warner G.:** *Harmonic analysis on semi-simple Lie groups. I*, Die Grundlehren der mathematischen Wissenschaften, Band 188. Springer-Verlag, New York-Heidelberg, (1972), xvi+529 pp.
- [19] **Weil A.:** *Basic number theory*, Third edition, Die Grundlehren der Mathematischen Wissenschaften, Band 144. Springer-Verlag, New York-Berlin, (1974), xviii+325 pp.

Université d’Orléans, Fédération Denis Poisson, Laboratoire MAPMO
B.P. 6759, 45067 Orléans cedex 2, France.

Université Pierre et Marie Curie, Laboratoire de Probabilités et Modèles Aléatoires,
4 place Jussieu, F-75252 Paris cedex 05, France.