# A slow transient diffusion in a drifted stable potential

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**Abstract**

We consider a diffusion process $X$ in a random potential $\mathbb{V}$ of the form $\mathbb{V}_x = \mathbb{S}_x - \delta x$ where $\delta$ is a positive drift and $\mathbb{S}$ is a strictly stable process of index $\alpha \in (1,2)$ with positive jumps. Then the diffusion is transient and $X_t / \log^\alpha t$ converges in law towards an exponential distribution. This behaviour contrasts with the case where $\mathbb{V}$ is a drifted Brownian motion and provides an example of a transient diffusion in a random potential which is as "slow" as in the recurrent setting.

## 1  Introduction

Let $(\mathbb{V}(x),\ x \in \mathbb{R})$ be a two-sided stochastic process defined on some probability space $(\Omega, \mathcal{F}, \mathbf{P})$. We call a diffusion in the random potential $\mathbb{V}$ an informal solution $X$ of the S.D.E:

$$\begin{cases} dX_t = d\beta_t - \frac{1}{2}\mathbb{V}'(X_t)dt \\ X_0 = 0, \end{cases}$$

where $\beta$ is a standard Brownian motion independent of $\mathbb{V}$. Of course, the process $\mathbb{V}$ may not be differentiable (for example when $\mathbb{V}$ is a Brownian motion) and we should formally consider $X$ as a diffusion whose conditional generator given $\mathbb{V}$ is

$$\frac{1}{2}e^{\mathbb{V}(x)}\frac{d}{dx}\left(e^{-\mathbb{V}(x)}\frac{d}{dx}\right).$$

Such a diffusion may be explicitly constructed from a Brownian motion through a random change of time and a random change of scale. This class of processes has

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been widely studied for the last twenty years and bears a close connection with the model of the random walk in random environment (RWRE), see [17] and [12] for a survey on RWRE and [11], [12] for the connection between the two models.

This model exhibits many interesting features. For instance, when the potential process $\mathbb{V}$ is a Brownian motion, the diffusion $X$ is recurrent and Brox [2] proved that $X_t / \log^2 t$ converges to a non-degenerate distribution. Thus, the diffusion is much "slower" than in the trivial case $\mathbb{V} = 0$ (then $X$ is simply a Brownian motion).

We point out that Brox's theorem is the analogue of Sinai's famous theorem for RWRE [13] (see also [4] and [8]). Just as for the RWRE, this result is a consequence of a so-called "localization phenomena": the diffusion is trapped in some valleys of its potential $\mathbb{V}$. Brox's theorem may also be extended to a wider class of potentials. For instance, when $\mathbb{V}$ is a strictly stable process of index $\alpha \in (0, 2]$, Schumacher [11] proved that

$$\frac{X_t}{\log^\alpha t} \xrightarrow[t \to \infty]{\text{law}} b_\infty,$$

where $b_\infty$ is a non-degenerate random variable, whose distribution depends on the parameters of the stable process $\mathbb{V}$.

There is also much interest concerning the behaviour of $X$ in the transient case. When the potential is a drifted Brownian motion *i.e.* $\mathbb{V}_x = \mathbb{B}_x - \frac{\kappa}{2}x$ where $\mathbb{B}$ is a two-sided Brownian motion and $\kappa > 0$, then the associated diffusion $X$ is transient toward $+\infty$ and its rate of growth is polynomial and depends on $\kappa$. Precisely, Kawazu and Tanaka [7] proved that

- If $0 < \kappa < 1$, then $\frac{1}{t^\kappa} X_t$ converges in law towards a Mittag-Leffler distribution of index $\kappa$.

- If $\kappa = 1$, then $\frac{\log t}{t} X_t$ converges in probability towards $\frac{1}{4}$.

- If $\kappa > 1$, then $\frac{1}{t} X_t$ converges almost surely towards $\frac{\kappa - 1}{4}$.

In particular, when $\kappa < 1$, the rate of growth of $X$ is sub-linear. Refined results on the rates of convergence for this process were later obtained by Tanaka [16] and Hu *et al.* [6].

In fact, this behaviour is not specific to diffusions in a drifted Brownian potential. More generally, it is proved in [15] that if $\mathbb{V}$ is a two-sided Lévy process with no positive jumps and if there exists $\kappa > 0$ such $\mathbf{E}[e^{\kappa \mathbb{V}_1}] = 1$, then the rate of growth of $X_t$ is linear when $\kappa > 1$ and of order $t^\kappa$ when $0 < \kappa < 1$ (see also [3] for a law of large numbers in a general Lévy potential). These results are the analogues of those previously obtained by Kesten *et al.* [9] for the discrete model of the RWRE.

In this paper, we study the asymptotic behaviour of a diffusion in a drifted stable potential. Precisely, let $(\mathbb{S}_x,\ x \in \mathbb{R})$ denote a two-sided càdlàg stable process with index $\alpha \in (1, 2)$. By two-sided, we mean that

(a) The process $(\mathbb{S}_x,\ x \geq 0)$ is strictly stable with index $\alpha \in (1, 2)$, in particular $\mathbb{S}_0 = 0$.

(b) For all $x_0 \in \mathbb{R}$, the process $(\mathbb{S}_{x+x_0} - \mathbb{S}_{x_0},\ x \in \mathbb{R})$ has the same law as $\mathbb{S}$.

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It is well known that the Lévy measure $\Pi$ of $\mathbb{S}$ has the form

$$\Pi(dx) = \left(c^+ \mathbf{1}_{\{x>0\}} + c^- \mathbf{1}_{\{x<0\}}\right) \frac{dx}{|x|^{\alpha+1}} \tag{1}$$

where $c^+$ and $c^-$ are two non-negative constants such that $c^+ + c^- > 0$. In particular, the process $(\mathbb{S}_x, \, x \geq 0)$ has no positive jumps (resp. no negative jumps) if and only if $c^+ = 0$ (resp. $c^- = 0$). Given $\delta > 0$, we consider a diffusion $X$ in the random potential

$$\mathbb{V}_x = \mathbb{S}_x - \delta x.$$

Since the index $\alpha$ of the stable process $\mathbb{S}$ is larger than 1, we have $\mathbf{E}[\mathbb{V}_x] = -\delta x$, and therefore

$$\lim_{x \to +\infty} \mathbb{V}_x = -\infty \quad \text{and} \quad \lim_{x \to -\infty} \mathbb{V}_x = +\infty \quad \text{almost surely.}$$

This implies that the random diffusion $X$ is transient toward $+\infty$. We already mentioned that, when $\mathbb{S}$ has no positive jumps (*i.e.* $c^+ = 0$), the rate of transience of $X$ is given in [15] and $X_t$ has a polynomial growth. Thus, we now assume that $\mathbb{S}$ possesses positive jumps.

**Theorem 1.** *Assume that $c^+ > 0$, then*

$$\frac{X_t}{\log^\alpha t} \xrightarrow[t \to \infty]{\text{law}} \mathcal{E}\left(\frac{c^+}{\alpha}\right),$$

*where $\mathcal{E}(c^+/\alpha)$ denotes an exponential law with parameter $c^+/\alpha$. This result also holds with $\sup_{s \leq t} X_s$ or $\inf_{s \geq t} X_s$ in place of $X_t$.*

The asymptotic behaviour of $X$ is in this case very different from the one observed when $\mathbb{V}$ is a drifted Brownian motion. Here, the rate of growth is very slow: it is the same as in the recurrent setting. We also note that neither the rate of growth nor the limiting law depend on the value of the drift parameter $\delta$.

Theorem 1 has a simple heuristic explanation: the "localisation phenomena" for the diffusion $X$ tells us that the time needed to reach a positive level $x$ is approximatively exponentially proportional to the biggest ascending barrier of $\mathbb{V}$ on the interval $[0, x]$. In the case of a Brownian potential, or more generally a spectrally negative Lévy potential, the addition of a negative drift somehow "kills" the ascending barriers, thus accelerating the diffusion and leading to a polynomial rate of transience. However, in our setting, the biggest ascending barrier on $[0, x]$ of the stable process $\mathbb{S}$ is of the same order as its biggest jump on this interval. Since, the addition of a drift has no influence on the jumps of the potential process, the time needed to reach level $x$ still remains of the same order as in the recurrent case (*i.e.* when the drift is zero) and yields a logarithmic rate of transience.

# 2 Proof of the theorem.

## 2.1 Representation of $X$ and of its hitting times.

In the remainder of this paper, we indifferently use the notation $\mathbb{V}_x$ or $\mathbb{V}(x)$. Let us first recall the classical representation of the diffusion $X$ in the random potential $\mathbb{V}$ from a Brownian motion through a random change of scale and a random change of time (see [2] or [12] for details). Let $(B_t,\ t \geq 0)$ denote a standard Brownian motion independent of $\mathbb{V}$ and let $\sigma$ stand for its hitting times:

$$\sigma(x) \stackrel{\text{def}}{=} \inf(t \geq 0\,,\, B_t = x).$$

Define the scale function of the diffusion $X$,

$$\mathbb{A}(x) \stackrel{\text{def}}{=} \int_0^x e^{\mathbb{V}_y} dy \quad \text{for } x \in \mathbb{R}. \tag{2}$$

Since $\lim_{x \to +\infty} \mathbb{V}_x/x = -\delta$ and $\lim_{x \to -\infty} \mathbb{V}_x/x = \delta$ almost surely, it is clear that

$$\mathbb{A}(\infty) = \lim_{x \to +\infty} \mathbb{A}(x) < \infty \quad \text{and} \quad \lim_{x \to -\infty} \mathbb{A}(x) = -\infty \quad \text{almost surely.}$$

Let $\mathbb{A}^{-1} : (-\infty, \mathbb{A}(\infty)) \mapsto \mathbb{R}$ denote the inverse of $\mathbb{A}$ and define

$$\mathbb{T}(t) \stackrel{\text{def}}{=} \int_0^t e^{-2\mathbb{V}(\mathbb{A}^{-1}(B_s))} ds \quad \text{for } 0 \leq t < \sigma(\mathbb{A}(\infty)).$$

Similarly, let $\mathbb{T}^{-1}$ denote the inverse of $\mathbb{T}$. According to Brox [2] (see also [12]), the diffusion $X$ in the random potential $\mathbb{V}$ may be represented in the form

$$X_t = \mathbb{A}^{-1}\left(B_{\mathbb{T}^{-1}(t)}\right). \tag{3}$$

It is now clear that, under our assumptions, the diffusion $X$ is transient toward $+\infty$. We will study $X$ via its hitting times $H$ defined by

$$H(r) \stackrel{\text{def}}{=} \inf(t \geq 0\,,\, X_t = r) \quad \text{for } r \geq 0.$$

Let $(L(t,x),\ t \geq 0,\ x \in \mathbb{R})$ stand for the bi-continuous version of the local time process of $B$. In view of (3), we can write

$$H(r) = \mathbb{T}\left(\sigma(\mathbb{A}(r))\right) = \int_0^{\sigma(\mathbb{A}(r))} e^{-2\mathbb{V}(\mathbb{A}^{-1}(B_s))} ds = \int_{-\infty}^{\mathbb{A}(r)} e^{-2\mathbb{V}(\mathbb{A}^{-1}(x))} L(\sigma(\mathbb{A}(r)), x) dx.$$

Making use of the change of variable $x = \mathbb{A}(y)$, we get

$$H(r) = \int_{-\infty}^r e^{-\mathbb{V}_y} L(\sigma(\mathbb{A}(r)), \mathbb{A}(y)) dy = I_1(r) + I_2(r) \tag{4}$$

where

$$I_1(r) \stackrel{\text{def}}{=} \int_0^r e^{-\mathbb{V}_y} L(\sigma(\mathbb{A}(r)), \mathbb{A}(y)) dy,$$

$$I_2(r) \stackrel{\text{def}}{=} \int_{-\infty}^0 e^{-\mathbb{V}_y} L(\sigma(\mathbb{A}(r)), \mathbb{A}(y)) dy.$$

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## 2.2 Proof of Theorem 1.

Given a càdlàg process $(Z_t , t \geq 0)$, we denote by $\Delta_t Z = Z_t - Z_{t-}$ the size of the jump at time $t$. We also use the notation $Z_t^\flat$ to denote the largest positive jump of $Z$ before time $t$,

$$Z_t^\flat \stackrel{\text{def}}{=} \sup(\Delta_s, \ 0 \leq s \leq t).$$

Let $Z_t^\#$ stand for the largest ascending barrier before time $t$, namely:

$$Z_t^\# \stackrel{\text{def}}{=} \sup_{0 \leq x \leq y \leq t} (Z_y - Z_x).$$

We also define the functionals:

$$\overline{Z}_t \stackrel{\text{def}}{=} \sup_{s \in [0,t]} Z_s \qquad \text{(running unilateral maximum)}$$

$$\underline{Z}_t \stackrel{\text{def}}{=} \inf_{s \in [0,t]} Z_s \qquad \text{(running unilateral minimum)}$$

$$Z_t^* \stackrel{\text{def}}{=} \sup_{s \in [0,t]} |Z_s| \qquad \text{(running bilateral supremum)}$$

We start with a simple lemma concerning the fluctuations of the potential process.

**Lemma 1.** *There exist two constants $c_1, c_2 > 0$ such that for all $a, x > 0$*

$$\mathbf{P}\{\mathbb{V}_x^\# \leq a\} \leq e^{-c_1 \frac{x}{a^\alpha}}, \tag{5}$$

*and whenever $\frac{a}{x}$ is sufficiently large,*

$$\mathbf{P}\{\mathbb{V}_x^* > a\} \leq c_2 \frac{x}{a^\alpha}. \tag{6}$$

*Proof.* Recall that $\mathbb{V}_x = \mathbb{S}_x - \delta x$. In view of the form of the density of the Lévy measure of $\mathbb{S}$ given in (1), we get

$$\mathbf{P}\{\mathbb{V}_x^\# \leq a\} \leq \mathbf{P}\{\mathbb{V}_x^\flat \leq a\} = \exp\left(-x \int_a^\infty \frac{c^+}{y^\alpha} dy\right) = \exp\left(-\frac{c^+}{\alpha} \frac{x}{a^\alpha}\right).$$

This yields (5). From the scaling property of the stable process $\mathbb{S}$, we also have

$$\mathbf{P}\{\mathbb{V}_x^* > a\} = \mathbf{P}\left\{x^{\frac{1}{\alpha}} \sup_{t \in [0,1]} |\mathbb{S}_t - \delta x^{1-\frac{1}{\alpha}} t| > a\right\} \leq \mathbf{P}\left\{\mathbb{S}_1^* > \frac{a}{x^{\frac{1}{\alpha}}} - \delta x^{1-\frac{1}{\alpha}}\right\}.$$

Notice further that $a/x^{1/\alpha} - \delta x^{1-1/\alpha} > a/(2x^{1/\alpha})$ whenever $a/x$ is large enough, therefore, making use of a classical estimate concerning the tail distribution of the stable process $\mathbb{S}$ (*c.f.* Proposition 4, p221 of [1]), we find that

$$\mathbf{P}\{\mathbb{V}_x^* > a\} \leq \mathbf{P}\left\{\mathbb{S}_1^* > \frac{a}{2x^{\frac{1}{\alpha}}}\right\} \leq \mathbf{P}\left\{\overline{\mathbb{S}}_1 > \frac{a}{2x^{\frac{1}{\alpha}}}\right\} + \mathbf{P}\left\{\underline{\mathbb{S}}_1 > \frac{a}{2x^{\frac{1}{\alpha}}}\right\} \leq c_2 \frac{x}{a^\alpha}.$$

$\square$

**Proposition 1.** *There exists a constant $c_3 > 0$ such that, for all $r$ sufficiently large and all $x \geq 0$,*

$$\mathbf{P}\{\mathbb{V}_r^\# \geq x + \log^4 r\} - c_3 e^{-\log^2 r} \leq \mathbf{P}\{\log I_1(r) \geq x\} \leq \mathbf{P}\{\mathbb{V}_r^\# \geq x - \log^4 r\} + c_3 e^{-\log^2 r}.$$

*Proof.* This estimate was first proved by Hu and Shi (see Lemma 4.1 of [5]) when the potential process is close to a standard Brownian motion. A similar result is given in Proposition 3.2 of [14] when $\mathbb{V}$ is a random walk in the domain of attraction of a stable law. As explained by Shi [12], the key idea is the combined use of Ray-Knight's Theorem and Laplace's method. However, in our setting, additional difficulties appear since the potential process is neither flat on integer interval nor continuous. We shall therefore give a complete proof but one can still look in [5] and [14] for additional details. Recall that

$$I_1(r) = \int_0^r e^{-\mathbb{V}_y} L(\sigma(\mathbb{A}(r)), \mathbb{A}(y)) dy,$$

where $L$ is the local time of the Brownian motion $B$ (independent of $\mathbb{V}$). Let $(U(t), \ t \geq 0)$ denote a two-dimensional squared Bessel process starting from zero, also independent of $\mathbb{V}$. According to the first Ray-Knight Theorem (*c.f.* Theorem 2.2 p455 of [10]), for any $x > 0$ the process $(L(\sigma(x), x - y), \ 0 \leq y \leq x)$ has the same law as $(U(y), \ 0 \leq y \leq x)$. Therefore, making use of the scaling property of the Brownian motion and the independence of $\mathbb{V}$ and $B$, for each fixed $r > 0$, the random variable $I_1(r)$ has the same law as

$$\widetilde{I}_1(r) \stackrel{\text{def}}{=} \mathbb{A}(r) \int_0^r e^{-\mathbb{V}_y} U\left(\frac{\mathbb{A}(r) - \mathbb{A}(y)}{\mathbb{A}(r)}\right) dy.$$

We simply need to prove the proposition for $\widetilde{I}_1$ instead of $I_1$. In the rest of the proof, we assume that $r$ is very large. We start with the upper bound. Define the event

$$\mathcal{E}_1 \stackrel{\text{def}}{=} \left\{ \sup_{t \in (0,1]} \frac{U(t)}{t \log\left(\frac{8}{t}\right)} \leq r \right\}.$$

According to Lemma 6.1 of [5], $\mathbf{P}\{\mathcal{E}_1^c\} \leq c_4 e^{-r/2}$ for some constant $c_4 > 0$. On $\mathcal{E}_1$, we have

$$
\begin{aligned}
\widetilde{I}_1(r) &\leq r \int_0^r e^{-\mathbb{V}_y} (\mathbb{A}(r) - \mathbb{A}(y)) \log\left(\frac{\mathbb{A}(r)}{\mathbb{A}(r) - \mathbb{A}(y)}\right) dy \\
&= r \int_0^r \left(\int_y^r e^{\mathbb{V}_z - \mathbb{V}_y} dz\right) \log\left(\frac{\mathbb{A}(r)}{\mathbb{A}(r) - \mathbb{A}(y)}\right) dy \\
&\leq r^2 e^{\mathbb{V}_r^\#} \int_0^r \log\left(\frac{\mathbb{A}(r)}{\mathbb{A}(r) - \mathbb{A}(y)}\right) dy.
\end{aligned}
$$

Notice also that $\mathbb{A}(r) = \int_0^r e^{\mathbb{V}_z} dz \leq r e^{\overline{\mathbb{V}}_r}$ and similarly $\mathbb{A}(r) - \mathbb{A}(y) \geq (r - y) e^{\underline{\mathbb{V}}_r}$. Therefore

$$
\begin{aligned}
\int_0^r \log\left(\frac{\mathbb{A}(r)}{\mathbb{A}(r) - \mathbb{A}(y)}\right) dy &\leq r(\overline{\mathbb{V}}_r - \underline{\mathbb{V}}_r) + \int_0^r \log\left(\frac{8r}{r - y}\right) dy \\
&= r(\overline{\mathbb{V}}_r - \underline{\mathbb{V}}_r + 1 + \log 8).
\end{aligned}
$$

Define the set $\mathcal{E}_2 \stackrel{\text{def}}{=} \{\overline{\mathbb{V}}_r - \underline{\mathbb{V}}_r \le e^{\log^3 r}\}$. In view of Lemma 1,

$$\mathbf{P}\{\mathcal{E}_2^c\} \le \mathbf{P}\left\{\mathbb{V}_r^* > \frac{1}{2}e^{\log^3 r}\right\} \le e^{-\log^2 r}.$$

Therefore, $\mathbf{P}\{(\mathcal{E}_1 \cap \mathcal{E}_2)^c\} \le 2e^{-\log^2 r}$ and on $\mathcal{E}_1 \cap \mathcal{E}_2$,

$$\widetilde{I}_1(r) \le r^3(e^{\log^3 r} + 1 + \log 8)e^{\mathbb{V}_r^\#} \le e^{\log^4 r + \mathbb{V}_r^\#}.$$

This completes the proof of the upper bound. We now deal with the lower bound. Define the sequence $(\gamma_k, \ k \ge 0)$ by induction

$$\begin{cases} \gamma_0 & \stackrel{\text{def}}{=} \ 0, \\ \gamma_{k+1} & \stackrel{\text{def}}{=} \ \inf(t > \gamma_n, \ |\mathbb{V}_t - \mathbb{V}_{\gamma_k}| \ge 1). \end{cases}$$

The sequence $(\gamma_{k+1} - \gamma_k, \ k \ge 0)$ is $i.i.d.$ and distributed as $\gamma_1 = \inf(t > 0, \ |\mathbb{V}_t| \ge 1)$. We denote by $[x]$ the integer part of $x$. We also use the notation $\epsilon \stackrel{\text{def}}{=} e^{-\log^3 r}$. Consider the following events

$$\begin{aligned} \mathcal{E}_3 & \stackrel{\text{def}}{=} \ \left\{\gamma_{[r^2]} > r\right\}, \\ \mathcal{E}_4 & \stackrel{\text{def}}{=} \ \left\{\gamma_k - \gamma_{k-1} \ge 2\epsilon \ \text{ for all } k = 1, 2 \ldots, [r^2]\right\}. \end{aligned}$$

In view of Cramer's large deviation Theorem and since $r$ is very large, we get that $\mathbf{P}\{\mathcal{E}_3^c\} \le e^{-r}$. We also have

$$\begin{aligned} \mathbf{P}\{\mathcal{E}_4^c\} \le \sum_{k=1}^{[r^2]} \mathbf{P}\{\gamma_k - \gamma_{k-1} < 2\epsilon\} & \le \ [r^2]\mathbf{P}\{\gamma_1 < 2\epsilon\} \\ & \le \ [r^2]\mathbf{P}\{\mathbb{V}_{2\epsilon}^* \ge 1\} \\ & \le \ e^{-\log^2 r}, \end{aligned}$$

where we used Lemma 1 for the last inequality. Define also

$$\mathcal{E}_5 \stackrel{\text{def}}{=} \{|\mathbb{V}_x - \mathbb{V}_r| < 1 \ \text{ for all } x \in [r - 2\epsilon, r]\}.$$

From time reversal, the processes $(\mathbb{V}_t, \ 0 \le t \le 2\epsilon)$ and $(\mathbb{V}_r - \mathbb{V}_{(r-t)^-}, 0 \le t \le 2\epsilon)$ have the same law. Thus,

$$\mathbf{P}\{\mathcal{E}_5^c\} \le \mathbf{P}\{\mathbb{V}_{2\epsilon}^* \ge 1\} \le e^{-\log^2 r}.$$

Setting $\mathcal{E}_6 \stackrel{\text{def}}{=} \mathcal{E}_3 \cap \mathcal{E}_4 \cap \mathcal{E}_5$, we get $\mathbf{P}\{\mathcal{E}_6^c\} \le 3e^{-\log^2 r}$. Moreover, it is easy to check (see figure 1) that on $\mathcal{E}_6$, we can always find $x_-, x_+$ such that:

$$\begin{cases} 0 \le x_- \le x_+ \le r - 2\epsilon, \\ \text{for any } a \in [x_-, x_- + \epsilon], \ |\mathbb{V}_{x_-} - \mathbb{V}_a| \le 2, \\ \text{for any } b \in [x_+, x_+ + \epsilon], \ |\mathbb{V}_{x_+} - \mathbb{V}_b| \le 2, \\ \mathbb{V}_{x_+} - \mathbb{V}_{x_-} \ge \mathbb{V}_r^\# - 4. \end{cases}$$
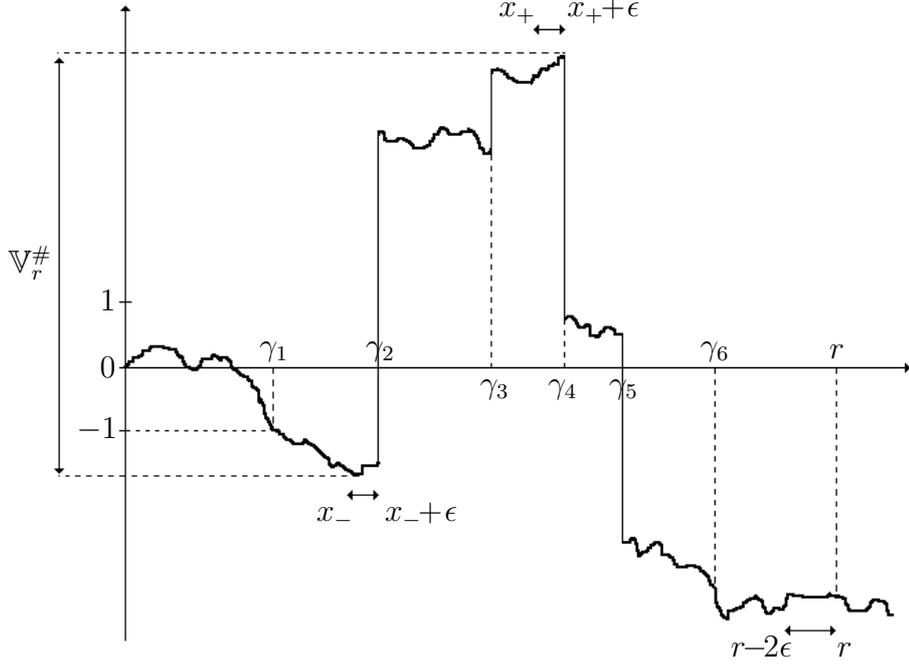
Figure 1: Sample path of $\mathbb{V}$ on $\mathcal{E}_6$.

Let us also define

$$\mathcal{E}_7 \stackrel{\text{def}}{=} \mathcal{E}_6 \cap \left\{ \inf_{y \in [x_-, x_-+\epsilon]} U\left( \frac{\mathbb{A}(r) - \mathbb{A}(y)}{\mathbb{A}(r)} \right) \geq \frac{\mathbb{A}(r) - \mathbb{A}(x_-)}{\mathbb{A}(r)} e^{-2\log^2 r} \right\},$$

$$\mathcal{E}_8 \stackrel{\text{def}}{=} \left\{ \mathbb{V}_r^\flat \geq 3\log^2 r \right\}.$$

We finally set $\mathcal{E}_9 \stackrel{\text{def}}{=} \mathcal{E}_7 \cap \mathcal{E}_8$. Then on $\mathcal{E}_9$, we have, for all $r$ large enough,

$$\begin{aligned}
\widetilde{I}_1(r) &\geq \mathbb{A}(r) \int_{x_-}^{x_-+\epsilon} e^{-\mathbb{V}_y} U\left( \frac{\mathbb{A}(r) - \mathbb{A}(y)}{\mathbb{A}(r)} \right) dy \\
&\geq e^{-\mathbb{V}_{x_-} - 2 - 2\log^2 r} \int_{x_-}^{x_-+\epsilon} (\mathbb{A}(r) - \mathbb{A}(x_-)) dy \\
&= e^{-\mathbb{V}_{x_-} - 2 - 2\log^2 r - \log^3 r} \int_{x_-}^{r} e^{\mathbb{V}_y} dy \\
&\geq e^{-\mathbb{V}_{x_-} - 2 - 2\log^2 r - \log^3 r} \int_{x_+}^{x_+ + \epsilon} e^{\mathbb{V}_y} dy \\
&\geq e^{\mathbb{V}_{x_+} - \mathbb{V}_{x_-} - 4 - 2\log^2 r - 2\log^3 r} \\
&\geq e^{\mathbb{V}_r^\# - \log^4 r}.
\end{aligned}$$

This proves the lower bound on $\mathcal{E}_9$. It simply remains to show that $\mathbf{P}\{\mathcal{E}_9^c\} \leq$

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$c_5 e^{-\log^2 r}$. According to Lemma 6.1 of [5], for any $0 < a < b$ and any $\eta > 0$, we have

$$\mathbf{P}\left\{ \inf_{a < t < b} U(t) \le \eta b \right\} \le 2\sqrt{\eta} + 2\exp\left( -\frac{\eta}{2(1 - a/b)} \right).$$

Therefore, making use of the independence of $\mathbb{V}$ and $U$, we find

$$\begin{aligned}
\mathbf{P}\{\mathcal{E}_9^c\} &\le \mathbf{P}\{\mathcal{E}_6^c\} + \mathbf{P}\{\mathcal{E}_8^c\} + \mathbf{P}\{\mathcal{E}_7^c \cap \mathcal{E}_6 \cap \mathcal{E}_8\} \\
&\le \mathbf{P}\{\mathcal{E}_6^c\} + \mathbf{P}\{\mathcal{E}_8^c\} + 2e^{-\log^2 r} + 2\mathbf{E}\left[ e^{\frac{1}{2}\mathbb{J}(r)e^{-2\log^2 r}} \mathbf{1}_{\mathcal{E}_6 \cap \mathcal{E}_8} \right],
\end{aligned}$$

where

$$\mathbb{J}(r) \stackrel{\text{def}}{=} \frac{\mathbb{A}(r) - \mathbb{A}(x_-)}{\mathbb{A}(x_- + \epsilon) - \mathbb{A}(x_-)}.$$

We have already proved that $\mathbf{P}\{\mathcal{E}_6^c\} \le 3e^{-\log^2 r}$. Using Lemma 1, we also check that $\mathbf{P}\{\mathcal{E}_8^c\} \le e^{-\log^2 r}$. Thus, it remains to show that

$$\mathbf{E}\left[ e^{\frac{1}{2}\mathbb{J}(r)e^{-2\log^2 r}} \mathbf{1}_{\mathcal{E}_6 \cap \mathcal{E}_8} \right] \le c_6 e^{-\log^2 r}. \tag{7}$$

Notice that, on $\mathcal{E}_6$,

$$\mathbb{A}(r) - \mathbb{A}(x_-) = \int_{x_-}^r e^{\mathbb{V}_y} dy \ge \int_{x_+}^{x_+ + \epsilon} e^{\mathbb{V}_y} dy \ge e^{\log^3 r + \mathbb{V}_{x_+} - 2},$$

and also

$$\mathbb{A}(x_- + \epsilon) - \mathbb{A}(x_-) = \int_{x_-}^{x_- + \epsilon} e^{\mathbb{V}_y} dy \le e^{\log^3 r + \mathbb{V}_{x_-} + 2}.$$

Therefore, on $\mathcal{E}_6 \cap \mathcal{E}_8$,

$$\mathbb{J}(r) \ge e^{\mathbb{V}_{x_+} - \mathbb{V}_{x_-} - 4} \ge e^{\mathbb{V}_r^\# - 8} \ge e^{\mathbb{V}_r^\flat - 8} \ge e^{3\log^2 r - 8}$$

which clearly yields (7) and the proof of the proposition is complete. $\square$

**Lemma 2.** *We have*

$$\frac{\mathbb{V}_r^\#}{r^{1/\alpha}} \xrightarrow[r \to \infty]{law} \mathbb{S}_1^\flat.$$

*Proof.* Let $f : [0,1] \mapsto \mathbb{R}$ be a deterministic càdlàg function. For $\lambda \ge 0$, define

$$f_\lambda(x) \stackrel{\text{def}}{=} f(x) - \lambda x.$$

We first show that

$$\lim_{\lambda \to \infty} f_\lambda^\#(1) = f^\flat(1). \tag{8}$$

It is clear that $f^\flat(1) = f_\lambda^\flat(1) \le f_\lambda^\#(1)$ for any $\lambda > 0$. Thus, we simply need to prove that $\limsup f_\lambda^\#(1) \le f^\flat(1)$. Let $\eta > 0$ and set

$$\begin{aligned}
A(\eta, \lambda) &\stackrel{\text{def}}{=} \sup\left( f_\lambda(y) - f_\lambda(x) ,\ 0 \le x \le y \le 1 \text{ and } y - x \le \eta \right), \\
B(\eta, \lambda) &\stackrel{\text{def}}{=} \sup\left( f_\lambda(y) - f_\lambda(x) ,\ 0 \le x \le y \le 1 \text{ and } y - x > \eta \right),
\end{aligned}$$

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so that

$$f_\lambda^\#(1) = \max(A(\eta, \lambda), B(\eta, \lambda)). \tag{9}$$

Notice that $A(\eta, \lambda) \leq A(\eta)$ where

$$A(\eta) \stackrel{\text{def}}{=} A(\eta, 0) = \sup\left(f(y) - f(x) \ , \ 0 \leq x \leq y \leq 1 \text{ and } y - x \leq \eta\right).$$

Since $f$ is càdlàg, we have $\lim_{\eta \to 0} A(\eta) = f^\natural(1)$. Thus, for any $\varepsilon > 0$, we can find $\eta_0 > 0$ small enough such that

$$\limsup_{\lambda \to \infty} A(\eta_0, \lambda) \leq f^\natural(1) + \varepsilon. \tag{10}$$

Notice also that

$$\begin{aligned}
B(\eta_0, \lambda) &\leq \sup\left(f(y) - f(x) - \eta_0\lambda \ , \ 0 \leq x \leq y \leq 1 \text{ and } y - x > \eta_0\right) \\
&\leq f^\#(1) - \eta_0\lambda
\end{aligned}$$

which implies

$$\lim_{\lambda \to \infty} B(\eta_0, \lambda) = -\infty. \tag{11}$$

The combination of (9), (10) and (11) yield (8). Making use of the scaling property of the stable process $\mathbb{S}$, for any fixed $r > 0$,

$$(\mathbb{V}_y \ , \ 0 \leq y \leq r) \stackrel{\text{law}}{=} (r^{1/\alpha}\mathbb{S}_y - \delta r y \ , \ 0 \leq y \leq 1).$$

Therefore, setting $\mathbb{R}(z) = (\mathbb{S}. - z\cdot)_1^\#$, we get the equality in law:

$$\frac{\mathbb{V}_r^\#}{r^{1/\alpha}} \stackrel{\text{law}}{=} \mathbb{R}(\delta r^{1-1/\alpha}). \tag{12}$$

Making use of (8), we see that $\mathbb{R}(z)$ converges almost surely towards $\mathbb{S}_1^\natural$ as $z$ goes to infinity. Since $\alpha > 1$ and $\delta > 0$, we also have $\delta r^{1-1/\alpha} \to \infty$ as $r$ goes to infinity and we conclude from (12) that

$$\frac{\mathbb{V}_r^\#}{r^{1/\alpha}} \xrightarrow[r \to \infty]{\text{law}} \mathbb{S}_1^\natural.$$

$\square$

*Proof of Theorem 1.* Recall that the random variable $\mathbb{S}_1^\natural$ denotes the largest positive jump of $\mathbb{S}$ over the interval $[0, 1]$. Thus, according to the density of the Lévy measure of $\mathbb{S}$,

$$\mathbf{P}\{\mathbb{S}_1^\natural \leq x\} = \exp\left(-\int_x^\infty \frac{c^+}{y^{\alpha+1}}dy\right) = \exp\left(-\frac{c^+}{\alpha y^\alpha}\right). \tag{13}$$

On the one hand, the combination of Lemma 1 and 2 readily shows that

$$\frac{\log(I_1(r))}{r^{1/\alpha}} \xrightarrow[r \to \infty]{\text{law}} \mathbb{S}_1^\natural. \tag{14}$$

On the other hand, the random variables $\mathbb{A}(\infty) = \lim_{x \to \infty} \mathbb{A}(x)$ and $\int_{-\infty}^0 e^{-\mathbb{V}_y}dy$ have the same law. Moreover, we already noticed that these random variables are

almost surely finite. Since the function $L(t, \cdot)$ is, for any fixed $t$, continuous with compact support, we get

$$I_2(r) = \int_{-\infty}^0 e^{-\mathbb{V}_y} L(\sigma(\mathbb{A}(r)), \mathbb{A}(y)) dy \leq \sup_{z \in (-\infty, 0]} L(\sigma(\mathbb{A}(\infty)), z) \int_{-\infty}^0 e^{-\mathbb{V}_y} dy < \infty.$$

Therefore,

$$\sup_{r \geq 0} I_2(r) < \infty \quad \text{almost surely.} \tag{15}$$

Combining (4), (14) and (15), we deduce that

$$\frac{\log(H(r))}{r^{1/\alpha}} \xrightarrow[r \to \infty]{\text{law}} \mathbb{S}_1^\natural$$

which, from the definition of the hitting times $H$, yields

$$\frac{\sup_{s \leq t} X_s}{\log^\alpha t} \xrightarrow[t \to \infty]{\text{law}} \left(\frac{1}{\mathbb{S}_1^\natural}\right)^\alpha.$$

According to (13), the random variable $(1/\mathbb{S}_1^\natural)^\alpha$ has an exponential distribution with parameter $c^+/\alpha$ so the proof of the theorem for $\sup_{s \leq t} X_s$ is complete. We finally use the classical argument given by Kawazu and Tanaka, p201 [7] to obtain the corresponding results for $X_t$ and $\inf_{s \geq t} X_s$. $\qquad \square$

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