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\mathcal{H}_∞ Filtering for a class of nonlinear stochastic systems

S. Halabi, H. Rafaralahy, M. Zasadzinski and H. Souley Ali

Abstract— In this paper, the purpose is to design an \mathcal{H}_∞ filter for a class of nonlinear stochastic systems such that the estimation error is exponentially mean-square stable with a prescribed \mathcal{H}_∞ norm criterion. The system is nonlinear and has multiplicative noises in both the state and measurement equations. Using a change of coordinates on the control input the problem is transformed into a robust stochastic filtering one. It is then solved via an LMI formulation.

Keywords— \mathcal{H}_∞ filter, stochastic systems, nonlinear systems, multiplicative noise, Lyapunov stochastic function, Itô formula.

INTRODUCTION

The nonlinear system is the more realistic representation of physical systems, so it is a good approach to study physical processes. The stochastic systems get a great importance during the last decades as shown by numerous references [1], [2], [3], [4], [5], [6], [7], [8].

Generally, the stochastic bilinear system designs a stochastic system with multiplicative noise instead of additive one [1], [2], [3]. The \mathcal{H}_∞ filtering for systems with multiplicative noise has been treated in many papers [1], [9], [10], [11].

The \mathcal{H}_∞ filtering has the advantage that no statistical assumption apart of boundedness on the exogenous signals is needed, which makes this technique useful in many practical applications. Some results on the \mathcal{H}_∞ filter design have been proposed in [12], [13], [14] and [15]. The \mathcal{H}_∞ filtering for continuous time linear stochastic systems can be found in [1].

In this paper, we consider a class of nonlinear stochastic systems with multiplicative noise and multiplicative control input. The measurements are subjected to multiplicative noise too. The nonlinearity considered is of the form $u_1(t)x(t)dw_1(t)$, i.e. the state $x(t)$ is multiplied simultaneously by the control input $u_1(t)$ and the noise $dw_1(t)$. Notice that, as in deterministic case, the nonlinearity in the control input can affect the observability of the system. The purpose is to design an \mathcal{H}_∞ filter for such a system. Using a change of variable [16] on the control input we transform the nonlinear stochastic system to a stochastic system with parameter uncertainty. The problem becomes to find an \mathcal{H}_∞ filter for the new stochastic uncertain system. Then the Itô formula and LMI method are used to find the filter to ensure the stability of the error.

Notations. Throughout the paper, \mathbf{E} represents expectation operator with respect to some probability measure \mathcal{P} . $\langle X, Y \rangle = X^T Y$ represents the inner product of the vectors $X, Y \in \mathbb{R}^n$. $\text{herm}(A)$ stands for $A + A^T$. $L_2(\Omega, \mathbb{R}^k)$ is the space of square-integrable \mathbb{R}^k -valued functions on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$ where Ω is the sample space, \mathcal{F} is a σ -algebra of subsets of the sample space called events

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and \mathcal{P} is the probability measure on \mathcal{F} . $(\mathcal{F}_t)_{t \geq 0}$ denote an increasing family of σ -algebras $(\mathcal{F}_t) \in \mathcal{F}$. We also denote by $\widehat{L}_2([0, \infty); \mathbb{R}^k)$ the space of non-anticipatory square-integrable stochastic process $f(\cdot) = (f(t))_{t \in [0, \infty)}$ in \mathbb{R}^k with respect to $(\mathcal{F}_t)_{t \in [0, \infty)}$ satisfying

$$\|f\|_{\widehat{L}_2}^2 = \mathbf{E} \left(\int_0^\infty \|f(t)\|^2 dt \right) < \infty$$

where $\|\cdot\|$ is the well-known Euclidean norm.

I. PROBLEM STATEMENT

Let us consider the following nonlinear stochastic system

$$\begin{cases} dx(t) &= (A_{t0}x(t) + u_1(t)A_{t1}x(t) + B_0v(t)) dt \\ &\quad + A_{w0}x(t) dw_0(t) + u_1(t)A_{w1}x(t) dw_1(t) \\ dy(t) &= Cx(t) dt + J_1x(t) dw_2(t) \end{cases} \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $y(t) \in \mathbb{R}^p$ is the output, $u_1(t) \in \mathbb{R}$ is the control input and $v(t) \in \mathbb{R}^q$ is the perturbation signal. The system is affected by two multiplicative noises in the state equation and by one multiplicative noise in the measurement equation. $w_i(t)$ for $i = 0, 1, 2$ are zero mean scalar Wiener processes verifying [4]

$$\mathbf{E}(dw_i(t)) = 0, \quad \mathbf{E}(dw_i(t)^2) = dt \text{ for } i = 0 \dots 2, \quad (2a)$$

$$\mathbf{E}(dw_0(t) dw_1(t)) = \mathbf{E}(dw_1(t) dw_0(t)) = \alpha_1 dt, \quad (2b)$$

$$\mathbf{E}(dw_0(t) dw_2(t)) = \mathbf{E}(dw_2(t) dw_0(t)) = \alpha_2 dt, \quad (2c)$$

$$\mathbf{E}(dw_1(t) dw_2(t)) = \mathbf{E}(dw_2(t) dw_1(t)) = \alpha_3 dt, \quad (2d)$$

$$\text{with } |\alpha_i| < 1 \text{ for } i = 1 \dots 3. \quad (2e)$$

As in the most cases for physical processes, we assume that the nonlinear stochastic system (1) has known bounded control input, i.e. $u_1(t) \in \Gamma \subset \mathbb{R}$, where

$$\Gamma = \{u_1(t) \in \mathbb{R} \mid u_{1\min} \leq u_1(t) \leq u_{1\max}\}. \quad (3)$$

In this paper, the aim is to design a stochastic filter in the following form

$$\begin{aligned} d\widehat{x}(t) &= (A_{t0}\widehat{x}(t) + u_1(t)A_{t1}\widehat{x}(t)) dt \\ &\quad + K_0(dy(t) - C\widehat{x}(t)dt) \\ &\quad + u_1(t)K_1(dy(t) - C\widehat{x}(t)dt) \end{aligned} \quad (4)$$

where K_0 and K_1 are the gain matrices to be designed.

Before formulating the problem, let us introduce the following definition and assumption.

Definition 1. [4], [6] *The stochastic system (1) with $v(t) \equiv 0$ is said to be exponentially mean-square stable if all initial states $x(0)$ yield*

$$\lim_{t \rightarrow \infty} \mathbf{E} \left(\|x(t)\|^2 \right) = 0, \quad \forall u_1(t) \in \Gamma. \quad (5)$$

Assumption 1. *The nonlinear stochastic system (1) is assumed to be exponentially mean-square stable.*

Define the estimation error as $e(t) = x(t) - \hat{x}(t)$, then we can state the problem as follows.

Problem 1. *Given a real $\gamma > 0$, the goal is to design an \mathcal{H}_∞ filter of the form of (4) such that the estimation error $e(t) = x(t) - \hat{x}(t)$ is exponentially mean-square stable and the following \mathcal{H}_∞ performance*

$$\|e\|_{\tilde{L}_2}^2 \leq \gamma \|v\|_{\tilde{L}_2}^2 \quad (6)$$

is achieved from the disturbance $v(t)$ to the estimation error $e(t)$.

The estimation error dynamics is given by

$$\begin{aligned} d e(t) &= (A_{t0} - K_0 C + u_1(t)(A_{t1} - K_1 C)) e(t) dt \\ &+ B_0 v(t) dt + A_{w0} x(t) d w_0(t) + u_1(t) A_{w1} x(t) d w_1(t) \\ &- (K_0 + u_1(t) K_1) J_1 x(t) d w_2(t). \end{aligned} \quad (7)$$

Now, let us introduce the following transformation matrix T_b ($|\alpha_3| \neq 1$, see (2e)) which is used to decorrelate the process $w_i(t)$ for $i = 0 \dots 2$ in system (1)

$$T_b = \begin{bmatrix} 1 & \frac{\alpha_1 - \alpha_2 \alpha_3}{1 - \alpha_3^2} & \alpha_2 \\ 0 & 1 & \alpha_3 \\ 0 & 0 & 1 \end{bmatrix}. \quad (8)$$

Since T_b is invertible, the state equation of (1) is equivalent to

$$\begin{aligned} d x(t) &= (A_{t0} x(t) + u_1(t) A_{t1} x(t) + B_0 v(t)) dt \\ &+ \begin{bmatrix} A_{w0} x(t) & u_1(t) A_{w1} x(t) & 0 \end{bmatrix} T_b \times T_b^{-1} \begin{bmatrix} d w_0(t) \\ d w_1(t) \\ d w_2(t) \end{bmatrix}, \end{aligned} \quad (9)$$

which can be rewritten as

$$\begin{aligned} d x(t) &= (A_{t0} x(t) + u_1(t) A_{t1} x(t) + B_0 v(t)) dt \\ &+ \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} d \bar{w}_0(t) \\ d \bar{w}_1(t) \\ d w_2(t) \end{bmatrix}, \end{aligned} \quad (10)$$

where

$$a = A_{w0} x(t), \quad (11a)$$

$$b = \left(\frac{\alpha_1 - \alpha_2 \alpha_3}{1 - \alpha_3^2} A_{w0} + u_1(t) A_{w1} \right) x(t), \quad (11b)$$

$$c = (\alpha_2 A_{w0} + \alpha_3 u_1(t) A_{w1}) x(t), \quad (11c)$$

$$\begin{aligned} d \bar{w}_0(t) &= d w_0(t) + \frac{\alpha_2 \alpha_3 - \alpha_1}{1 - \alpha_3^2} d w_1(t) \\ &+ \frac{\alpha_1 \alpha_3 - \alpha_2}{1 - \alpha_3^2} d w_2(t), \end{aligned} \quad (11d)$$

$$d \bar{w}_1(t) = d w_1(t) - \alpha_3 d w_2(t), \quad (11e)$$

and the error dynamics (7) becomes

$$\begin{aligned} d e(t) &= (A_{t0} - K_0 C + u_1(t)(A_{t1} - K_1 C)) e(t) dt \\ &+ B_0 v(t) dt + \begin{bmatrix} a & b & d \end{bmatrix} \begin{bmatrix} d \bar{w}_0(t) \\ d \bar{w}_1(t) \\ d w_2(t) \end{bmatrix}. \end{aligned} \quad (12)$$

where $a, b, d \bar{w}_0(t)$ and $d \bar{w}_1(t)$ are given by (11) and d is given by

$$d = (\alpha_2 A_{w0} + \alpha_3 u_1(t) A_{w1} - (K_0 + u_1(t) K_1) J_1) x(t). \quad (13)$$

The processes $\bar{w}_0(t), \bar{w}_1(t)$ and $w_2(t)$ verify

$$\mathbf{E}(d \bar{w}_i(t)) = 0, \quad \text{for } i = 0, 1, \mathbf{E}(d w_2(t)) = 0, \quad (14a)$$

$$\mathbf{E}(d \bar{w}_0(t) d \bar{w}_1(t)) = \mathbf{E}(d \bar{w}_1(t) d \bar{w}_0(t)) = 0, \quad (14b)$$

$$\mathbf{E}(d \bar{w}_0(t) d w_2(t)) = \mathbf{E}(d w_2(t) d \bar{w}_0(t)) = 0, \quad (14c)$$

$$\mathbf{E}(d \bar{w}_1(t) d w_2(t)) = \mathbf{E}(d w_2(t) d \bar{w}_1(t)) = 0, \quad (14d)$$

$$\mathbf{E}(d \bar{w}_0(t)^2) = \bar{\alpha} dt, \quad (14e)$$

$$\mathbf{E}(d \bar{w}_1(t)^2) = \alpha dt, \quad (14f)$$

$$\mathbf{E}(d w_2(t)^2) = dt, \quad (14g)$$

with

$$\begin{aligned} \bar{\alpha} &= 1 - \frac{2(\alpha_1 + \alpha_2)}{1 + \alpha_3} + \frac{(1 + \alpha_3^2)(\alpha_1^2 + \alpha_2^2) - 4\alpha_1\alpha_2\alpha_3}{(1 - \alpha_3^2)^2}, \\ \alpha &= (1 - 3\alpha_3^2). \end{aligned}$$

Let us consider the following augmented state vector

$$\xi(t) = \begin{bmatrix} x(t) \\ e(t) \end{bmatrix}. \quad (15)$$

The dynamics of the augmented system is given by

$$\begin{aligned} d \xi(t) &= ((\mathcal{A}_{t0} + u_1(t) \mathcal{A}_{t1}) \xi(t) + \mathcal{B}_0 v(t)) dt \\ &+ \mathcal{A}_{w0} \xi(t) d \bar{w}_0(t) + (\mathcal{A}_{w10} + u_1(t) \mathcal{A}_{w11}) \xi(t) d \bar{w}_1(t) \\ &+ (\mathcal{A}_{w20} + u_1(t) \mathcal{A}_{w21}) \xi(t) d w_2(t) \end{aligned} \quad (16)$$

with

$$\begin{aligned} \mathcal{A}_{ti} &= \begin{bmatrix} A_{ti} & 0 \\ 0 & A_{ti} - K_i C \end{bmatrix}, \quad \text{for } i = 0, 1, \\ \mathcal{B}_0 &= \begin{bmatrix} B_0 \\ B_0 \end{bmatrix}, \quad \mathcal{A}_{w0} = \begin{bmatrix} A_{w0} & 0 \\ A_{w0} & 0 \end{bmatrix}, \\ \mathcal{A}_{w10} &= \begin{bmatrix} \frac{\alpha_1 - \alpha_2 \alpha_3}{1 - \alpha_3^2} A_{w0} & 0 \\ \frac{\alpha_1 - \alpha_2 \alpha_3}{1 - \alpha_3^2} A_{w0} & 0 \end{bmatrix}, \quad \mathcal{A}_{w11} = \begin{bmatrix} A_{w1} & 0 \\ A_{w1} & 0 \end{bmatrix}, \\ \mathcal{A}_{w20} &= \begin{bmatrix} \alpha_2 A_{w0} & 0 \\ \alpha_2 A_{w0} - K_0 J_1 & 0 \end{bmatrix}, \\ \mathcal{A}_{w21} &= \begin{bmatrix} \alpha_3 A_{w1} & 0 \\ \alpha_3 A_{w1} - K_1 J_1 & 0 \end{bmatrix}. \end{aligned}$$

As in [16], let us introduce a change of variable on the control $u_1(t)$ as follows

$$u_1(t) = \varphi + \sigma \varepsilon(t), \quad (17)$$

where $\varphi \in \mathbb{R}$ and $\sigma \in \mathbb{R}$ are given by

$$\varphi = \frac{1}{2}(u_{1\min} + u_{1\max}), \sigma = \frac{1}{2}(u_{1\max} - u_{1\min}). \quad (18)$$

The new ‘‘uncertain’’ variable is $\varepsilon(t) \in \bar{\Gamma} \subset \mathbb{R}$ where $\bar{\Gamma}$ is defined by

$$\bar{\Gamma} = \{\varepsilon(t) \in \mathbb{R} \mid \varepsilon_{\min} = -1 \leq \varepsilon(t) \leq \varepsilon_{\max} = 1\}. \quad (19)$$

So the augmented system (16) becomes

$$\begin{aligned} d\xi(t) = & \left(\left(\hat{A}_{t0} + \Delta \hat{A}_{t0}(t) \right) \xi(t) + \hat{B}_0 v(t) \right) dt \\ & + \hat{A}_{w0} d\bar{w}_0(t) + \left(\hat{A}_{w1} + \Delta \hat{A}_{w1}(t) \right) \xi(t) d\bar{w}_1(t) \\ & + \left(\hat{A}_{w2} + \Delta \hat{A}_{w2}(t) \right) \xi(t) d w_2(t) \end{aligned} \quad (20)$$

where

$$\begin{aligned} \hat{A}_{t0} &= \mathcal{A}_{t0} + \varphi \mathcal{A}_{t1}, & \Delta \hat{A} &= H_1 \Delta_\xi(\varepsilon(t)) H_t, \\ \hat{B} &= \mathcal{B}, & \hat{A}_{w0} &= \mathcal{A}_{w0}, \\ \hat{A}_{w1} &= \mathcal{A}_{w10} + \varphi \mathcal{A}_{w11}, & \Delta \hat{A}_{w1} &= H_2 \Delta_\xi(\varepsilon(t)) H_w, \\ \hat{A}_{w2} &= \mathcal{A}_{w20} + \varphi \mathcal{A}_{w21}, & \Delta \hat{A}_{w2} &= H_3 \Delta_\xi(\varepsilon(t)) H_w, \end{aligned}$$

with

$$\begin{aligned} H_1 &= \sigma \mathcal{A}_{t1}, & H_t &= I_{2n}, \\ H_2 &= \begin{bmatrix} \sigma \mathcal{A}_{w1} \\ \sigma \mathcal{A}_{w1} \end{bmatrix}, & H_w &= \begin{bmatrix} I_n & 0 \end{bmatrix}, \\ H_3 &= \begin{bmatrix} \sigma \alpha_3 \mathcal{A}_{w1} \\ \sigma \alpha_3 \mathcal{A}_{w1} - \sigma K_1 J_1 \end{bmatrix}, & \Delta_\xi(\varepsilon(t)) &= \varepsilon(t). \end{aligned}$$

Notice that $\Delta_\xi(\varepsilon(t))$ satisfies

$$\|\Delta_\xi(\varepsilon(t))\| \leq 1. \quad (21)$$

II. ANALYSIS OF THE EXPONENTIAL MEAN-SQUARE STABILITY OF THE AUGMENTED SYSTEM

In this part, the conditions to ensure the exponential mean-square stability of system (20) with $v(t) \equiv 0$ are derived in terms of LMI. For this purpose, consider the following Lyapunov function candidate

$$V(\xi(t)) = \xi^T(t) \mathcal{P} \xi(t) \quad \text{with } \mathcal{P} = \mathcal{P}^T > 0. \quad (22)$$

Using the Itô formula [4], [5], [6], [7], [8] and the majoration lemma [17], the following lemma is given to ensure the exponential mean-square stability of the augmented system (20)

Lemma 1. *The system (20) is exponentially mean-square stable if there exists a matrix $\mathcal{P} = \mathcal{P}^T > 0$ and some given positive real μ_1, μ_2 and μ_3 , such that*

$$\begin{bmatrix} (1,1) & \mathcal{P} H_1 & \alpha^{\frac{1}{2}} \hat{A}_{w0}^T \mathcal{P} & \alpha^{\frac{1}{2}} \hat{A}_{w1}^T \mathcal{P} & 0 & \hat{A}_{w2}^T \mathcal{P} & 0 \\ H_1^T \mathcal{P} & -\mu_1 I_{2n} & 0 & 0 & 0 & 0 & 0 \\ \alpha^{\frac{1}{2}} \hat{A}_{w0}^T \mathcal{P} & 0 & -\mathcal{P} & 0 & 0 & 0 & 0 \\ \alpha^{\frac{1}{2}} \hat{A}_{w1}^T \mathcal{P} & 0 & 0 & -\mathcal{P} & \mathcal{P} H_2 & 0 & 0 \\ 0 & 0 & 0 & H_2^T \mathcal{P} & -\mu_2 I_{2n} & 0 & 0 \\ \hat{A}_{w2}^T \mathcal{P} & 0 & 0 & 0 & 0 & -\mathcal{P} & \mathcal{P} H_3 \\ 0 & 0 & 0 & 0 & 0 & H_3^T \mathcal{P} & -\mu_3 I_{2n} \end{bmatrix} < 0, \quad (23)$$

where

$$(1,1) = \mathcal{P} \hat{A}_{t0} + \hat{A}_{t0}^T \mathcal{P} + \mu_1 H_t^T H_t + (\alpha \mu_2 + \mu_3) H_w^T H_w$$

Proof. Consider the Lyapunov function candidate (22), and applying Itô formula to the system (20), we get

$$dV(\xi(t)) = \mathcal{L}V(\xi(t)) dt + 2\xi^T(t) \mathcal{P} \Phi(t) \xi(t) \quad (24)$$

where

$$\begin{aligned} \Phi(t) = & \hat{A}_{w0} d\bar{w}_0(t) + \left(\hat{A}_{w1} + \Delta \hat{A}_{w1}(t) \right) d\bar{w}_1(t) \\ & + \left(\hat{A}_{w2} + \Delta \hat{A}_{w2}(t) \right) d w_2(t), \end{aligned} \quad (25)$$

and

$$\begin{aligned} \mathcal{L}V(\xi(t)) dt = & 2\xi^T(t) \mathcal{P} \left(\hat{A}_{t0} + \Delta \hat{A}_{t0}(t) \right) \xi(t) dt \\ & + \xi^T(t) \langle \mathcal{P} \Phi(t), \Phi(t) \rangle \xi(t). \end{aligned} \quad (26)$$

By replacing (25) and (26) in (24), it becomes

$$\begin{aligned} dV(\xi(t)) = & 2\xi^T(t) \mathcal{P} \left(\hat{A}_{t0} + \Delta \hat{A}_{t0}(t) \right) \xi(t) dt \\ & + \xi^T(t) \hat{A}_{w0}^T \mathcal{P} \hat{A}_{w0} \xi(t) d\bar{w}_0(t)^2 \\ & + \xi^T(t) \left(\hat{A}_{w1} + \Delta \hat{A}_{w1}(t) \right)^T \mathcal{P} \\ & \quad \times \left(\hat{A}_{w1} + \Delta \hat{A}_{w1}(t) \right) \xi(t) d\bar{w}_1(t)^2 \\ & + \xi^T(t) \left(\hat{A}_{w2} + \Delta \hat{A}_{w2}(t) \right)^T \mathcal{P} \\ & \quad \times \left(\hat{A}_{w2} + \Delta \hat{A}_{w2}(t) \right) \xi(t) d w_2(t)^2 \\ & + \Psi(t), \end{aligned} \quad (27)$$

where

$$\begin{aligned} \Psi(t) = & \xi^T(t) \hat{A}_{w0}^T \mathcal{P} \left(\hat{A}_{w1} + \Delta \hat{A}_{w1}(t) \right) \xi(t) d\bar{w}_0(t) d\bar{w}_1(t) \\ & + \xi^T(t) \left(\hat{A}_{w1} + \Delta \hat{A}_{w1}(t) \right)^T \mathcal{P} \hat{A}_{w0} \xi(t) d\bar{w}_1(t) d\bar{w}_0(t) \\ & + \xi^T(t) \hat{A}_{w0}^T \mathcal{P} \left(\hat{A}_{w2} + \Delta \hat{A}_{w2}(t) \right) \xi(t) d\bar{w}_0(t) d w_2(t) \\ & + \xi^T(t) \left(\hat{A}_{w2} + \Delta \hat{A}_{w2}(t) \right)^T \mathcal{P} \hat{A}_{w0} \xi(t) d w_2(t) d\bar{w}_0(t) \\ & + \xi^T(t) \left(\hat{A}_{w1} + \Delta \hat{A}_{w1}(t) \right)^T \mathcal{P} \\ & \quad \times \left(\hat{A}_{w2} + \Delta \hat{A}_{w2}(t) \right) \xi(t) d\bar{w}_1(t) d w_2(t) \\ & + \xi^T(t) \left(\hat{A}_{w2} + \Delta \hat{A}_{w2}(t) \right)^T \mathcal{P} \\ & \quad \times \left(\hat{A}_{w1} + \Delta \hat{A}_{w1}(t) \right) \xi(t) d w_2(t) d\bar{w}_1(t) \\ & + 2\xi^T(t) \mathcal{P} \hat{A}_{w0} \xi(t) d\bar{w}_0(t) \\ & + 2\xi^T(t) \mathcal{P} \left(\hat{A}_{w1} + \Delta \hat{A}_{w1}(t) \right) \xi(t) d\bar{w}_1(t) \\ & + 2\xi^T(t) \mathcal{P} \left(\hat{A}_{w2} + \Delta \hat{A}_{w2}(t) \right) \xi(t) d w_2(t). \end{aligned} \quad (28)$$

From the majoration lemma [17], we have

$$2\xi^T(t)\mathcal{P}\Delta\hat{A}_{t0}(t)\xi(t) \leq \xi^T(t) (\mu_1^{-1}\mathcal{P}H_1H_1^T\mathcal{P} + \mu_1H_t^TH_t) \xi(t), \quad (29)$$

$$\left(\hat{A}_{w1} + \Delta\hat{A}_{w1}(t)\right)^T \mathcal{P} \left(\hat{A}_{w1} + \Delta\hat{A}_{w1}(t)\right) \leq \hat{A}_{w1}^T(\mathcal{P}^{-1} - \mu_2^{-1}H_2H_2^T)^{-1}\hat{A}_{w1} + \mu_2H_w^TH_w. \quad (30)$$

and

$$\left(\hat{A}_{w2} + \Delta\hat{A}_{w2}(t)\right)^T \mathcal{P} \left(\hat{A}_{w2} + \Delta\hat{A}_{w2}(t)\right) \leq \hat{A}_{w2}^T(\mathcal{P}^{-1} - \mu_3^{-1}H_3H_3^T)^{-1}\hat{A}_{w2} + \mu_3H_w^TH_w. \quad (31)$$

Using the Schur lemma, lemma 1 gives

$$\begin{aligned} &\mathcal{P}\hat{A}_{t0} + \hat{A}_{t0}^T\mathcal{P} + \mu_1^{-1}\mathcal{P}H_1H_1^T\mathcal{P} + \mu_1H_t^TH_t + \bar{\alpha}\hat{A}_{w0}^T\mathcal{P}\hat{A}_{w0} \\ &+ \alpha \left(\hat{A}_{w1}^T(\mathcal{P}^{-1} - \mu_2^{-1}H_2H_2^T)^{-1}\hat{A}_{w1} + \mu_2H_w^TH_w\right) \\ &+ \hat{A}_{w2}^T(\mathcal{P}^{-1} - \mu_3^{-1}H_3H_3^T)^{-1}\hat{A}_{w2} + \mu_3H_w^TH_w = -\mathcal{K} < 0. \end{aligned} \quad (32)$$

Now, taking the expectation of (27) (see [7]) and using the relations (14) and the inequalities (29)–(30), then $\mathbf{E}(dV(\xi(t)))$ can be bounded as

$$\mathbf{E}(dV(\xi(t))) \leq \mathbf{E}(\xi^T(t)(-\mathcal{K})\xi(t) dt) + \underbrace{\mathbf{E}(\Psi(t))}_{=0}. \quad (33)$$

Note that $\lambda_{\min}(\mathcal{K}) > 0$ where λ_{\min} is the smallest eigenvalue of \mathcal{K} . Then the inequality (33) yields to

$$\mathbf{E}(dV(\xi(t))) \leq -\lambda_{\min}(\mathcal{K}) \mathbf{E}(\|\xi(t)\|^2 dt). \quad (34)$$

Let $\beta > 0$ be given by (see [18])

$$\beta = \frac{\lambda_{\min}(\mathcal{K})}{\lambda_{\max}(\mathcal{P})} \quad (35)$$

and using the integration-by-part formula, we derive that

$$d(e^{\beta t}V(\xi(t))) = e^{\beta t}(\beta V(\xi(t))) dt + dV(\xi(t)) \quad (36)$$

or, from (34), using

$$V(\xi(t)) = \xi^T(t)\mathcal{P}\xi(t) \leq \lambda_{\max}(\mathcal{P}) \|\xi(t)\|^2$$

and taking the expectation of (36), we have

$$\mathbf{E}(dV(\xi(t))) \leq \mathbf{E}\left(e^{\beta t} \left((\beta\lambda_{\max}(\mathcal{P}) - \lambda_{\min}(\mathcal{K})) \|\xi(t)\|^2\right) dt\right). \quad (37)$$

or, according to the β value, it can be shown that

$$\beta\lambda_{\max}(\mathcal{P}) - \lambda_{\min}(\mathcal{K}) = 0, \quad (38)$$

and so (36) is equivalent to (see (33))

$$\mathbf{E}(dV(\xi(t))) \leq 0. \quad (39)$$

Integrating both sides from 0 to $t > 0$ and using Fubini's theorem [19], we have

$$e^{\beta t} \mathbf{E}(\xi^T(t)\mathcal{P}\xi(t)) \leq e^{\beta \times 0} \mathbf{E}(\xi^T(0)\mathcal{P}\xi(0)), \quad (40)$$

which can be rewritten as

$$\lambda_{\min}(\mathcal{P}) \mathbf{E}(\|\xi(t)\|^2) \leq \mathbf{E}(\xi^T(t)\mathcal{P}\xi(t)) \leq ce^{\beta t}, \quad (41)$$

where $c = \mathbf{E}(\xi^T(0)\mathcal{P}\xi(0))$ is a given constant.

Finally, from (41) the following inequality is obtained

$$\mathbf{E}(\|\xi(t)\|^2) \leq \frac{c}{\lambda_{\min}(\mathcal{P})} e^{-\beta t}, \quad (42)$$

and so we have

$$\lim_{t \rightarrow \infty} \mathbf{E}(\|\xi(t)\|^2) = 0. \quad (43)$$

The exponential mean-square stability of the augmented system is proved. \bullet

III. SYNTHESIS OF THE FILTER AND THE \mathcal{H}_∞ CRITERION ACHIEVEMENT

To study the \mathcal{H}_∞ criterion, we add the output $e(t)$ to system (20) (with $v(t) \neq 0$) which becomes

$$\begin{aligned} d\xi(t) &= \left(\left(\hat{A}_{t0} + \Delta\hat{A}_{t0}(t) \right) \xi(t) + \hat{B}_0 v(t) \right) dt \\ &+ \hat{A}_{w0} d\bar{w}_0(t) + \left(\hat{A}_{w1} + \Delta\hat{A}_{w1}(t) \right) d\bar{w}_1(t) \\ &+ \left(\hat{A}_{w2} + \Delta\hat{A}_{w2}(t) \right) \xi(t) dw_2(t) \\ e(t) &= \hat{C}\xi(t) \end{aligned} \quad (44)$$

where $\hat{C} = \begin{bmatrix} 0 & I_n \end{bmatrix}$.

Then, the following theorem is given to solve problem 1.

Theorem 1. *The \mathcal{H}_∞ stochastic filtering problem 1 is solved for the system (1) with the filter (4) such that the dynamics of the estimation error (7) is exponentially mean-square stable and verifies the \mathcal{H}_∞ performance (6) if there exist reals $\mu_1 > 0, \mu_2 > 0$ and $\mu_3 > 0$ and matrices $P_1 = P_1^T > 0 \in \mathbb{R}^{n \times n}, P_2 = P_2^T > 0 \in \mathbb{R}^{n \times n}, P_3 \in \mathbb{R}^{n \times n}, Y_{20} \in \mathbb{R}^{n \times p}, Y_{21} \in \mathbb{R}^{n \times p}, Y_{30} \in \mathbb{R}^{n \times p}$ and $Y_{31} \in \mathbb{R}^{n \times p}$, such that the following LMI holds*

$$\begin{bmatrix} (1,1) & (1,2) & (P_1+P_3)B_0 & \sigma P_1 A_{t1} & (1,5) & (1,6) & (1,7) \\ (1,2)^T & (2,2) & (P_3^T+P_2)B_0 & \sigma P_3^T A_{t1} & (2,5) & 0 & 0 \\ B^T(P_1+P_3^T) & B^T(P_3+P_2) & -\gamma^2 I_q & 0 & 0 & 0 & 0 \\ \sigma A_{t1}^T P_1 & \sigma A_{t1}^T P_3 & 0 & -\mu_1 I_n & 0 & 0 & 0 \\ (1,5)^T & (2,5)^T & 0 & 0 & -\mu_1 I_n & 0 & 0 \\ (1,6)^T & 0 & 0 & 0 & 0 & -P_1 & -P_3 \\ (1,7)^T & 0 & 0 & 0 & 0 & -P_3^T & -P_2 \\ (1,8)^T & 0 & 0 & 0 & 0 & 0 & 0 \\ (1,9)^T & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ (1,11)^T & 0 & 0 & 0 & 0 & 0 & 0 \\ (1,12)^T & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ (1,8) & (1,9) & 0 & (1,11) & (1,12) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -P_1 & -P_3 & (8,10) & 0 & 0 & 0 & 0 \\ -P_3^T & -P_2 & (9,10) & 0 & 0 & 0 & 0 \\ (8,10)^T & (9,10)^T & -\mu_2 I_n & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -P_1 & -P_3 & (11,13) & 0 \\ 0 & 0 & 0 & -P_3^T & -P_2 & (12,13) & 0 \\ 0 & 0 & 0 & (11,13)^T & (12,13)^T & -\mu_3 I_n & 0 \end{bmatrix} < 0, \quad (45)$$

where

$$\begin{aligned} (1,1) &= \text{herm}(P_1 A_\varphi) + (\mu_1 + \alpha\mu_2 + \mu_3)I_n, \\ (1,2) &= P_3 A_\varphi + A_\varphi^T P_3 - Y_{30} C - \varphi Y_{31} C, \end{aligned}$$

$$\begin{aligned}
(2, 2) &= \text{herm}(P_2 A_\alpha - Y_{20} C - \varphi Y_{21} C) + (\mu_1 + 1) I_n, \\
(1, 5) &= \sigma(P_3 A_{t1} - Y_{31} C), \\
(2, 5) &= \sigma(P_2 A_{t1} - Y_{21} C), \\
(1, 6) &= \bar{\alpha}^{\frac{1}{2}} A_{w0}^T (P_1 + P_3^T), \\
(1, 7) &= \bar{\alpha}^{\frac{1}{2}} A_{w0}^T (P_3 + P_2), \\
(1, 8) &= \alpha^{\frac{1}{2}} (f A_{w0}^T (P_1 + P_3^T) - J_1^T Y_{30}^T), \\
(1, 9) &= \alpha^{\frac{1}{2}} (f A_{w0}^T (P_3 + P_2) - J_1^T Y_{20}^T), \\
(1, 11) &= (\alpha_2 A_{w0}^T + \alpha_3 A_{w1}^T) (P_1 + P_3^T) - (Y_{30} + Y_{31}) J_1, \\
(1, 12) &= (\alpha_2 A_{w0}^T + \alpha_3 A_{w1}^T) (P_3 + P_2) - (Y_{20} + Y_{21}) J_1, \\
(8, 10) &= \sigma(P_1 + P_3) A_{w1}, \\
(9, 10) &= \sigma(P_3^T + P_2) A_{w1}, \\
(11, 13) &= \sigma \alpha_3 (P_1 + P_3) A_{w1} - \sigma Y_{31} J_1, \\
(12, 13) &= \sigma \alpha_3 (P_3^T + P_2) A_{w1} - \sigma Y_{21} J_1, \\
A_\varphi &= A_{t0} + \varphi A_{t1}, \\
f &= \frac{\alpha_1 - \alpha_2 \alpha_3}{1 - \alpha_3^2} + \alpha_2,
\end{aligned}$$

and such that the gain matrices K_0 and K_1 are solution of the following equation

$$\begin{bmatrix} P_2 \\ P_3 \end{bmatrix} \begin{bmatrix} K_0 & K_1 \end{bmatrix} = \begin{bmatrix} Y_{20} & Y_{21} \\ Y_{30} & Y_{31} \end{bmatrix}. \quad (46)$$

Proof. Consider Lyapunov function (22) and the following Lyapunov matrix $\mathcal{P} = \mathcal{P}^T > 0$

$$\mathcal{P} = \begin{bmatrix} P_1 & P_3 \\ P_3^T & P_2 \end{bmatrix} \text{ with } P_1, P_2, P_3 \in \mathbb{R}^{n \times n}. \quad (47)$$

Lemma 1 ensures the exponential mean square stability for the system (44) with $v(t) \equiv 0$.

Now we will study the perturbation attenuation for $v(t) \neq 0$. For this purpose, we consider $\xi(0) = 0$ and the following performance index which is function of the perturbation signal $v(t)$ and not of the initial condition $\xi(0)$

$$J_{\xi v} = \int_0^\infty \mathbf{E} \left(\left(\xi^T(t) \widehat{C}^T \widehat{C} \xi(t) - \gamma^2 v^T(t) v(t) \right) dt \right). \quad (48)$$

We can write $J_{\xi v}$ as follow

$$\begin{aligned}
J_{\xi v} &= \int_0^\infty \mathbf{E} \left(\left(\xi^T(t) \widehat{C}^T \widehat{C} \xi(t) - \gamma^2 v^T(t) v(t) \right) dt + dV(\xi(t)) \right) \\
&\quad - \mathbf{E}(V(\xi(t))_{t=\infty}) + \mathbf{E}(V(\xi(t))_{t=0}). \quad (49)
\end{aligned}$$

So, since $\mathbf{E}(V(\xi(t))_{t=0}) = 0$ because $\xi(0) = 0$ and $\mathbf{E}(V(\xi(t))_{t=\infty}) \geq 0$, this implies

$$\begin{aligned}
J_{\xi v} &\leq \int_0^\infty \mathbf{E} \left(\left(\xi^T(t) \widehat{C}^T \widehat{C} \xi(t) - \gamma^2 v^T(t) v(t) \right) dt \right. \\
&\quad \left. + dV(\xi(t)) \right). \quad (50)
\end{aligned}$$

Now if the LMI (45) holds, then applying Schur lemma [20] yields

$$\underbrace{\begin{bmatrix} \Theta & \mathcal{P} \widehat{B}_0 \\ \widehat{B}_0^T \mathcal{P} & 0 \end{bmatrix}}_{\Pi} + \begin{bmatrix} \widehat{C}^T \widehat{C} & 0 \\ 0 & -\gamma^2 I_q \end{bmatrix} < 0 \quad (51)$$

with

$$\begin{aligned}
\Theta &= \mathcal{P} \widehat{A}_{t0} + \widehat{A}_{t0}^T \mathcal{P} + \mu_1^{-1} \mathcal{P} H_1 H_1^T \mathcal{P} + \mu_1 H_t^T H_t \\
&\quad + \bar{\alpha} \widehat{A}_{w0}^T \mathcal{P} \widehat{A}_{w0} + \alpha \widehat{A}_{w1}^T (\mathcal{P}^{-1} - \mu_2^{-1} H_2 H_2^T)^{-1} \widehat{A}_{w1} \\
&\quad + (\alpha \mu_2 + \mu_3) H_w^T H_w + \widehat{A}_{w2}^T (\mathcal{P}^{-1} - \mu_3^{-1} H_3 H_3^T)^{-1} \widehat{A}_{w2}
\end{aligned}$$

Therefore

$$\begin{aligned}
J_{\xi v} &\leq \int_0^\infty \mathbf{E} \left(\left[\xi(t)^T \quad v(t)^T \right] \Pi \begin{bmatrix} \xi(t) \\ v(t) \end{bmatrix} dt \right. \\
&\quad \left. + \left[\xi(t)^T \quad v(t)^T \right] \begin{bmatrix} \widehat{C}^T \widehat{C} & 0 \\ 0 & -\gamma^2 I_q \end{bmatrix} \begin{bmatrix} \xi(t) \\ v(t) \end{bmatrix} dt \right) < 0,
\end{aligned}$$

so, if the LMI (45) holds the exponential mean-square stability and the \mathcal{H}_∞ performance are satisfied. •

IV. NUMERICAL EXAMPLE

In this part we consider a numerical example such that the matrices of the system (1) are given by

$$A_{t0} = \begin{bmatrix} -1.5 & 1 & -1 \\ 0.5 & -2.5 & 1 \\ 0 & -0.6 & -3.5 \end{bmatrix}, \quad B_0 = \begin{bmatrix} -0.1 & 0.3 \\ -1 & 0.2 \\ 0.6 & 0.5 \end{bmatrix},$$

$$A_{t1} = \begin{bmatrix} -0.01 & 0.1 & 0 \\ 0 & -0.05 & 0 \\ 0.15 & 0 & -0.02 \end{bmatrix},$$

$$A_{w0} = \begin{bmatrix} 1 & 0 & 0.2 \\ 0.5 & 0.3 & -0.1 \\ -0.2 & 0 & 0.2 \end{bmatrix},$$

$$A_{w1} = \begin{bmatrix} -0.03 & 0 & -0.03 \\ 0 & -0.01 & 0 \\ 0.04 & 0 & -0.07 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$$J_1 = \begin{bmatrix} -0.03 & 0 & -0.03 \\ 0 & -0.01 & 0 \end{bmatrix}.$$

The control $u_1(t)$ is defined as in (3), with

$$u_{1 \min} = -5 \leq u_1(t) \leq u_{1 \max} = 6,$$

and the initial state $\xi(0) = \begin{bmatrix} x^T(0) & e^T(0) \end{bmatrix}$ is

$$\xi^T(0) = \begin{bmatrix} -1 & 0.5 & 1 & -1 & 0.5 & 1 \end{bmatrix}.$$

The correlation coefficients α_i between the Wiener processes are given by the following matrix

$$\begin{bmatrix} 1 & \alpha_1 & \alpha_2 \\ \alpha_1 & 1 & \alpha_3 \\ \alpha_2 & \alpha_3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0.0165 & 0.0129 \\ 0.0165 & 1 & -0.004 \\ 0.0129 & -0.004 & 1 \end{bmatrix}.$$

We can easily verify that the system is mean-square stable by using the same change of variable as in (17) and (18) or by using [4]. From theorem 1, for $\gamma = 1.7$, the solution of LMI (45) is obtained with $\mu_1 = 0.284$, $\mu_2 = 0.057$ and $\mu_3 = 0.004$.

Then the gain matrices K_0 and K_1 are

$$K_0 = \begin{bmatrix} -6.359 & 116.110 \\ -6.550 & 73.058 \\ 0.969 & -25.807 \end{bmatrix}, K_1 = \begin{bmatrix} -0.149 & 0.023 \\ -0.029 & -0.083 \\ 0.164 & 0.0141 \end{bmatrix}.$$

The following figures show the simulation results of the augmented system (44). The actual state $x(t)$ is plotted in figure 1. The estimation error $e(t)$ and the disturbance signal $v(t)$ are given in figure 2.

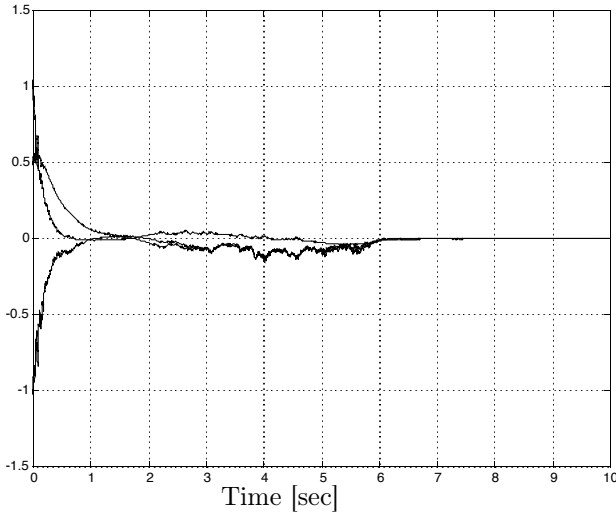


Fig. 1. The actual state $x(t)$.

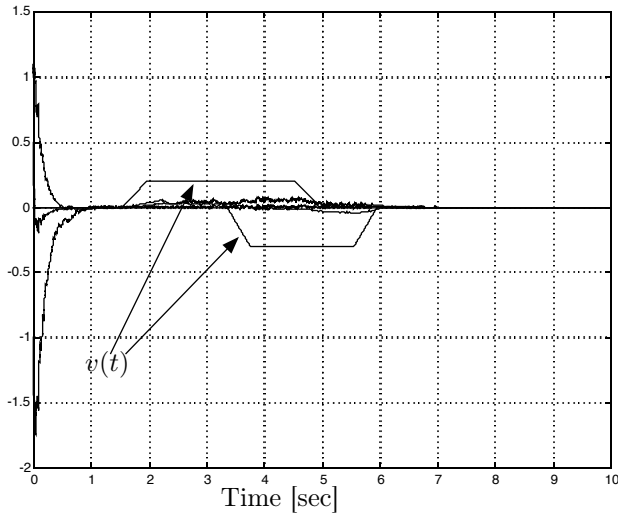


Fig. 2. The estimation error $e(t)$ and the disturbance $v(t)$.

V. CONCLUSION

This paper has provided a solution to the \mathcal{H}_∞ filtering problem for a class of nonlinear stochastic systems affected by three correlated Wiener processes, two of them in the state equation and the last one in the measurement equation. The filtering of the nonlinear stochastic system is transformed into a robust stochastic filtering one using a

change of coordinates on the control input. Then the solution is expressed in terms of LMI coupled to an equality constraint.

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