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ON THE STRUCTURE OF TRIANGULATED CATEGORIES WITH FINITELY MANY INDECOMPOSABLES

CLAIRE AMIOT

Abstract. We study the problem of classifying triangulated categories with finite-dimensional morphism spaces and finitely many indecomposables over an algebraically closed field. We obtain a new proof of the following result due to Xiao and Zhu: the Auslander-Reiten quiver of such a category is of the form $\mathbb{Z}\Delta/G$ where $\Delta$ is a disjoint union of simply laced Dynkin diagrams and $G$ a weakly admissible group of automorphisms of $\mathbb{Z}\Delta$. Then we prove that for ‘most’ groups $G$, the category $T$ is standard, i.e. $k$-linearly equivalent to an orbit category $D^b(\text{mod } k\Delta)/\Phi$. This happens in particular when $T$ is maximal $d$-Calabi-Yau with $d \geq 2$. Moreover, if $T$ is standard and algebraic, we can even construct a triangle equivalence between $T$ and the corresponding orbit category. Finally we give a sufficient condition for the category of projectives of a Frobenius category to be triangulated. This allows us to construct non standard 1-Calabi-Yau categories using deformed preprojective algebras of generalized Dynkin type.

Introduction

Let $k$ be an algebraically closed field and $T$ a small $k$-linear triangulated category (see [31]) with split idempotents. We assume that

a) $T$ is Hom-finite, i.e. the space $\text{Hom}_T(X,Y)$ is finite-dimensional for all objects $X, Y$ of $T$.

It follows that indecomposable objects of $T$ have local endomorphism rings and that each object of $T$ decomposes into a finite direct sum of indecomposables [9, 3.3]. We assume moreover that

b) $T$ is locally finite, i.e. for each indecomposable $X$ of $T$, there are at most finitely many isoclasses of indecomposables $Y$ such that $\text{Hom}_T(X,Y) \neq 0$.

It was shown in [32] that condition $b)$ implies its dual. Condition $b)$ holds in particular if we have

b’) $T$ is additively finite, i.e. there are only finitely many isomorphism classes of indecomposables in $T$.

The study of particular classes of such triangulated categories $T$ has a long history. Let us briefly recall some of its highlights:

1) If $A$ is a representation-finite selfinjective algebra, then the stable category $T$ of finite-dimensional (right) $A$-modules satisfies our assumptions and is additively finite. The structure of the underlying $k$-linear category of $T$ was determined by C. Riedtmann in [24], [25], [26] and [27].

2) In [4], D. Happel showed that the bounded derived category of the category of finite-dimensional representations of a representation-finite quiver is locally finite and described its underlying $k$-linear category.

3) The stable category $\mathcal{CM}(R)$ of Cohen-Macaulay modules over a commutative complete local Gorenstein isolated singularity $R$ of dimension $d$ is a Hom-finite triangulated category which is $(d-1)$-Calabi-Yau (cf. for example [17] and [14]). In [2], M. Auslander and I. Reiten showed that if the dimension of $R$ is 1, then the category $\mathcal{CM}(R)$ is additively finite and computed the shape of components of its Auslander-Reiten quiver.
The cluster category $C_Q$ of a finite quiver $Q$ without oriented cycles was introduced in \cite{7} if $Q$ is an orientation of a Dynkin diagram of type $A$ and in \cite{6} in the general case. The category $C_Q$ is triangulated \cite{19} and, if $Q$ is representation-finite, satisfies\(a)\) and\(b')\).

In a recent article \cite{32}, J. Xiao and B. Zhu determined the structure of the Auslander-Reiten quiver of a locally finite triangulated category. In this paper, we obtain the same result with a new proof in section 4, namely that each connected component of the Auslander-Reiten quiver of the category $T$ is of the form $\mathbb{Z}\Delta/G$, where $\Delta$ is a simply laced Dynkin diagram and $G$ is trivial or a weakly admissible group of automorphisms. We are interested in the $k$-linear structure of $T$. If the Auslander-Reiten quiver of $T$ is of the form $\mathbb{Z}\Delta$, we show that the category $T$ is standard, i.e. it is equivalent to the mesh category $k(\mathbb{Z}\Delta)$. Then in section 5, we prove that $T$ is standard if the number of vertices of $\Gamma = \mathbb{Z}\Delta/G$ is strictly greater than the number of isoclasses of indecomposables of $\mod k\Delta$. In the last section, using \cite{4} we construct examples of non standard triangulated categories such that $\Gamma = \mathbb{Z}\Delta/\tau$.

Finally, in the standard cases, we are interested in the triangulated structure of $T$. For this, we need to make additional assumptions on $T$. If the Auslander-Reiten quiver $\mathcal{Q}$ is of the form $\mathbb{Z}\Delta$, and if $T$ is the base of a tower of triangulated categories \cite{18}, we show that there is a triangle equivalence between $T$ and the derived category $D^{b}(\mod k\Delta)$. For the additively finite cases, we have to assume that $T$ is standard and algebraic in the sense of \cite{20}. We then show that $T$ is (algebraically) triangle equivalent to the orbit category of $D^{b}(\mod k\Delta)$ under the action of a weakly admissible group of automorphisms. In particular, for each $d \geq 2$, the algebraic triangulated categories with finitely many indecomposables which are maximal Calabi-Yau of CY-dimension $d$ are parametrized by the simply laced Dynkin diagrams.

Our results apply in particular to many stable categories $\mod A$ of representation-finite selfinjective algebras $A$. These algebras were classified up to stable equivalence by C. Riedtmann \cite{25} and H. Asashiba \cite{1}. In \cite{3}, J. Bialkowski and A. Skowroński give a necessary and sufficient condition on these algebras so that their stable categories $\mod A$ are Calabi-Yau. In \cite{12} and \cite{13}, T. Holm and P. Jørgensen prove that certain stable categories $\mod A$ are in fact $d$-cluster categories. These results can also be proved using our corollary 7.0.6.

This paper is organized as follows: In section 1, we prove that $T$ has Auslander-Reiten triangles. Section 2 is dedicated to definitions about stable valued translation quivers and admissible automorphisms groups \cite{12}, \cite{13}, \cite{8}. We show in section 3 that the Auslander-Reiten quiver of $T$ is a stable valued quiver and in section 4, we reprove the result of J. Xiao and B. Zhu \cite{32}: The Auslander-Reiten quiver is a disjoint union of quivers $\mathbb{Z}\Delta/G$, where $\Delta$ is a Dynkin quiver of type $A$, $D$, or $E$, and $G$ a weakly admissible group of automorphisms. In section 5, we construct a covering functor $D^{b}(\mod k\Delta) \to T$ using Riedtmann’s method \cite{24}. Then, in section 6, we exhibit some combinatorial cases in which $T$ has to be standard, in particular when $T$ is maximal $d$-Calabi-Yau with $d \geq 2$. Section 7 is dedicated to the algebraic case. If $T$ is algebraic and standard, we can construct a triangle equivalence between $T$ and an orbit category. If $P$ is a $k$-category such that $\mod P$ is a Frobenius category satisfying certain conditions, we will prove in section 8 that $P$ has naturally a triangulated structure. This allows us to deduce in section 9 that the category $\proj P^{l}(\Delta)$ of the projective modules over a deformed preprojective algebra of generalized Dynkin type $\Delta$ is naturally triangulated and to reduce the classification of the additively finite triangulated categories which are 1-Calabi-Yau to that of the deformed preprojective algebras in the sense of \cite{4}. 
ON THE STRUCTURE OF TRIANGULATED CATEGORIES

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Notation and terminology

We work over an algebraically closed field \( k \). By a triangulated category, we mean a \( k \)-linear triangulated category \( T \). We write \( S \) for the suspension functor of \( T \) and \( U \xrightarrow{u} V \xrightarrow{v} W \xrightarrow{w} SU \) for a distinguished triangle. We say that \( T \) is Hom-finite if for each pair \( X, Y \) of objects in \( T \), the space \( \text{Hom}_T(X,Y) \) is finite-dimensional over \( k \). The category \( T \) will be called a Krull-Schmidt category if each object is isomorphic to a finite direct sum of indecomposable objects and the endomorphism ring of an indecomposable object is a local ring. This implies that idempotents of \( T \) split [11, I 3.2]. The category \( T \) will be called locally finite if for each indecomposable \( X \) of \( T \), there are only finitely many isoclasses of indecomposables \( Y \) such that \( \text{Hom}_T(X,Y) \neq 0 \). This property is selfdual by [32, prop 1.1].

The Serre functor will be denoted by \( \nu \) (see definition in section 1). The Auslander-Reiten translation will always be denoted by \( \tau \) (section 1).

Let \( T \) and \( T' \) be two triangulated categories. An \( S \)-functor \((F,\phi)\) is given by a \( k \)-linear functor \( F : T \to T' \) and a functor isomorphism \( \phi \) between the functors \( F \circ S \) and \( S' \circ F \), where \( S \) is the suspension of \( T \) and \( S' \) the suspension of \( T' \). The notion of \( \nu \)-functor, or \( \tau \)-functor is then clear. A triangle functor is an \( S \)-functor \((F,\phi)\) such that for each triangle \( U \xrightarrow{u} V \xrightarrow{v} W \xrightarrow{w} SU \) of \( T \), the sequence \( SU \xrightarrow{F} FU \xrightarrow{FU} FW \xrightarrow{\phi \circ Fw} S'FU \) is a triangle of \( T' \).

The category \( T \) is Calabi-Yau if there exists an integer \( d > 0 \) such that we have a triangle functor isomorphism between \( S^d \) and \( \nu \). We say that \( T \) is maximal \( d \)-Calabi-Yau if \( T \) is \( d \)-Calabi-Yau and if for each covering functor \( T' \to T \) with \( T' \) \( d \)-Calabi-Yau, we have a \( k \)-linear equivalence between \( T \) and \( T' \).

For an additive \( k \)-category \( E \), we write \( \text{mod} E \) for the category of contravariant finitely presented functors from \( E \) to \( \text{mod} k \) (section 3), and if the projectives of \( \text{mod} E \) coincide with the injectives, \( \text{mod} E \) will be the stable category.

1. Serre duality and Auslander-Reiten triangles

1.1. Serre duality. Recall from [23] that a Serre functor for \( T \) is an autoequivalence \( \nu : T \to T \) together with an isomorphism \( D\text{Hom}_T(\cdot,\cdot) \simeq \text{Hom}_T(\cdot,\nu \cdot) \) for each \( X \in T \), where \( D \) is the duality \( \text{Hom}_X(\cdot,\cdot) \).

Theorem 1.1.1. Let \( T \) be a Krull-Schmidt, locally finite triangulated category. Then \( T \) has a Serre functor \( \nu \).

Proof. Let \( X \) be an object of \( T \). We write \( X^\wedge \) for the functor \( \text{Hom}_T(\cdot,X) \) and \( F \) for the functor \( D\text{Hom}_T(\cdot,\cdot) \). Using the lemma [23, I.1.6] we just have to show that \( F \) is representable. Indeed, the category \( T^{\text{op}} \) is locally finite as well. The proof is in two steps.

Step 1: The functor \( F \) is finitely presented.

Let \( Y_1,\ldots,Y_r \) be representatives of the isoclasses of indecomposable objects of \( T \) such that \( FY_i \) is not zero. The space \( \text{Hom}(Y_1^\wedge,F) \) is finite-dimensional over \( k \). Indeed it is isomorphic to \( FY_i \) by the Yoneda lemma. Therefore, the functor \( \text{Hom}(Y_1^\wedge,F) \otimes_k Y_1^\wedge \) is representable. We get an epimorphism from a representable
functor to $F$:

$$
\bigoplus_{i=1}^r \text{Hom}(Y_i^\wedge, F) \otimes_k Y_i^\wedge \to F.
$$

By applying the same argument to its kernel we get a projective presentation of $F$ of the form $U^\wedge \to V^\wedge \to F \to 0$, with $U$ and $V$ in $\mathcal{T}$.

**Step 2:** A cohomological functor $H : \mathcal{T}^{\text{op}} \to \text{mod } k$ is representable if and only if it is finitely presented.

Let $U^\wedge \xrightarrow{u^\wedge} V^\wedge \xrightarrow{v^\wedge} H \xrightarrow{w^\wedge} 0$ be a presentation of $H$. We form an exact sequence

$$
U^\wedge \xrightarrow{u^\wedge} V^\wedge \xrightarrow{v^\wedge} W^\wedge \xrightarrow{w^\wedge} (SU)^\wedge.
$$

Since the composition of $\phi$ with $u^\wedge$ is zero and $H$ is cohomological, the morphism $\phi$ factors through $v^\wedge$. But $H$ is the cokernel of $u^\wedge$, so $v^\wedge$ factors through $\phi$. We obtain a commutative diagram:

$$
\begin{array}{ccc}
U^\wedge & \xrightarrow{u^\wedge} & V^\wedge \\
\uparrow & & \downarrow \phi \\
H & \xrightarrow{i^\wedge} & V^\wedge \\
\downarrow & & \downarrow \phi' \\
W^\wedge & \xrightarrow{w^\wedge} & (SU)^\wedge.
\end{array}
$$

The equality $\phi' \circ i \circ \phi = \phi' \circ v^\wedge = \phi$ implies that $\phi' \circ i$ is the identity of $H$ because $\phi$ is an epimorphism. We deduce that $H$ is a direct factor of $W^\wedge$. The composition $i \circ \phi' = e^\wedge$ is an idempotent. Then $e \in \text{End}(W)$ splits and we get $H = W^\wedge$ for a direct factor $W'$ of $W$.

☐

### 1.2. Auslander-Reiten triangles.

**Definition 1.2.1.** [[10]] A triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} SX$ of $\mathcal{T}$ is called an Auslander-Reiten triangle or AR-triangle if the following conditions are satisfied:

1. **(AR1)** $X$ and $Z$ are indecomposable objects;
2. **(AR2)** $w \neq 0$;
3. **(AR3)** if $f : W \to Z$ is not a retraction, there exists $f' : W \to Y$ such that $vf' = f$;
4. **(AR3')** if $g : X \to V$ is not a section, there exists $g' : Y \to V$ such that $g'u = g$.

Let us recall that, if (AR1) and (AR2) hold, the conditions (AR3) and (AR3') are equivalent. We say that a triangulated category $\mathcal{T}$ has Auslander-Reiten triangles if, for any indecomposable object $Z$ of $\mathcal{T}$, there exists an AR-triangle ending at $Z$: $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} SX$. In this case, the AR-triangle is unique up to triangle isomorphism inducing the identity of $Z$.

The following proposition is proved in [[23, Proposition I.2.3]]

**Proposition 1.2.1.** The category $\mathcal{T}$ has Auslander-Reiten triangles.

The composition $\tau = S^{-1}u$ is called the Auslander-Reiten translation. An AR-triangle of $\mathcal{T}$ ending at $Z$ has the form:

$$
\tau Z \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \nu Z.
$$
2. **Translation quivers.** In this section, we recall some definitions and notations concerning quivers [8]. A quiver \( Q = (Q_0, Q_1, s, t) \) is given by the set \( Q_0 \) of its vertices, the set \( Q_1 \) of its arrows, a source map \( s \) and a tail map \( t \). If \( x \in Q_0 \) is a vertex, we denote by \( x^+ \) the set of direct successors of \( x \), and by \( x^- \) the set of its direct predecessors. We say that \( Q \) is **locally finite** if for each vertex \( x \in Q_0 \), there are finitely many arrows ending at \( x \) and starting at \( x \) (in this case, \( x^+ \) and \( x^- \) are finite sets). The quiver \( Q \) is said to be **without double arrows**, if two different arrows cannot have the same tail and source.

**Definition 2.1.1.** A **stable translation quiver** \((Q, \tau)\) is a locally finite quiver without double arrows with a bijection \( \tau : Q_0 \to Q_0 \) such that \((\tau x)^+ = x^- \) for each vertex \( x \). For each arrow \( \alpha : x \to y \), let \( \sigma \alpha \) the unique arrow \( \tau y \to x \).

Note that a stable translation quiver can have loops.

**Definition 2.1.2.** A **valued translation quiver** \((Q, \tau, a)\) is a stable translation quiver \((Q, \tau)\) with a map \( a : Q_1 \to \mathbb{N} \) such that \( a(\alpha) = a(\sigma \alpha) \) for each arrow \( \alpha \). If \( \alpha \) is an arrow from \( x \) to \( y \), we write \( a_{xy} \) instead of \( a(\alpha) \).

**Definition 2.1.3.** Let \( \Delta \) be an oriented tree. The **repetition of \( \Delta \)** is the quiver \( Z\Delta \) defined as follows:

- \((Z\Delta)_0 = Z \times \Delta_0 \)
- \((Z\Delta)_1 = Z \times \Delta_1 \cup \sigma(Z \times \Delta_1) \) with arrows \((n, \alpha) : (n, x) \to (n, y) \) and \( \sigma(n, \alpha) : (n-1, y) \to (n, x) \) for each arrow \( \alpha : x \to y \) of \( \Delta \).

The quiver \( Z\Delta \) with the translation \( \tau(n, x) = (n-1, x) \) is clearly a stable translation quiver which does not depend (up to isomorphism) on the orientation of \( \Delta \) (see [24]).

2.2. **Groups of weakly admissible automorphisms.**

**Definition 2.2.1.** An automorphism group \( G \) of a quiver is said to be **admissible** [24] if no orbit of \( G \) intersects a set of the form \( \{x\} \cup x^+ \) or \( \{x\} \cup x^- \) in more than one point. It said to be weakly admissible [8] if, for each \( g \in G - \{1\} \) and for each \( x \in Q_0 \), we have \( x^+ \cap (gx)^+ = \emptyset \).

Note that an admissible automorphism group is a weakly admissible automorphism group. Let us fix a numbering and an orientation of the simply-laced Dynkin trees.

\[
\begin{align*}
\mathbb{A}_n : \quad & 1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n-1 \longrightarrow n \\
\mathbb{D}_n : \quad & 1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n-2 \\
\mathbb{E}_n : \quad & 1 \longrightarrow 2 \longleftarrow 3 \longrightarrow 5 \longrightarrow \cdots \longrightarrow n
\end{align*}
\]

Let \( \Delta \) be a Dynkin tree. We define an automorphism \( S \) of \( Z\Delta \) as follows:
- if \( \Delta = \mathbb{A}_n \), then \( S(p, q) = (p + q, n + 1 - q) \);
• if $\Delta = \mathbb{D}_n$ with $n$ even, then $S = \tau^{-n+1}$;
• if $\Delta = \mathbb{D}_n$ with $n$ odd, then $S = \tau^{-n+1} \phi$ where $\phi$ is the automorphism of $\mathbb{D}_n$ which exchanges $n$ and $n-1$;
• if $\Delta = \mathbb{E}_6$, then $S = \phi \tau^{-6}$ where $\phi$ is the automorphism of $\mathbb{E}_6$ which exchanges 2 and 5, and 1 and 6;
• if $\Delta = \mathbb{E}_7$, then $S = \tau^{-9}$;
• and if $\Delta = \mathbb{E}_8$, then $S = \tau^{-15}$.

In [24, Anhang 2], Riedtmann describes all admissible automorphism groups of Dynkin diagrams. Here is a more precise result:

**Theorem 2.2.1.** Let $\Delta$ be a Dynkin tree and $G$ a non trivial group of weakly admissible automorphisms of $\mathbb{Z} \Delta$. Then $G$ is isomorphic to $\mathbb{Z}$, and here is a list of its possible generators:

- if $\Delta = \mathbb{A}_n$ with $n$ odd, possible generators are $\tau^r$ and $\phi \tau^r$ with $r \geq 1$, where $\phi = \tau^{\frac{-r}{2}} S$ is an automorphism of $\mathbb{Z} \Delta$ of order 2;
- if $\Delta = \mathbb{A}_n$ with $n$ even, then possible generators are $\rho^r$, where $r \geq 1$ and where $\rho = \tau^{\frac{1}{2}} S$. (Since $\rho^2 = \tau^{-1}$, $\tau^r$ is a possible generator.)
- if $\Delta = \mathbb{D}_n$ with $n \geq 5$, then possible generators are $\tau^r$ and $\tau^r \phi$, where $r \geq 1$ and where $\phi = (n-1,n)$ is the automorphism of $\mathbb{D}_n$ exchanging $n$ and $n-1$.
- if $\Delta = \mathbb{D}_4$, then possible generators are $\phi \tau^r$, where $r \geq 1$ and where $\phi$ belongs to $\mathcal{S}_3$ the permutation group on 3 elements seen as subgroup of automorphisms of $\mathbb{D}_4$.
- if $\Delta = \mathbb{E}_6$, then possible generators are $\tau^r$ and $\phi \tau^r$, where $r \geq 1$ and where $\phi$ is the automorphism of $\mathbb{E}_6$ exchanging 2 and 5, and 1 and 6.
- if $\Delta = \mathbb{E}_n$ with $n = 7,8$, possible generators are $\tau^r$, where $r \geq 1$.

The unique weakly admissible automorphism group which is not admissible exists for $\mathbb{A}_n$, $n$ even, and is generated by $\rho$.

3. **Property of the Auslander-Reiten translation**

We define the Auslander-Reiten quiver $\Gamma_T$ of the category $T$ as a valued quiver $(\Gamma, a)$). The vertices are the isoclasses of indecomposable objects. Given two indecomposable objects $X$ and $Y$ of $T$, we draw one arrow from $x = [X]$ to $y = [Y]$ if the vector space $R(X,Y)/R^2(X,Y)$ is not zero, where $R(?, ?)$ is the radical of the bifunctor $\text{Hom}_T(?, ?)$. A morphism of $R(X,Y)$ which does not vanish in the quotient $R(X,Y)/R^2(X,Y)$ will be called irreducible. Then we put

$$a_{xy} = \text{dim}_k R(X,Y)/R^2(X,Y).$$

Remark that the fact that $T$ is locally finite implies that its AR-quiver is locally finite. The aim of this section is to show that $\Gamma_T$ with the translation $\tau$ defined in the first part is a valued translation quiver. In other words, we want to show the proposition:

**Proposition 3.0.2.** If $X$ and $Y$ are indecomposable objects of $T$, we have the equality

$$\text{dim}_k R(X,Y)/R^2(X,Y) = \text{dim}_k R(\tau Y,X)/R^2(\tau Y,X).$$

Let us recall some definitions [11].

**Definition 3.0.2.** A morphism $g : Y \to Z$ is called sink morphism if the following hold:

1. $g$ is not a retraction;
2. if $h : M \to Z$ is not a retraction, then $h$ factors through $g$;
3. if $u$ is an endomorphism of $Y$ which satisfies $gu = u$, then $u$ is an automorphism.
Proposition 4.1.1. Let such that \( T \). Let \( \dim \).

Lemma 4.1.2. Theorem 4.0.4. Zhu ([33]): quiver.

Lemma 3.0.3. R.

\[
\text{Let } g : X \to Y \text{ be a morphism from } Y \text{ to } Z, \text{ where } Z \text{ is indecomposable and } Y = \bigoplus_{i=1}^{r} Y_{n_{i}}. \text{ is the decomposition of } Y \text{ into indecomposables. Then the morphism } g \text{ is a sink morphism if and only if the following hold:}
\]

1. For each \( i = 1, \ldots, r \) and \( j = 1, \ldots, n_{i} \), the morphism \( g_{i,j} \) belongs to the radical \( R(Y_{i}, Z) \).

2. For each \( i = 1, \ldots, r \), the family \( (g_{i,j})_{j=1,\ldots,n_{i}} \) forms a \( k \)-basis of the space \( R(Y_{i}, Z)/R^{2}(Y_{i}, Z) \).

3. If \( h \in \Hom_{T}(Y', Z) \) is irreducible and \( Y' \) indecomposable, then \( h \) factors through \( g \) and \( Y' \) is isomorphic to \( Y_{i} \) for some \( i \).

Using this lemma, it is easy to see that proposition 3.0.2 holds. Thus, the Auslander-Reiten quiver \( \Gamma_{T} = (\Gamma, \tau, a) \) of the category \( T \) is a valued translation quiver.

4. Structure of the Auslander-Reiten quiver

This section is dedicated to an other proof of a theorem due to J. Xiao and B. Zhu ([33]):

Theorem 4.0.4. \([33]\) Let \( T \) be a Krull-Schmidt, locally finite triangulated category. Let \( \Gamma \) be a connected component of the AR-quiver of \( T \). Then there exists a Dynkin tree \( \Delta \) of type \( \mathbb{A}, \mathbb{D} \) or \( \mathbb{E} \), a weakly admissible automorphism group \( G \) of \( \mathbb{Z}\Delta \) and an isomorphism of valued translation quiver

\[
\theta : \Gamma \to \mathbb{Z}\Delta/G.
\]

The underlying graph of the tree \( \Delta \) is unique up to isomorphism (it is called the type of \( \Gamma \)), and the group \( G \) is unique up to conjugacy in \( \text{Aut}(\mathbb{Z}\Delta) \).

In particular, if \( T \) has an infinite number of isoclasses of indecomposable objects, then \( G \) is trivial, and \( \Gamma \) is the repetition quiver \( \mathbb{Z}\Delta \).

4.1. Auslander-Reiten quivers with a loop. In this section, we suppose that the Auslander-Reiten quiver of \( T \) contains a loop, i.e. there exists an arrow with the same tail and source. Thus, we suppose that there exists an indecomposable \( X \) of \( T \) such that

\[
\dim_{k} R(X, X)/R^{2}(X, X) \geq 1.
\]

Proposition 4.1.1. Let \( X \) be an indecomposable object of \( T \). Suppose that we have \( \dim_{k} R(X, X)/R^{2}(X, X) \geq 1 \). Then \( \tau X \) is isomorphic to \( X \).

To prove this, we need a lemma.

Lemma 4.1.2. Let \( X_{1} \xrightarrow{f_{1}} X_{2} \xrightarrow{f_{2}} \cdots \xrightarrow{f_{n}} X_{n+1} \) be a sequence of irreducible morphisms between indecomposable objects with \( n \geq 2 \). If the composition \( f_{n} \circ f_{n-1} \cdots f_{1} \) is zero, then there exists an \( i \) such that \( \tau^{-1} X_{i} \) is isomorphic to \( X_{i+2} \).
Proof. The proof proceeds by induction on \( n \). Let us show the assertion for \( n = 2 \).

Suppose \( X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \) is a sequence such that \( f_2 \circ f_1 = 0 \). We can then construct an AR-triangle:

\[
\begin{align*}
X_1 \xrightarrow{(f_1,f_2)^T} X_2 & \oplus X \xrightarrow{(g_1,g_2)} \tau^{-1} X_1 & \xrightarrow{\beta} SX_1 \\
\end{align*}
\]

The composition \( f_2 \circ f_1 \) is zero, thus the morphism \( f_2 \) factors through \( g_1 \). As the morphisms \( g_1 \) and \( f_2 \) are irreducible, we conclude that \( \beta \) is a retraction, and \( X_3 \) a direct summand of \( \tau^{-1} X_1 \). But \( X_1 \) is indecomposable, so \( \beta \) is an isomorphism between \( X_3 \) and \( \tau^{-1} X_1 \).

Now suppose that the property holds for an integer \( n - 1 \) and that we have \( f_n \circ f_{n-1} \cdots f_1 = 0 \). If the composition \( f_{n-1} \cdots f_1 \) is zero, the proposition holds by induction. So we can suppose that for \( i \leq n - 2 \), the objects \( \tau^{-1} X_i \) and \( X_{i+2} \) are not isomorphic. We show now by induction on \( i \) that for each \( i \leq n - 1 \), there exists a map \( \beta_i : \tau^{-1} X_i \to X_{i+1} \) such that \( f_n \cdots f_{i+1} = \beta_{i+1} g_i \) where \( g_i : X_{i+1} \to \tau^{-1} X_i \) is an irreducible morphism. For \( i = 1 \), we construct an AR-triangle:

\[
\begin{align*}
X_1 \xrightarrow{(f_1,f_2)^T} X_2 & \oplus X \xrightarrow{(g_1,g_2)} \tau^{-1} X_1 & \xrightarrow{\beta} SX_1 \\
\end{align*}
\]

As the composition \( f_n \cdots f_1 \) is zero, we have the factorization \( f_n \cdots f_2 = \beta_1 g_1 \).

Now for \( i \), as \( \tau^{-1} X_{i-1} \) is not isomorphic to \( X_{i+1} \), there exists an AR-triangle of the form:

\[
\begin{align*}
X_1 \xrightarrow{(g_{i-1},f_1,f_2)^T} \tau^{-1} X_{i-1} \oplus X_{i+1} \oplus X_1 \xrightarrow{(g_{i+1},g_i,g_1)} \tau^{-1} X_2 & \xrightarrow{\beta} SX_i \\
\end{align*}
\]

By induction, \( -\beta_{i-1} g_{i-1} + f_n \cdots f_{i+1} f_i \) is zero, thus \( f_n \cdots f_{i+1} \) factors through \( g_i \). This property is true for \( i = n - 1 \), so we have a map \( \beta_{n-1} : \tau^{-1} X_{n-1} \to X_{n+1} \) such that \( \beta_{n-1} g_{n-1} = f_n \). As \( g_{n-1} \) and \( f_n \) are irreducible, we conclude that \( \beta_{n-1} \) is an isomorphism between \( X_{n+1} \) and \( \tau^{-1} X_{n-1} \).

Now we are able to prove proposition 4.1.1. There exists an irreducible map \( f : X \to X \). Suppose that \( X \) and \( \tau X \) are not isomorphic. Then from the previous lemma, the endomorphism \( f^n \) is non zero for each \( n \). But since \( T \) is a Krull-Schmidt, locally finite category, a power of the radical \( \mathcal{R}(X,X) \) vanishes. This is a contradiction.

4.2. Proof of theorem 4.0.4. Let \( \tilde{\Gamma} = (\tilde{\Gamma},\tilde{\alpha},\tilde{\omega}) \) be the valued translation quiver obtained from \( \Gamma \) by removing the loops, i.e.
we have \( \tilde{\Gamma}_0 = \Gamma_0, \tilde{\Gamma}_1 = \{ \alpha \in \Gamma_1 \text{ such that } s(\alpha) \neq t(\alpha) \} \), and \( \tilde{\omega} = a_{\tilde{f}_1} \).

Lemma 4.2.1. The quaver \( \tilde{\Gamma} = (\tilde{\Gamma},\tilde{\alpha},\tilde{\omega}) \) with the translation \( \tau \) is a valued translation quiver without loop.

Proof. We have to check that the map \( \sigma \) is well-defined. But from proposition 4.1.1 if \( \alpha \) is a loop on a vertex \( x \), \( \sigma(\alpha) \) is the unique arrow from \( \tau x = x \) to \( x \), i.e.
subpaths of the form \( \alpha \sigma \) set. Then for each \( n \), there is an indecomposable sequence \((X, Y, Z)_n\).

Proof. Let \( \Gamma \) be an AR-triangle. We write \((X, Y, Z)_n\) such that the composition \(f_i : X_{i-1} \to X_i\) such that the composition \(f_nf_{n-1} \cdots f_1\) does not vanish. But the functor \(\text{Hom}_T(X_0, ?)\) has finite support. Thus there is an indecomposable \( Y \) which appears an infinite number of times in the sequence \((X_i)_i\). But since \( R^N(Y, Y) \) vanishes for an \( N \), we have a contradiction. \( \square \)

Let \( S_\Delta \) be the system of representatives of isoclasses of indecomposables of \( T \). For an indecomposable \( Y \) of \( T \), we put

\[
 l(Y) = \sum_{M \in S} \dim_k \text{Hom}_T(M, Y).
\]

This sum is finite since \( T \) is locally finite.

Lemma 4.2.3. For \( x \) in \( \Delta_0 \), we write \( d_x = l(\overline{x}) \). Then for each \( x \in \Delta_0 \), we have:

\[
 \sum_{y \in \Delta_0} d_y C_{xy} = 2.
\]

Proof. Let \( X \) and \( U \) be indecomposables of \( T \). Let

\[
 X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} SX
\]

be an AR-triangle. We write \((U, ?)\) for the cohomological functor \(\text{Hom}_T(U, ?)\). Thus, we have a long exact sequence:

\[
(U, S^{-1}Z) \xrightarrow{S^{-1}w} (U, X) \xrightarrow{u^*} (U, Y) \xrightarrow{v^*} (U, Z) \xrightarrow{w^*} (U, SX).
\]
Let $S_Z(U)$ be the image of the map $w_*$. We have the exact sequence:

$$0 \rightarrow S_{S^*Z}(U) \rightarrow (U, X) \overset{w_*}{\rightarrow} (U, Y) \overset{v_*}{\rightarrow} (U, Z) \overset{u_*}{\rightarrow} S_Z(U) \rightarrow 0.$$ 

Thus we have the following equality:

$$\dim_k S_Z(U) + \dim_k S_{S^*Z}(U) + \dim_k (U, Y) = \dim_k (U, X) + \dim_k (U, Z).$$

If $U$ is not isomorphic to $Z$, each map from $U$ to $Z$ is radical, thus $S_Z(U)$ is zero. If $U$ is isomorphic to $Z$, the map $w_*$ factors through the radical of $\text{End}(Z)$, so $S_Z(Z)$ is isomorphic to $k$. Then summing the previous equality when $U$ runs over $S$, we get:

$$l(X) + l(Z) = l(Y) + 2.$$ 

Clearly $l$ is $\tau$-invariant, thus $l(Z)$ equals $l(X)$. If the decomposition of $Y$ is of the form $\bigoplus_{i=1}^r Y_i^{n_i}$, we get:

$$l(Y) = \sum_i n_i l(Y_i) = \sum_{i, X \rightarrow Y_i \in \mathcal{F}} a_{XY} l(Y_i) + a_{XX} l(X).$$

We deduce the formula:

$$2 = (2 - a_{XX}) l(X) - \sum_{i, X \rightarrow Y_i \in \mathcal{F}} a_{XY} l(Y_i).$$

Let $x$ be a vertex of the tree $\Delta$ and $\pi$ its image in $\tilde{\Gamma}$. Then an arrow $\pi \rightarrow Y$ in $\tilde{\Gamma}$ comes from an arrow $(x, 0) \rightarrow (y, 0)$ in $\mathbb{Z}\Delta$ or from an arrow $(x, 0) \rightarrow (y, -1)$ in $\mathbb{Z}\Delta$, i.e. from an arrow $(y, 0) \rightarrow (x, 0)$. Indeed the projection $\mathbb{Z}\Delta \rightarrow \mathbb{Z}\Delta/G$ is a covering. From this we deduce the following equality:

$$2 = (2 - a_{\pi\pi}) d_x - \sum_{y, x \rightarrow y \in \Delta} a_{\pi\pi} d_y - \sum_{y, y \rightarrow x \in \Delta} a_{\pi\pi} d_y = \sum_{y \in \Delta} d_y C_{xy}. \tag{4.1.1}$$

Now we can prove theorem 4.0.4. The matrix $C$ is a ‘generalized Cartan matrix’. The previous lemma gives us a subadditive function which is not additive. Thus by [12], the underlying graph of $C$ is of ‘generalized Dynkin type’. As $C$ is symmetric, the graph is necessarily of type $\mathbb{A}$, $\mathbb{D}$, $\mathbb{E}$, or $\mathbb{L}$. But this graph is the graph $\Delta$ with the valuation $a$. We are done in the cases $\mathbb{A}$, $\mathbb{D}$, or $\mathbb{E}$.

The case $\mathbb{L}_n$ occurs when the AR-quiver contains at least one loop. We can see $\mathbb{L}_n$ as $\mathbb{A}_n$ with valuations on the vertices with a loop. Then, it is obvious that the automorphism groups of $\mathbb{Z}\mathbb{L}_n$ are generated by $\tau^r$ for an $r \geq 1$. But proposition 4.1.1 tells us that a vertex $x$ with a loop satisfies $\tau x = x$. Thus $G$ is generated by $\tau$ and the AR-quiver has the following form:

$$1 \overset{1}{\leftarrow} 2 \overset{2}{\rightarrow} 3 \cdots \overset{n}{\rightarrow} \text{n}.$$ 

This quiver is isomorphic to the quiver $\mathbb{Z}\mathbb{A}_{2n}/G$ where $G$ is the group generated by the automorphism $\tau^n S = \rho$.

The suspension functor $S$ sends the indecomposables on indecomposables, thus it can be seen as an automorphism of the AR-quiver. It is exactly the automorphism $S$ defined in section 2.2.

As shown in [13], it follows from the results of [10] that for each Dynkin tree $\Delta$ and for each weakly admissible group of automorphisms $G$ of $\mathbb{Z}\Delta$, there exists a locally finite triangulated category $\mathcal{T}$ such that $\Gamma_{\mathcal{T}} \simeq \mathbb{Z}\Delta/G$. This category is of the form $\mathcal{T} = D^b(\text{mod } k\Delta)/\varphi$ where $\varphi$ is an auto-equivalence of $D^b(\text{mod } k\Delta)$. 


5. Construction of a covering functor

From now, we suppose that the AR-quiver $\Gamma$ of $\mathcal{T}$ is connected. We know its structure. It is natural to ask: Is the category $\mathcal{T}$ standard, i.e. equivalent as a $k$-linear category to the mesh category $k(\Gamma)$? First, in this part we construct a covering functor $F : k(\mathbb{Z}\Delta) \to \mathcal{T}$.

5.1. Construction. We write $\pi : Z\Delta \to \Gamma$ for the canonical projection. As $G$ is a weakly admissible group, this projection verifies the following property: if $x$ is a vertex of $Z\Delta$, the number of arrows of $Z\Delta$ with source $x$ is equal to the number of arrows of $Z\Delta/G$ with source $\pi x$. Let $S$ be a system of representatives of the isoclasses of indecomposables of $\mathcal{T}$. We write $\text{ind}\mathcal{T}$ for the full subcategory of $\mathcal{T}$ whose set of objects is $S$. For a tree $\Delta$, we write $k(Z\Delta)$ for the mesh category (see [24]). Using the same proof as Riedtmann [24], one shows the following theorem:

**Theorem 5.1.1.** There exists a $k$-linear functor $F : k(Z\Delta) \to \text{ind}\mathcal{T}$ which is surjective and induces bijections:

$$\bigoplus_{Fz = Fy} \text{Hom}_{k(Z\Delta)}(x, z) \to \text{Hom}_{\mathcal{T}}(Fx, Fy),$$

for all vertices $x$ and $y$ of $Z\Delta$.

5.2. Infinite case. If the category $\mathcal{T}$ is locally finite not finite, the constructed functor $F$ is immediately fully faithful. Thus we get the corollary.

**Corollary 5.2.1.** If $\text{ind}\mathcal{T}$ is not finite, then we have a $k$-linear equivalence between $\mathcal{T}$ and the mesh category $k(Z\Delta)$.

5.3. Uniqueness criterion. The covering functor $F$ can be see as a $k$-linear functor from the derived category $\mathcal{D}^b(\text{mod} k\Delta)$ to the category $\mathcal{T}$. By construction, it satisfies the following property called the AR-property:

For each AR-triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} SX$ of $\mathcal{D}^b(\text{mod} k\Delta)$, there exists a triangle of $\mathcal{T}$ of the form $FX \xrightarrow{Ff} FY \xrightarrow{Fg} FZ \xrightarrow{F} SFX$.

In fact, thanks to this property, $F$ is determined by its restriction to the subcategory $\text{proj} k\Delta = k(\Delta)$, i.e. we have the following lemma:

**Lemma 5.3.1.** Let $F$ and $G$ be $k$-linear functors from $\mathcal{D}^b(\text{mod} k\Delta)$ to $\mathcal{T}$. Suppose that $F$ and $G$ satisfy the AR-property and that the restrictions $F|_{k(\Delta)}$ and $G|_{k(\Delta)}$ are isomorphic. Then the functors $F$ and $G$ are isomorphic as $k$-linear functors.

**Proof.** It is easy to construct this isomorphism by induction using the (TR3) axiom of the triangulated categories (see [22]). \hfill \Box

6. Particular cases of $k$-linear equivalence

From now we suppose that the category $\mathcal{T}$ is finite, i.e. $\mathcal{T}$ has finitely many isoclasses of indecomposables objects.

6.1. Equivalence criterion. Let $\Gamma$ be the AR-quiver of $\mathcal{T}$ and suppose that it is isomorphic to $\mathbb{Z}\Delta/G$. Let $\varphi$ be a generator of $G$. It induces an automorphism in the mesh category $k(Z\Delta)$ that we still denote by $\varphi$. Then we have the following equivalence criterion:

**Proposition 6.1.1.** The categories $k(\Gamma)$ and $\text{ind}\mathcal{T}$ are equivalent as $k$-categories if and only if there exists a covering functor $F : k(Z\Delta) \to \text{ind}\mathcal{T}$ and an isomorphism of functors $\Phi : F \circ \varphi \to F$. 

The proof consists in constructing a $k$-linear equivalence between ind$\mathcal{T}$ and the orbit category $k(\Delta)/\varphi^2$ using the universal property of the orbit category (see [12]), and then constructing an equivalence between $k(\Delta)/\varphi^2$ and $k(\Gamma)$.

6.2. Cylindric case for $\mathbb{A}_n$.

**Theorem 6.2.1.** If $\Delta = \mathbb{A}_n$ and $\varphi = \tau^r$ for some $r \geq 1$, then there exists a functor isomorphism $\Phi : F \circ \varphi \to F$, i.e. for each object $x$ of $k(\Delta)$ there exists an automorphism $\Phi_x$ of $Fx$ such that for each arrow $\alpha : x \to y$ of $\Delta$, the following diagram commutes:

$$
\begin{array}{ccc}
Fx & \xrightarrow{\Phi_x} & Fx \\
F\alpha \downarrow & & \downarrow F\varphi \alpha \\
Fy & \xrightarrow{\Phi_y} & Fy.
\end{array}
$$

To prove this, we need the following lemma:

**Lemma 6.2.2.** Let $\alpha : x \to y$ be an arrow of $\mathbb{A}_n$ and let $c$ be a path from $x$ to $\tau^r y$, $r \in \mathbb{Z}$, which is not zero in the mesh category $k(\mathbb{A}_n)$. Then $c$ can be written $c' \alpha$ where $c'$ is a path from $y$ to $\tau y$ (up to sign).

**Proof.** There is a path from $x$ to $\tau^r y$, thus, we have $\text{Hom}_{k(\Delta)}(x, \tau^r y) \cong k$, and $x$ and $\tau^r y$ are opposite vertices of a ‘rectangle’ in $\mathbb{A}_n$. This implies that there exists a path from $x$ to $\tau y$ beginning by $\alpha$. □

**Proof. (of theorem 6.2.1)** Combining proposition 6.1.1 and lemma 5.3.1, we have just to construct an isomorphism between the restriction of $F$ and $F \circ \varphi$ to a subquiver $\mathbb{A}_n$.

Let us fix a full subquiver of $\mathbb{A}_n$ of the following form:

$$
x_1 \xrightarrow{\alpha_1} x_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} x_n
$$

such that $x_1, \ldots, x_n$ are representatives of the $r$-orbits in $\mathbb{A}_n$. We define the $(\Phi_{x_i})_{i=1}^n$ by induction. We fix $\Phi_{x_1} = Id_{Fx_1}$. Now suppose we have constructed some automorphisms $\Phi_{x_1}, \ldots, \Phi_{x_i}$ such that for each $j \leq i$ the following diagram is commutative:

$$
\begin{array}{ccc}
Fx_{j-1} & \xrightarrow{\Phi_{x_{j-1}}} & Fx_{j-1} \\
F\alpha_{j-1} \downarrow & & \downarrow F\varphi \alpha_{j-1} \\
Fx_j & \xrightarrow{\Phi_{x_j}} & Fx_j.
\end{array}
$$

The composition $(F \varphi \alpha_i) \circ \Phi_{x_i}$ is in the morphism space $\text{Hom}_T(Fx_i, Fx_{i+1})$, which is isomorphic, by theorem 5.1.1, to the space

$$
\bigoplus_{z=Fx_{i+1}} \text{Hom}_{k(\Delta)}(x_i, z).
$$

Thus we can write

$$(F \varphi \alpha_i)\Phi_{x_i} = \lambda F\alpha_i + \sum_{z \neq Fx_{i+1}} F\beta_z$$

where $\beta_z$ belongs to $\text{Hom}_{k(\Delta)}(x_i, z)$ and $Fz = Fx_{i+1}$. But $Fz$ is equal to $Fx_{i+1}$ if and only if $z$ is of the form $\tau^l x_{i+1}$ for an $l$ in $\mathbb{Z}$. By the lemma, we can write $\beta_z = \beta'_l \alpha_i$. Thus we have the equality:

$$(F \varphi \alpha_i)\Phi_{x_i} = F(\lambda Id_{x_{i+1}} + \sum_{z} \beta'_l)F\alpha_i.$$
The scalar $\lambda$ is not zero. Indeed, $\Phi_{x_i}$ is an automorphism, thus the image of $(F\varphi \alpha_i)\Phi_{x_i}$ is not zero in the quotient 
\[ R(Fx_i, Fx_{i+1})/R^2(Fx_i, Fx_{i+1}). \]

Thus $\Phi_{x_{i+1}} = F(\lambda Id_{x_{i+1}} + \sum z_\alpha \beta'_\alpha)$ is an automorphism of $Fx_{i+1}$ which verifies the commutation relation
\[ (F\varphi \alpha_i) \circ \Phi_{x_i} = \Phi_{x_{i+1}} \circ F\alpha_i. \]

\[ \square \]

6.3. Other standard cases. In the mesh category $k(Z\Delta)$, where $\Delta$ is a Dynkin tree, the length of the non zero paths is bounded. Thus there exist automorphisms $\varphi$ such that, for an arrow $\alpha : x \to y$ of $\Delta$, the paths from $x$ to $\varphi^r y$ vanish in the mesh category for all $r \neq 0$. In other words, for each arrow $\alpha : x \to y$ of $Z\Delta$, we have:
\[ \text{Hom}_{k(Z\Delta)/\varphi^r}(x, y) = \bigoplus_{r \in \mathbb{Z}} \text{Hom}_{k(Z\Delta)}(x, \varphi^r y) = \text{Hom}_{k(Z\Delta)}(x, y) \simeq k, \]
where $k(Z\Delta)/\varphi^2$ is the orbit category (see section 5.1).

Lemma 6.3.1. Let $T$ be a finite triangulated category with AR-quiver $\Gamma = Z\Delta/G$. Let $\varphi$ be a generator of $G$ and suppose that $\varphi$ verifies for each arrow $x \to y$ of $Z\Delta$
\[ \bigoplus_{r \in \mathbb{Z}} \text{Hom}_{k(Z\Delta)}(x, \varphi^r y) = \text{Hom}_{k(Z\Delta)}(x, y) \simeq k. \]

Let $F : k(Z\Delta) \to T$ and $G : k(Z\Delta) \to T$ be covering functors satisfying the AR-property. Suppose that $F$ and $G$ agree up to isomorphism on the objects of $k(Z\Delta)$. Then $F$ and $G$ are isomorphic as $k$-linear functors.

Proof. Using lemma 5.3.1, we have just to construct an isomorphism between the functors restricted to $\Delta$. Let $\alpha : x \to y$ be an arrow of $\Delta$. Using the equation $5.1.1$ and the hypothesis, we have the following isomorphisms:
\[ \text{Hom}_\Gamma(Fx, FY) \simeq \bigoplus_{Fz = FY} \text{Hom}_{k(Z\Delta)}(x, z) \simeq \bigoplus_{r \in \mathbb{Z}} \text{Hom}_{k(Z\Delta)}(x, \varphi^r y) \simeq k, \]
and then
\[ \text{Hom}_\Gamma(Gx, GY) \simeq \text{Hom}_\Gamma(Fx, FY) \simeq k. \]

Thus there exists a scalar $\lambda$ such that $G\alpha = \lambda F\alpha$. This scalar does not vanish since $F$ and $G$ are covering functors. As $\Delta$ is a tree, we can find some $\lambda_x$ for $x \in \Delta$ by induction such that
\[ G\alpha = \lambda_x \lambda_y^{-1} F\alpha. \]

Now it is easy to check that $\Phi_{x} = \lambda_x Id_{Fx}$ is the functor isomorphism. \[ \square \]

This lemma gives us an isomorphism between the functors $F$ and $G$ $\varphi$. Moreover, using the same argument, one can show that the covering functor $F$ is an $S$-functor and a $\tau$-functor.

For each Dynkin tree $\Delta$ we can determine the automorphisms $\varphi$ which satisfy this combinatorial property. Using the preceding lemma and the equivalence criterion we deduce the following theorem:

Theorem 6.3.2. Let $T$ be a finite triangulated category with AR-quiver $\Gamma = Z\Delta/G$. Let $\varphi$ be a generator of $G$. If one of these cases holds,
- $\Delta = \mathbb{A}_n$ with $n$ odd and $G$ is generated by $\tau'$ or $\varphi = \tau' \phi$ with $r \geq n-1$ and $\phi = \tau \frac{n+1}{2} S$;
- $\Delta = \mathbb{A}_n$ with $n$ even and $G$ is generated by $\rho'$ with $r \geq n-1$ and $\rho = \tau \frac{n}{2} S$;
Theorem 7.0.5. The following result which is proved in section 7.3:

- $\Delta = \mathbb{D}_n$ with $n \geq 5$ and $G$ is generated by $\tau^r$ or $\tau^r \phi$ with $r \geq n - 2$ and $\phi$ as in theorem 2.2.4.
- $\Delta = \mathbb{D}_4$ and $G$ is generated by $\phi \tau^r$, where $r \geq 2$ and $\phi$ runs over $\sigma_3$.
- $\Delta = \mathbb{E}_6$ and $G$ is generated by $\tau^r$ or $\tau^r \phi$ where $r \geq 5$ and $\phi$ is as in theorem 2.2.4.
- $\Delta = \mathbb{E}_7$ and $G$ is generated by $\tau^r$, $r \geq 8$.
- $\Delta = \mathbb{E}_8$ and $G$ is generated by $\tau^r$, $r \geq 14$.

then $T$ is standard, i.e. the categories $\mathcal{T}$ and $k(\Gamma)$ are equivalent as $k$-linear categories.

Corollary 7.0.6. To \cite[Cor 8.4]{[19]}: a category respects $F$ if $\Phi$ is an auto-equivalence of $\Delta$.

Corollary 7.0.6. A finite maximal $d$-Calabi-Yau (see \cite{[19], [8]}) triangulated category $\mathcal{T}$, with $d \geq 2$, is standard, i.e. there exists a $k$-linear equivalence between $\mathcal{T}$ the orbit category $\mathcal{D}^b(\mathbb{mod} k\Delta)/\tau^{-1}S^d\nu^{-1}$ where $\Delta$ is Dynkin of type $A$, $D$ or $E$.

7. Algebraic case

For some automorphism groups $G$, we know the $k$-linear structure of $T$. But what about the triangulated structure? We can only give an answer adding hypothesis on the triangulated structure. In this section, we distinguish two cases:

- If $T$ is locally finite, not finite, we have the following theorem which is proved in section 7.3.

Theorem 7.0.4. Let $\mathcal{T}$ be a connected locally finite triangulated category with infinitely many indecomposables. If $\mathcal{T}$ is the base of a tower of triangulated categories \cite{[18]}, then $\mathcal{T}$ is triangle equivalent to $\mathcal{D}(\mathbb{mod} k\Delta)$ for some Dynkin diagram $\Delta$.

Now if $T$ is a finite standard category which is algebraic, i.e. $T$ is triangle equivalent to $\mathcal{E}$ for some $k$-linear Frobenius category $\mathcal{E}$ (\cite{[10], 3.6]), then we have the following result which is proved in section 7.3:

Theorem 7.0.5. Let $\mathcal{T}$ be a finite triangulated category, which is connected, algebraic and standard. Then, there exists a Dynkin diagram $\Delta$ of type $A$, $D$ or $E$ and an auto-equivalence $\Phi$ of $\mathcal{D}(\mathbb{mod} k\Delta)$ such that $\mathcal{T}$ is triangle equivalent to the orbit category $\mathcal{D}(\mathbb{mod} k\Delta)/\Phi$.

This theorem combined with corollary 6.3.3 yields the following result (compare to \cite[Cor 8.4]{[19]}):

Corollary 7.0.6. If $T$ is a finite algebraic maximal $d$-Calabi-Yau category with $d \geq 2$, then $\mathcal{T}$ is triangle equivalent to the orbit category $\mathcal{D}(\mathbb{mod} k\Delta)/\tau^{-1}S^d\nu^{-1}$ for some Dynkin diagram $\Delta$.

7.1. $\partial$-functor. We recall the following definition from \cite{[18]} and \cite{[30]}.

Definition 7.1.1. Let $\mathcal{H}$ be an exact category and $\mathcal{T}$ a triangulated category. A $\partial$-functor $(I, \partial) : \mathcal{H} \rightarrow \mathcal{T}$ is given by:

- an additive $k$-linear functor $I : \mathcal{H} \rightarrow \mathcal{T}$;
- for each conflation $\epsilon : X \rightarrow Y \rightarrow Z$ of $\mathcal{H}$, a morphism $\partial \epsilon : IZ \rightarrow SI X$ functorial in $\epsilon$ such that $I I \epsilon IY \rightarrow IZ$ is a triangle of $\mathcal{T}$.

For each exact category $\mathcal{H}$, the inclusion $I : \mathcal{H} \rightarrow \mathcal{D}(\mathcal{H})$ can be completed to a $\partial$-functor $(I, \partial)$ in a unique way. Let $\mathcal{T}$ and $\mathcal{T}'$ be triangulated categories. If $(F, \varphi) : \mathcal{T} \rightarrow \mathcal{T}'$ is an $S$-functor and $(I, \partial) : \mathcal{H} \rightarrow \mathcal{T}$ is a $\partial$-functor, we say that $F$ respects $\partial$ if $(F \circ I, \varphi(F\partial)) : \mathcal{H} \rightarrow \mathcal{T}'$ is a $\partial$-functor. Obviously each triangle functor respects $\partial$. 
Proposition 7.1.1. Let $\mathcal{H}$ be a $k$-linear hereditary abelian category and let $(I, \partial) : \mathcal{H} \to \mathcal{T}$ be a $\partial$-functor. Then there exists a unique (up to isomorphism) $k$-linear $S$-functor $F : \mathcal{D}^b(\mathcal{H}) \to \mathcal{T}$ which respects $\partial$.

Proof. On $\mathcal{H}$ (which can be seen as a full subcategory of $\mathcal{D}^b(\mathcal{H})$), the functor $F$ is uniquely determined. We want $F$ to be an $S$-functor, so $F$ is uniquely determined on $S^n\mathcal{H}$ for $n \in \mathbb{Z}$ too. Since $\mathcal{H}$ is hereditary, each object of $\mathcal{D}^b(\mathcal{H})$ is isomorphic to a direct sum of stalk complexes, i.e. complexes concentrated in a single degree. Thus, the functor $F$ is uniquely determined on the objects. Now, let $X$ and $Y$ be stalk complexes of $\mathcal{D}^b(\mathcal{H})$ and $f : X \to Y$ a non-zero morphism. We can suppose that $X$ is in $\mathcal{H}$ and $Y$ is in $S^n\mathcal{H}$. If $n \neq 0, 1$, $f$ is necessarily zero. If $n = 0$, then $f$ is a morphism in $\mathcal{H}$ and $Ff$ is uniquely determined. If $n = 1$, $f$ is an element of $\text{Ext}^1_H(X, S^{-1}Y)$, so gives us a conflation \[ \varepsilon : S^{-1}Y \rightarrowrightarrow E \rightarrowrightarrow X \] in $\mathcal{H}$. The functor $F$ respects $\partial$, thus $Ff$ has to be equal to $\varphi \circ \partial \varepsilon$ where $\varphi$ is the natural isomorphism between $SFS^{-1}Y$ and $FY$. Since $\partial$ is functorial, $F$ is a functor. The result follows. \[\square\]

A priori this functor is not a triangle functor. We recall a theorem proved by B. Keller [18, cor 2.7].

Theorem 7.1.2. Let $\mathcal{H}$ be a $k$-linear exact category, and $\mathcal{T}$ be the base of a tower of triangulated categories $[18]$. Let $(I, \partial) : \mathcal{H} \to \mathcal{T}$ be a $\partial$-functor such that for each $n < 0$, and all objects $X$ and $Y$ of $\mathcal{H}$, the space $\text{Hom}_\mathcal{T}(IX, S^nY)$ vanishes. Then there exists a triangle functor $F : \mathcal{D}^b(\mathcal{H}) \to \mathcal{T}$ such that the following diagram commutes up to isomorphism of $\partial$-functors:

\[
\begin{array}{ccc}
\mathcal{H} & \xrightarrow{(I, \partial)} & \mathcal{D}^b(\mathcal{H}) \\
\downarrow & & \downarrow F \\
\mathcal{T} & & \\
\end{array}
\]

From theorem 7.1.2 and the proposition above we deduce the following corollary:

Corollary 7.1.3. (compare to [29]) Let $\mathcal{T}$, $\mathcal{H}$ and $(I, \partial) : \mathcal{H} \to \mathcal{T}$ be as in theorem 7.1.3. If $\mathcal{H}$ is hereditary, then the unique functor $F : \mathcal{D}^b(\mathcal{H}) \to \mathcal{T}$ which respects $\partial$ is a triangle functor.

7.2. Proof of theorem 7.0.4. Let $F$ be the $k$-linear equivalence constructed in theorem 7.1.3 between an algebraic triangulated category $\mathcal{T}$ and $\mathcal{D}^b(\mathcal{H})$ where $\mathcal{H} = \text{mod} \ k\Delta$ and $\Delta$ is a simply laced Dynkin graph. As we saw in section 3, the covering functor is an $S$-functor.

The category $\mathcal{H}$ is the heart of the standard $t$-structure on $\mathcal{D}^b(\mathcal{H})$. The image of this $t$-structure through $F$ is a $t$-structure on $\mathcal{T}$. Indeed, $F$ is an $S$-equivalence, so the conditions (i) and (ii) from [3] Def 1.3.1 hold obviously. And since $\mathcal{H}$ is hereditary, for an object $X$ of $\mathcal{D}^b(\mathcal{H})$, the morphism $X \to S\tau_{\leq 0}X$ of the triangle

\[
\tau_{\leq 0}X \torightarrow X \torightarrow \tau_{\geq 0}X \torightarrow S\tau_{\leq 0}X
\]

vanishes. Thus the image of this triangle through $F$ is a triangle of $\mathcal{T}$ and condition (iii) of [3] Def 1.3.1 holds. Then we get a $t$-structure on $\mathcal{T}$ whose heart is $\mathcal{H}$.

It results from [3, Prop 1.2.4] that the inclusion of the heart of a $t$-structure can be uniquely completed to a $\partial$-functor. Thus we obtain a $\partial$-functor $(F_0, \partial) : \mathcal{H} \to \mathcal{T}$ with $F_0 = F|_\mathcal{H}$.

The functor $F$ is an $S$-equivalence. Thus for each $n < 0$, and all objects $X$ and $Y$ of $\mathcal{H}$, the space $\text{Hom}_\mathcal{T}(FX, S^nFY)$ vanishes. Now we can apply theorem 7.1.3.
and we get the following commutative diagram:

\[
\begin{array}{ccc}
H & \rightarrow & D^b(H), \\
(\mathcal{F}, \partial) & \downarrow & \uparrow \\
T & \rightarrow & F
\end{array}
\]

where \( F \) is the \( S \)-equivalence and \( G \) is a triangle functor. Note that a priori \( F \) is an \( S \)-functor which does not respect \( \partial \). The functors \( F|_{\gamma} \) and \( G|_{\gamma} \) are isomorphic. The functor \( F \) is an \( S \)-functor thus we have an isomorphism \( F|_{\gamma} \simeq G|_{\gamma} \) for each \( n \in \mathbb{Z} \). Thus the functor \( G \) is essentially surjective. Since \( H \) is the category \( \mod k\Delta \), to show that \( G \) is fully faithful, we have just to show that for each \( p \in \mathbb{Z} \),

\[ \text{Hom}_{D^b(H)}(A, S^pA) \xrightarrow{\sim} \text{Hom}_T(GA, S^pGA) \]

where \( A \) is the free module \( k\Delta \). For \( p = 0 \), this is clear because \( A \) is in \( H \). And for \( p \neq 0 \) both sides vanish.

Thus \( G \) is a triangle equivalence between \( D^b(H) \) and \( T \).

### 7.3. Finite algebraic standard case.

For a small dg category \( A \), we denote by \( \mathcal{O}(A) \) the category of dg \( A \)-modules, by \( D(A) \) the derived category of \( A \) and by \( \text{per} A \) the perfect derived category of \( A \), i.e. the smallest triangulated subcategory of \( D(A) \) which is stable under passage to direct factors and contains the free \( A \)-modules \( A(?, A) \), where \( A \) runs through the objects of \( A \). Recall that a small triangulated category is algebraic if it is triangle equivalent to \( \text{per} A \) for a dg category \( A \). For two small dg categories \( A \) and \( B \), a triangle functor \( \text{per} A \rightarrow \text{per} B \) is algebraic if it is isomorphic to the functor \( F_X = L \otimes_k \Delta X \) associated with a dg bimodule \( X \), i.e. an object of the derived category \( D(A^{\text{op}} \otimes B) \).

Let \( \Phi \) be an algebraic autoequivalence of \( D^b(\mod k\Delta) \) such that the orbit category \( D^b(\mod k\Delta)/\Phi \) is triangulated. Let \( Y \) be a dg \( k\Delta\Delta \)-bimodule such that \( \Phi = F_Y \). In section 9.3 of [19], it was shown that there is a canonical triangle equivalence between this orbit category and the perfect derived category of a certain small dg category. Thus, the orbit category is algebraic, and endowed with a canonical triangle equivalence to the perfect derived category of a small dg category. Moreover, by the construction in [loc. cit.], the projection functor

\[ \pi : D^b(\mod k\Delta) \rightarrow D^b(\mod k\Delta)/\Phi \]

is algebraic.

The proof of theorem 7.0.5 is based on the following universal property of the triangulated orbit category \( D^b(\mod k\Delta)/\Phi \). For the proof, we refer to section 9.3 of [19].

**Proposition 7.3.1.** Let \( B \) be a small dg category and

\[ F_X = L \otimes_k \Delta X : D^b(\mod k\Delta) \rightarrow \text{per} B \]

an algebraic triangle functor given by a dg \( k\Delta\Delta \)-bimodule \( X \). Suppose that there is an isomorphism between \( Y \otimes_k \Delta \Delta X \) and \( X \) in the derived bimodule category \( D(k\Delta^{\text{op}} \otimes B) \). Then the functor \( F_X \) factors, up to isomorphism of triangle functors, through the projection

\[ \pi : D^b(\mod k\Delta) \rightarrow D^b(\mod k\Delta)/\Phi. \]

Moreover, the induced triangle functor is algebraic.

Let us recall a lemma of Van den Bergh [21]:
Lemma 7.3.2. Let $Q$ be a quiver without oriented cycles and $A$ be a dg category. We denote by $k(Q)$ the category of paths of $Q$ and by $\text{Can} : C A \to DA$ the canonical functor. Then we have the following properties:

a) Each functor $F : k(Q) \to DA$ lifts, up to isomorphism, to a functor $\tilde{F} : k(Q) \to CA$ which verifies the following property: For each vertex $j$ of $Q$, the induced morphism

$$\bigoplus_i \tilde{F}_i \to \tilde{F}_j,$$

where $i$ runs through the immediate predecessors of $j$, is a monomorphism which splits as a morphism of graded $A$-modules.

b) Let $F$ and $G$ be functors from $k(Q)$ to $CA$, and suppose that $F$ satisfies the property of a). Then any morphism of functors $\varphi : \text{Can} \circ F \to \text{Can} \circ G$ lifts to a morphism $\tilde{\varphi} : F \to G$.

Proof. a) For each vertex $i$ of $Q$, the object $Fi$ is isomorphic in $DA$ to its cofibrant resolution $X_i$. Thus for each arrow $\alpha : i \to j$, $F$ induces a morphism $f_{\alpha} : X_i \to X_j$ which can be lifted to $CA$ since the $X_i$ are cofibrant. Since $Q$ has no oriented cycle, it is easy to choose the $f_{\alpha}$ such that the property is satisfied.

b) For each vertex $i$ of $Q$, we may assume that $Fi$ is cofibrant. Then we can lift $\varphi_i : \text{Can} \circ Fi \to \text{Can} \circ Gi$ to $\psi_i : Fi \to Gi$. For each arrow $\alpha$ of $Q$, the square

$$
\begin{array}{ccc}
Fi & \xrightarrow{F_i} & Fj \\
\downarrow{\psi_i} & & \downarrow{\psi_j} \\
Gi & \xrightarrow{G_i} & Gj
\end{array}
$$

is commutative in $DA$. Thus the square

$$
\begin{array}{ccc}
\bigoplus_i Fi & \xrightarrow{\bigoplus_i F_i} & \bigoplus_j Fj \\
\downarrow{\bigoplus_i \psi_i} & & \downarrow{\bigoplus_j \psi_j} \\
\bigoplus_j Gi & \xrightarrow{\bigoplus_j G_i} & \bigoplus_j Gj
\end{array}
$$

is commutative up to nullhomotopic morphism $h : \bigoplus_i Fi \to Gj$. Since the morphism $f : \bigoplus_i Fi \to Fj$ is split mono in the category of graded $A$-modules, $h$ extends along $f$ and we can modify $\psi_j$ so that the square becomes commutative in $CA$. The quiver $Q$ does not have oriented cycles, so we can construct $\tilde{\varphi}$ by induction. \qed

Proof. (of theorem 7.0.3) The category $T$ is small and algebraic, thus we may assume that $T = \text{per} A$ for some small dg category $A$. Let $F : D^b(\text{mod} k\Delta) \to T$ be the covering functor of theorem 5.1.1. Let $\Phi$ be an auto-equivalence of $D^b(\text{mod} k\Delta)$ such that the AR-quiver of the orbit category $D^b(\text{mod} k\Delta)/\Phi$ is isomorphic (as translation quiver) to the AR-quiver of $T$. We may assume that $\Phi = -L_{\Delta} Y$ for an object $Y$ of $D(k\Delta_{op} \otimes k\Delta)$. The orbit category $D^b(\text{mod} k\Delta)/\Phi$ is algebraic, thus it is $\text{per} B$ for some dg category $B$.

The functor $F|_{k\Delta}$ lifts by lemma 7.3.2 to a functor $\tilde{F}$ from $k(\Delta)$ to $CA$. This means that the object $X = \tilde{F}(k\Delta)$ has a structure of dg $k\Delta_{op} \otimes A$-module. We denote by $X$ the image of this object in $D(k\Delta_{op} \otimes A)$.

The functors $F$ and $-L_{\Delta} X$ become isomorphic when restricted to $k(\Delta)$. Moreover $-L_{\Delta} X$ satisfies the AR-property since it is a triangulated functor. Thus by lemma 5.3.1 they are isomorphic as $k$-linear functors. So we have the following
The category $T$ is standard, thus there exists an isomorphism of $k$-linear functors:

$$c : - \otimes_{k\Delta} X \to - \otimes_{k\Delta} Y \otimes_{k\Delta} X.$$

The functor $-\otimes_{k\Delta} X$ restricted to the category $k(\Delta)$ satisfies the property of $a)$ of lemma 7.3. Thus we can apply $b)$ and lift $c|_{k(\Delta)}$ to an isomorphism $\tilde{c}$ between $X$ and $Y \otimes_{k\Delta} X$ as dg-$k\Delta^{op} \otimes A$-modules.

By the universal property of the orbit category, the bimodule $X$ endowed with the isomorphism $\tilde{c}$ yields a triangle functor from $D^b(mod k\Delta)/\Phi$ to $T$ which comes from a bimodule $Z$ in $D(B^{op} \otimes A)$.

The functor $-\otimes_{k\Delta} Z$ is essentially surjective. Let us show that it is fully faithful. For $M$ and $N$ objects of $D^b(mod k\Delta)$ we have the following commutative diagram:

$$\begin{array}{c}
\bigoplus_{n \in \mathbb{Z}} \text{Hom}_D(M, \Phi^n N) \\
\pi \Downarrow \\
\text{Hom}_{D/\Phi}(\pi M, \pi N) \\
\Downarrow \\
\text{Hom}_T(FM, FN),
\end{array}$$

where $D$ means $D^b(mod k\Delta)$. The two diagonal morphisms are isomorphisms, thus so is the horizontal morphism. This proves that $-\otimes_{k\Delta} Z$ is a triangle equivalence between the orbit category $D^b(mod k\Delta)/\Phi$ and $T$. $\square$

8. Triangulated structure on the category of projectives

Let $k$ be a algebraically closed field and $\mathcal{P}$ a $k$-linear category with split idempotents. The category $\text{mod} \mathcal{P}$ of contravariant finitely presented functors from $\mathcal{P}$ to $mod k$ is exact. As the idempotents split, the projectives of $\text{mod} \mathcal{P}$ coincide with the representables. Thus the Yoneda functor gives a natural equivalence between $\mathcal{P}$ and $\text{proj} \mathcal{P}$. Assume besides that $\text{mod} \mathcal{P}$ has a structure of Frobenius category. The stable category $\text{mod} \mathcal{P}$ is a triangulated category, we write $\Sigma$ for the suspension functor.

Let $S$ be an auto-equivalence of $\mathcal{P}$. It can be extended to an exact functor from $\text{mod} \mathcal{P}$ to $\text{mod} \mathcal{P}$ and thus to a triangle functor of $\text{mod} \mathcal{P}$. The aim of this part is to find a necessary condition on the functor $S$ such that the category $(\mathcal{P}, S)$ has a triangulated structure. Heller already showed [14, thm 16.4] that if there exists an isomorphism of triangle functors between $S$ and $\Sigma^3$, then $\mathcal{P}$ has a pretriangulated structure. But he did not succeed in proving the octahedral axiom. We are going to impose a stronger condition on the functor $S$ and prove the following theorem:
Theorem 8.0.3. Assume there exists an exact sequence of exact functors from $\text{mod} \mathcal{P}$ to $\text{mod} \mathcal{P}$:

$$0 \rightarrow \text{Id} \rightarrow X^0 \rightarrow X^1 \rightarrow X^2 \rightarrow S \rightarrow 0,$$

where the $X^i$, $i = 0, 1, 2$, take values in $\text{proj} \mathcal{P}$. Then the category $\mathcal{P}$ has a structure of triangulated category with suspension functor $S$.

For an $M$ in $\text{mod} \mathcal{P}$, denote $T_M : X^0 M \rightarrow X^1 M \rightarrow X^2 M \rightarrow SX^0 M$ a standard triangle. A triangle of $\mathcal{P}$ will be a sequence $X : P \xrightarrow{u} Q \xrightarrow{v} R \xrightarrow{w} SP$ which is isomorphic to a standard triangle $T_M$ for an $M$ in $\text{mod} \mathcal{P}$.

8.1. $\mathcal{S}$-complexes, $\Phi$-$\mathcal{S}$-complexes and standard triangles. Let $\text{AcP}(\text{mod} \mathcal{P})$ be the category of acyclic complexes with projective components. It is a Frobenius category whose projective-injectives are the contractible complexes, i.e. the complexes homotopic to zero. The functor $Z^0 : \text{AcP}(\text{mod} \mathcal{P}) \to \text{mod} \mathcal{P}$ which sends a complex

$$\cdots \rightarrow X^{-1} \xrightarrow{x^{-1}} X^0 \xrightarrow{x^0} X^1 \xrightarrow{x^1} \cdots$$

to the kernel of $x^0$ is an exact functor. It sends the projective-injectives to projective-injectives and induces a triangle equivalence between $\text{AcP}(\text{mod} \mathcal{P})$ and $\text{mod} \mathcal{P}$.

Definition 8.1.1. An object of $\text{AcP}(\text{mod} \mathcal{P})$ is called an $\mathcal{S}$-complex if it is $\mathcal{S}$-periodic, i.e. if it has the following form:

$$\cdots \rightarrow P \xrightarrow{u} Q \xrightarrow{v} R \xrightarrow{w} SP \xrightarrow{Su} SQ \xrightarrow{Sv} \cdots.$$

The category $\mathcal{S}$-$\text{comp}$ of $\mathcal{S}$-complexes with $\mathcal{S}$-periodic morphisms is a non full subcategory of $\text{AcP}(\text{mod} \mathcal{P})$. It is a Frobenius category. The projective-injectives are the $\mathcal{S}$-contractibles, i.e. the complexes homotopic to zero with an $\mathcal{S}$-periodic homotopy. Using the functor $Z^0$, we get an exact functor from $\mathcal{S}$-$\text{comp}$ to $\text{mod} \mathcal{P}$ which induces a triangle functor:

$$Z^0 : \mathcal{S}$-$\text{comp} \rightarrow \text{mod} \mathcal{P}.$$

Fix a sequence as in theorem 8.0.3. Clearly, it induces for each object $M$ of $\text{mod} \mathcal{P}$, a functorial isomorphism in $\text{mod} \mathcal{P}$, $\Phi_M : \Sigma^3 M \rightarrow SM$.

Let $Y$ be an $\mathcal{S}$-complex,

$$\cdots \rightarrow P \xrightarrow{u} Q \xrightarrow{v} R \xrightarrow{w} SP \xrightarrow{Su} SQ \xrightarrow{Sv} \cdots.$$

Let $M$ be the kernel of $u$. Then $Y$ induces an isomorphism $\theta$ (in $\text{mod} \mathcal{P}$) between $\Sigma^3 M$ and $SM$. If $\theta$ is equal to $\Phi_M$, we will say that $X$ is a $\Phi$-$\mathcal{S}$-complex.

Let $M$ be an object of $\text{mod} \mathcal{P}$. The standard triangle $T_M$ can be seen as a $\Phi$-$\mathcal{S}$-complex:

$$\cdots \rightarrow X^0 M \rightarrow X^1 M \rightarrow X^2 M \rightarrow SX^0 M \rightarrow SX^1 M \rightarrow \cdots.$$

The functor $T$ which sends an object $M$ of $\text{mod} \mathcal{P}$ to the $\mathcal{S}$-complex $T_M$ is exact since the $X^i$ are exact. It satisfies the relation $Z^0 \circ T \simeq \text{Id}_{\text{mod} \mathcal{P}}$. Moreover, as it preserves the projective-injectives, it induces a triangle functor:

$$T : \text{mod} \mathcal{P} \rightarrow \mathcal{S}$-$\text{comp}.$$
8.2. Properties of the functors \( Z^0 \) and \( T \).

**Lemma 8.2.1.** An \( S \)-complex which is homotopy-equivalent to a \( \Phi \)-\( S \)-complex is a \( \Phi \)-\( S \)-complex.

**Proof.** Let \( X : P \xrightarrow{u} Q \xrightarrow{v} R \xrightarrow{w} SP \) be an \( S \)-complex homotopy-equivalent to the \( \Phi \)-\( S \)-complex \( X' : P' \xrightarrow{u'} Q' \xrightarrow{v'} R' \xrightarrow{w'} SP' \). Let \( M \) be the kernel of \( u \) and \( M' \) the kernel of \( u' \). By assumption, there exists a \( S \)-periodic homotopy equivalence \( f \) from \( X \) to \( X' \), which induces a morphism \( g = Z^0 f : M \to M' \). Thus, we get the following commutative diagram:

![Diagram](attachment:image.png)

The morphism \( g \) is an isomorphism of \( \text{mod} \mathcal{P} \) since \( f \) is an isomorphism of \( S \text{-comp} \). Thus the morphisms \( \Sigma^3 g \) and \( Sg \) are isomorphisms of \( \text{mod} \mathcal{P} \). The following equality in \( \text{mod} \mathcal{P} \)

\[
\theta = (Sg)^{-1} \Phi_{M'} \Sigma^3 g = \Phi_M
\]

shows that the complex \( X \) is a \( \Phi \)-\( S \)-complex. \( \square \)

**Lemma 8.2.2.** Let

\[
X : P \xrightarrow{u} Q \xrightarrow{v} R \xrightarrow{w} SP \quad \text{and} \quad X' : P' \xrightarrow{u'} Q' \xrightarrow{v'} R' \xrightarrow{w'} SP'
\]

be two \( \Phi \)-\( S \)-complexes. Suppose that we have a commutative square:

![Diagram](attachment:image.png)

Then, there exists a morphism \( f^2 : R \to R' \) such that \((f^0, f^1, f^2)\) extends to an \( S \)-periodic morphism from \( X \) to \( X' \).

**Proof.** Let \( M \) be the kernel of \( u \), \( M' \) be the kernel of \( u' \) and \( f : M \to M' \) be the morphism induced by the commutative square. As \( R \) and \( R' \) are projective-injective objects, we can find a morphism \( g^2 : R \to R' \) such that the following square commutes:

![Diagram](attachment:image.png)
The morphism $g^2$ induces a morphism $g : SM \to SM'$ such that the following square is commutative in $\text{mod} \mathcal{P}$:

$$
\begin{array}{ccc}
\Sigma^3 M & \xrightarrow{\Phi_M} & SM \\
\downarrow \Sigma^3 f & & \downarrow g \\
\Sigma^3 M' & \xrightarrow{\Phi_{M'}} & SM'.
\end{array}
$$

Thus the morphisms $Sf$ and $g$ are equal in $\text{mod} \mathcal{P}$, i.e. there exists a projective-injective $I$ of $\text{mod} \mathcal{P}$ and morphisms $\alpha : SM \to I$ and $\beta : I \to SM'$ such that $g - Sf = \beta \alpha$. Let $p$ (resp. $p'$) be the epimorphism from $R$ onto $SM$ (resp. from $R'$ onto $SM'$). Then, as $I$ is projective, $\beta$ factors through $p'$.

We put $f^2 = g^2 - \gamma \alpha p$. Then obviously, we have the equalities $f^2 v = v' f^1$ and $w' f^2 = Sf^0 w$. Thus the morphism $(f^0, f^1, f^2)$ extends to a morphism of $S$-comp. \qed

**Proposition 8.2.3.** The functor $Z^0 : \Phi \text{-}$-$\text{comp} \to \text{mod} \mathcal{P}$ is full and essentially surjective. Its kernel is an ideal whose square vanishes.

**Proof.** The functor $Z^0$ is essentially surjective since we have the relation $Z^0 \circ T = \text{Id}_{\text{mod} \mathcal{P}}$.

Let us show that $Z^0$ is full. Let

$$
X : P \xrightarrow{u} Q \xrightarrow{v} R \xrightarrow{w} SP \quad \text{and} \quad X' : P' \xrightarrow{u'} Q' \xrightarrow{v'} R' \xrightarrow{w'} SP'
$$

be two $\Phi$-$\text{comp}$ complexes. Let $M$ (resp. $M'$) be the kernel of $u$ (resp. $u'$). As $P$, $Q$, $P'$ and $Q'$ are projective-injective, there exist morphisms $f^0 : P \to P'$ and $f^1 : Q \to Q'$ such that the following diagram commutes:

$$
\begin{array}{ccc}
M & \xrightarrow{f} & P \\
\downarrow & & \downarrow f^0 \\
M' & \xrightarrow{f^1} & P'.
\end{array}
$$

Now the result follows from lemma 8.2.2.
Now let \( f : X \to X' \) be a morphism in the kernel of \( Z^0 \). Up to homotopy, we can suppose that \( f \) has the following form:

\[
\begin{array}{cccc}
P & \overset{u}{\longrightarrow} & Q & \overset{v}{\longrightarrow} & R & \overset{w}{\longrightarrow} & SP \\
0 & & 0 & & f^1 & & 0 \\
P' & \overset{u'}{\longrightarrow} & Q' & \overset{v'}{\longrightarrow} & R' & \overset{w'}{\longrightarrow} & SP'.
\end{array}
\]

As the composition \( w'f^2 \) vanishes and as \( Q' \) is projective-injective, \( f^2 \) factors through \( v' \). For the same argument, \( f'^2 \) factors through \( u' \). If \( f \) and \( f' \) are composable morphisms of the kernel of \( Z^0 \), we get the following diagram:

\[
\begin{array}{cccc}
P & \overset{u}{\longrightarrow} & Q & \overset{v}{\longrightarrow} & R & \overset{w}{\longrightarrow} & SP \\
0 & & 0 & & h^2 & & 0 \\
P' & \overset{u'}{\longrightarrow} & Q' & \overset{v'}{\longrightarrow} & R' & \overset{w'}{\longrightarrow} & SP'.
\end{array}
\]

The composition \( f'f \) vanishes obviously.

**Corollary 8.2.4.** A \( \Phi \)-\( S \)-complex morphism \( f \) which induces an isomorphism \( Z^0(f) \) in \( \text{mod}\, \mathcal{P} \) is an homotopy-equivalence.

This corollary comes from the previous theorem and from the following lemma.

**Lemma 8.2.5.** Let \( F : \mathcal{C} \to \mathcal{C}' \) be a full functor between two additive categories. If the kernel of \( F \) is an ideal whose square vanishes, then \( F \) detects isomorphisms.

**Proof.** Let \( u \in \text{Hom}_\mathcal{C}(A, B) \) be a morphism in \( \mathcal{C} \) such that \( Fu \) is an isomorphism. Since the functor \( F \) is full, there exists \( v \in \text{Hom}_\mathcal{C}(B, A) \) such that \( Fv = (Fu)^{-1} \). The morphism \( w = uv - Id_B \) is in the kernel of \( F \), thus \( w^2 \) vanishes. Then the morphism \( v(Id_B - w) \) is a right inverse of \( u \). In the same way we show that \( u \) has a left inverse, so \( u \) is an isomorphism.

**Proposition 8.2.6.** The category of \( \Phi \)-\( S \)-complexes is equivalent to the category of \( S \)-complexes which are homotopy-equivalent to standard triangles.

**Proof.** Since standard triangles are \( \phi \)-\( S \)-complexes, each \( S \)-complex that is homotopy equivalent to a standard triangle is a \( \Phi \)-\( S \)-complex (lemma 8.2.3).

Let \( X : P \overset{u}{\longrightarrow} Q \overset{v}{\longrightarrow} R \overset{w}{\longrightarrow} SP \) be a \( \Phi \)-\( S \)-complex. Let \( M \) be the kernel of \( u \). Then there exist morphisms \( f^1 : P \to X^0M \) and \( f^1 : Q \to X^1M \) such that the following diagram is commutative:

\[
\begin{array}{ccc}
M & \overset{f^0}{\longrightarrow} & P \\
\downarrow & & \downarrow f^1 \\
M & \longrightarrow & X^0M \\
\end{array}
\]

\[
\begin{array}{ccc}
M & \overset{f^0}{\longrightarrow} & X^0M \\
\downarrow & & \downarrow f^1 \\
M & \longrightarrow & X^1M.
\end{array}
\]

We can complete (lemma 8.2.3) \( f \) into an \( S \)-periodic morphism from \( X \) in \( T_M \). The morphism \( f \) satisfies \( Z^0f = Id_M \), so \( Z^0(T_M) \) and \( Z^0(X) \) are equal in \( \text{mod}\, \mathcal{P} \). By the corollary, \( T_M \) and \( X \) are homotopy-equivalent. Thus the inclusion functor \( T \) is essentially surjective.

\[\square\]
These two diagrams summarize the results of this section:

8.3. **Proof of theorem 8.0.3.** We are going to show that the $\Phi$-$S$-complexes form a system of triangles of the category $\mathcal{P}$. We use triangle axioms as in [22].

**TR0:** For each object $M$ of $\mathcal{P}$, the $S$-complex $\begin{array}{ccc} M & \longrightarrow & M \\ & \searrow & \\
0 & \longrightarrow & SM \end{array}$ is homotopy-equivalent to the zero complex, so is a $\Phi$-$S$-complex.

**TR1:** Let $u : P \rightarrow Q$ be a morphism of $\mathcal{P}$, and let $M$ be its kernel. We can find morphisms $f^0$ and $f^1$ so as to obtain a commutative square:

We form the following push-out:

$$
\begin{array}{ccc}
0 & \longrightarrow & \text{Coker}a \\
\gamma & & \\
0 & \longrightarrow & \text{Coker}u
\end{array}
\begin{array}{ccc}
X^0M & \longrightarrow & X^1M \\
\gamma & & \\
X^2M & \longrightarrow & \Sigma X^0M
\end{array}
\begin{array}{ccc}
\text{Coker}a & \longrightarrow & SM \\
\Sigma \gamma & & \\
\Sigma \text{Coker}a & \longrightarrow & SM
\end{array}
\begin{array}{ccc}
P & \longrightarrow & Q \\
\gamma & & \\
\Sigma \text{Coker}u & \longrightarrow & SM
\end{array}
\begin{array}{ccc}
R & \longrightarrow & SM \\
\gamma & & \\
R & \longrightarrow & SM
\end{array}
$$

It induces a triangle morphism of the triangular category $\text{mod}\mathcal{P}$:
The morphism \( \gamma \) is an isomorphism in \( \text{mod} \mathcal{P} \) since \( \text{Coker } a \) and \( \text{Coker } u \) are canonically isomorphic to \( \Sigma^2 M \) in \( \text{mod} \mathcal{P} \). By the five lemma, \( X^2 M \to R \) is an isomorphism in \( \text{mod} \mathcal{P} \). Since \( X^2 M \) is projective-injective, so is \( R \). Thus the complex \( P \xrightarrow{u} Q \xrightarrow{v} R \xrightarrow{w} \mathcal{S} \mathcal{P} \) is an \( S \)-complex. Then we have to see that it is a \( \Phi \)-\( S \)-complex. Let \( \theta \) be the isomorphism between \( \mathcal{S} \mathcal{M} \) and \( \Sigma^2 M \) induced by this complex. We write \( \alpha \) (resp. \( \beta \)) for the canonical isomorphism in \( \text{mod} \mathcal{P} \) between \( \Sigma^2 M \) and \( \text{Coker } a \) (resp. \( \text{Coker } u \)). From the commutative diagram:

\[
\begin{array}{ccccccc}
\Sigma^2 M & \xrightarrow{\alpha} & X^2 M & \xrightarrow{\beta} & SM & \xrightarrow{\Phi_M} & \Sigma M \\
\downarrow{\gamma} & & \downarrow{\Phi} & & \downarrow{\Sigma} & & \downarrow{\Sigma} \\
\text{Coker } u & \xrightarrow{\theta} & R & \xrightarrow{\phi} & SM & \xrightarrow{\Sigma \phi} & \Sigma M \\
\end{array}
\]

we deduce the equality \( \theta = (\Sigma \beta)^{-1} \gamma \Sigma \alpha \Phi_M = \Phi_M \) in \( \text{mod} \mathcal{P} \). The constructed \( S \)-complex is a \( \Phi \)-\( S \)-complex.

**TR2:** Let \( X : P \xrightarrow{u} Q \xrightarrow{v} R \xrightarrow{w} \mathcal{S} \mathcal{P} \) be a \( \Phi \)-\( S \)-complex. It is homotopy-equivalent to a standard triangle \( T_M \). Thus the \( S \)-complex

\[
X' : Q \xrightarrow{-v} R \xrightarrow{-w} \mathcal{S} \mathcal{P} \xrightarrow{-\phi} \mathcal{S} \mathcal{Q}
\]

is homotopy-equivalent to \( T_M[1] \). Since \( T \) is a triangle functor, the objects \( T \Sigma M \) and \( T_M[1] \) are isomorphic in the stable category \( S \text{-comp} \), i.e. they are homotopy-equivalent. Thus, by lemma 8.2.1, \( T_M[1] \) is a \( \Phi \)-\( S \)-complex and then so is \( X' \).

**TR3:** This axiom is a direct consequence of lemma 8.2.2.

**TR4:** Let \( X \) and \( X' \) be two \( \Phi \)-\( S \)-complexes and suppose we have a commutative diagram:

\[
\begin{array}{ccccccc}
X : P & \xrightarrow{u} & Q & \xrightarrow{v} & R & \xrightarrow{w} & \mathcal{S} \mathcal{P} \\
\downarrow{f^0} & & \downarrow{f^1} & & \downarrow{\mathcal{S} f^0} & & \\
X' : P' & \xrightarrow{u'} & Q' & \xrightarrow{v'} & R' & \xrightarrow{w'} & \mathcal{S} \mathcal{P}'.
\end{array}
\]

Let \( M \) (resp. \( M' \)) be the kernel of \( u \) (resp. \( u' \)), and \( g : M \to M' \) the induced morphism. The morphism \( Tg : TM \to TM' \) induces a \( S \)-complex morphism \( \tilde{g} = (g^0, g^1, g^2) \) between \( X \) and \( X' \).

We are going to show that we can find a morphism \( f^2 : R \to R' \) such that \( (f^0, f^1, f^2) \) can be extended in an \( S \)-complex morphism that is homotopic to \( \tilde{g} \). As \( (g^0, g^1) \) and \( (f^0, f^1) \) induce the same morphism \( g \) in the kernels, we have some morphisms \( h^1 : Q \to P' \) and \( h^2 : R \to Q' \) such that \( f^0 - g^0 = h^1 u \) and \( f^1 - g^1 = u' h^1 + h^2 v \). We put \( f^2 = g^2 + v' h^2 \). We have the following equalities:

\[
\begin{align*}
f^2 v &= g^2 v + v' h^2 v & \text{and} & \quad w' f^2 = w' g^2 \\
&= v' (g^1 + h^2 v) & \text{and} & \quad (\mathcal{S} g^0') w \\
&= v' (f^1 - u' h^1) & \text{and} & \quad (\mathcal{S} f^0 - \mathcal{S} h^1 Su) w \\
&= v' f^1 & \text{and} & \quad (\mathcal{S} f^0) w
\end{align*}
\]
Thus \((f^0, f^1, f^2)\) can be extended to an \(S\)-periodic morphism \(\tilde{f}\) which is \(S\)-homotopic to \(\tilde{g}\). Their respective cones \(C(f)\) and \(C(g)\) are isomorphic as \(S\)-complexes. Moreover, since \(\tilde{g}\) is a composition of \(Tg : T_M \to T_{M'}\) with homotopy-equivalences, the cones \(C(g)\) and \(C(Tg)\) are homotopy-equivalent.

In \(\text{mod} \mathcal{P}\), we have a triangle

\[
\begin{array}{c}
M \xrightarrow{g} M' \xrightarrow{C(g)} \Sigma M.
\end{array}
\]

Since \(T\) is a triangle functor, the sequence

\[
\begin{array}{c}
T_M \xrightarrow{Tg} T_{M'} \xrightarrow{T(C(g))} T_{\Sigma M}
\end{array}
\]

is a triangle in \(S\text{-}\text{comp}\). But we know that \(T_M \xrightarrow{Tg} T_{M'} \xrightarrow{C(Tg)} T_{\Sigma M}\) is a triangle in \(S\text{-}\text{comp}\). Thus the objects \(C(Tg)\) and \(T_{C(g)}\) are isomorphic in \(S\text{-}\text{comp}\), i.e. homotopy-equivalent. Thus, the cone \(C(\tilde{f})\) of \(\tilde{f}\) is a \(\Phi\)-\(S\)-complex by lemma 8.2.1.

9. Application to the deformed preprojective algebras

In this section, we apply the theorem 8.0.3 to show that the category of finite dimensional projective modules over a deformed preprojective algebra of generalized Dynkin type (see [4]) is triangulated. This will give us some examples of non-standard triangulated categories with finitely many indecomposables.

9.1. Preprojective algebra of generalized Dynkin type. Recall the notations of [4]. Let \(\Delta\) be a generalized Dynkin graph of type \(A_n, D_n (n \geq 4), E_n (n = 6, 7, 8),\) or \(L_n\). Let \(Q_\Delta\) be the following associated quiver:

\[
\begin{array}{c}
\Delta = A_n (n \geq 1) : & 0 \xrightarrow{a_0} 1 \xrightarrow{a_1} 2 \xrightarrow{a_2} \cdots \xrightarrow{a_{n-2}} n - 2 \xrightarrow{a_{n-2}} n - 1 \\
\Delta = D_n (n \geq 4) : & 0 \\
& \xrightarrow{a_0} 2 \xrightarrow{a_2} 3 \xrightarrow{a_3} \cdots \xrightarrow{a_{n-2}} n - 2 \xrightarrow{a_{n-2}} n - 1 \\
\Delta = E_n (n = 6, 7, 8) : & 1 \xrightarrow{a_1} 2 \xrightarrow{a_2} 3 \xrightarrow{a_3} 4 \xrightarrow{a_4} \cdots \xrightarrow{a_{n-2}} n - 2 \xrightarrow{a_{n-2}} n - 1 \\
\Delta = L_n (n \geq 1) : & \epsilon = \sum_{i=1}^n a_i, \text{ for each vertex } i \text{ of } Q_\Delta.
\end{array}
\]

The preprojective algebra \(P(\Delta)\) associated to the graph \(\Delta\) is the quotient of the path algebra \(kQ_\Delta\) by the relations:

\[
\sum_{i,a} a_{i} \epsilon, \text{ for each vertex } i \text{ of } Q_\Delta.
\]
The following proposition is classical \[4\] prop 2.1.

**Proposition 9.1.1.** The preprojective algebra \(P(\Delta)\) is finite dimensional and self-injective. Its Nakayama permutation \(\nu\) is the identity for \(\Delta = A_{n}, D_{2n}, E_{7}, E_{8}\) and \(L_{n}\), and is of order 2 in all other cases.

9.2. Deformed preprojective algebras of generalized Dynkin type. Let us recall the definition of deformed preprojective algebra introduced by \[4\]. Let \(\Delta\) be a graph of generalized Dynkin type. We define an associated algebra \(R(\Delta)\) as follows:

\[
\begin{align*}
R(A_{n}) &= k; \\
R(D_{n}) &= k\langle x, y \rangle/(x^2, y^2, (x+y)^{n-2}); \\
R(E_{n}) &= k\langle x, y \rangle/(x^2, y^3, (x+y)^{n-3}); \\
R(L_{n}) &= k[x]/(x^{2n}).
\end{align*}
\]

Further, we fix an exceptional vertex in each graph as follows (with the notations of the previous section):

\[
\begin{align*}
0 & \text{ for } \Delta = A_{n} \text{ or } L_{n}, \\
2 & \text{ for } \Delta = D_{n}, \\
3 & \text{ for } \Delta = E_{n}.
\end{align*}
\]

Let \(f\) be an element of the square \(\text{rad}^2 R(\Delta)\) of the radical of \(R(\Delta)\). The **deformed preprojective algebra** \(P^{f}(\Delta)\) is the quotient of the path algebra \(kQ_{\Delta}\) by the relations:

\[
\sum_{\mathbf{a} = i} a_{\mathbf{a}} \quad \text{ for each non exceptional vertex } i \text{ of } Q,
\]

and

\[
\begin{align*}
a_{0}\bar{a}_{0} &= 0, \\
\bar{a}_{0}a_{0} + \bar{a}_{1}a_{1} + a_{2}\bar{a}_{2} + f(\bar{a}_{0}a_{0}, \bar{a}_{1}a_{1}), \text{ and } (\bar{a}_{0}a_{0} + \bar{a}_{1}a_{1})^{n-2} & \text{ for } \Delta = A_{n}, \\
\bar{a}_{0}a_{0} + \bar{a}_{2}a_{2} + a_{3}\bar{a}_{3} + f(\bar{a}_{0}a_{0}, \bar{a}_{2}a_{2}), \text{ and } (\bar{a}_{0}a_{0} + \bar{a}_{2}a_{2})^{n-3} & \text{ for } \Delta = E_{n}, \\
e^{2} + a_{0}\bar{e} + e f(e), \text{ and } e^{2n} & \text{ for } \Delta = L_{n}.
\end{align*}
\]

Note that if \(f\) is zero, we get the preprojective algebra \(P(\Delta)\).

9.3. Corollaries of \[4\]. The following proposition \[4\] prop 3.4 shows that the category \(\text{proj} P^{f}(\Delta)\) of finite-dimensional projective modules over a deformed preprojective algebra satisfies the hypothesis of theorem \[8\] 8.0.3.

**Proposition 9.3.1.** Let \(A = P^{f}(\Delta)\) be a deformed preprojective algebra. Then there exists an exact sequence of \(A\cdot A\)-bimodules

\[
0 \rightarrow \kappa_{A_{n-1}} A \rightarrow P_{2} \rightarrow P_{1} \rightarrow P_{0} \rightarrow A \rightarrow 0,
\]

where \(\Phi\) is an automorphism of \(A\) and where the \(P_{i}\)’s are projective as bimodules. Moreover, for each idempotent \(e_{i}\) of \(A\), we have \(\Phi(e_{i}) = e_{\nu(i)}\).

So we can easily deduce the corollary:

**Corollary 9.3.2.** Let \(P^{f}(\Delta)\) be a deformed preprojective algebra of generalized Dynkin type. Then the category \(\text{proj} P^{f}(\Delta)\) of finite dimensional projective modules is triangulated. The suspension is the Nakayama functor.

Indeed, if \(P_{i} = e_{i}A\) is a projective indecomposable, then \(P_{i} \otimes_{A} A_{\Phi}\) is equal to \(\Phi(e_{i})A = e_{\nu(i)}A\) thus to \(\nu(P_{i})\).

Now we are able to answer to the question of the previous part and find a triangulated category with finitely many indecomposables which is not standard. The proof of the following theorem comes essentially from the theorem \[4\] thm 1.3.
Theorem 9.3.3. Let $k$ be an algebraically closed field of characteristic 2. Then there exist $k$-linear triangulated categories with finitely many indecomposables which are not standard.

Proof. By theorem [4, thm 1.3], we know that there exist basic deformed preprojective algebras of generalized Dynkin type $P_f(\Delta)$ which are not isomorphic to $P(\Delta)$. Thus the categories $\text{proj} \ P_f(\Delta)$ and $\text{proj} \ P(\Delta)$ cannot be equivalent. But both are triangulated by corollary 9.3.2 and have the same AR-quiver $Z\Delta/\tau = Q_{\Delta}$. □

Conversely, we have the following theorem:

Theorem 9.3.4. Let $\mathcal{T}$ be a finite 1-Calabi-Yau triangulated category. Then $\mathcal{T}$ is equivalent to $\text{proj} \ \Lambda$ as $k$-category, where $\Lambda$ is a deformed preprojective algebra of generalized Dynkin type.

Proof. Let $M_1, \ldots, M_n$ be representatives of the isoclasses of indecomposable objects of $\mathcal{T}$. The $k$-algebra $\Lambda = \text{End}(\bigoplus_{i=1}^n M_i)$ is basic, finite-dimensional and selfinjective since $\mathcal{T}$ has a Serre duality. It is easy to see that $\mathcal{T}$ and $\text{proj} \ \Lambda$ are equivalent as $k$-categories.

Let $\text{mod} \ \Lambda$ be the category of finitely presented $\Lambda$-modules. It is a Frobenius category. Denote by $\Sigma$ the suspension functor of the triangulated category $\text{mod} \ \Lambda$. The category $\mathcal{T}$ is 1-Calabi-Yau, that is to say that the suspension functor $S$ of the triangulated category $\mathcal{T}$ and the Serre functor $\nu$ are isomorphic. But in $\text{mod} \ \Lambda$, the functors $S$ and $\Sigma^3$ are isomorphic. Thus, for each non-projective simple $\Lambda$-module $M$ we have an isomorphism $\Sigma^3 M \cong \nu M$. By [4, thm 1.2], we get immediately the result. □

References