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ON THE STRUCTURE OF TRIANGULATED CATEGORIES WITH FINITELY MANY INDECOMPOSABLES

CLAIRE AMIOT

ABSTRACT. We study the problem of classifying triangulated categories with finite-dimensional morphism spaces and finitely many indecomposables over an algebraically closed field. We obtain a new proof of the following result due to Xiao and Zhu: the Auslander-Reiten quiver of such a category is of the form $\mathbb{Z}\Delta/G$ where Δ is a disjoint union of simply laced Dynkin diagrams and G a weakly admissible group of automorphisms of $\mathbb{Z}\Delta$. Then we prove that for ‘most’ groups G , the category \mathcal{T} is standard, *i.e.* k -linearly equivalent to an orbit category $\mathcal{D}^b(\text{mod } k\Delta)/\Phi$. This happens in particular when \mathcal{T} is maximal d -Calabi-Yau with $d \geq 2$. Moreover, if \mathcal{T} is standard and algebraic, we can even construct a triangle equivalence between \mathcal{T} and the corresponding orbit category. Finally we give a sufficient condition for the category of projectives of a Frobenius category to be triangulated. This allows us to construct non standard 1-Calabi-Yau categories using deformed preprojective algebras of generalized Dynkin type.

INTRODUCTION

Let k be an algebraically closed field and \mathcal{T} a small k -linear triangulated category (see [31]) with split idempotents. We assume that

a) \mathcal{T} is *Hom-finite*, *i.e.* the space $\text{Hom}_{\mathcal{T}}(X, Y)$ is finite-dimensional for all objects X, Y of \mathcal{T} .

It follows that indecomposable objects of \mathcal{T} have local endomorphism rings and that each object of \mathcal{T} decomposes into a finite direct sum of indecomposables [9, 3.3]. We assume moreover that

b) \mathcal{T} is *locally finite*, *i.e.* for each indecomposable X of \mathcal{T} , there are at most finitely many isoclasses of indecomposables Y such that $\text{Hom}_{\mathcal{T}}(X, Y) \neq 0$.

It was shown in [32] that condition b) implies its dual. Condition b) holds in particular if we have

b') \mathcal{T} is *additively finite*, *i.e.* there are only finitely many isomorphism classes of indecomposables in \mathcal{T} .

The study of particular classes of such triangulated categories \mathcal{T} has a long history. Let us briefly recall some of its highlights:

(1) If A is a representation-finite selfinjective algebra, then the stable category \mathcal{T} of finite-dimensional (right) A -modules satisfies our assumptions and is additively finite. The structure of the underlying k -linear category of \mathcal{T} was determined by C. Riedtmann in [24], [25], [26] and [27].

(2) In [10], D. Happel showed that the bounded derived category of the category of finite-dimensional representations of a representation-finite quiver is locally finite and described its underlying k -linear category.

(3) The stable category $\underline{\text{CM}}(R)$ of Cohen-Macaulay modules over a commutative complete local Gorenstein isolated singularity R of dimension d is a Hom-finite triangulated category which is $(d - 1)$ -Calabi-Yau (cf. for example [17] and [34]). In [2], M. Auslander and I. Reiten showed that if the dimension of R is 1, then the category $\underline{\text{CM}}(R)$ is additively finite and computed the shape of components of its Auslander-Reiten quiver.

(4) The cluster category \mathcal{C}_Q of a finite quiver Q without oriented cycles was introduced in [7] if Q is an orientation of a Dynkin diagram of type \mathbb{A} and in [6] in the general case. The category \mathcal{C}_Q is triangulated [19] and, if Q is representation-finite, satisfies *a*) and *b'*).

In a recent article [32], J. Xiao and B. Zhu determined the structure of the Auslander-Reiten quiver of a locally finite triangulated category. In this paper, we obtain the same result with a new proof in section 4, namely that each connected component of the Auslander-Reiten quiver of the category \mathcal{T} is of the form $\mathbb{Z}\Delta/G$, where Δ is a simply laced Dynkin diagram and G is trivial or a weakly admissible group of automorphisms. We are interested in the k -linear structure of \mathcal{T} . If the Auslander-Reiten quiver of \mathcal{T} is of the form $\mathbb{Z}\Delta$, we show that the category \mathcal{T} is standard, *i.e.* it is equivalent to the mesh category $k(\mathbb{Z}\Delta)$. Then in section 6, we prove that \mathcal{T} is standard if the number of vertices of $\Gamma = \mathbb{Z}\Delta/G$ is strictly greater than the number of isoclasses of indecomposables of $\text{mod } k\Delta$. In the last section, using [4] we construct examples of non standard triangulated categories such that $\Gamma = \mathbb{Z}\Delta/\tau$.

Finally, in the standard cases, we are interested in the triangulated structure of \mathcal{T} . For this, we need to make additional assumptions on \mathcal{T} . If the Auslander-Reiten quiver is of the form $\mathbb{Z}\Delta$, and if \mathcal{T} is the base of a tower of triangulated categories [18], we show that there is a triangle equivalence between \mathcal{T} and the derived category $\mathcal{D}^b(\text{mod } k\Delta)$. For the additively finite cases, we have to assume that \mathcal{T} is standard and algebraic in the sense of [20]. We then show that \mathcal{T} is (algebraically) triangle equivalent to the orbit category of $\mathcal{D}^b(\text{mod } k\Delta)$ under the action of a weakly admissible group of automorphisms. In particular, for each $d \geq 2$, the algebraic triangulated categories with finitely many indecomposables which are maximal Calabi-Yau of CY-dimension d are parametrized by the simply laced Dynkin diagrams.

Our results apply in particular to many stable categories $\text{mod } A$ of representation-finite selfinjective algebras A . These algebras were classified up to stable equivalence by C. Riedtmann [25] [27] and H. Asashiba [1]. In [5], J. Białkowski and A. Skowroński give a necessary and sufficient condition on these algebras so that their stable categories $\text{mod } A$ are Calabi-Yau. In [15] and [16], T. Holm and P. Jørgensen prove that certain stable categories $\text{mod } A$ are in fact d -cluster categories. These results can also be proved using our corollary 7.0.6.

This paper is organized as follows: In section 1, we prove that \mathcal{T} has Auslander-Reiten triangles. Section 2 is dedicated to definitions about stable valued translation quivers and admissible automorphisms groups [12], [13], [8]. We show in section 3 that the Auslander-Reiten quiver of \mathcal{T} is a stable valued quiver and in section 4, we reprove the result of J. Xiao and B. Zhu [32]: The Auslander-Reiten quiver is a disjoint union of quivers $\mathbb{Z}\Delta/G$, where Δ is a Dynkin quiver of type \mathbb{A}, \mathbb{D} or \mathbb{E} , and G a weakly admissible group of automorphisms. In section 5, we construct a covering functor $\mathcal{D}^b(\text{mod } k\Delta) \rightarrow \mathcal{T}$ using Riedtmann's method [24]. Then, in section 6, we exhibit some combinatorial cases in which \mathcal{T} has to be standard, in particular when \mathcal{T} is maximal d -Calabi-Yau with $d \geq 2$. Section 7 is dedicated to the algebraic case. If \mathcal{T} is algebraic and standard, we can construct a triangle equivalence between \mathcal{T} and an orbit category. If \mathcal{P} is a k -category such that $\text{mod } \mathcal{P}$ is a Frobenius category satisfying certain conditions, we will prove in section 8 that \mathcal{P} has naturally a triangulated structure. This allows us to deduce in section 9 that the category $\text{proj } P^f(\Delta)$ of the projective modules over a deformed preprojective algebra of generalized Dynkin type [4] is naturally triangulated and to reduce the classification of the additively finite triangulated categories which are 1-Calabi-Yau to that of the deformed preprojective algebras in the sense of [4].

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NOTATION AND TERMINOLOGY

We work over an algebraically closed field k . By a *triangulated category*, we mean a k -linear triangulated category \mathcal{T} . We write S for the suspension functor of \mathcal{T} and $U \xrightarrow{u} V \xrightarrow{v} W \xrightarrow{w} SU$ for a distinguished triangle. We say that \mathcal{T} is *Hom-finite* if for each pair X, Y of objects in \mathcal{T} , the space $\mathrm{Hom}_{\mathcal{T}}(X, Y)$ is finite-dimensional over k . The category \mathcal{T} will be called a *Krull-Schmidt* category if each object is isomorphic to a finite direct sum of indecomposable objects and the endomorphism ring of an indecomposable object is a local ring. This implies that idempotents of \mathcal{T} split [11, I 3.2]. The category \mathcal{T} will be called *locally finite* if for each indecomposable X of \mathcal{T} , there are only finitely many isoclasses of indecomposables Y such that $\mathrm{Hom}_{\mathcal{T}}(X, Y) \neq 0$. This property is selfdual by [32, prop 1.1].

The *Serre functor* will be denoted by ν (see definition in section 1). The *Auslander-Reiten translation* will always be denoted by τ (section 1).

Let \mathcal{T} and \mathcal{T}' be two triangulated categories. An *S-functor* (F, ϕ) is given by a k -linear functor $F : \mathcal{T} \rightarrow \mathcal{T}'$ and a functor isomorphism ϕ between the functors $F \circ S$ and $S' \circ F$, where S is the suspension of \mathcal{T} and S' the suspension of \mathcal{T}' . The notion of ν -functor, or τ -functor is then clear. A *triangle functor* is an *S-functor* (F, ϕ) such that for each triangle $U \xrightarrow{u} V \xrightarrow{v} W \xrightarrow{w} SU$ of \mathcal{T} , the sequence

$$FU \xrightarrow{Fu} FV \xrightarrow{Fv} FW \xrightarrow{\phi_U \circ Fw} S'FU$$

is a triangle of \mathcal{T}' . The category \mathcal{T} is *Calabi-Yau* if there exists an integer $d > 0$ such that we have a triangle functor isomorphism between S^d and ν . We say that \mathcal{T} is *maximal d-Calabi-Yau* if \mathcal{T} is *d-Calabi-Yau* and if for each covering functor $\mathcal{T}' \rightarrow \mathcal{T}$ with \mathcal{T}' *d-Calabi-Yau*, we have a k -linear equivalence between \mathcal{T} and \mathcal{T}' .

For an additive k -category \mathcal{E} , we write $\mathrm{mod} \mathcal{E}$ for the category of contravariant finitely presented functors from \mathcal{E} to $\mathrm{mod} k$ (section 8), and if the projectives of $\mathrm{mod} \mathcal{E}$ coincide with the injectives, $\underline{\mathrm{mod}} \mathcal{E}$ will be the *stable category*.

1. SERRE DUALITY AND AUSLANDER-REITEN TRIANGLES

1.1. Serre duality. Recall from [23] that a *Serre functor* for \mathcal{T} is an autoequivalence $\nu : \mathcal{T} \rightarrow \mathcal{T}$ together with an isomorphism $D\mathrm{Hom}_{\mathcal{T}}(X, ?) \simeq \mathrm{Hom}_{\mathcal{T}}(?, \nu X)$ for each $X \in \mathcal{T}$, where D is the duality $\mathrm{Hom}_k(?, k)$.

Theorem 1.1.1. *Let \mathcal{T} be a Krull-Schmidt, locally finite triangulated category. Then \mathcal{T} has a Serre functor ν .*

Proof. Let X be an object of \mathcal{T} . We write X^\wedge for the functor $\mathrm{Hom}_{\mathcal{T}}(?, X)$ and F for the functor $D\mathrm{Hom}_{\mathcal{T}}(X, ?)$. Using the lemma [23, I.1.6] we just have to show that F is representable. Indeed, the category \mathcal{T}^{op} is locally finite as well. The proof is in two steps.

Step 1: The functor F is finitely presented.

Let Y_1, \dots, Y_r be representatives of the isoclasses of indecomposable objects of \mathcal{T} such that FY_i is not zero. The space $\mathrm{Hom}(Y_i^\wedge, F)$ is finite-dimensional over k . Indeed it is isomorphic to FY_i by the Yoneda lemma. Therefore, the functor $\mathrm{Hom}(Y_i^\wedge, F) \otimes_k Y_i^\wedge$ is representable. We get an epimorphism from a representable

functor to F :

$$\bigoplus_{i=1}^r \text{Hom}(Y_i^\wedge, F) \otimes_k Y_i^\wedge \longrightarrow F.$$

By applying the same argument to its kernel we get a projective presentation of F of the form $U^\wedge \longrightarrow V^\wedge \longrightarrow F \longrightarrow 0$, with U and V in \mathcal{T} .

Step 2: A cohomological functor $H : \mathcal{T}^{op} \rightarrow \mathbf{mod} k$ is representable if and only if it is finitely presented.

Let $U^\wedge \xrightarrow{u^\wedge} V^\wedge \xrightarrow{\phi} H \longrightarrow 0$ be a presentation of H . We form a triangle $U \xrightarrow{u} V \xrightarrow{v} W \xrightarrow{w} SU$. We get an exact sequence

$$U^\wedge \xrightarrow{u^\wedge} V^\wedge \xrightarrow{v^\wedge} W^\wedge \xrightarrow{w^\wedge} (SU)^\wedge.$$

Since the composition of ϕ with u^\wedge is zero and H is cohomological, the morphism ϕ factors through v^\wedge . But H is the cokernel of u^\wedge , so v^\wedge factors through ϕ . We obtain a commutative diagram:

$$\begin{array}{ccccc} U^\wedge & \xrightarrow{u^\wedge} & V^\wedge & \xrightarrow{v^\wedge} & W^\wedge & \xrightarrow{w^\wedge} & (SU)^\wedge \\ & & \downarrow \phi & \nearrow i & \downarrow \phi' & & \\ & & H & & & & \end{array}$$

The equality $\phi' \circ i \circ \phi = \phi' \circ v^\wedge = \phi$ implies that $\phi' \circ i$ is the identity of H because ϕ is an epimorphism. We deduce that H is a direct factor of W^\wedge . The composition $i \circ \phi' = e^\wedge$ is an idempotent. Then $e \in \text{End}(W)$ splits and we get $H = W'^\wedge$ for a direct factor W' of W . □

1.2. Auslander-Reiten triangles.

Definition 1.2.1. [10] A triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} SX$ of \mathcal{T} is called an Auslander-Reiten triangle or AR-triangle if the following conditions are satisfied:

(AR1) X and Z are indecomposable objects;

(AR2) $w \neq 0$;

(AR3) if $f : W \rightarrow Z$ is not a retraction, there exists $f' : W \rightarrow Y$ such that $vf' = f$;

(AR3') if $g : X \rightarrow V$ is not a section, there exists $g' : Y \rightarrow V$ such that $g'u = g$.

Let us recall that, if (AR1) and (AR2) hold, the conditions (AR3) and (AR3') are equivalent. We say that a triangulated category \mathcal{T} has Auslander-Reiten triangles if, for any indecomposable object Z of \mathcal{T} , there exists an AR-triangle ending at Z : $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} SX$. In this case, the AR-triangle is unique up to triangle isomorphism inducing the identity of Z .

The following proposition is proved in [23, Proposition I.2.3]

Proposition 1.2.1. *The category \mathcal{T} has Auslander-Reiten triangles.*

The composition $\tau = S^{-1}\nu$ is called the Auslander-Reiten translation. An AR-triangle of \mathcal{T} ending at Z has the form:

$$\tau Z \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \nu Z.$$

2. VALUED TRANSLATION QUIVERS AND AUTOMORPHISM GROUPS

2.1. Translation quivers. In this section, we recall some definitions and notations concerning quivers [8]. A quiver $Q = (Q_0, Q_1, s, t)$ is given by the set Q_0 of its vertices, the set Q_1 of its arrows, a source map s and a tail map t . If $x \in Q_0$ is a vertex, we denote by x^+ the set of direct successors of x , and by x^- the set of its direct predecessors. We say that Q is *locally finite* if for each vertex $x \in Q_0$, there are finitely many arrows ending at x and starting at x (in this case, x^+ and x^- are finite sets). The quiver Q is said to be *without double arrows*, if two different arrows cannot have the same tail and source.

Definition 2.1.1. A stable translation quiver (Q, τ) is a locally finite quiver without double arrows with a bijection $\tau : Q_0 \rightarrow Q_0$ such that $(\tau x)^+ = x^-$ for each vertex x . For each arrow $\alpha : x \rightarrow y$, let $\sigma\alpha$ the unique arrow $\tau y \rightarrow x$.

Note that a stable translation quiver can have loops.

Definition 2.1.2. A valued translation quiver (Q, τ, a) is a stable translation quiver (Q, τ) with a map $a : Q_1 \rightarrow \mathbb{N}$ such that $a(\alpha) = a(\sigma\alpha)$ for each arrow α . If α is an arrow from x to y , we write a_{xy} instead of $a(\alpha)$.

Definition 2.1.3. Let Δ be an oriented tree. The repetition of Δ is the quiver $\mathbb{Z}\Delta$ defined as follows:

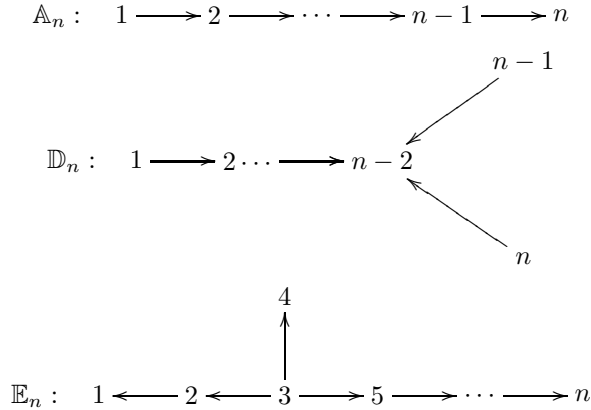
- $(\mathbb{Z}\Delta)_0 = \mathbb{Z} \times \Delta_0$
- $(\mathbb{Z}\Delta)_1 = \mathbb{Z} \times \Delta_1 \cup \sigma(\mathbb{Z} \times \Delta_1)$ with arrows $(n, \alpha) : (n, x) \rightarrow (n, y)$ and $\sigma(n, \alpha) : (n-1, y) \rightarrow (n, x)$ for each arrow $\alpha : x \rightarrow y$ of Δ .

The quiver $\mathbb{Z}\Delta$ with the translation $\tau(n, x) = (n-1, x)$ is clearly a stable translation quiver which does not depend (up to isomorphism) on the orientation of Δ (see [24]).

2.2. Groups of weakly admissible automorphisms.

Definition 2.2.1. An automorphism group G of a quiver is said to be admissible [24] if no orbit of G intersects a set of the form $\{x\} \cup x^+$ or $\{x\} \cup x^-$ in more than one point. It is said to be weakly admissible [8] if, for each $g \in G - \{1\}$ and for each $x \in Q_0$, we have $x^+ \cap (gx)^+ = \emptyset$.

Note that an admissible automorphism group is a weakly admissible automorphism group. Let us fix a numbering and an orientation of the simply-laced Dynkin trees.



Let Δ be a Dynkin tree. We define an automorphism S of $\mathbb{Z}\Delta$ as follows:

- if $\Delta = \mathbb{A}_n$, then $S(p, q) = (p+q, n+1-q)$;

- if $\Delta = \mathbb{D}_n$ with n even, then $S = \tau^{-n+1}$;
- if $\Delta = \mathbb{D}_n$ with n odd, then $S = \tau^{-n+1}\phi$ where ϕ is the automorphism of \mathbb{D}_n which exchanges n and $n-1$;
- if $\Delta = \mathbb{E}_6$, then $S = \phi\tau^{-6}$ where ϕ is the automorphism of \mathbb{E}_6 which exchanges 2 and 5, and 1 and 6;
- if $\Delta = \mathbb{E}_7$, then $S = \tau^{-9}$;
- and if $\Delta = \mathbb{E}_8$, then $S = \tau^{-15}$.

In [24, Anhang 2], Riedtmann describes all admissible automorphism groups of Dynkin diagrams. Here is a more precise result:

Theorem 2.2.1. *Let Δ be a Dynkin tree and G a non trivial group of weakly admissible automorphisms of $\mathbb{Z}\Delta$. Then G is isomorphic to \mathbb{Z} , and here is a list of its possible generators:*

- if $\Delta = \mathbb{A}_n$ with n odd, possible generators are τ^r and $\phi\tau^r$ with $r \geq 1$, where $\phi = \tau^{\frac{n+1}{2}}S$ is an automorphism of $\mathbb{Z}\Delta$ of order 2;
- if $\Delta = \mathbb{A}_n$ with n even, then possible generators are ρ^r , where $r \geq 1$ and where $\rho = \tau^{\frac{n}{2}}S$. (Since $\rho^2 = \tau^{-1}$, τ^r is a possible generator.)
- if $\Delta = \mathbb{D}_n$ with $n \geq 5$, then possible generators are τ^r and $\tau^r\phi$, where $r \geq 1$ and where $\phi = (n-1, n)$ is the automorphism of \mathbb{D}_n exchanging n and $n-1$.
- if $\Delta = \mathbb{D}_4$, then possible generators are $\phi\tau^r$, where $r \geq 1$ and where ϕ belongs to \mathfrak{S}_3 the permutation group on 3 elements seen as subgroup of automorphisms of \mathbb{D}_4 .
- if $\Delta = \mathbb{E}_6$, then possible generators are τ^r and $\phi\tau^r$, where $r \geq 1$ and where ϕ is the automorphism of \mathbb{E}_6 exchanging 2 and 5, and 1 and 6.
- if $\Delta = \mathbb{E}_n$ with $n = 7, 8$, possible generators are τ^r , where $r \geq 1$.

The unique weakly admissible automorphism group which is not admissible exists for \mathbb{A}_n , n even, and is generated by ρ .

3. PROPERTY OF THE AUSLANDER-REITEN TRANSLATION

We define the Auslander-Reiten quiver $\Gamma_{\mathcal{T}}$ of the category \mathcal{T} as a valued quiver (Γ, a) . The vertices are the isoclasses of indecomposable objects. Given two indecomposable objects X and Y of \mathcal{T} , we draw one arrow from $x = [X]$ to $y = [Y]$ if the vector space $\mathcal{R}(X, Y)/\mathcal{R}^2(X, Y)$ is not zero, where $\mathcal{R}(?, ?)$ is the radical of the bifunctor $\text{Hom}_{\mathcal{T}}(?, ?)$. A morphism of $\mathcal{R}(X, Y)$ which does not vanish in the quotient $\mathcal{R}(X, Y)/\mathcal{R}^2(X, Y)$ will be called *irreducible*. Then we put

$$a_{xy} = \dim_k \mathcal{R}(X, Y)/\mathcal{R}^2(X, Y).$$

Remark that the fact that \mathcal{T} is locally finite implies that its AR-quiver is locally finite. The aim of this section is to show that $\Gamma_{\mathcal{T}}$ with the translation τ defined in the first part is a valued translation quiver. In other words, we want to show the proposition:

Proposition 3.0.2. *If X and Y are indecomposable objects of \mathcal{T} , we have the equality*

$$\dim_k \mathcal{R}(X, Y)/\mathcal{R}^2(X, Y) = \dim_k \mathcal{R}(\tau Y, X)/\mathcal{R}^2(\tau Y, X).$$

Let us recall some definitions [11].

Definition 3.0.2. *A morphism $g : Y \rightarrow Z$ is called sink morphism if the following hold*

- (1) g is not a retraction;
- (2) if $h : M \rightarrow Z$ is not a retraction, then h factors through g ;
- (3) if u is an endomorphism of Y which satisfies $gu = u$, then u is an automorphism.

Dually, a morphism $f : X \rightarrow Y$ is called source morphism if the following hold:

- (1) f is not a section;
- (2) if $h : X \rightarrow M$ is not a section, then h factors through f ;
- (3) if u is an endomorphism of Y which satisfies $uf = f$, then u is an automorphism.

These conditions imply that X and Z are indecomposable. Obviously, if

$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} SX$ is an AR-triangle, then u is a source morphism and v is a sink morphism. Conversely, if $v \in \text{Hom}_{\mathcal{T}}(Y, Z)$ is a sink morphism (or if $u \in \text{Hom}_{\mathcal{T}}(X, Y)$ is a source morphism), then there exists an AR-triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} SX$ (see [11, I 4.5]).

The following lemma (and the dual statement) is proved in [28, 2.2.5].

Lemma 3.0.3. *Let g be a morphism from Y to Z , where Z is indecomposable and $Y = \bigoplus_{i=1}^r Y_i^{n_i}$ is the decomposition of Y into indecomposables. Then the morphism g is a sink morphism if and only if the following hold:*

- (1) For each $i = 1, \dots, r$ and $j = 1, \dots, n_i$, the morphism $g_{i,j}$ belongs to the radical $\mathcal{R}(Y_i, Z)$.
- (2) For each $i = 1, \dots, r$, the family $(\bar{g}_{i,j})_{j=1, \dots, n_i}$ forms a k -basis of the space $\mathcal{R}(Y_i, Z)/\mathcal{R}^2(Y_i, Z)$.
- (3) If $h \in \text{Hom}_{\mathcal{T}}(Y', Z)$ is irreducible and Y' indecomposable, then h factors through g and Y' is isomorphic to Y_i for some i .

Using this lemma, it is easy to see that proposition 3.0.2 holds. Thus, the Auslander-Reiten quiver $\Gamma_{\mathcal{T}} = (\Gamma, \tau, a)$ of the category \mathcal{T} is a valued translation quiver.

4. STRUCTURE OF THE AUSLANDER-REITEN QUIVER

This section is dedicated to an other proof of a theorem due to J. Xiao and B. Zhu ([33]):

Theorem 4.0.4. [33] *Let \mathcal{T} be a Krull-Schmidt, locally finite triangulated category. Let Γ be a connected component of the AR-quiver of \mathcal{T} . Then there exists a Dynkin tree Δ of type \mathbb{A} , \mathbb{D} or \mathbb{E} , a weakly admissible automorphism group G of $\mathbb{Z}\Delta$ and an isomorphism of valued translation quiver*

$$\theta : \Gamma \xrightarrow{\sim} \mathbb{Z}\Delta/G.$$

The underlying graph of the tree Δ is unique up to isomorphism (it is called the type of Γ), and the group G is unique up to conjugacy in $\text{Aut}(\mathbb{Z}\Delta)$.

In particular, if \mathcal{T} has an infinite number of isoclasses of indecomposable objects, then G is trivial, and Γ is the repetition quiver $\mathbb{Z}\Delta$.

4.1. Auslander-Reiten quivers with a loop. In this section, we suppose that the Auslander-Reiten quiver of \mathcal{T} contains a loop, i.e. there exists an arrow with same tail and source. Thus, we suppose that there exists an indecomposable X of \mathcal{T} such that

$$\dim_k \mathcal{R}(X, X)/\mathcal{R}^2(X, X) \geq 1.$$

Proposition 4.1.1. *Let X be an indecomposable object of \mathcal{T} . Suppose that we have $\dim_k \mathcal{R}(X, X)/\mathcal{R}^2(X, X) \geq 1$. Then τX is isomorphic to X .*

To prove this, we need a lemma.

Lemma 4.1.2. *Let $X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \dots \xrightarrow{f_n} X_{n+1}$ be a sequence of irreducible morphisms between indecomposable objects with $n \geq 2$. If the composition $f_n \circ f_{n-1} \cdots \circ f_1$ is zero, then there exists an i such that $\tau^{-1} X_i$ is isomorphic to X_{i+2} .*

Proof. The proof proceeds by induction on n . Let us show the assertion for $n = 2$. Suppose $X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3$ is a sequence such that $f_2 \circ f_1 = 0$. We can then construct an AR-triangle:

$$\begin{array}{ccccc} X_1 & \xrightarrow{(f_1, f_1)^T} & X_2 \oplus X & \xrightarrow{(g_1, g_2)} & \tau^{-1} X_1 & \longrightarrow & SX_1 \\ & & \downarrow (f_2, 0) & \swarrow \beta & & & \\ & & X_3 & & & & \end{array}$$

The composition $f_2 \circ f_1$ is zero, thus the morphism f_2 factors through g_1 . As the morphisms g_1 and f_2 are irreducible, we conclude that β is a retraction, and X_3 a direct summand of $\tau^{-1} X_1$. But X_1 is indecomposable, so β is an isomorphism between X_3 and $\tau^{-1} X_1$.

Now suppose that the property holds for an integer $n - 1$ and that we have $f_n f_{n-1} \cdots f_1 = 0$. If the composition $f_{n-1} \cdots f_1$ is zero, the proposition holds by induction. So we can suppose that for $i \leq n - 2$, the objects $\tau^{-1} X_i$ and X_{i+2} are not isomorphic. We show now by induction on i that for each $i \leq n - 1$, there exists a map $\beta_i : \tau^{-1} X_i \rightarrow X_{n+1}$ such that $f_n \cdots f_{i+1} = \beta_i g_i$ where $g_i : X_{i+1} \rightarrow \tau^{-1} X_i$ is an irreducible morphism. For $i = 1$, we construct an AR-triangle:

$$\begin{array}{ccccc} X_1 & \xrightarrow{(f_1, f_1')^T} & X_2 \oplus X_1' & \xrightarrow{(g_1, g_1')} & \tau^{-1} X_1 & \longrightarrow & SX_1 \\ & & \downarrow (f_n \cdots f_2, 0) & \swarrow \beta_1 & & & \\ & & X_{n+1} & & & & \end{array}$$

As the composition $f_n \cdots f_1$ is zero, we have the factorization $f_n \cdots f_2 = \beta_1 g_1$.

Now for i , as $\tau^{-1} X_{i-1}$ is not isomorphic to X_{i+1} , there exists an AR-triangle of the form:

$$\begin{array}{ccccc} X_i & \xrightarrow{(g_{i-1}, f_i, f_i')^T} & \tau^{-1} X_{i-1} \oplus X_{i+1} \oplus X_i' & \xrightarrow{(g_i', g_i, g_i')} & \tau^{-1} X_i & \longrightarrow & SX_i \\ & & \downarrow (-\beta_{i-1}, f_n \cdots f_{i+1}, 0) & \swarrow \beta_i & & & \\ & & X_{n+1} & & & & \end{array}$$

By induction, $-\beta_{i-1} g_{i-1} + f_n \cdots f_{i+1} f_i$ is zero, thus $f_n \cdots f_{i+1}$ factors through g_i . This property is true for $i = n - 1$, so we have a map $\beta_{n-1} : \tau^{-1} X_{n-1} \rightarrow X_{n+1}$ such that $\beta_{n-1} g_{n-1} = f_n$. As g_{n-1} and f_n are irreducible, we conclude that β_{n-1} is an isomorphism between X_{n+1} and $\tau^{-1} X_{n-1}$. \square

Now we are able to prove proposition 4.1.1. There exists an irreducible map $f : X \rightarrow X$. Suppose that X and τX are not isomorphic. Then from the previous lemma, the endomorphism f^n is non zero for each n . But since \mathcal{T} is a Krull-Schmidt, locally finite category, a power of the radical $\mathcal{R}(X, X)$ vanishes. This is a contradiction.

4.2. Proof of theorem 4.0.4. Let $\tilde{\Gamma} = (\tilde{\Gamma}_0, \tilde{\Gamma}_1, \tilde{a})$ be the valued translation quiver obtained from Γ by removing the loops, *i.e.* we have $\tilde{\Gamma}_0 = \Gamma_0$, $\tilde{\Gamma}_1 = \{\alpha \in \Gamma_1 \text{ such that } s(\alpha) \neq t(\alpha)\}$, and $\tilde{a} = a|_{\tilde{\Gamma}_1}$.

Lemma 4.2.1. *The quiver $\tilde{\Gamma} = (\tilde{\Gamma}_0, \tilde{\Gamma}_1, \tilde{a})$ with the translation τ is a valued translation quiver without loop.*

Proof. We have to check that the map σ is well-defined. But from proposition 4.1.1, if α is a loop on a vertex x , $\sigma(\alpha)$ is the unique arrow from $\tau x = x$ to x , *i.e.*

$\sigma(\alpha) = \alpha$. Thus $\tilde{\Gamma}$ is obtained from Γ by removing some σ -orbits and it keeps the structure of stable valued translation quiver. \square

Now, we can apply Riedtmann's Struktursatz [24] and the result of Happel-Preiser-Ringel [13]. There exist a tree Δ and an admissible automorphism group G (which may be trivial) of $\mathbb{Z}\Delta$ such that $\tilde{\Gamma}$ is isomorphic to $\mathbb{Z}\Delta/G$ as a valued translation quiver. The underlying graph of the tree Δ is then unique up to isomorphism and the group G is unique up to conjugacy in $\text{Aut}(\mathbb{Z}\Delta)$. Let x be a vertex of Δ . We write \bar{x} for the image of x by the map:

$$\Delta \longrightarrow \mathbb{Z}\Delta \xrightarrow{\pi} \mathbb{Z}\Delta/G \simeq \tilde{\Gamma} \hookrightarrow \Gamma.$$

Let $C : \Delta_0 \times \Delta_0 \rightarrow \mathbb{Z}$ be the matrix defined as follows:

- $C(x, y) = -a_{\bar{x}\bar{y}}$ (resp. $-a_{\bar{y}\bar{x}}$) if there exists an arrow from x to y (resp. from y to x) in Δ ,
- $C(x, x) = 2 - a_{\bar{x}\bar{x}}$,
- $C(x, y) = 0$ otherwise.

The matrix C is symmetric; it is a 'generalized Cartan matrix' in the sense of [12]. If we remove the loops from the 'underlying graph of C ' (in the sense of [12]), we get the underlying graph of Δ .

In order to apply the result of Happel-Preiser-Ringel [12, section 2], we have to show:

Lemma 4.2.2. *The set Δ_0 of vertices of Δ is finite.*

Proof. Riedtmann's construction of Δ is the following. We fix a vertex x_0 in $\tilde{\Gamma}_0$. Then the vertices of Δ are the paths of $\tilde{\Gamma}$ beginning on x_0 and which do not contain subpaths of the form $\alpha\sigma(\alpha)$, where α is in $\tilde{\Gamma}_1$. Now suppose that Δ_0 is an infinite set. Then for each n , there exists a sequence:

$$x_0 \xrightarrow{\alpha_1} x_1 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_{n-1}} x_{n-1} \xrightarrow{\alpha_n} x_n$$

such that $\tau x_{i+2} \neq x_i$. Then there exist some indecomposables X_0, \dots, X_n such that the vector space $\mathcal{R}(X_{i-1}, X_i)/\mathcal{R}^2(X_{i-1}, X_i)$ is not zero. Thus from the lemma 4.1.2, there exists irreducible morphisms $f_i : X_{i-1} \rightarrow X_i$ such that the composition $f_n f_{n-1} \dots f_1$ does not vanish. But the functor $\text{Hom}_{\mathcal{T}}(X_0, ?)$ has finite support. Thus there is an indecomposable Y which appears an infinite number of times in the sequence $(X_i)_i$. But since $\mathcal{R}^N(Y, Y)$ vanishes for an N , we have a contradiction. \square

Let \mathcal{S} a system of representatives of isoclasses of indecomposables of \mathcal{T} . For an indecomposable Y of \mathcal{T} , we put

$$l(Y) = \sum_{M \in \mathcal{S}} \dim_k \text{Hom}_{\mathcal{T}}(M, Y).$$

This sum is finite since \mathcal{T} is locally finite.

Lemma 4.2.3. *For x in Δ_0 , we write $d_x = l(\bar{x})$. Then for each $x \in \Delta_0$, we have:*

$$\sum_{y \in \Delta_0} d_y C_{xy} = 2.$$

Proof. Let X and U be indecomposables of \mathcal{T} . Let

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} SX$$

be an AR-triangle. We write $(U, ?)$ for the cohomological functor $\text{Hom}_{\mathcal{T}}(U, ?)$. Thus, we have a long exact sequence:

$$(U, S^{-1}Z) \xrightarrow{S^{-1}w_*} (U, X) \xrightarrow{u_*} (U, Y) \xrightarrow{v_*} (U, Z) \xrightarrow{w_*} (U, SX).$$

Let $S_Z(U)$ be the image of the map w_* . We have the exact sequence:

$$0 \longrightarrow S_{S^{-1}Z}(U) \longrightarrow (U, X) \xrightarrow{u_*} (U, Y) \xrightarrow{v_*} (U, Z) \xrightarrow{w_*} S_Z(U) \longrightarrow 0.$$

Thus we have the following equality:

$$\dim_k S_Z(U) + \dim_k S_{S^{-1}Z}(U) + \dim_k (U, Y) = \dim_k (U, X) + \dim_k (U, Z).$$

If U is not isomorphic to Z , each map from U to Z is radical, thus $S_Z(U)$ is zero. If U is isomorphic to Z , the map w_* factors through the radical of $\mathbf{End}(Z)$, so $S_Z(Z)$ is isomorphic to k . Then summing the previous equality when U runs over S , we get:

$$l(X) + l(Z) = l(Y) + 2.$$

Clearly l is τ -invariant, thus $l(Z)$ equals $l(X)$. If the decomposition of Y is of the form $\bigoplus_{i=1}^r Y_i^{n_i}$, we get:

$$l(Y) = \sum_i n_i l(Y_i) = \sum_{i, X \rightarrow Y_i \in \tilde{\Gamma}} a_{XY_i} l(Y_i) + a_{XX} l(X).$$

We deduce the formula:

$$2 = (2 - a_{XX})l(X) - \sum_{i, X \rightarrow Y_i \in \tilde{\Gamma}} a_{XY_i} l(Y_i).$$

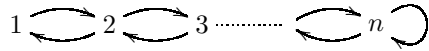
Let x be a vertex of the tree Δ and \bar{x} its image in $\tilde{\Gamma}$. Then an arrow $\bar{x} \rightarrow Y$ in $\tilde{\Gamma}$ comes from an arrow $(x, 0) \rightarrow (y, 0)$ in $\mathbb{Z}\Delta$ or from an arrow $(x, 0) \rightarrow (y, -1)$ in $\mathbb{Z}\Delta$, *i.e.* from an arrow $(y, 0) \rightarrow (x, 0)$. Indeed the projection $\mathbb{Z}\Delta \rightarrow \mathbb{Z}\Delta/G$ is a covering. From this we deduce the following equality:

$$2 = (2 - a_{\bar{x}\bar{x}})d_x - \sum_{y, x \rightarrow y \in \Delta} a_{\bar{x}\bar{y}}d_y - \sum_{y, y \rightarrow x \in \Delta} a_{\bar{y}\bar{x}}d_y = \sum_{y \in \Delta_0} d_y C_{xy}.$$

□

Now we can prove theorem 4.0.4. The matrix C is a ‘generalized Cartan matrix’. The previous lemma gives us a subadditive function which is not additive. Thus by [12], the underlying graph of C is of ‘generalized Dynkin type’. As C is symmetric, the graph is necessarily of type \mathbb{A} , \mathbb{D} , \mathbb{E} , or \mathbb{L} . But this graph is the graph Δ with the valuation a . We are done in the cases \mathbb{A} , \mathbb{D} , or \mathbb{E} .

The case \mathbb{L}_n occurs when the AR-quiver contains at least one loop. We can see \mathbb{L}_n as \mathbb{A}_n with valuations on the vertices with a loop. Then, it is obvious that the automorphism groups of $\mathbb{Z}\mathbb{L}_n$ are generated by τ^r for an $r \geq 1$. But proposition 4.1.1 tell us that a vertex x with a loop satisfies $\tau x = x$. Thus G is generated by τ and the AR-quiver has the following form:



This quiver is isomorphic to the quiver $\mathbb{Z}\mathbb{A}_{2n}/G$ where G is the group generated by the automorphism $\tau^n S = \rho$.

The suspension functor S sends the indecomposables on indecomposables, thus it can be seen as an automorphism of the AR-quiver. It is exactly the automorphism S defined in section 2.2.

As shown in [33], it follows from the results of [19] that for each Dynkin tree Δ and for each weakly admissible group of automorphisms G of $\mathbb{Z}\Delta$, there exists a locally finite triangulated category \mathcal{T} such that $\Gamma_{\mathcal{T}} \simeq \mathbb{Z}\Delta/G$. This category is of the form $\mathcal{T} = \mathcal{D}^b(\text{mod } k\Delta)/\varphi$ where φ is an auto-equivalence of $\mathcal{D}^b(\text{mod } k\Delta)$.

5. CONSTRUCTION OF A COVERING FUNCTOR

From now, we suppose that the AR-quiver Γ of \mathcal{T} is connected. We know its structure. It is natural to ask: Is the category \mathcal{T} *standard*, *i.e.* equivalent as a k -linear category to the mesh category $k(\Gamma)$? First, in this part we construct a covering functor $F : k(\mathbb{Z}\Delta) \rightarrow \mathcal{T}$.

5.1. Construction. We write $\pi : \mathbb{Z}\Delta \rightarrow \Gamma$ for the canonical projection. As G is a weakly admissible group, this projection verifies the following property: if x is a vertex of $\mathbb{Z}\Delta$, the number of arrows of $\mathbb{Z}\Delta$ with source x is equal to the number of arrows of $\mathbb{Z}\Delta/G$ with source πx . Let \mathcal{S} be a system of representatives of the isoclasses of indecomposables of \mathcal{T} . We write $\text{ind}\mathcal{T}$ for the full subcategory of \mathcal{T} whose set of objects is \mathcal{S} . For a tree Δ , we write $k(\mathbb{Z}\Delta)$ for the mesh category (see [24]). Using the same proof as Riedtmann [24], one shows the following theorem:

Theorem 5.1.1. *There exists a k -linear functor $F : k(\mathbb{Z}\Delta) \rightarrow \text{ind}\mathcal{T}$ which is surjective and induces bijections:*

$$\bigoplus_{Fz=FY} \text{Hom}_{k(\mathbb{Z}\Delta)}(x, z) \rightarrow \text{Hom}_{\mathcal{T}}(Fx, Fy),$$

for all vertices x and y of $\mathbb{Z}\Delta$.

5.2. Infinite case. If the category \mathcal{T} is locally finite not finite, the constructed functor F is immediately fully faithful. Thus we get the corollary.

Corollary 5.2.1. *If $\text{ind}\mathcal{T}$ is not finite, then we have a k -linear equivalence between \mathcal{T} and the mesh category $k(\mathbb{Z}\Delta)$.*

5.3. Uniqueness criterion. The covering functor F can be seen as a k -linear functor from the derived category $\mathcal{D}^b(\text{mod } k\Delta)$ to the category \mathcal{T} . By construction, it satisfies the following property called the *AR-property*:

For each AR-triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} SX$ of $\mathcal{D}^b(\text{mod } k\Delta)$, there exists a triangle of \mathcal{T} of the form $FX \xrightarrow{Ff} FY \xrightarrow{Fg} FZ \xrightarrow{\epsilon} SFX$.

In fact, thanks to this property, F is determined by its restriction to the subcategory $\text{proj } k\Delta = k(\Delta)$, *i.e.* we have the following lemma:

Lemma 5.3.1. *Let F and G be k -linear functors from $\mathcal{D}^b(\text{mod } k\Delta)$ to \mathcal{T} . Suppose that F and G satisfy the AR-property and that the restrictions $F|_{k(\Delta)}$ and $G|_{k(\Delta)}$ are isomorphic. Then the functors F and G are isomorphic as k -linear functors.*

Proof. It is easy to construct this isomorphism by induction using the (TR3) axiom of the triangulated categories (see [22]). \square

6. PARTICULAR CASES OF k -LINEAR EQUIVALENCE

From now we suppose that the category \mathcal{T} is finite, *i.e.* \mathcal{T} has finitely many isoclasses of indecomposable objects.

6.1. Equivalence criterion. Let Γ be the AR-quiver of \mathcal{T} and suppose that it is isomorphic to $\mathbb{Z}\Delta/G$. Let φ be a generator of G . It induces an automorphism in the mesh category $k(\mathbb{Z}\Delta)$ that we still denote by φ . Then we have the following equivalence criterion:

Proposition 6.1.1. *The categories $k(\Gamma)$ and $\text{ind}\mathcal{T}$ are equivalent as k -categories if and only if there exists a covering functor $F : k(\mathbb{Z}\Delta) \rightarrow \text{ind}\mathcal{T}$ and an isomorphism of functors $\Phi : F \circ \varphi \rightarrow F$.*

The proof consists in constructing a k -linear equivalence between $\text{ind}\mathcal{T}$ and the orbit category $k(\mathbb{Z}\Delta)/\varphi^{\mathbb{Z}}$ using the universal property of the orbit category (see [19]), and then constructing an equivalence between $k(\mathbb{Z}\Delta)/\varphi^{\mathbb{Z}}$ and $k(\Gamma)$.

6.2. Cylindric case for \mathbb{A}_n .

Theorem 6.2.1. *If $\Delta = \mathbb{A}_n$ and $\varphi = \tau^r$ for some $r \geq 1$, then there exists a functor isomorphism $\Phi : F \circ \varphi \rightarrow F$, i.e. for each object x of $k(\mathbb{Z}\Delta)$ there exists an automorphism Φ_x of Fx such that for each arrow $\alpha : x \rightarrow y$ of $\mathbb{Z}\Delta$, the following diagram commutes:*

$$\begin{array}{ccc} Fx & \xrightarrow{\Phi_x} & Fx \\ F\alpha \downarrow & & \downarrow F\varphi\alpha \\ Fy & \xrightarrow{\Phi_y} & Fy. \end{array}$$

To prove this, we need the following lemma:

Lemma 6.2.2. *Let $\alpha : x \rightarrow y$ be an arrow of $\mathbb{Z}\mathbb{A}_n$ and let c be a path from x to $\tau^r y$, $r \in \mathbb{Z}$, which is not zero in the mesh category $k(\mathbb{Z}\mathbb{A}_n)$. Then c can be written $c'\alpha$ where c' is a path from y to $\tau^r y$ (up to sign).*

Proof. There is a path from x to $\tau^r y$, thus, we have $\text{Hom}_{k(\mathbb{Z}\Delta)}(x, \tau^r y) \simeq k$, and x and $\tau^r y$ are opposite vertices of a ‘rectangle’ in $\mathbb{Z}\mathbb{A}_n$. This implies that there exists a path from x to $\tau^r y$ beginning by α . \square

Proof. (of theorem 6.2.1) Combining proposition 6.1.1 and lemma 5.3.1, we have just to construct an isomorphism between the restriction of F and $F \circ \varphi$ to a subquiver \mathbb{A}_n .

Let us fix a full subquiver of $\mathbb{Z}\mathbb{A}_n$ of the following form:

$$x_1 \xrightarrow{\alpha_1} x_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} x_n$$

such that $x_1 \dots, x_n$ are representatives of the τ -orbits in $\mathbb{Z}\mathbb{A}_n$. We define the $(\Phi_{x_i})_{i=1 \dots n}$ by induction. We fix $\Phi_{x_1} = \text{Id}_{Fx_1}$. Now suppose we have constructed some automorphisms $\Phi_{x_1}, \dots, \Phi_{x_i}$ such that for each $j \leq i$ the following diagram is commutative:

$$\begin{array}{ccc} Fx_{j-1} & \xrightarrow{\Phi_{x_{j-1}}} & Fx_{j-1} \\ F\alpha_{j-1} \downarrow & & \downarrow F\varphi\alpha_{j-1} \\ Fx_j & \xrightarrow{\Phi_{x_j}} & Fx_j. \end{array}$$

The composition $(F\varphi\alpha_i) \circ \Phi_{x_i}$ is in the morphism space $\text{Hom}_{\mathcal{T}}(Fx_i, Fx_{i+1})$, which is isomorphic, by theorem 5.1.1, to the space

$$\bigoplus_{Fz=Fx_{i+1}} \text{Hom}_{k(\mathbb{Z}\Delta)}(x_i, z).$$

Thus we can write

$$(F\varphi\alpha_i)\Phi_{x_i} = \lambda F\alpha_i + \sum_{z \neq x_{i+1}} F\beta_z$$

where β_z belongs to $\text{Hom}_{k(\mathbb{Z}\Delta)}(x_i, z)$ and $Fz = Fx_{i+1}$. But Fz is equal to Fx_{i+1} if and only if z is of the form $\tau^{rl}x_{i+1}$ for an l in \mathbb{Z} . By the lemma, we can write $\beta_z = \beta'_z\alpha_i$. Thus we have the equality:

$$(F\varphi\alpha_i)\Phi_{x_i} = F(\lambda \text{Id}_{x_{i+1}} + \sum_z \beta'_z)F\alpha_i.$$

The scalar λ is not zero. Indeed, Φ_{x_i} is an automorphism, thus the image of $(F\varphi\alpha_i)\Phi_{x_i}$ is not zero in the quotient

$$\mathcal{R}(Fx_i, Fx_{i+1})/\mathcal{R}^2(Fx_i, Fx_{i+1}).$$

Thus $\Phi_{x_{i+1}} = F(\lambda Id_{x_{i+1}} + \sum_z \beta'_z)$ is an automorphism of Fx_{i+1} which verifies the commutation relation

$$(F\varphi\alpha_i) \circ \Phi_{x_i} = \Phi_{x_{i+1}} \circ F\alpha_i.$$

□

6.3. Other standard cases. In the mesh category $k(\mathbb{Z}\Delta)$, where Δ is a Dynkin tree, the length of the non zero paths is bounded. Thus there exist automorphisms φ such that, for an arrow $\alpha : x \rightarrow y$ of Δ , the paths from x to $\varphi^r y$ vanish in the mesh category for all $r \neq 0$. In other words, for each arrow $\alpha : x \rightarrow y$ of $\mathbb{Z}\Delta$, we have:

$$\text{Hom}_{k(\mathbb{Z}\Delta)/\varphi^{\mathbb{Z}}}(x, y) = \bigoplus_{r \in \mathbb{Z}} \text{Hom}_{k(\mathbb{Z}\Delta)}(x, \varphi^r y) = \text{Hom}_{k(\mathbb{Z}\Delta)}(x, y) \simeq k,$$

where $k(\mathbb{Z}\Delta)/\varphi^{\mathbb{Z}}$ is the orbit category (see section 6.1).

Lemma 6.3.1. *Let \mathcal{T} be a finite triangulated category with AR-quiver $\Gamma = \mathbb{Z}\Delta/G$. Let φ be a generator of G and suppose that φ verifies for each arrow $x \rightarrow y$ of $\mathbb{Z}\Delta$*

$$\bigoplus_{r \in \mathbb{Z}} \text{Hom}_{k(\mathbb{Z}\Delta)}(x, \varphi^r y) = \text{Hom}_{k(\mathbb{Z}\Delta)}(x, y) \simeq k.$$

Let $F : k(\mathbb{Z}\Delta) \rightarrow \mathcal{T}$ and $G : k(\mathbb{Z}\Delta) \rightarrow \mathcal{T}$ be covering functors satisfying the AR-property. Suppose that F and G agree up to isomorphism on the objects of $k(\mathbb{Z}\Delta)$. Then F and G are isomorphic as k -linear functors.

Proof. Using lemma 5.3.1, we have just to construct an isomorphism between the functors restricted to Δ . Let $\alpha : x \rightarrow y$ be an arrow of Δ . Using theorem 5.1.1 and the hypothesis, we have the following isomorphisms:

$$\text{Hom}_{\mathcal{T}}(Fx, Fy) \simeq \bigoplus_{Fz=Fx} \text{Hom}_{k(\mathbb{Z}\Delta)}(x, z) \simeq \bigoplus_{r \in \mathbb{Z}} \text{Hom}_{k(\mathbb{Z}\Delta)}(x, \varphi^r y) \simeq k$$

and then

$$\text{Hom}_{\mathcal{T}}(Gx, Gy) \simeq \text{Hom}_{\mathcal{T}}(Fx, Fy) \simeq k.$$

Thus there exists a scalar λ such that $G\alpha = \lambda F\alpha$. This scalar does not vanish since F and G are covering functors. As Δ is a tree, we can find some λ_x for $x \in \Delta$ by induction such that

$$G\alpha = \lambda_x \lambda_y^{-1} F\alpha.$$

Now it is easy to check that $\Phi_x = \lambda_x Id_{Fx}$ is the functor isomorphism. □

This lemma gives us an isomorphism between the functors F and $F \circ \varphi$. Moreover, using the same argument, one can show that the covering functor F is an S -functor and a τ -functor.

For each Dynkin tree Δ we can determine the automorphisms φ which satisfy this combinatorial property. Using the preceding lemma and the equivalence criterion we deduce the following theorem:

Theorem 6.3.2. *Let \mathcal{T} be a finite triangulated category with AR-quiver $\Gamma = \mathbb{Z}\Delta/G$. Let φ be a generator of G . If one of these cases holds,*

- $\Delta = \mathbb{A}_n$ with n odd and G is generated by τ^r or $\varphi = \tau^r \phi$ with $r \geq \frac{n-1}{2}$ and $\phi = \tau^{\frac{n+1}{2}} S$;
- $\Delta = \mathbb{A}_n$ with n even and G is generated by ρ^r with $r \geq n-1$ and $\rho = \tau^{\frac{n}{2}} S$;

- $\Delta = \mathbb{D}_n$ with $n \geq 5$ and G is generated by τ^r or $\tau^r \phi$ with $r \geq n - 2$ and ϕ as in theorem 2.2.1;
- $\Delta = \mathbb{D}_4$ and G is generated by $\phi \tau^r$, where $r \geq 2$ and ϕ runs over σ_3 ;
- $\Delta = \mathbb{E}_6$ and G is generated by τ^r or $\tau^r \phi$ where $r \geq 5$ and ϕ is as in theorem 2.2.1;
- $\Delta = \mathbb{E}_7$ and G is generated by τ^r , $r \geq 8$;
- $\Delta = \mathbb{E}_8$ and G is generated by τ^r , $r \geq 14$.

then \mathcal{T} is standard, i.e. the categories \mathcal{T} and $k(\Gamma)$ are equivalent as k -linear categories.

Corollary 6.3.3. *A finite maximal d -Calabi-Yau (see [19, 8]) triangulated category \mathcal{T} , with $d \geq 2$, is standard, i.e. there exists a k -linear equivalence between \mathcal{T} the orbit category $\mathcal{D}^b(\text{mod } k\Delta)/\tau^{-1}S^{d-1}$ where Δ is Dynkin of type \mathbb{A} , \mathbb{D} or \mathbb{E}*

7. ALGEBRAIC CASE

For some automorphism groups G , we know the k -linear structure of \mathcal{T} . But what about the triangulated structure? We can only give an answer adding hypothesis on the triangulated structure. In this section, we distinguish two cases:

If \mathcal{T} is locally finite, not finite, we have the following theorem which is proved in section 7.2:

Theorem 7.0.4. *Let \mathcal{T} be a connected locally finite triangulated category with infinitely many indecomposables. If \mathcal{T} is the base of a tower of triangulated categories [18], then \mathcal{T} is triangle equivalent to $\mathcal{D}^b(\text{mod } k\Delta)$ for some Dynkin diagram Δ .*

Now if \mathcal{T} is a finite standard category which is algebraic, i.e. \mathcal{T} is triangle equivalent to $\underline{\mathcal{E}}$ for some k -linear Frobenius category \mathcal{E} ([20, 3.6]), then we have the following result which is proved in section 7.3:

Theorem 7.0.5. *Let \mathcal{T} be a finite triangulated category, which is connected, algebraic and standard. Then, there exists a Dynkin diagram Δ of type \mathbb{A} , \mathbb{D} or \mathbb{E} and an auto-equivalence Φ of $\mathcal{D}^b(\text{mod } k\Delta)$ such that \mathcal{T} is triangle equivalent to the orbit category $\mathcal{D}^b(\text{mod } k\Delta)/\Phi$.*

This theorem combined with corollary 6.3.3 yields the following result (compare to [19, Cor 8.4]):

Corollary 7.0.6. *If \mathcal{T} is a finite algebraic maximal d -Calabi-Yau category with $d \geq 2$, then \mathcal{T} is triangle equivalent to the orbit category $\mathcal{D}^b(\text{mod } k\Delta)/S^{d-1}$ for some Dynkin diagram Δ .*

7.1. ∂ -functor. We recall the following definition from [18] and [30].

Definition 7.1.1. *Let \mathcal{H} be an exact category and \mathcal{T} a triangulated category. A ∂ -functor $(I, \partial) : \mathcal{H} \rightarrow \mathcal{T}$ is given by:*

- an additive k -linear functor $I : \mathcal{H} \rightarrow \mathcal{T}$;
- for each conflation $\epsilon : X \xrightarrow{i} Y \xrightarrow{p} Z$ of \mathcal{H} , a morphism $\partial\epsilon : IZ \rightarrow SIX$ functorial in ϵ such that $IX \xrightarrow{Ii} IY \xrightarrow{Ip} IZ \xrightarrow{\partial\epsilon} SIX$ is a triangle of \mathcal{T} .

For each exact category \mathcal{H} , the inclusion $I : \mathcal{H} \rightarrow \mathcal{D}^b(\mathcal{H})$ can be completed to a ∂ -functor (I, ∂) in a unique way. Let \mathcal{T} and \mathcal{T}' be triangulated categories. If $(F, \varphi) : \mathcal{T} \rightarrow \mathcal{T}'$ is an S -functor and $(I, \partial) : \mathcal{H} \rightarrow \mathcal{T}$ is a ∂ -functor, we say that F respects ∂ if $(F \circ I, \varphi(F\partial)) : \mathcal{H} \rightarrow \mathcal{T}'$ is a ∂ -functor. Obviously each triangle functor respects ∂ .

Proposition 7.1.1. *Let \mathcal{H} be a k -linear hereditary abelian category and let $(I, \partial) : \mathcal{H} \rightarrow \mathcal{T}$ be a ∂ -functor. Then there exists a unique (up to isomorphism) k -linear S -functor $F : \mathcal{D}^b(\mathcal{H}) \rightarrow \mathcal{T}$ which respects ∂ .*

Proof. On \mathcal{H} (which can be seen as a full subcategory of $\mathcal{D}^b(\mathcal{H})$), the functor F is uniquely determined. We want F to be an S -functor, so F is uniquely determined on $S^n\mathcal{H}$ for $n \in \mathbb{Z}$ too. Since \mathcal{H} is hereditary, each object of $\mathcal{D}^b(\mathcal{H})$ is isomorphic to a direct sum of stalk complexes, *i.e.* complexes concentrated in a single degree. Thus, the functor F is uniquely determined on the objects. Now, let X and Y be stalk complexes of $\mathcal{D}^b(\mathcal{H})$ and $f : X \rightarrow Y$ a non-zero morphism. We can suppose that X is in \mathcal{H} and Y is in $S^n\mathcal{H}$. If $n \neq 0, 1$, f is necessarily zero. If $n = 0$, then f is a morphism in \mathcal{H} and Ff is uniquely determined. If $n = 1$, f is an element of $\text{Ext}_{\mathcal{H}}^1(X, S^{-1}Y)$, so gives us a conflation $\epsilon : S^{-1}Y \xrightarrow{i} E \xrightarrow{p} X$ in \mathcal{H} . The functor F respects ∂ , thus Ff has to be equal to $\varphi \circ \partial\epsilon$ where φ is the natural isomorphism between $SFS^{-1}Y$ and FY . Since ∂ is functorial, F is a functor. The result follows. \square

A priori this functor is not a triangle functor. We recall a theorem proved by B. Keller [18, cor 2.7].

Theorem 7.1.2. *Let \mathcal{H} be a k -linear exact category, and \mathcal{T} be the base of a tower of triangulated categories [18]. Let $(I, \partial) : \mathcal{H} \rightarrow \mathcal{T}$ be a ∂ -functor such that for each $n < 0$, and all objects X and Y of \mathcal{H} , the space $\text{Hom}_{\mathcal{T}}(IX, S^n IY)$ vanishes. Then there exists a triangle functor $F : \mathcal{D}^b(\mathcal{H}) \rightarrow \mathcal{T}$ such that the following diagram commutes up to isomorphism of ∂ -functors:*

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\quad} & \mathcal{D}^b(\mathcal{H}) \\ & \searrow (I, \partial) & \swarrow F \\ & \mathcal{T} & \end{array}$$

From theorem 7.1.2, and the proposition above we deduce the following corollary:

Corollary 7.1.3. *(compare to [29]) Let \mathcal{T} , \mathcal{H} and $(I, \partial) : \mathcal{H} \rightarrow \mathcal{T}$ be as in theorem 7.1.2. If \mathcal{H} is hereditary, then the unique functor $F : \mathcal{D}^b(\mathcal{H}) \rightarrow \mathcal{T}$ which respects ∂ is a triangle functor.*

7.2. Proof of theorem 7.0.4. Let F be the k -linear equivalence constructed in theorem 5.1.1 between an algebraic triangulated category \mathcal{T} and $\mathcal{D}^b(\mathcal{H})$ where $\mathcal{H} = \text{mod } k\Delta$ and Δ is a simply laced Dynkin graph. As we saw in section 6, the covering functor is an S -functor.

The category \mathcal{H} is the heart of the standard t -structure on $\mathcal{D}^b(\mathcal{H})$. The image of this t -structure through F is a t -structure on \mathcal{T} . Indeed, F is an S -equivalence, so the conditions (i) and (ii) from [3, Def 1.3.1] hold obviously. And since \mathcal{H} is hereditary, for an object X of $\mathcal{D}^b(\mathcal{H})$, the morphism $X \rightarrow S\tau_{\leq 0}X$ of the triangle

$$\tau_{\leq 0}X \longrightarrow X \longrightarrow \tau_{\geq 0}X \longrightarrow S\tau_{\leq 0}X$$

vanishes. Thus the image of this triangle through F is a triangle of \mathcal{T} and condition (iii) of [3, Def 1.3.1] holds. Then we get a t -structure on \mathcal{T} whose heart is \mathcal{H} .

It results from [3, Prop 1.2.4] that the inclusion of the heart of a t -structure can be uniquely completed to a ∂ -functor. Thus we obtain a ∂ -functor $(F_0, \partial) : \mathcal{H} \rightarrow \mathcal{T}$ with $F_0 = F|_{\mathcal{H}}$.

The functor F is an S -equivalence. Thus for each $n < 0$, and all objects X and Y of \mathcal{H} , the space $\text{Hom}_{\mathcal{T}}(FX, S^n FY)$ vanishes. Now we can apply theorem 7.1.2

and we get the following commutative diagram:

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\quad} & \mathcal{D}^b(\mathcal{H}), \\ & \searrow (F_0, \partial) & \nearrow G \\ & \mathcal{T} & \nearrow F \end{array}$$

where F is the S -equivalence and G is a triangle functor. Note that a priori F is an S -functor which does not respect ∂ . The functors $F|_{\mathcal{H}}$ and $G|_{\mathcal{H}}$ are isomorphic. The functor F is an S -functor thus we have an isomorphism $F|_{S^n \mathcal{H}} \simeq G|_{S^n \mathcal{H}}$ for each $n \in \mathbb{Z}$. Thus the functor G is essentially surjective. Since \mathcal{H} is the category $\text{mod } k\Delta$, to show that G is fully faithful, we have just to show that for each $p \in \mathbb{Z}$, there is an isomorphism induced by G

$$\text{Hom}_{\mathcal{D}^b(\mathcal{H})}(A, S^p A) \xrightarrow{\sim} \text{Hom}_{\mathcal{T}}(GA, S^p GA)$$

where A is the free module $k\Delta$. For $p = 0$, this is clear because A is in \mathcal{H} . And for $p \neq 0$ both sides vanish.

Thus G is a triangle equivalence between $\mathcal{D}^b(\mathcal{H})$ and \mathcal{T} .

7.3. Finite algebraic standard case. For a small dg category \mathcal{A} , we denote by \mathcal{CA} the category of dg \mathcal{A} -modules, by \mathcal{DA} the derived category of \mathcal{A} and by $\text{per } \mathcal{A}$ the *perfect derived category* of \mathcal{A} , *i.e.* the smallest triangulated subcategory of \mathcal{DA} which is stable under passage to direct factors and contains the free \mathcal{A} -modules $\mathcal{A}(?, A)$, where A runs through the objects of \mathcal{A} . Recall that a small triangulated category is *algebraic* if it is triangle equivalent to $\text{per } \mathcal{A}$ for a dg category \mathcal{A} . For two small dg categories \mathcal{A} and \mathcal{B} , a triangle functor $\text{per } \mathcal{A} \rightarrow \text{per } \mathcal{B}$ is *algebraic* if it is isomorphic to the functor

$$F_X = ? \otimes_{\mathcal{A}}^L X$$

associated with a dg bimodule X , *i.e.* an object of the derived category $\mathcal{D}(\mathcal{A}^{op} \otimes \mathcal{B})$.

Let Φ be an algebraic autoequivalence of $\mathcal{D}^b(\text{mod } k\Delta)$ such that the orbit category $\mathcal{D}^b(\text{mod } k\Delta)/\Phi$ is triangulated. Let Y be a dg $k\Delta$ - $k\Delta$ -bimodule such that $\Phi = F_Y$. In section 9.3 of [19], it was shown that there is a canonical triangle equivalence between this orbit category and the perfect derived category of a certain small dg category. Thus, the orbit category is algebraic, and endowed with a canonical triangle equivalence to the perfect derived category of a small dg category. Moreover, by the construction in [loc. cit.], the projection functor

$$\pi : \mathcal{D}^b(\text{mod } k\Delta) \rightarrow \mathcal{D}^b(\text{mod } k\Delta)/\Phi$$

is algebraic.

The proof of theorem 7.0.5 is based on the following universal property of the triangulated orbit category $\mathcal{D}^b(\text{mod } k\Delta)/\Phi$. For the proof, we refer to section 9.3 of [19].

Proposition 7.3.1. *Let \mathcal{B} be a small dg category and*

$$F_X = ? \otimes_{k\Delta}^L X : \mathcal{D}^b(\text{mod } k\Delta) \rightarrow \text{per } \mathcal{B}$$

an algebraic triangle functor given by a dg $k\Delta$ - \mathcal{A} -bimodule X . Suppose that there is an isomorphism between $Y \otimes_{k\Delta}^L X$ and X in the derived bimodule category $\mathcal{D}(k\Delta^{op} \otimes \mathcal{B})$. Then the functor F_X factors, up to isomorphism of triangle functors, through the projection

$$\pi : \mathcal{D}^b(\text{mod } k\Delta) \rightarrow \mathcal{D}^b(\text{mod } k\Delta)/\Phi.$$

Moreover, the induced triangle functor is algebraic.

Let us recall a lemma of Van den Bergh [21]:

Lemma 7.3.2. *Let Q be a quiver without oriented cycles and \mathcal{A} be a dg category. We denote by $k(Q)$ the category of paths of Q and by $\text{Can} : \mathcal{CA} \rightarrow \mathcal{DA}$ the canonical functor. Then we have the following properties:*

a) *Each functor $F : k(Q) \rightarrow \mathcal{DA}$ lifts, up to isomorphism, to a functor $\tilde{F} : k(Q) \rightarrow \mathcal{CA}$ which verifies the following property: For each vertex j of Q , the induced morphism*

$$\bigoplus_i \tilde{F}i \rightarrow \tilde{F}j,$$

where i runs through the immediate predecessors of j , is a monomorphism which splits as a morphism of graded \mathcal{A} -modules.

b) *Let F and G be functors from $k(Q)$ to \mathcal{CA} , and suppose that F satisfies the property of a). Then any morphism of functors $\varphi : \text{Can} \circ F \rightarrow \text{Can} \circ G$ lifts to a morphism $\tilde{\varphi} : F \rightarrow G$.*

Proof. a) For each vertex i of Q , the object Fi is isomorphic in \mathcal{DA} to its cofibrant resolution X_i . Thus for each arrow $\alpha : i \rightarrow j$, F induces a morphism $f_\alpha : X_i \rightarrow X_j$ which can be lifted to \mathcal{CA} since the X_i are cofibrant. Since Q has no oriented cycle, it is easy to choose the f_α such that the property is satisfied.

b) For each vertex i of Q , we may assume that Fi is cofibrant. Then we can lift $\varphi_i : \text{Can} \circ Fi \rightarrow \text{Can} \circ Gi$ to $\psi_i : Fi \rightarrow Gi$. For each arrow α of Q , the square

$$\begin{array}{ccc} Fi & \xrightarrow{F_\alpha} & Fj \\ \downarrow \psi_i & & \downarrow \psi_j \\ Gi & \xrightarrow{G_\alpha} & Gj \end{array}$$

is commutative in \mathcal{DA} . Thus the square

$$\begin{array}{ccc} \bigoplus_i Fi & \xrightarrow{F_\alpha} & Fj \\ \downarrow (\psi_i) & & \downarrow \psi_j \\ \bigoplus_i Gi & \xrightarrow{G_\alpha} & Gj \end{array}$$

is commutative up to nullhomotopic morphism $h : \bigoplus_i Fi \rightarrow Gj$. Since the morphism $f : \bigoplus_i Fi \rightarrow Fj$ is split mono in the category of graded \mathcal{A} -modules, h extends along f and we can modify ψ_j so that the square becomes commutative in \mathcal{CA} . The quiver Q does not have oriented cycles, so we can construct $\tilde{\varphi}$ by induction. \square

Proof. (of theorem 7.0.5) The category \mathcal{T} is small and algebraic, thus we may assume that $\mathcal{T} = \text{per } \mathcal{A}$ for some small dg category \mathcal{A} . Let $F : \mathcal{D}^b(\text{mod } k\Delta) \rightarrow \mathcal{T}$ be the covering functor of theorem 5.1.1. Let Φ be an auto-equivalence of $\mathcal{D}^b(\text{mod } k\Delta)$ such that the AR-quiver of the orbit category $\mathcal{D}^b(\text{mod } k\Delta)/\Phi$ is isomorphic (as translation quiver) to the AR-quiver of \mathcal{T} . We may assume that $\Phi = - \overset{L}{\otimes}_{k\Delta} Y$ for an object Y of $\mathcal{D}(k\Delta^{op} \otimes k\Delta)$. The orbit category $\mathcal{D}^b(\text{mod } k\Delta)/\Phi$ is algebraic, thus it is $\text{per } \mathcal{B}$ for some dg category \mathcal{B} .

The functor $F|_{k(\Delta)}$ lifts by lemma 7.3.2 to a functor \tilde{F} from $k(\Delta)$ to \mathcal{CA} . This means that the object $X = \tilde{F}(k\Delta)$ has a structure of dg $k\Delta^{op} \otimes \mathcal{A}$ -module. We denote by X the image of this object in $\mathcal{D}(k\Delta^{op} \otimes \mathcal{A})$.

The functors F and $- \overset{L}{\otimes}_{k\Delta} X$ become isomorphic when restricted to $k(\Delta)$. Moreover $- \overset{L}{\otimes}_{k\Delta} X$ satisfies the AR-property since it is a triangulated functor. Thus by lemma 5.3.1, they are isomorphic as k -linear functors. So we have the following

diagram:

$$\begin{array}{ccc} \mathcal{D}^b(\text{mod } k\Delta) & \xrightarrow{-\otimes_{k\Delta}^L X} & \text{per } \mathcal{A} = \mathcal{T} \\ \text{\scriptsize \circlearrowleft} & & \\ & & -\otimes_{k\Delta}^L Y \end{array}$$

The category \mathcal{T} is standard, thus there exists an isomorphism of k -linear functors:

$$c : -\otimes_{k\Delta}^L X \longrightarrow -\otimes_{k\Delta}^L Y \otimes_{k\Delta}^L X.$$

The functor $-\otimes_{k\Delta}^L X$ restricted to the category $k(\Delta)$ satisfies the property of *a*) of lemma 7.3.2. Thus we can apply *b*) and lift $c|_{k(\Delta)}$ to an isomorphism \tilde{c} between X and $Y \otimes_{k\Delta}^L X$ as $\text{dg-}k\Delta^{\text{op}} \otimes \mathcal{A}$ -modules.

By the universal property of the orbit category, the bimodule X endowed with the isomorphism \tilde{c} yields a triangle functor from $\mathcal{D}^b(\text{mod } k\Delta)/\Phi$ to \mathcal{T} which comes from a bimodule Z in $\mathcal{D}(\mathcal{B}^{\text{op}} \otimes \mathcal{A})$.

$$\begin{array}{ccc} \mathcal{D}^b(\text{mod } k\Delta) & \xrightarrow{-\otimes_{k\Delta}^L X} & \text{per } \mathcal{A} = \mathcal{T} \\ \text{\scriptsize \circlearrowleft} & & \\ \downarrow \pi & \dashrightarrow & \text{\scriptsize \circlearrowleft} \\ \mathcal{D}^b(\text{mod } k\Delta)/\Phi = \text{per } \mathcal{B} & & -\otimes_{k\Delta}^L Z \end{array}$$

The functor $-\otimes_{k\Delta}^L Z$ is essentially surjective. Let us show that it is fully faithful. For M and N objects of $\mathcal{D}^b(\text{mod } k\Delta)$ we have the following commutative diagram:

$$\begin{array}{ccc} & \bigoplus_{n \in \mathbb{Z}} \text{Hom}_{\mathcal{D}}(M, \Phi^n N) & \\ \pi \swarrow & & \searrow -\otimes_{k\Delta}^L X = F \\ \text{Hom}_{\mathcal{D}/\Phi}(\pi M, \pi N) & \xrightarrow{-\otimes_{k\Delta}^L Z} & \text{Hom}_{\mathcal{T}}(FM, FN), \end{array}$$

where \mathcal{D} means $\mathcal{D}^b(\text{mod } k\Delta)$. The two diagonal morphisms are isomorphisms, thus so is the horizontal morphism. This proves that $-\otimes_{k\Delta}^L Z$ is a triangle equivalence between the orbit category $\mathcal{D}^b(\text{mod } k\Delta)/\Phi$ and \mathcal{T} . \square

8. TRIANGULATED STRUCTURE ON THE CATEGORY OF PROJECTIVES

Let k be an algebraically closed field and \mathcal{P} a k -linear category with split idempotents. The category $\text{mod } \mathcal{P}$ of contravariant finitely presented functors from \mathcal{P} to $\text{mod } k$ is exact. As the idempotents split, the projectives of $\text{mod } \mathcal{P}$ coincide with the representables. Thus the Yoneda functor gives a natural equivalence between \mathcal{P} and $\text{proj } \mathcal{P}$. Assume besides that $\text{mod } \mathcal{P}$ has a structure of Frobenius category. The stable category $\underline{\text{mod}} \mathcal{P}$ is a triangulated category, we write Σ for the suspension functor.

Let S be an auto-equivalence of \mathcal{P} . It can be extended to an exact functor from $\text{mod } \mathcal{P}$ to $\text{mod } \mathcal{P}$ and thus to a triangle functor of $\underline{\text{mod}} \mathcal{P}$. The aim of this part is to find a necessary condition on the functor S such that the category (\mathcal{P}, S) has a triangulated structure. Heller already showed [14, thm 16.4] that if there exists an isomorphism of triangle functors between S and Σ^3 , then \mathcal{P} has a pretriangulated structure. But he did not succeed in proving the octahedral axiom. We are going to impose a stronger condition on the functor S and prove the following theorem:

Theorem 8.0.3. *Assume there exists an exact sequence of exact functors from $\text{mod } \mathcal{P}$ to $\text{mod } \mathcal{P}$:*

$$0 \longrightarrow Id \longrightarrow X^0 \longrightarrow X^1 \longrightarrow X^2 \longrightarrow S \longrightarrow 0 ,$$

where the X^i , $i = 0, 1, 2$, take values in $\text{proj } \mathcal{P}$. Then the category \mathcal{P} has a structure of triangulated category with suspension functor S .

For an M in $\text{mod } \mathcal{P}$, denote $T_M : X^0 M \longrightarrow X^1 M \longrightarrow X^2 M \longrightarrow SX^0 M$ a standard triangle. A triangle of \mathcal{P} will be a sequence $X : P \xrightarrow{u} Q \xrightarrow{v} R \xrightarrow{w} SP$ which is isomorphic to a standard triangle T_M for an M in $\text{mod } \mathcal{P}$.

8.1. S-complexes, Φ -S-complexes and standard triangles. Let $\mathcal{Acp}(\text{mod } \mathcal{P})$ be the category of acyclic complexes with projective components. It is a Frobenius category whose projective-injectives are the contractible complexes, *i.e.* the complexes homotopic to zero. The functor $Z^0 : \mathcal{Acp}(\text{mod } \mathcal{P}) \rightarrow \text{mod } \mathcal{P}$ which sends a complex

$$\cdots \longrightarrow X^{-1} \xrightarrow{x^{-1}} X^0 \xrightarrow{x^0} X^1 \xrightarrow{x^1} \cdots$$

to the kernel of x^0 is an exact functor. It sends the projective-injectives to projective-injectives and induces a triangle equivalence between $\underline{\mathcal{Acp}(\text{mod } \mathcal{P})}$ and $\underline{\text{mod } \mathcal{P}}$.

Definition 8.1.1. *An object of $\mathcal{Acp}(\text{mod } \mathcal{P})$ is called an S -complex if it is S -periodic, *i.e.* if it has the following form:*

$$\cdots \longrightarrow P \xrightarrow{u} Q \xrightarrow{v} R \xrightarrow{w} SP \xrightarrow{Su} SQ \longrightarrow \cdots .$$

The category $S\text{-comp}$ of S -complexes with S -periodic morphisms is a non full subcategory of $\mathcal{Acp}(\text{mod } \mathcal{P})$. It is a Frobenius category. The projective-injectives are the S -contractibles, *i.e.* the complexes homotopic to zero with an S -periodic homotopy. Using the functor Z^0 , we get an exact functor from $S\text{-comp}$ to $\text{mod } \mathcal{P}$ which induces a triangle functor:

$$\underline{Z^0} : \underline{S\text{-comp}} \longrightarrow \underline{\text{mod } \mathcal{P}} .$$

Fix a sequence as in theorem 8.0.3. Clearly, it induces for each object M of $\underline{\text{mod } \mathcal{P}}$, a functorial isomorphism in $\underline{\text{mod } \mathcal{P}}$, $\Phi_M : \Sigma^3 M \longrightarrow SM$.

Let Y be an S -complex,

$$Y : \cdots \longrightarrow P \xrightarrow{u} Q \xrightarrow{v} R \xrightarrow{w} SP \xrightarrow{Su} SQ \longrightarrow \cdots .$$

Let M be the kernel of u . Then Y induces an isomorphism θ (in $\underline{\text{mod } \mathcal{P}}$) between $\Sigma^3 M$ and SM . If θ is equal to Φ_M , we will say that Y is a Φ - S -complex.

Let M be an object of $\text{mod } \mathcal{P}$. The standard triangle T_M can be seen as a Φ - S -complex:

$$\cdots \longrightarrow X^0 M \longrightarrow X^1 M \longrightarrow X^2 M \longrightarrow SX^0 M \longrightarrow SX^1 M \longrightarrow \cdots .$$

The functor T which sends an object M of $\text{mod } \mathcal{P}$ to the S -complex T_M is exact since the X^i are exact. It satisfies the relation $Z^0 \circ T \simeq Id_{\text{mod } \mathcal{P}}$. Moreover, as it preserves the projective-injectives, it induces a triangle functor:

$$T : \underline{\text{mod } \mathcal{P}} \rightarrow \underline{S\text{-comp}} .$$

8.2. Properties of the functors Z^0 and T .

Lemma 8.2.1. *An S -complex which is homotopy-equivalent to a Φ - S -complex is a Φ - S -complex.*

Proof. Let $X : P \xrightarrow{u} Q \xrightarrow{v} R \xrightarrow{w} SP$ be an S -complex homotopy-equivalent to the Φ - S -complex $X' : P' \xrightarrow{u'} Q' \xrightarrow{v'} R' \xrightarrow{w'} SP'$. Let M be the kernel of u and M' the kernel of u' . By assumption, there exists a S -periodic homotopy equivalence f from X to X' , which induces a morphism $g = Z^0 f : M \rightarrow M'$. Thus, we get the following commutative diagram:

$$\begin{array}{ccccccc}
 & & P & \longrightarrow & Q & \longrightarrow & R & \longrightarrow & SP \\
 & & \downarrow f^0 & & \downarrow f^1 & & \downarrow f^2 & & \downarrow Sf^0 \\
 M & \nearrow & & & & & \Sigma^3 M & \xrightarrow{\theta} & SM & \nearrow & & & \\
 \downarrow g & & P' & \longrightarrow & Q' & \longrightarrow & R' & \longrightarrow & SP' & & & & \\
 M' & \nearrow & & & & & \Sigma^3 M' & \xrightarrow{\Phi_{M'}} & SM' & \nearrow & & & \\
 & & & & & & \downarrow \Sigma^3 g & & \downarrow Sg & & & &
 \end{array}$$

The morphism g is an isomorphism of $\underline{\text{mod}}\mathcal{P}$ since f is an isomorphism of $\underline{S\text{-comp}}$. Thus the morphisms $\Sigma^3 g$ and Sg are isomorphisms of $\underline{\text{mod}}\mathcal{P}$. The following equality in $\underline{\text{mod}}\mathcal{P}$

$$\theta = (Sg)^{-1} \Phi_{M'} \Sigma^3 g = \Phi_M$$

shows that the complex X is a Φ - S -complex. \square

Lemma 8.2.2. *Let*

$$X : P \xrightarrow{u} Q \xrightarrow{v} R \xrightarrow{w} SP \quad \text{and} \quad X' : P' \xrightarrow{u'} Q' \xrightarrow{v'} R' \xrightarrow{w'} SP'$$

be two Φ - S -complexes. Suppose that we have a commutative square:

$$\begin{array}{ccc}
 P & \xrightarrow{u} & Q \\
 f^0 \downarrow & & \downarrow f^1 \\
 P' & \xrightarrow{u'} & Q'
 \end{array}$$

Then, there exists a morphism $f^2 : R \rightarrow R'$ such that (f^0, f^1, f^2) extends to an S -periodic morphism from X to X' .

Proof. Let M be the kernel of u , M' be the kernel of u' and $f : M \rightarrow M'$ be the morphism induced by the commutative square. As R and R' are projective-injective objects, we can find a morphism $g^2 : R \rightarrow R'$ such that the following square commutes:

$$\begin{array}{ccc}
 Q & \xrightarrow{v} & R \\
 f^1 \downarrow & & \downarrow g^2 \\
 Q' & \xrightarrow{v'} & R'
 \end{array}$$

The morphism g^2 induces a morphism $g : SM \rightarrow SM'$ such that the following square is commutative in $\underline{\text{mod}}\mathcal{P}$:

$$\begin{array}{ccc} \Sigma^3 M & \xrightarrow{\Phi_M} & SM \\ \Sigma^3 f \downarrow & & \downarrow g \\ \Sigma^3 M' & \xrightarrow{\Phi_{M'}} & SM'. \end{array}$$

Thus the morphisms Sf and g are equal in $\underline{\text{mod}}\mathcal{P}$, i.e. there exists a projective-injective I of $\text{mod}\mathcal{P}$ and morphisms $\alpha : SM \rightarrow I$ and $\beta : I \rightarrow SM'$ such that $g - Sf = \beta\alpha$. Let p (resp. p') be the epimorphism from R onto SM (resp. from R' onto SM'). Then, as I is projective, β factors through p' .

$$\begin{array}{ccccc} Q & \xrightarrow{v} & R & \xrightarrow{w} & SP \\ \downarrow f^1 & & \downarrow g^2 & & \downarrow Sf^0 \\ Q' & \xrightarrow{v'} & R' & \xrightarrow{w'} & SP' \\ & & \downarrow p' & & \downarrow p' \\ & & SM' & & \end{array}$$

$\begin{array}{c} \nearrow p \\ SM \\ \searrow p' \\ \downarrow \beta \\ I \\ \downarrow \alpha \\ SM' \\ \nearrow p' \end{array}$

We put $f^2 = g^2 - \gamma\alpha p$. Then obviously, we have the equalities $f^2 v = v' f^1$ and $w' f^2 = Sf^0 w$. Thus the morphism (f^0, f^1, f^2) extends to a morphism of $S\text{-comp}$. \square

Proposition 8.2.3. *The functor $\underline{Z}^0 : \underline{\Phi}\text{-}S\text{-comp} \rightarrow \underline{\text{mod}}\mathcal{P}$ is full and essentially surjective. Its kernel is an ideal whose square vanishes.*

Proof. The functor \underline{Z}^0 is essentially surjective since we have the relation $\underline{Z}^0 \circ \underline{T} = \text{Id}_{\underline{\text{mod}}\mathcal{P}}$.

Let us show that \underline{Z}^0 is full. Let

$$X : P \xrightarrow{u} Q \xrightarrow{v} R \xrightarrow{w} SP \quad \text{and} \quad X' : P' \xrightarrow{u'} Q' \xrightarrow{v'} R' \xrightarrow{w'} SP'$$

be two Φ - S -complexes. Let M (resp. M') be the kernel of u (resp. u'). As P , Q , P' and Q' are projective-injective, there exist morphisms $f^0 : P \rightarrow P'$ and $f^1 : Q \rightarrow Q'$ such that the following diagram commutes:

$$\begin{array}{ccccc} M & \longrightarrow & P & \xrightarrow{u} & Q \\ f \downarrow & & \downarrow f^0 & & \downarrow f^1 \\ M' & \longrightarrow & P' & \xrightarrow{u'} & Q'. \end{array}$$

Now the result follows from lemma 8.2.2.

Now let $\underline{f} : X \rightarrow X'$ be a morphism in the kernel of \underline{Z}^0 . Up to homotopy, we can suppose that \underline{f} has the following form:

$$\begin{array}{ccccccc} P & \xrightarrow{u} & Q & \xrightarrow{v} & R & \xrightarrow{w} & SP \\ \downarrow 0 & & \downarrow 0 & & \downarrow f^2 & & \downarrow 0 \\ P' & \xrightarrow{u'} & Q' & \xrightarrow{v'} & R' & \xrightarrow{w'} & SP' \end{array}$$

As the composition $w'f^2$ vanishes and as Q' is projective-injective, f^2 factors through v' . For the same argument, f^2 factors through w . If \underline{f} and \underline{f}' are composable morphisms of the kernel of \underline{Z}^0 , we get the following diagram:

$$\begin{array}{ccccccc} P & \xrightarrow{u} & Q & \xrightarrow{v} & R & \xrightarrow{w} & SP \\ \downarrow 0 & & \downarrow 0 & & \downarrow f^2 & & \downarrow 0 \\ P' & \xrightarrow{u'} & Q' & \xrightarrow{v'} & R' & \xrightarrow{w'} & SP' \\ \downarrow 0 & & \downarrow 0 & & \downarrow f'^2 & & \downarrow 0 \\ P'' & \xrightarrow{u''} & Q'' & \xrightarrow{v''} & R'' & \xrightarrow{w''} & SP'' \end{array}$$

(Dotted arrows h^2 and h'^3 indicate factorizations through Q' and R'' respectively.)

The composition $\underline{f}'\underline{f}$ vanishes obviously. □

Corollary 8.2.4. *A Φ - S -complex morphism f which induces an isomorphism $\underline{Z}^0(f)$ in $\mathbf{mod}\mathcal{P}$ is an homotopy-equivalence.*

This corollary comes from the previous theorem and from the following lemma.

Lemma 8.2.5. *Let $F : \mathcal{C} \rightarrow \mathcal{C}'$ be a full functor between two additive categories. If the kernel of F is an ideal whose square vanishes, then F detects isomorphisms.*

Proof. Let $u \in \mathbf{Hom}_{\mathcal{C}}(A, B)$ be a morphism in \mathcal{C} such that Fu is an isomorphism. Since the functor F is full, there exists v in $\mathbf{Hom}_{\mathcal{C}}(B, A)$ such that $Fv = (Fu)^{-1}$. The morphism $w = uv - Id_B$ is in the kernel of F , thus w^2 vanishes. Then the morphism $v(Id_B - w)$ is a right inverse of u . In the same way we show that u has a left inverse, so u is an isomorphism. □

Proposition 8.2.6. *The category of Φ - S -complexes is equivalent to the category of S -complexes which are homotopy-equivalent to standard triangles.*

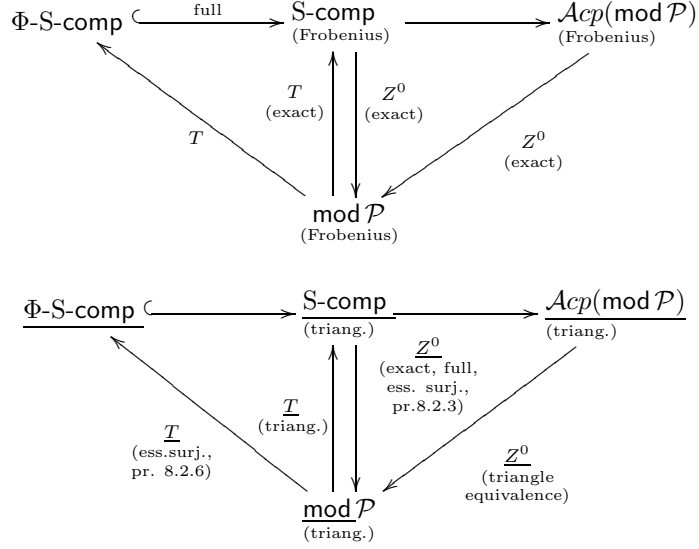
Proof. Since standard triangles are ϕ - S -complexes, each S -complex that is homotopy equivalent to a standard triangle is a Φ - S -complex (lemma 8.2.1).

Let $X : P \xrightarrow{u} Q \xrightarrow{v} R \xrightarrow{w} SP$ be a Φ - S -complex. Let M be the kernel of u . Then there exist morphisms $f^0 : P \rightarrow X^0M$ and $f^1 : Q \rightarrow X^1M$ such that the following diagram is commutative:

$$\begin{array}{ccccc} M & \xrightarrow{\quad} & P & \xrightarrow{u} & Q \\ \parallel & & \downarrow f^0 & & \downarrow f^1 \\ M & \xrightarrow{\quad} & X^0M & \xrightarrow{\quad} & X^1M \end{array}$$

We can complete (lemma 8.2.2) f into an S -periodic morphism from X in T_M . The morphism f satisfies $Z^0f = Id_M$, so $\underline{Z}^0(T_M)$ and $Z^0(X)$ are equal in $\mathbf{mod}\mathcal{P}$. By the corollary, T_M and X are homotopy-equivalent. Thus the inclusion functor T is essentially surjective. □

These two diagrams summarize the results of this section:



8.3. Proof of theorem 8.0.3. We are going to show that the Φ - S -complexes form a system of triangles of the category \mathcal{P} . We use triangle axioms as in [22].

TR0: For each object M of \mathcal{P} , the S -complex $M \xrightarrow{=} M \longrightarrow 0 \longrightarrow SM$ is homotopy-equivalent to the zero complex, so is a Φ - S -complex.

TR1: Let $u : P \rightarrow Q$ be a morphism of \mathcal{P} , and let M be its kernel. We can find morphisms f^0 and f^1 so as to obtain a commutative square:

$$\begin{array}{ccccc}
 & X^0M & \xrightarrow{a} & X^1M & \xrightarrow{b} & X^2M \\
 & \nearrow & & \searrow & & \nearrow \\
 M & & & & & \text{Coker } a \\
 \parallel & \downarrow f^0 & & \downarrow f^1 & & \downarrow \gamma \\
 M & & P & \xrightarrow{u} & Q & \\
 & & & & \searrow & \downarrow \\
 & & & & & \text{Coker } u.
 \end{array}$$

We form the following push-out:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \text{Coker } a & \longrightarrow & X^2M & \longrightarrow & SM \longrightarrow 0 \\
 & & \downarrow \gamma & \text{PO} & \downarrow & & \parallel \\
 0 & \longrightarrow & \text{Coker } u & \longrightarrow & R & \longrightarrow & SM \longrightarrow 0.
 \end{array}$$

It induces a triangle morphism of the triangular category $\underline{\text{mod}} \mathcal{P}$:

$$\begin{array}{ccccccc}
 \text{Coker } a & \longrightarrow & X^2M & \longrightarrow & SM & \longrightarrow & \Sigma \text{Coker } a \\
 \downarrow \gamma & & \downarrow & & \parallel & & \downarrow \Sigma \gamma \\
 \text{Coker } u & \longrightarrow & R & \longrightarrow & SM & \longrightarrow & \Sigma \text{Coker } u.
 \end{array}$$

Thus (f^0, f^1, f^2) can be extended to an S -periodic morphism \tilde{f} which is S -homotopic to \tilde{g} . Their respective cones $C(\tilde{f})$ and $C(\tilde{g})$ are isomorphic as S -complexes. Moreover, since \tilde{g} is a composition of $Tg : T_M \rightarrow T_{M'}$ with homotopy-equivalences, the cones $C(\tilde{g})$ and $C(Tg)$ are homotopy-equivalent.

In $\underline{\text{mod}}\mathcal{P}$, we have a triangle

$$M \xrightarrow{g} M' \longrightarrow C(g) \longrightarrow \Sigma M.$$

Since \underline{T} is a triangle functor, the sequence

$$T_M \xrightarrow{Tg} T_{M'} \longrightarrow T_{C(g)} \longrightarrow T_{\Sigma M}$$

is a triangle in $\underline{S\text{-comp}}$. But we know that

$$T_M \xrightarrow{Tg} T_{M'} \longrightarrow C(Tg) \longrightarrow T_M[1]$$

is a triangle in $\underline{S\text{-comp}}$. Thus the objects $C(Tg)$ and $T_{C(g)}$ are isomorphic in $\underline{S\text{-comp}}$, *i.e.* homotopy-equivalent. Thus, the cone $C(\tilde{f})$ of \tilde{f} is a Φ - S -complex by lemma 8.2.1.

9. APPLICATION TO THE DEFORMED PREPROJECTIVE ALGEBRAS

In this section, we apply the theorem 8.0.3 to show that the category of finite dimensional projective modules over a deformed preprojective algebra of generalized Dynkin type (see [4]) is triangulated. This will give us some examples of non standard triangulated categories with finitely many indecomposables.

9.1. Preprojective algebra of generalized Dynkin type. Recall the notations of [4]. Let Δ be a generalized Dynkin graph of type \mathbb{A}_n , \mathbb{D}_n ($n \geq 4$), \mathbb{E}_n ($n = 6, 7, 8$), or \mathbb{L}_n . Let Q_Δ be the following associated quiver:

$$\Delta = \mathbb{A}_n \ (n \geq 1) : \quad 0 \begin{array}{c} \xrightarrow{a_0} \\ \xleftarrow{\bar{a}_0} \end{array} 1 \begin{array}{c} \xrightarrow{a_1} \\ \xleftarrow{\bar{a}_1} \end{array} 2 \cdots \cdots n-2 \begin{array}{c} \xrightarrow{a_{n-2}} \\ \xleftarrow{\bar{a}_{n-2}} \end{array} n-1$$

$$\Delta = \mathbb{D}_n \ (n \geq 4) : \quad \begin{array}{c} 0 \\ \begin{array}{c} \swarrow a_0 \\ \searrow \bar{a}_0 \end{array} \\ 2 \begin{array}{c} \xrightarrow{a_2} \\ \xleftarrow{\bar{a}_2} \end{array} 3 \cdots \cdots n-2 \begin{array}{c} \xrightarrow{a_{n-2}} \\ \xleftarrow{\bar{a}_{n-2}} \end{array} n-1 \\ \begin{array}{c} \swarrow a_1 \\ \searrow \bar{a}_1 \end{array} \\ 1 \end{array}$$

$$\Delta = \mathbb{E}_n \ (n = 6, 7, 8) : \quad \begin{array}{c} 0 \\ \begin{array}{c} \uparrow \bar{a}_0 \\ \downarrow a_0 \end{array} \\ 1 \begin{array}{c} \xrightarrow{a_1} \\ \xleftarrow{\bar{a}_1} \end{array} 2 \begin{array}{c} \xrightarrow{a_2} \\ \xleftarrow{\bar{a}_2} \end{array} 3 \begin{array}{c} \xrightarrow{a_3} \\ \xleftarrow{\bar{a}_3} \end{array} 4 \cdots \cdots n-2 \begin{array}{c} \xrightarrow{a_{n-2}} \\ \xleftarrow{\bar{a}_{n-2}} \end{array} n-1 \end{array}$$

$$\Delta = \mathbb{L}_n \ (n \geq 1) : \quad \epsilon = \bar{\epsilon} \begin{array}{c} \circlearrowleft \\ \circlearrowright \end{array} 0 \begin{array}{c} \xrightarrow{a_0} \\ \xleftarrow{\bar{a}_0} \end{array} 1 \begin{array}{c} \xrightarrow{a_1} \\ \xleftarrow{\bar{a}_1} \end{array} 2 \cdots \cdots n-2 \begin{array}{c} \xrightarrow{a_{n-2}} \\ \xleftarrow{\bar{a}_{n-2}} \end{array} n-1 .$$

The *preprojective algebra* $P(\Delta)$ associated to the graph Δ is the quotient of the path algebra kQ_Δ by the relations:

$$\sum_{sa=i} a\bar{a}, \quad \text{for each vertex } i \text{ of } Q_\Delta.$$

The following proposition is classical [4, prop 2.1].

Proposition 9.1.1. *The preprojective algebra $P(\Delta)$ is finite dimensional and self-injective. Its Nakayama permutation ν is the identity for $\Delta = \mathbb{A}_1, \mathbb{D}_{2n}, \mathbb{E}_7, \mathbb{E}_8$ and \mathbb{L}_n , and is of order 2 in all other cases.*

9.2. Deformed preprojective algebras of generalized Dynkin type. Let us recall the definition of deformed preprojective algebra introduced by [4]. Let Δ be a graph of generalized Dynkin type. We define an associated algebra $R(\Delta)$ as follows:

$$\begin{aligned} R(\mathbb{A}_n) &= k; \\ R(\mathbb{D}_n) &= k\langle x, y \rangle / (x^2, y^2, (x+y)^{n-2}); \\ R(\mathbb{E}_n) &= k\langle x, y \rangle / (x^2, y^3, (x+y)^{n-3}); \\ R(\mathbb{L}_n) &= k[x] / (x^{2n}). \end{aligned}$$

Further, we fix an exceptional vertex in each graph as follows (with the notations of the previous section):

$$\begin{aligned} 0 &\text{ for } \Delta = \mathbb{A}_n \text{ or } \mathbb{L}_n, \\ 2 &\text{ for } \Delta = \mathbb{D}_n, \\ 3 &\text{ for } \Delta = \mathbb{E}_n. \end{aligned}$$

Let f be an element of the square $\text{rad}^2 R(\Delta)$ of the radical of $R(\Delta)$. The *deformed preprojective algebra* $P^f(\Delta)$ is the quotient of the path algebra kQ_Δ by the relations:

$$\sum_{sa=i} a\bar{a}, \quad \text{for each non exceptional vertex } i \text{ of } Q,$$

and

$$\begin{aligned} a_0\bar{a}_0 & && \text{for } \Delta = \mathbb{A}_n; \\ \bar{a}_0a_0 + \bar{a}_1a_1 + a_2\bar{a}_2 + f(\bar{a}_0a_0, \bar{a}_1a_1), \text{ and } (\bar{a}_0a_0 + \bar{a}_1a_1)^{n-2} & && \text{for } \Delta = \mathbb{D}_n; \\ \bar{a}_0a_0 + \bar{a}_2a_2 + a_3\bar{a}_3 + f(\bar{a}_0a_0, \bar{a}_2a_2), \text{ and } (\bar{a}_0a_0 + \bar{a}_2a_2)^{n-3} & && \text{for } \Delta = \mathbb{E}_n; \\ \epsilon^2 + a_0\bar{a}_0 + \epsilon f(\epsilon), \text{ and } \epsilon^{2n} & && \text{for } \Delta = \mathbb{L}_n. \end{aligned}$$

Note that if f is zero, we get the preprojective algebra $P(\Delta)$.

9.3. Corollaries of [4]. The following proposition [4, prop 3.4] shows that the category $\text{proj } P^f(\Delta)$ of finite-dimensional projective modules over a deformed preprojective algebra satisfies the hypothesis of theorem 8.0.3.

Proposition 9.3.1. *Let $A = P^f(\Delta)$ be a deformed preprojective algebra. Then there exists an exact sequence of A - A -bimodules*

$$0 \longrightarrow {}_1A_{\Phi^{-1}} \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0,$$

where Φ is an automorphism of A and where the P_i 's are projective as bimodules. Moreover, for each idempotent e_i of A , we have $\Phi(e_i) = e_{\nu(i)}$.

So we can easily deduce the corollary:

Corollary 9.3.2. *Let $P^f(\Delta)$ be a deformed preprojective algebra of generalized Dynkin type. Then the category $\text{proj } P^f(\Delta)$ of finite dimensional projective modules is triangulated. The suspension is the Nakayama functor.*

Indeed, if $P_i = e_iA$ is a projective indecomposable, then $P_i \otimes_A A_\Phi$ is equal to $\Phi(e_i)A = e_{\nu(i)}A$ thus to $\nu(P_i)$.

Now we are able to answer to the question of the previous part and find a triangulated category with finitely many indecomposables which is not standard. The proof of the following theorem comes essentially from the theorem [4, thm 1.3].

Theorem 9.3.3. *Let k be an algebraically closed field of characteristic 2. Then there exist k -linear triangulated categories with finitely many indecomposables which are not standard.*

Proof. By theorem [4, thm 1.3], we know that there exist basic deformed preprojective algebras of generalized Dynkin type $P^f(\Delta)$ which are not isomorphic to $P(\Delta)$. Thus the categories $\text{proj } P^f(\Delta)$ and $\text{proj } P(\Delta)$ can not be equivalent. But both are triangulated by corollary 9.3.2 and have the same AR-quiver $\mathbb{Z}\Delta/\tau = Q_\Delta$. \square

Conversely, we have the following theorem:

Theorem 9.3.4. *Let \mathcal{T} be a finite 1-Calabi-Yau triangulated category. Then \mathcal{T} is equivalent to $\text{proj } \Lambda$ as k -category, where Λ is a deformed preprojective algebra of generalized Dynkin type.*

Proof. Let M_1, \dots, M_n be representatives of the isoclasses of indecomposable objects of \mathcal{T} . The k -algebra $\Lambda = \text{End}(\bigoplus_{i=1}^n M_i)$ is basic, finite-dimensional and selfinjective since \mathcal{T} has a Serre duality. It is easy to see that \mathcal{T} and $\text{proj } \Lambda$ are equivalent as k -categories.

Let $\text{mod } \Lambda$ be the category of finitely presented Λ -modules. It is a Frobenius category. Denote by Σ the suspension functor of the triangulated category $\overline{\text{mod } \Lambda}$. The category \mathcal{T} is 1-Calabi-Yau, that is to say that the suspension functor S of the triangulated category \mathcal{T} and the Serre functor ν are isomorphic. But in $\overline{\text{mod } \Lambda}$, the functors S and Σ^3 are isomorphic. Thus, for each non projective simple Λ -module M we have an isomorphism $\Sigma^3 M \simeq \nu M$. By [4, thm 1.2], we get immediately the result. \square

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