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ON THE STRUCTURE OF TRIANGULATED CATEGORIES WITH FINITELY MANY INDECOMPOSABLES

CLAIRE AMIOT

Abstract. We study the problem of classifying triangulated categories with finite-dimensional morphism spaces and finitely many indecomposables over an algebraically closed field. We obtain a new proof of the following result due to Xiao and Zhu: the Auslander-Reiten quiver of such a category is of the form $\mathbb{Z}\Delta/G$ where Δ is a disjoint union of simply laced Dynkin diagrams and G a weakly admissible group of automorphisms of $\mathbb{Z}\Delta$. Then we prove that for 'most' groups G, the category $\mathcal{T}$ is standard, *i.e.* k-linearly equivalent to an orbit category $\mathcal{D}^b(\mathsf{mod}\, k\Delta)/\Phi$. This happens in particular when $\mathcal{T}$ is maximal d-Calabi-Yau with $d \geq 2$. Moreover, if $\mathcal{T}$ is standard and algebraic, we can even construct a triangle equivalence between $\mathcal{T}$ and the corresponding orbit category. Finally we give a sufficient condition for the category of projectives of a Frobenius category to be triangulated. This allows us to construct non standard 1-Calabi-Yau categories using deformed preprojective algebras of generalized Dynkin type.

Introduction

Let k be an algebraically closed field and $\mathcal{T}$ a small k-linear triangulated category (see [31]) with split idempotents. We assume that

a) $\mathcal{T}$ is Hom-*finite*, *i.e.* the space $\mathsf{Hom}_{\mathcal{T}}(X, Y)$ is finite-dimensional for all objects X, Y of $\mathcal{T}$.

It follows that indecomposable objects of $\mathcal{T}$ have local endomorphism rings and that each object of $\mathcal{T}$ decomposes into a finite direct sum of indecomposables [9, 3.3]. We assume moreover that

b) $\mathcal{T}$ is *locally finite*, *i.e.* for each indecomposable X of $\mathcal{T}$, there are at most finitely many isoclasses of indecomposables Y such that $\mathsf{Hom}_{\mathcal{T}}(X, Y) \neq 0$.

It was shown in [32] that condition b) implies its dual. Condition b) holds in particular if we have

b') $\mathcal{T}$ is *additively finite*, *i.e.* there are only finitely many isomorphism classes of indecomposables in $\mathcal{T}$.

The study of particular classes of such triangulated categories $\mathcal{T}$ has a long history. Let us briefly recall some of its highlights:

(1) If A is a representation-finite selfinjective algebra, then the stable category $\mathcal{T}$ of finite-dimensional (right) A-modules satisfies our assumptions and is additively finite. The structure of the underlying k-linear category of $\mathcal{T}$ was determined by C. Riedtmann in [24], [25], [26] and [27].

(2) In [10], D. Happel showed that the bounded derived category of the category of finite-dimensional representations of a representation-finite quiver is locally finite and described its underlying k-linear category.

(3) The stable category $\underline{CM}(R)$ of Cohen-Macaulay modules over a commutative complete local Gorenstein isolated singularity R of dimension d is a Hom-finite triangulated category which is $(d-1)$-Calabi-Yau (cf. for example [17] and [34]). In [2], M. Auslander and I. Reiten showed that if the dimension of R is 1, then the category $\underline{CM}(R)$ is additively finite and computed the shape of components of its Auslander-Reiten quiver.

(4) The cluster category $\mathcal{C}_Q$ of a finite quiver Q without oriented cycles was introduced in [7] if Q is an orientation of a Dynkin diagram of type $\mathbb{A}$ and in [6] in the general case. The category $\mathcal{C}_Q$ is triangulated [19] and, if Q is representation-finite, satisfies a) and b').

In a recent article [32], J. Xiao and B. Zhu determined the structure of the Auslander-Reiten quiver of a locally finite triangulated category. In this paper, we obtain the same result with a new proof in section 4, namely that each connected component of the Auslander-Reiten quiver of the category $\mathcal{T}$ is of the form $\mathbb{Z}\Delta/G$, where Δ is a simply laced Dynkin diagram and G is trivial or a weakly admissible group of automorphisms. We are interested in the k-linear structure of $\mathcal{T}$. If the Auslander-Reiten quiver of $\mathcal{T}$ is of the form $\mathbb{Z}\Delta$, we show that the category $\mathcal{T}$ is standard, *i.e.* it is equivalent to the mesh category $k(\mathbb{Z}\Delta)$. Then in section 6, we prove that $\mathcal{T}$ is standard if the number of vertices of $\Gamma = \mathbb{Z}\Delta/G$ is strictly greater than the number of isoclasses of indecomposables of $\mathsf{mod}\, k\Delta$. In the last section, using [4] we construct examples of non standard triangulated categories such that $\Gamma = \mathbb{Z}\Delta/\tau$.

Finally, in the standard cases, we are interested in the triangulated structure of $\mathcal{T}$. For this, we need to make additional assumptions on $\mathcal{T}$. If the Auslander-Reiten quiver is of the form $\mathbb{Z}\Delta$, and if $\mathcal{T}$ is the base of a tower of triangulated categories [18], we show that there is a triangle equivalence between $\mathcal{T}$ and the derived category $\mathcal{D}^b(\mathsf{mod}\, k\Delta)$. For the additively finite cases, we have to assume that $\mathcal{T}$ is standard and algebraic in the sense of [20]. We then show that $\mathcal{T}$ is (algebraically) triangle equivalent to the orbit category of $\mathcal{D}^b(\mathsf{mod}\, k\Delta)$ under the action of a weakly admissible group of automorphisms. In particular, for each $d \geq 2$, the algebraic triangulated categories with finitely many indecomposables which are maximal Calabi-Yau of CY-dimension d are parametrized by the simply laced Dynkin diagrams.

Our results apply in particular to many stable categories $\underline{\mathsf{mod}}\, A$ of representation-finite selfinjective algebras A. These algebras were classified up to stable equivalence by C. Riedtmann [25] [27] and H. Asashiba [1]. In [5], J. Białkowski and A. Skowroński give a necessary and sufficient condition on these algebras so that their stable categories $\underline{\mathsf{mod}}\, A$ are Calabi-Yau. In [15] and [16], T. Holm and P. Jørgensen prove that certain stable categories $\underline{\mathsf{mod}}\, A$ are in fact d-cluster categories. These results can also be proved using our corollary 7.0.6.

This paper is organized as follows: In section 1, we prove that $\mathcal{T}$ has Auslander-Reiten triangles. Section 2 is dedicated to definitions about stable valued translation quivers and admissible automorphisms groups [12], [13], [8]. We show in section 3 that the Auslander-Reiten quiver of $\mathcal{T}$ is a stable valued quiver and in section 4, we reprove the result of J. Xiao and B. Zhu [32]: The Auslander-Reiten quiver is a disjoint union of quivers $\mathbb{Z}\Delta/G$, where Δ is a Dynkin quiver of type $\mathbb{A}, \mathbb{D}$ or $\mathbb{E}$, and G a weakly admissible group of automorphisms. In section 5, we construct a covering functor $\mathcal{D}^b(\mathsf{mod}\, k\Delta) \to \mathcal{T}$ using Riedtmann's method [24]. Then, in section 6, we exhibit some combinatorial cases in which $\mathcal{T}$ has to be standard, in particular when $\mathcal{T}$ is maximal d-Calabi-Yau with $d \geq 2$. Section 7 is dedicated to the algebraic case. If $\mathcal{T}$ is algebraic and standard, we can construct a triangle equivalence between $\mathcal{T}$ and an orbit category. If $\mathcal{P}$ is a k-category such that $\mathsf{mod}\, \mathcal{P}$ is a Frobenius category satisfying certain conditions, we will prove in section 8 that $\mathcal{P}$ has naturally a triangulated structure. This allows us to deduce in section 9 that the category $\mathsf{proj}\, P^f(\Delta)$ of the projective modules over a deformed preprojective algebra of generalized Dynkin type [4] is naturally triangulated and to reduce the classification of the additively finite triangulated categories which are 1-Calabi-Yau to that of the deformed preprojective algebras in the sense of [4].

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Notation and terminology

We work over an algebraically closed field k. By a *triangulated category*, we mean a k-linear triangulated category $\mathcal{T}$. We write S for the suspension functor of $\mathcal{T}$ and $U \xrightarrow{u} V \xrightarrow{v} W \xrightarrow{w} SU$ for a distinguished triangle. We say that $\mathcal{T}$ is Hom-*finite* if for each pair X, Y of objects in $\mathcal{T}$, the space $\mathsf{Hom}_{\mathcal{T}}(X,Y)$ is finite-dimensional over k. The category $\mathcal{T}$ will be called a *Krull-Schmidt* category if each object is isomorphic to a finite direct sum of indecomposable objects and the endomorphism ring of an indecomposable object is a local ring. This implies that idempotents of $\mathcal{T}$ split [11, I 3.2]. The category $\mathcal{T}$ will be called *locally finite* if for each indecomposable X of $\mathcal{T}$, there are only finitely many isoclasses of indecomposables Y such that $\mathsf{Hom}_{\mathcal{T}}(X,Y) \neq 0$. This property is selfdual by [32, prop 1.1].

The *Serre functor* will be denoted by ν (see definition in section 1). The *Auslander-Reiten translation* will always be denoted by τ (section 1).

Let $\mathcal{T}$ and $\mathcal{T}'$ be two triangulated categories. An S-*functor* (F,ϕ) is given by a k-linear functor $F : \mathcal{T} \to \mathcal{T}'$ and a functor isomorphism ϕ between the functors $F \circ S$ and $S' \circ F$, where S is the suspension of $\mathcal{T}$ and S' the suspension of $\mathcal{T}'$. The notion of ν-functor, or τ-functor is then clear. A *triangle functor* is an S-functor (F,ϕ) such that for each triangle $U \xrightarrow{u} V \xrightarrow{v} W \xrightarrow{w} SU$ of $\mathcal{T}$, the sequence $FU \xrightarrow{Fu} FV \xrightarrow{Fv} FW \xrightarrow{\phi_U \circ Fw} S'FU$ is a triangle of $\mathcal{T}'$.

The category $\mathcal{T}$ is *Calabi-Yau* if there exists an integer $d > 0$ such that we have a triangle functor isomorphism between S^d and ν. We say that $\mathcal{T}$ is *maximal d-Calabi-Yau* if $\mathcal{T}$ is d-Calabi-Yau and if for each covering functor $\mathcal{T}' \to \mathcal{T}$ with $\mathcal{T}'$ d-Calabi-Yau, we have a k-linear equivalence between $\mathcal{T}$ and $\mathcal{T}'$.

For an additive k-category $\mathcal{E}$, we write $\mathsf{mod}\,\mathcal{E}$ for the category of contravariant finitely presented functors from $\mathcal{E}$ to $\mathsf{mod}\,k$ (section 8), and if the projectives of $\mathsf{mod}\,\mathcal{E}$ coincide with the injectives, $\underline{\mathsf{mod}}\,\mathcal{E}$ will be the *stable category.*

1. Serre duality and Auslander-Reiten triangles

1.1. **Serre duality.** Recall from [23] that a *Serre functor* for $\mathcal{T}$ is an autoequivalence $\nu : \mathcal{T} \to \mathcal{T}$ together with an isomorphism $D\mathsf{Hom}_{\mathcal{T}}(X,?) \simeq \mathsf{Hom}_{\mathcal{T}}(?,\nu X)$ for each $X \in \mathcal{T}$, where D is the duality $\mathsf{Hom}_k(?,k)$.

Theorem 1.1.1. *Let $\mathcal{T}$ be a Krull-Schmidt, locally finite triangulated category. Then $\mathcal{T}$ has a Serre functor ν.*

Proof. Let X be an object of $\mathcal{T}$. We write $X^\wedge$ for the functor $\mathsf{Hom}_{\mathcal{T}}(?,X)$ and F for the functor $D\mathsf{Hom}_{\mathcal{T}}(X,?)$. Using the lemma [23, I.1.6] we just have to show that F is representable. Indeed, the category $\mathcal{T}^{op}$ is locally finite as well. The proof is in two steps.

Step 1: The functor F is finitely presented.

Let $Y_1,\dots,Y_r$ be representatives of the isoclasses of indecomposable objects of $\mathcal{T}$ such that FY_i is not zero. The space $\mathsf{Hom}(Y_i^\wedge, F)$ is finite-dimensional over k. Indeed it is isomorphic to FY_i by the Yoneda lemma. Therefore, the functor $\mathsf{Hom}(Y_i^\wedge, F) \otimes_k Y_i^\wedge$ is representable. We get an epimorphism from a representable

functor to F:

$$\bigoplus_{i=1}^{r} \mathsf{Hom}(Y_i^\wedge, F) \otimes_k Y_i^\wedge \longrightarrow F.$$

By applying the same argument to its kernel we get a projective presentation of F of the form $U^\wedge \longrightarrow V^\wedge \longrightarrow F \longrightarrow 0$, with U and V in $\mathcal{T}$.

Step 2: A cohomological functor $H : \mathcal{T}^{op} \to \mathsf{mod}\, k$ is representable if and only if it is finitely presented.

Let $U^\wedge \xrightarrow{u^\wedge} V^\wedge \xrightarrow{\phi} H \longrightarrow 0$ be a presentation of H. We form a triangle $U \xrightarrow{u} V \xrightarrow{v} W \xrightarrow{w} SU$. We get an exact sequence

$$U^\wedge \xrightarrow{u^\wedge} V^\wedge \xrightarrow{v^\wedge} W^\wedge \xrightarrow{w^\wedge} (SU)^\wedge .$$

Since the composition of ϕ with $u^\wedge$ is zero and H is cohomological, the morphism ϕ factors through $v^\wedge$. But H is the cokernel of $u^\wedge$, so $v^\wedge$ factors through ϕ. We obtain a commutative diagram:

$$\begin{tikzcd} U^\wedge \arrow[r, "u^\wedge"] & V^\wedge \arrow[r, "v^\wedge"] \arrow[d, "\phi"'] & W^\wedge \arrow[r, "w^\wedge"] \arrow[dl, shift left, "\phi'"] & SU. \\ & H \arrow[ur, "i"] & & \end{tikzcd}$$

The equality $\phi' \circ i \circ \phi = \phi' \circ v^\wedge = \phi$ implies that $\phi' \circ i$ is the identity of H because ϕ is an epimorphism. We deduce that H is a direct factor of $W^\wedge$. The composition $i \circ \phi' = e^\wedge$ is an idempotent. Then $e \in \mathsf{End}(W)$ splits and we get $H = W'^\wedge$ for a direct factor W' of W.

□

1.2. **Auslander-Reiten triangles.**

Definition 1.2.1. [10] *A triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} SX$ of $\mathcal{T}$ is called an* Auslander-Reiten triangle *or* AR-triangle *if the following conditions are satisfied:*

(AR1) X and Z are indecomposable objects;

(AR2) $w \neq 0$;

(AR3) if $f : W \to Z$ is not a retraction, there exists $f' : W \to Y$ such that $vf' = f$;

(AR3') if $g : X \to V$ is not a section, there exists $g' : Y \to V$ such that $g'u = g$.

Let us recall that, if (AR1) and (AR2) hold, the conditions (AR3) and (AR3') are equivalent. We say that a triangulated category $\mathcal{T}$ *has Auslander-Reiten triangles* if, for any indecomposable object Z of $\mathcal{T}$, there exists an AR-triangle ending at Z: $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} SX$. In this case, the AR-triangle is unique up to triangle isomorphism inducing the identity of Z.

The following proposition is proved in [23, Proposition I.2.3]

Proposition 1.2.1. *The category $\mathcal{T}$ has Auslander-Reiten triangles.*

The composition $\tau = S^{-1}\nu$ is called the Auslander-Reiten translation. An AR-triangle of $\mathcal{T}$ ending at Z has the form:

$$\tau Z \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \nu Z.$$

2. Valued translation quivers and automorphism groups

2.1. **Translation quivers.** In this section, we recall some definitions and notations concerning quivers [8]. A quiver $Q = (Q_0, Q_1, s, t)$ is given by the set Q_0 of its vertices, the set Q_1 of its arrows, a source map s and a tail map t. If $x \in Q_0$ is a vertex, we denote by x^+ the set of direct successors of x, and by x^- the set of its direct predecessors. We say that Q is *locally finite* if for each vertex $x \in Q_0$, there are finitely many arrows ending at x and starting at x (in this case, x^+ and x^- are finite sets). The quiver Q is said to be *without double arrows*, if two different arrows cannot have the same tail and source.

Definition 2.1.1. *A* stable translation quiver (Q, τ) *is a locally finite quiver without double arrows with a bijection $\tau : Q_0 \to Q_0$ such that $(\tau x)^+ = x^-$ for each vertex x. For each arrow $\alpha : x \to y$, let $\sigma\alpha$ the unique arrow $\tau y \to x$.*

Note that a stable translation quiver can have loops.

Definition 2.1.2. *A* valued translation quiver (Q, τ, a) *is a stable translation quiver (Q, τ) with a map $a : Q_1 \to \mathbb{N}$ such that $a(\alpha) = a(\sigma\alpha)$ for each arrow α. If α is an arrow from x to y, we write a_{xy} instead of $a(\alpha)$.*

Definition 2.1.3. *Let Δ be an oriented tree. The* repetition of Δ *is the quiver $\mathbb{Z}\Delta$ defined as follows:*

- $(\mathbb{Z}\Delta)_0 = \mathbb{Z} \times \Delta_0$
- $(\mathbb{Z}\Delta)_1 = \mathbb{Z} \times \Delta_1 \cup \sigma(\mathbb{Z} \times \Delta_1)$ *with arrows $(n, \alpha) : (n, x) \to (n, y)$ and $\sigma(n, \alpha) : (n-1, y) \to (n, x)$ for each arrow $\alpha : x \to y$ of Δ.*

The quiver $\mathbb{Z}\Delta$ with the translation $\tau(n, x) = (n-1, x)$ is clearly a stable translation quiver which does not depend (up to isomorphism) on the orientation of Δ (see [24]).

2.2. **Groups of weakly admissible automorphisms.**

Definition 2.2.1. *An automorphism group G of a quiver is said to be* admissible [24] *if no orbit of G intersects a set of the form $\{x\} \cup x^+$ or $\{x\} \cup x^-$ in more than one point. It said to be* weakly admissible [8] *if, for each $g \in G - \{1\}$ and for each $x \in Q_0$, we have $x^+ \cap (gx)^+ = \emptyset$.*

Note that an admissible automorphism group is a weakly admissible automorphism group. Let us fix a numbering and an orientation of the simply-laced Dynkin trees.

$$\mathbb{A}_n : \quad 1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n-1 \longrightarrow n$$

$$\begin{tikzcd}
 & & & n-1 \arrow[dl] \\
\mathbb{D}_n : \quad 1 \arrow[r] & 2 \cdots \arrow[r] & n-2 & \\
 & & & n \arrow[ul]
\end{tikzcd}$$

$$\begin{tikzcd}
 & & 4 & & & \\
\mathbb{E}_n : \quad 1 & 2 \arrow[l] & 3 \arrow[l] \arrow[u] \arrow[r] & 5 \arrow[r] & \cdots \arrow[r] & n
\end{tikzcd}$$

Let Δ be a Dynkin tree. We define an automorphism S of $\mathbb{Z}\Delta$ as follows:

- if $\Delta = \mathbb{A}_n$, then $S(p, q) = (p + q, n + 1 - q)$;

- if $\Delta = \mathbb{D}_n$ with n even, then $S = \tau^{-n+1}$;
- if $\Delta = \mathbb{D}_n$ with n odd, then $S = \tau^{-n+1}\phi$ where ϕ is the automorphism of $\mathbb{D}_n$ which exchanges n and $n-1$;
- if $\Delta = \mathbb{E}_6$, then $S = \phi\tau^{-6}$ where ϕ is the automorphism of $\mathbb{E}_6$ which exchanges 2 and 5, and 1 and 6;
- if $\Delta = \mathbb{E}_7$, then $S = \tau^{-9}$;
- and if $\Delta = \mathbb{E}_8$, then $S = \tau^{-15}$.

In [24, Anhang 2], Riedtmann describes all admissible automorphism groups of Dynkin diagrams. Here is a more precise result:

Theorem 2.2.1. *Let Δ be a Dynkin tree and G a non trivial group of weakly admissible automorphisms of $\mathbb{Z}\Delta$. Then G is isomorphic to $\mathbb{Z}$, and here is a list of its possible generators:*

- *if $\Delta = \mathbb{A}_n$ with n odd, possible generators are τ^r and $\phi\tau^r$ with $r \geq 1$, where $\phi = \tau^{\frac{n+1}{2}}S$ is an automorphism of $\mathbb{Z}\Delta$ of order 2;*
- *if $\Delta = \mathbb{A}_n$ with n even, then possible generators are ρ^r, where $r \geq 1$ and where $\rho = \tau^{\frac{n}{2}}S$. (Since $\rho^2 = \tau^{-1}$, τ^r is a possible generator.)*
- *if $\Delta = \mathbb{D}_n$ with $n \geq 5$, then possible generators are τ^r and $\tau^r\phi$, where $r \geq 1$ and where $\phi = (n-1, n)$ is the automorphism of $\mathbb{D}_n$ exchanging n and $n-1$.*
- *if $\Delta = \mathbb{D}_4$, then possible generators are $\phi\tau^r$, where $r \geq 1$ and where ϕ belongs to $\mathfrak{S}_3$ the permutation group on 3 elements seen as subgroup of automorphisms of $\mathbb{D}_4$.*
- *if $\Delta = \mathbb{E}_6$, then possible generators are τ^r and $\phi\tau^r$, where $r \geq 1$ and where ϕ is the automorphism of $\mathbb{E}_6$ exchanging 2 and 5, and 1 and 6.*
- *if $\Delta = \mathbb{E}_n$ with $n = 7, 8$, possible generators are τ^r, where $r \geq 1$.*

The unique weakly admissible automorphism group which is not admissible exists for $\mathbb{A}_n$, n even, and is generated by ρ.

3. Property of the Auslander-Reiten translation

We define the Auslander-Reiten quiver $\Gamma_{\mathcal{T}}$ of the category $\mathcal{T}$ as a valued quiver (Γ, a). The vertices are the isoclasses of indecomposable objects. Given two indecomposable objects X and Y of $\mathcal{T}$, we draw one arrow from $x = [X]$ to $y = [Y]$ if the vector space $\mathcal{R}(X,Y)/\mathcal{R}^2(X,Y)$ is not zero, where $\mathcal{R}(?,?)$ is the radical of the bifunctor $\mathsf{Hom}_{\mathcal{T}}(?,?)$. A morphism of $\mathcal{R}(X,Y)$ which does not vanish in the quotient $\mathcal{R}(X,Y)/\mathcal{R}^2(X,Y)$ will be called *irreducible*. Then we put

$$a_{xy} = \dim_k \mathcal{R}(X,Y)/\mathcal{R}^2(X,Y).$$

Remark that the fact that $\mathcal{T}$ is locally finite implies that its AR-quiver is locally finite. The aim of this section is to show that $\Gamma_{\mathcal{T}}$ with the translation τ defined in the first part is a valued translation quiver. In other words, we want to show the proposition:

Proposition 3.0.2. *If X and Y are indecomposable objects of $\mathcal{T}$, we have the equality*

$$\dim_k \mathcal{R}(X,Y)/\mathcal{R}^2(X,Y) = \dim_k \mathcal{R}(\tau Y, X)/\mathcal{R}^2(\tau Y, X).$$

Let us recall some definitions [11].

Definition 3.0.2. *A morphism $g : Y \to Z$ is called* sink morphism *if the following hold*

(1) g is not a retraction;

(2) if $h : M \to Z$ is not a retraction, then h factors through g;

(3) if u is an endomorphism of Y which satisfies $gu = u$, then u is an automorphism.

Dually, a morphism $f : X \to Y$ is called source morphism *if the following hold:*
(1) f is not a section;
(2) if $h : X \to M$ is not a section, then h factors through f;
(3) if u is an endomorphism of Y which satisfies $uf = f$, then u is an automorphism.

These conditions imply that X and Z are indecomposable. Obviously, if $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} SX$ is an AR-triangle, then u is a source morphism and v is a sink morphism. Conversely, if $v \in \mathsf{Hom}_{\mathcal{T}}(Y, Z)$ is a sink morphism (or if $u \in \mathsf{Hom}_{\mathcal{T}}(X, Y)$ is a source morphism), then there exists an AR-triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} SX$ (see [11, I 4.5]).

The following lemma (and the dual statement) is proved in [28, 2.2.5].

Lemma 3.0.3. *Let g be a morphism from Y to Z, where Z is indecomposable and $Y = \bigoplus_{i=1}^{r} Y_i^{n_i}$ is the decomposition of Y into indecomposables. Then the morphism g is a sink morphism if and only if the following hold:*
(1) For each $i = 1, \dots, r$ and $j = 1, \dots, n_i$, the morphism $g_{i,j}$ belongs to the radical $\mathcal{R}(Y_i, Z)$.
(2) For each $i = 1, \dots, r$, the family $(\overline{g}_{i,j})_{j=1,\dots,n_i}$ forms a k-basis of the space $\mathcal{R}(Y_i, Z)/\mathcal{R}^2(Y_i, Z)$.
(3) If $h \in \mathsf{Hom}_{\mathcal{T}}(Y', Z)$ is irreducible and Y' indecomposable, then h factors through g and Y' is isomorphic to Y_i for some i.

Using this lemma, it is easy to see that proposition 3.0.2 holds. Thus, the Auslander-Reiten quiver $\Gamma_{\mathcal{T}} = (\Gamma, \tau, a)$ of the category $\mathcal{T}$ is a valued translation quiver.

4. Structure of the Auslander-Reiten quiver

This section is dedicated to an other proof of a theorem due to J. Xiao and B. Zhu ([33]):

Theorem 4.0.4. [33] *Let $\mathcal{T}$ be a Krull-Schmidt, locally finite triangulated category. Let Γ be a connected component of the AR-quiver of $\mathcal{T}$. Then there exists a Dynkin tree Δ of type $\mathbb{A}$, $\mathbb{D}$ or $\mathbb{E}$, a weakly admissible automorphism group G of $\mathbb{Z}\Delta$ and an isomorphism of valued translation quiver*

$$\theta : \Gamma \xrightarrow{\sim} \mathbb{Z}\Delta/G.$$

The underlying graph of the tree Δ is unique up to isomorphism (it is called the type of Γ*), and the group G is unique up to conjugacy in $Aut(\mathbb{Z}\Delta)$.*

In particular, if $\mathcal{T}$ has an infinite number of isoclasses of indecomposable objects, then G is trivial, and Γ is the repetition quiver $\mathbb{Z}\Delta$.

4.1. **Auslander-Reiten quivers with a loop.** In this section, we suppose that the Auslander-Reiten quiver of $\mathcal{T}$ contains a loop, *i.e.* there exists an arrow with same tail and source. Thus, we suppose that there exists an indecomposable X of $\mathcal{T}$ such that

$$\dim_k \mathcal{R}(X, X)/\mathcal{R}^2(X, X) \geq 1.$$

Proposition 4.1.1. *Let X be an indecomposable object of $\mathcal{T}$. Suppose that we have $\dim_k \mathcal{R}(X, X)/\mathcal{R}^2(X, X) \geq 1$. Then τX is isomorphic to X.*

To prove this, we need a lemma.

Lemma 4.1.2. *Let $X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_n} X_{n+1}$ be a sequence of irreducible morphisms between indecomposable objects with $n \geq 2$. If the composition $f_n \circ f_{n-1} \cdots f_1$ is zero, then there exists an i such that $\tau^{-1} X_i$ is isomorphic to X_{i+2}.*

Proof. The proof proceeds by induction on n. Let us show the assertion for $n = 2$. Suppose $X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3$ is a sequence such that $f_2 \circ f_1 = 0$. We can then construct an AR-triangle:

$$\begin{tikzcd} X_1 \arrow[r, "{(f_1,f)^T}"] & X_2 \oplus X \arrow[r, "{(g_1,g_2)}"] \arrow[d, "{(f_2,0)}"] & \tau^{-1}X_1 \arrow[r] \arrow[dl, dotted, "\beta"] & SX_1 \\ & X_3 & & \end{tikzcd}$$

The composition $f_2 \circ f_1$ is zero, thus the morphism f_2 factors through g_1. As the morphisms g_1 and f_2 are irreducible, we conclude that β is a retraction, and X_3 a direct summand of $\tau^{-1}X_1$. But X_1 is indecomposable, so β is an isomorphism between X_3 and $\tau^{-1}X_1$.

Now suppose that the property holds for an integer $n-1$ and that we have $f_n f_{n-1} \cdots f_1 = 0$. If the composition $f_{n-1} \cdots f_1$ is zero, the proposition holds by induction. So we can suppose that for $i \leq n-2$, the objects $\tau^{-1}X_i$ and X_{i+2} are not isomorphic. We show now by induction on i that for each $i \leq n-1$, there exists a map $\beta_i : \tau^{-1}X_i \to X_{n+1}$ such that $f_n \cdots f_{i+1} = \beta_i g_i$ where $g_i : X_{i+1} \to \tau^{-1}X_i$ is an irreducible morphism. For $i = 1$, we construct an AR-triangle:

$$\begin{tikzcd} X_1 \arrow[r, "{(f_1,f_1')^T}"] & X_2 \oplus X_1' \arrow[r, "{(g_1,g_1')}"] \arrow[d, "{(f_n\cdots f_2,0)}"'] & \tau^{-1}X_1 \arrow[r] \arrow[dl, dotted, "\beta_1"] & SX_1 \\ & X_{n+1} & & \end{tikzcd}$$

As the composition $f_n \cdots f_1$ is zero, we have the factorization $f_n \cdots f_2 = \beta_1 g_1$.

Now for i, as $\tau^{-1}X_{i-1}$ is not isomorphic to X_{i+1}, there exists an AR-triangle of the form:

$$\begin{tikzcd} X_i \arrow[r, "{(g_{i-1},f_i,f_i')^T}"] & \tau^{-1}X_{i-1} \oplus X_{i+1} \oplus X_i' \arrow[r, "{(g_i'',g_i,g_i')}"] \arrow[d, "{(-\beta_{i-1},f_n\cdots f_{i+1},0)}"'] & \tau^{-1}X_i \arrow[r] \arrow[dl, dotted, "\beta_i"] & SX_i \\ & X_{n+1} & & \end{tikzcd}$$

By induction, $-\beta_{i-1}g_{i-1} + f_n \cdots f_{i+1} f_i$ is zero, thus $f_n \cdots f_{i+1}$ factors through g_i. This property is true for $i = n-1$, so we have a map $\beta_{n-1} : \tau^{-1}X_{n-1} \to X_{n+1}$ such that $\beta_{n-1}g_{n-1} = f_n$. As g_{n-1} and f_n are irreducible, we conclude that β_{n-1} is an isomorphism between X_{n+1} and $\tau^{-1}X_{n-1}$. □

Now we are able to prove proposition 4.1.1. There exists an irreducible map $f : X \to X$. Suppose that X and τX are not isomorphic. Then from the previous lemma, the endomorphism f^n is non zero for each n. But since $\mathcal{T}$ is a Krull-Schmidt, locally finite category, a power of the radical $\mathcal{R}(X, X)$ vanishes. This is a contradiction.

4.2. **Proof of theorem 4.0.4.** Let $\tilde{\Gamma} = (\tilde{\Gamma}_0, \tilde{\Gamma}_1, \tilde{a})$ be the valued translation quiver obtained from Γ by removing the loops, *i.e.* we have $\tilde{\Gamma}_0 = \Gamma_0$, $\tilde{\Gamma}_1 = \{\alpha \in \Gamma_1$ such that $s(\alpha) \neq t(\alpha)\}$, and $\tilde{a} = a_{|_{\tilde{\Gamma}_1}}$.

Lemma 4.2.1. *The quiver $\tilde{\Gamma} = (\tilde{\Gamma}_0, \tilde{\Gamma}_1, \tilde{a})$ with the translation τ is a valued translation quiver without loop.*

Proof. We have to check that the map σ is well-defined. But from proposition 4.1.1, if α is a loop on a vertex x, $\sigma(\alpha)$ is the unique arrow from $\tau x = x$ to x, *i.e.*

$\sigma(\alpha) = \alpha$. Thus $\tilde{\Gamma}$ is obtained from Γ by removing some σ-orbits and it keeps the structure of stable valued translation quiver. $\square$

Now, we can apply Riedtmann's Struktursatz [24] and the result of Happel-Preiser-Ringel [13]. There exist a tree Δ and an admissible automorphism group G (which may be trivial) of $\mathbb{Z}\Delta$ such that $\tilde{\Gamma}$ is isomorphic to $\mathbb{Z}\Delta/G$ as a valued translation quiver. The underlying graph of the tree Δ is then unique up to isomorphism and the group G is unique up to conjugacy in $\mathrm{Aut}(\mathbb{Z}\Delta)$. Let x be a vertex of Δ. We write $\overline{x}$ for the image of x by the map:

$$\Delta \longrightarrow \mathbb{Z}\Delta \xrightarrow{\pi} \mathbb{Z}\Delta/G \simeq \tilde{\Gamma} \hookrightarrow \Gamma.$$

Let $C : \Delta_0 \times \Delta_0 \to \mathbb{Z}$ be the matrix defined as follows:

- $C(x, y) = -a_{\overline{x}\,\overline{y}}$ (resp. $-a_{\overline{y}\,\overline{x}}$) if there exists an arrow from x to y (resp. from y to x) in Δ,
- $C(x, x) = 2 - a_{\overline{x}\,\overline{x}}$,
- $C(x, y) = 0$ otherwise.

The matrix C is symmetric; it is a 'generalized Cartan matrix' in the sense of [12]. If we remove the loops from the 'underlying graph of C' (in the sense of [12]), we get the underlying graph of Δ.

In order to apply the result of Happel-Preiser-Ringel [12, section 2], we have to show:

Lemma 4.2.2. *The set Δ_0 of vertices of Δ is finite.*

Proof. Riedtmann's construction of Δ is the following. We fix a vertex x_0 in $\tilde{\Gamma}_0$. Then the vertices of Δ are the paths of $\tilde{\Gamma}$ beginning on x_0 and which do not contain subpaths of the form $\alpha\sigma(\alpha)$, where α is in $\tilde{\Gamma}_1$. Now suppose that Δ_0 is an infinite set. Then for each n, there exists a sequence:

$$x_0 \xrightarrow{\alpha_1} x_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} x_{n-1} \xrightarrow{\alpha_n} x_n$$

such that $\tau x_{i+2} \neq x_i$. Then there exist some indecomposables $X_0, \ldots, X_n$ such that the vector space $\mathcal{R}(X_{i-1}, X_i)/\mathcal{R}^2(X_{i-1}, X_i)$ is not zero. Thus from the lemma 4.1.2, there exists irreducible morphisms $f_i : X_{i-1} \to X_i$ such that the composition $f_n f_{n-1} \cdots f_1$ does not vanish. But the functor $\mathsf{Hom}_{\mathcal{T}}(X_0, ?)$ has finite support. Thus there is an indecomposable Y which appears an infinite number of times in the sequence $(X_i)_i$. But since $\mathcal{R}^N(Y, Y)$ vanishes for an N, we have a contradiction. $\square$

Let $\mathcal{S}$ a system of representatives of isoclasses of indecomposables of $\mathcal{T}$. For an indecomposable Y of $\mathcal{T}$, we put

$$l(Y) = \sum_{M \in \mathcal{S}} \dim_k \mathsf{Hom}_{\mathcal{T}}(M, Y).$$

This sum is finite since $\mathcal{T}$ is locally finite.

Lemma 4.2.3. *For x in Δ_0, we write $d_x = l(\overline{x})$. Then for each $x \in \Delta_0$, we have:*

$$\sum_{y \in \Delta_0} d_y C_{xy} = 2.$$

Proof. Let X and U be indecomposables of $\mathcal{T}$. Let

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} SX$$

be an AR-triangle. We write $(U, ?)$ for the cohomological functor $\mathsf{Hom}_{\mathcal{T}}(U, ?)$. Thus, we have a long exact sequence:

$$(U, S^{-1}Z) \xrightarrow{S^{-1}w_*} (U, X) \xrightarrow{u_*} (U, Y) \xrightarrow{v_*} (U, Z) \xrightarrow{w_*} (U, SX).$$

Let $S_Z(U)$ be the image of the map w_*. We have the exact sequence:

$$0 \longrightarrow S_{S^{-1}Z}(U) \longrightarrow (U,X) \xrightarrow{u_*} (U,Y) \xrightarrow{v_*} (U,Z) \xrightarrow{w_*} S_Z(U) \longrightarrow 0.$$

Thus we have the following equality:

$$\dim_k S_Z(U) + \dim_k S_{S^{-1}Z}(U) + \dim_k(U,Y) = \dim_k(U,X) + \dim_k(U,Z).$$

If U is not isomorphic to Z, each map from U to Z is radical, thus $S_Z(U)$ is zero. If U is isomorphic to Z, the map w_* factors through the radical of $\mathsf{End}(Z)$, so $S_Z(Z)$ is isomorphic to k. Then summing the previous equality when U runs over $\mathcal{S}$, we get:

$$l(X) + l(Z) = l(Y) + 2.$$

Clearly l is τ-invariant, thus $l(Z)$ equals $l(X)$. If the decomposition of Y is of the form $\bigoplus_{i=1}^r Y_i^{n_i}$, we get:

$$l(Y) = \sum_i n_i l(Y_i) = \sum_{i, X \to Y_i \in \tilde{\Gamma}} a_{XY_i} l(Y_i) + a_{XX} l(X).$$

We deduce the formula:

$$2 = (2 - a_{XX}) l(X) - \sum_{i, X \to Y_i \in \tilde{\Gamma}} a_{XY_i} l(Y_i).$$

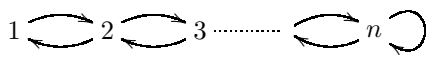
Let x be a vertex of the tree Δ and $\overline{x}$ its image in $\tilde{\Gamma}$. Then an arrow $\overline{x} \to Y$ in $\tilde{\Gamma}$ comes from an arrow $(x,0) \to (y,0)$ in $\mathbb{Z}\Delta$ or from an arrow $(x,0) \to (y,-1)$ in $\mathbb{Z}\Delta$, *i.e.* from an arrow $(y,0) \to (x,0)$. Indeed the projection $\mathbb{Z}\Delta \to \mathbb{Z}\Delta/G$ is a covering. From this we deduce the following equality:

$$2 = (2 - a_{\overline{x}\,\overline{x}}) d_x - \sum_{y, x \to y \in \Delta} a_{\overline{x}\,\overline{y}} d_y - \sum_{y, y \to x \in \Delta} a_{\overline{y}\,\overline{x}} d_y = \sum_{y \in \Delta_0} d_y C_{xy}.$$

□

Now we can prove theorem 4.0.4. The matrix C is a 'generalized Cartan matrix'. The previous lemma gives us a subadditive function which is not additive. Thus by [12], the underlying graph of C is of 'generalized Dynkin type'. As C is symmetric, the graph is necessarily of type $\mathbb{A}$, $\mathbb{D}$, $\mathbb{E}$, or $\mathbb{L}$. But this graph is the graph Δ with the valuation a. We are done in the cases $\mathbb{A}$, $\mathbb{D}$, or $\mathbb{E}$.

The case $\mathbb{L}_n$ occurs when the AR-quiver contains at least one loop. We can see $\mathbb{L}_n$ as $\mathbb{A}_n$ with valuations on the vertices with a loop. Then, it is obvious that the automorphism groups of $\mathbb{Z}\mathbb{L}_n$ are generated by τ^r for an $r \geq 1$. But proposition 4.1.1 tell us that a vertex x with a loop satisfies $\tau x = x$. Thus G is generated by τ and the AR-quiver has the following form:



This quiver is isomorphic to the quiver $\mathbb{Z}\mathbb{A}_{2n}/G$ where G is the group generated by the automorphism $\tau^n S = \rho$.

The suspension functor S sends the indecomposables on indecomposables, thus it can be seen as an automorphism of the AR-quiver. It is exactly the automorphism S defined in section 2.2.

As shown in [33], it follows from the results of [19] that for each Dynkin tree Δ and for each weakly admissible group of automorphisms G of $\mathbb{Z}\Delta$, there exists a locally finite triangulated category $\mathcal{T}$ such that $\Gamma_{\mathcal{T}} \simeq \mathbb{Z}\Delta/G$. This category is of the form $\mathcal{T} = \mathcal{D}^b(\mathsf{mod}\, k\Delta)/\varphi$ where φ is an auto-equivalence of $\mathcal{D}^b(\mathsf{mod}\, k\Delta)$.

5. Construction of a covering functor

From now, we suppose that the AR-quiver Γ of $\mathcal{T}$ is connected. We know its structure. It is natural to ask: Is the category $\mathcal{T}$ *standard*, *i.e.* equivalent as a k-linear category to the mesh category $k(\Gamma)$? First, in this part we construct a covering functor $F : k(\mathbb{Z}\Delta) \to \mathcal{T}$.

5.1. **Construction.** We write $\pi : \mathbb{Z}\Delta \to \Gamma$ for the canonical projection. As G is a weakly admissible group, this projection verifies the following property: if x is a vertex of $\mathbb{Z}\Delta$, the number of arrows of $\mathbb{Z}\Delta$ with source x is equal to the number of arrows of $\mathbb{Z}\Delta/G$ with source πx. Let $\mathcal{S}$ be a system of representatives of the isoclasses of indecomposables of $\mathcal{T}$. We write ind$\mathcal{T}$ for the full subcategory of $\mathcal{T}$ whose set of objects is $\mathcal{S}$. For a tree Δ, we write $k(\mathbb{Z}\Delta)$ for the mesh category (see [24]). Using the same proof as Riedtmann [24], one shows the following theorem:

Theorem 5.1.1. *There exists a k-linear functor $F : k(\mathbb{Z}\Delta) \to$ ind$\mathcal{T}$ which is surjective and induces bijections:*

$$\bigoplus_{Fz=Fy} \mathsf{Hom}_{k(\mathbb{Z}\Delta)}(x,z) \to \mathsf{Hom}_{\mathcal{T}}(Fx,Fy),$$

for all vertices x and y of $\mathbb{Z}\Delta$.

5.2. **Infinite case.** If the category $\mathcal{T}$ is locally finite not finite, the constructed functor F is immediately fully faithful. Thus we get the corollary.

Corollary 5.2.1. *If ind$\mathcal{T}$ is not finite, then we have a k-linear equivalence between $\mathcal{T}$ and the mesh category $k(\mathbb{Z}\Delta)$.*

5.3. **Uniqueness criterion.** The covering functor F can be see as a k-linear functor from the derived category $\mathcal{D}^b(\mathsf{mod}\, k\Delta)$ to the category $\mathcal{T}$. By construction, it satisfies the following property called the *AR-property*:

For each AR-triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} SX$ of $\mathcal{D}^b(\mathsf{mod}\, k\Delta)$, there exists a triangle of $\mathcal{T}$ of the form $FX \xrightarrow{Ff} FY \xrightarrow{Fg} FZ \xrightarrow{\epsilon} SFX$.

In fact, thanks to this property, F is determined by its restriction to the subcategory $\mathsf{proj}\, k\Delta = k(\Delta)$, *i.e.* we have the following lemma:

Lemma 5.3.1. *Let F and G be k-linear functors from $\mathcal{D}^b(\mathsf{mod}\, k\Delta)$ to $\mathcal{T}$. Suppose that F and G satisfy the AR-property and that the restrictions $F_{|_{k(\Delta)}}$ and $G_{|_{k(\Delta)}}$ are isomorphic. Then the functors F and G are isomorphic as k-linear functors.*

Proof. It is easy to construct this isomorphism by induction using the (TR3) axiom of the triangulated categories (see [22]). □

6. Particular cases of k-linear equivalence

From now we suppose that the category $\mathcal{T}$ is finite, *i.e.* $\mathcal{T}$ has finitely many isoclasses of indecomposable objects.

6.1. **Equivalence criterion.** Let Γ be the AR-quiver of $\mathcal{T}$ and suppose that it is isomorphic to $\mathbb{Z}\Delta/G$. Let φ be a generator of G. It induces an automorphism in the mesh category $k(\mathbb{Z}\Delta)$ that we still denote by φ. Then we have the following equivalence criterion:

Proposition 6.1.1. *The categories $k(\Gamma)$ and ind$\mathcal{T}$ are equivalent as k-categories if and only if there exists a covering functor $F : k(\mathbb{Z}\Delta) \to$ ind$\mathcal{T}$ and an isomorphism of functors $\Phi : F \circ \varphi \to F$.*

The proof consists in constructing a k-linear equivalence between $\mathrm{ind}\mathcal{T}$ and the orbit category $k(\mathbb{Z}\Delta)/\varphi^{\mathbb{Z}}$ using the universal property of the orbit category (see [19]), and then constructing an equivalence between $k(\mathbb{Z}\Delta)/\varphi^{\mathbb{Z}}$ and $k(\Gamma)$.

6.2. **Cylindric case for $\mathbb{A}_n$.**

Theorem 6.2.1. *If $\Delta = \mathbb{A}_n$ and $\varphi = \tau^r$ for some $r \geq 1$, then there exists a functor isomorphism $\Phi : F \circ \varphi \to F$, i.e. for each object x of $k(\mathbb{Z}\Delta)$ there exists an automorphism Φ_x of Fx such that for each arrow $\alpha : x \to y$ of $\mathbb{Z}\Delta$, the following diagram commutes:*

$$\begin{tikzcd} Fx \arrow[r, "\Phi_x"] \arrow[d, "F\alpha"'] & Fx \arrow[d, "F\varphi\alpha"] \\ Fy \arrow[r, "\Phi_y"] & Fy. \end{tikzcd}$$

To prove this, we need the following lemma:

Lemma 6.2.2. *Let $\alpha : x \to y$ be an arrow of $\mathbb{Z}\mathbb{A}_n$ and let c be a path from x to $\tau^r y$, $r \in \mathbb{Z}$, which is not zero in the mesh category $k(\mathbb{Z}\mathbb{A}_n)$. Then c can be written $c'\alpha$ where c' is a path from y to $\tau^r y$ (up to sign).*

Proof. There is a path from x to $\tau^r y$, thus, we have $\mathsf{Hom}_{k(\mathbb{Z}\Delta)}(x, \tau^r y) \simeq k$, and x and $\tau^r y$ are opposite vertices of a 'rectangle' in $\mathbb{Z}\mathbb{A}_n$. This implies that there exists a path from x to $\tau^r y$ beginning by α. □

Proof. (of theorem 6.2.1) Combining proposition 6.1.1 and lemma 5.3.1, we have just to construct an isomorphism between the restriction of F and $F \circ \varphi$ to a subquiver $\mathbb{A}_n$.

Let us fix a full subquiver of $\mathbb{Z}\mathbb{A}_n$ of the following form:

$$x_1 \xrightarrow{\alpha_1} x_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} x_n$$

such that $x_1 \ldots, x_n$ are representatives of the τ-orbits in $\mathbb{Z}\mathbb{A}_n$. We define the $(\Phi_{x_i})_{i=1\ldots n}$ by induction. We fix $\Phi_{x_1} = Id_{Fx_1}$. Now suppose we have constructed some automorphisms $\Phi_{x_1}, \ldots, \Phi_{x_i}$ such that for each $j \leq i$ the following diagram is commutative:

$$\begin{tikzcd} Fx_{j-1} \arrow[r, "\Phi_{x_{j-1}}"] \arrow[d, "F\alpha_{j-1}"'] & Fx_{j-1} \arrow[d, "F\varphi\alpha_{j-1}"] \\ Fx_j \arrow[r, "\Phi_{x_j}"] & Fx_j. \end{tikzcd}$$

The composition $(F\varphi\alpha_i) \circ \Phi_{x_i}$ is in the morphism space $\mathsf{Hom}_{\mathcal{T}}(Fx_i, Fx_{i+1})$, which is isomorphic, by theorem 5.1.1, to the space

$$\bigoplus_{Fz = Fx_{i+1}} \mathsf{Hom}_{k(\mathbb{Z}\Delta)}(x_i, z).$$

Thus we can write

$$(F\varphi\alpha_i)\Phi_{x_i} = \lambda F\alpha_i + \sum_{z \neq x_{i+1}} F\beta_z$$

where β_z belongs to $\mathsf{Hom}_{k(\mathbb{Z}\Delta)}(x_i, z)$ and $Fz = Fx_{i+1}$. But Fz is equal to Fx_{i+1} if and only if z is of the form $\tau^{rl} x_{i+1}$ for an l in $\mathbb{Z}$. By the lemma, we can write $\beta_z = \beta'_z \alpha_i$. Thus we have the equality:

$$(F\varphi\alpha_i)\Phi_{x_i} = F(\lambda Id_{x_{i+1}} + \sum_z \beta'_z) F\alpha_i.$$

The scalar λ is not zero. Indeed, Φ_{x_i} is an automorphism, thus the image of $(F\varphi\alpha_i)\Phi_{x_i}$ is not zero in the quotient

$$\mathcal{R}(Fx_i, Fx_{i+1})/\mathcal{R}^2(Fx_i, Fx_{i+1}).$$

Thus $\Phi_{x_{i+1}} = F(\lambda Id_{x_{i+1}} + \sum_z \beta'_z)$ is an automorphism of Fx_{i+1} which verifies the commutation relation

$$(F\varphi\alpha_i) \circ \Phi_{x_i} = \Phi_{x_{i+1}} \circ F\alpha_i.$$

□

6.3. **Other standard cases.** In the mesh category $k(\mathbb{Z}\Delta)$, where Δ is a Dynkin tree, the length of the non zero paths is bounded. Thus there exist automorphisms φ such that, for an arrow $\alpha : x \to y$ of Δ, the paths from x to $\varphi^r y$ vanish in the mesh category for all $r \neq 0$. In other words, for each arrow $\alpha : x \to y$ of $\mathbb{Z}\Delta$, we have:

$$\mathsf{Hom}_{k(\mathbb{Z}\Delta)/\varphi^{\mathbb{Z}}}(x, y) = \bigoplus_{r\in\mathbb{Z}} \mathsf{Hom}_{k(\mathbb{Z}\Delta)}(x, \varphi^r y) = \mathsf{Hom}_{k(\mathbb{Z}\Delta)}(x, y) \simeq k,$$

where $k(\mathbb{Z}\Delta)/\varphi^{\mathbb{Z}}$ is the orbit category (see section 6.1).

Lemma 6.3.1. *Let $\mathcal{T}$ be a finite triangulated category with AR-quiver $\Gamma = \mathbb{Z}\Delta/G$. Let φ be a generator of G and suppose that φ verifies for each arrow $x \to y$ of $\mathbb{Z}\Delta$*

$$\bigoplus_{r\in\mathbb{Z}} \mathsf{Hom}_{k(\mathbb{Z}\Delta)}(x, \varphi^r y) = \mathsf{Hom}_{k(\mathbb{Z}\Delta)}(x, y) \simeq k.$$

Let $F : k(\mathbb{Z}\Delta) \to \mathcal{T}$ and $G : k(\mathbb{Z}\Delta) \to \mathcal{T}$ be covering functors satisfying the AR-property. Suppose that F and G agree up to isomorphism on the objects of $k(\mathbb{Z}\Delta)$. Then F and G are isomorphic as k-linear functors.

Proof. Using lemma 5.3.1, we have just to construct an isomorphism between the functors restricted to Δ. Let $\alpha : x \to y$ be an arrow of Δ. Using theorem 5.1.1 and the hypothesis, we have the following isomorphisms:

$$\mathsf{Hom}_{\mathcal{T}}(Fx, Fy) \simeq \bigoplus_{Fz=Fy} \mathsf{Hom}_{k(\mathbb{Z}\Delta)}(x, z) \simeq \bigoplus_{r\in\mathbb{Z}} \mathsf{Hom}_{k(\mathbb{Z}\Delta)}(x, \varphi^r y) \simeq k$$

and then

$$\mathsf{Hom}_{\mathcal{T}}(Gx, Gy) \simeq \mathsf{Hom}_{\mathcal{T}}(Fx, Fy) \simeq k.$$

Thus there exists a scalar λ such that $G\alpha = \lambda F\alpha$. This scalar does not vanish since F and G are covering functors. As Δ is a tree, we can find some λ_x for $x \in \Delta$ by induction such that

$$G\alpha = \lambda_x \lambda_y^{-1} F\alpha.$$

Now it is easy to check that $\Phi_x = \lambda_x Id_{Fx}$ is the functor isomorphism. □

This lemma gives us an isomorphism between the functors F and $F\circ\varphi$, Moreover, using the same argument, one can show that the covering functor F is an S-functor and a τ-functor.

For each Dynkin tree Δ we can determine the automorphisms φ which satisfy this combinatorial property. Using the preceding lemma and the equivalence criterion we deduce the following theorem:

Theorem 6.3.2. *Let $\mathcal{T}$ be a finite triangulated category with AR-quiver $\Gamma = \mathbb{Z}\Delta/G$. Let φ be a generator of G. If one of these cases holds,*

- *$\Delta = \mathbb{A}_n$ with n odd and G is generated by τ^r or $\varphi = \tau^r\phi$ with $r \geq \frac{n-1}{2}$ and $\phi = \tau^{\frac{n+1}{2}}S$;*
- *$\Delta = \mathbb{A}_n$ with n even and G is generated by ρ^r with $r \geq n-1$ and $\rho = \tau^{\frac{n}{2}}S$;*

- $\Delta = \mathbb{D}_n$ *with* $n \geq 5$ *and* G *is generated by* τ^r *or* $\tau^r\phi$ *with* $r \geq n-2$ *and* ϕ *as in theorem 2.2.1;*
- $\Delta = \mathbb{D}_4$ *and* G *is generated by* $\phi\tau^r$*, where* $r \geq 2$ *and* ϕ *runs over* σ_3*;*
- $\Delta = \mathbb{E}_6$ *and* G *is generated by* τ^r *or* $\tau^r\phi$ *where* $r \geq 5$ *and* ϕ *is as in theorem 2.2.1;*
- $\Delta = \mathbb{E}_7$ *and* G *is generated by* τ^r*,* $r \geq 8$*;*
- $\Delta = \mathbb{E}_8$ *and* G *is generated by* τ^r*,* $r \geq 14$*.*

then T *is standard, i.e. the categories* $\mathcal{T}$ *and* $k(\Gamma)$ *are equivalent as* k*-linear categories.*

Corollary 6.3.3. *A finite maximal* d*-Calabi-Yau (see* [19, 8]*) triangulated category* $\mathcal{T}$*, with* $d \geq 2$*, is standard,* i.e. *there exists a* k*-linear equivalence between* $\mathcal{T}$ *the orbit category* $\mathcal{D}^b(\mathsf{mod}\, k\Delta)/\tau^{-1}S^{d-1}$ *where* Δ *is Dynkin of type* $\mathbb{A}$*,* $\mathbb{D}$ *or* $\mathbb{E}$

7. Algebraic case

For some automorphism groups G, we know the k-linear structure of $\mathcal{T}$. But what about the triangulated structure? We can only give an answer adding hypothesis on the triangulated structure. In this section, we distinguish two cases:

If $\mathcal{T}$ is locally finite, not finite, we have the following theorem which is proved in section 7.2:

Theorem 7.0.4. *Let* $\mathcal{T}$ *be a connected locally finite triangulated category with infinitely many indecomposables. If* $\mathcal{T}$ *is the base of a tower of triangulated categories* [18]*, then* $\mathcal{T}$ *is triangle equivalent to* $\mathcal{D}^b(\mathsf{mod}\, k\Delta)$ *for some Dynkin diagram* Δ*.*

Now if $\mathcal{T}$ is a finite standard category which is algebraic, *i.e.* $\mathcal{T}$ is triangle equivalent to $\underline{\mathcal{E}}$ for some k-linear Frobenius category $\mathcal{E}$ ([20, 3.6]), then we have the following result which is proved in section 7.3:

Theorem 7.0.5. *Let* $\mathcal{T}$ *be a finite triangulated category, which is connected, algebraic and standard. Then, there exists a Dynkin diagram* Δ *of type* $\mathbb{A}$*,* $\mathbb{D}$ *or* $\mathbb{E}$ *and an auto-equivalence* Φ *of* $\mathcal{D}^b(\mathsf{mod}\, k\Delta)$ *such that* $\mathcal{T}$ *is triangle equivalent to the orbit category* $\mathcal{D}^b(\mathsf{mod}\, k\Delta)/\Phi$*.*

This theorem combined with corollary 6.3.3 yields the following result (compare to [19, Cor 8.4]):

Corollary 7.0.6. *If* $\mathcal{T}$ *is a finite algebraic maximal* d*-Calabi-Yau category with* $d \geq 2$*, then* $\mathcal{T}$ *is triangle equivalent to the orbit category* $\mathcal{D}^b(\mathsf{mod}\, k\Delta)/S^d\nu^{-1}$ *for some Dynkin diagram* Δ*.*

7.1. ∂-functor.

We recall the following definition from [18] and [30].

Definition 7.1.1. *Let* $\mathcal{H}$ *be an exact category and* $\mathcal{T}$ *a triangulated category. A* ∂-functor $(I, \partial) : \mathcal{H} \to \mathcal{T}$ *is given by:*

- *an additive* k*-linear functor* $I : \mathcal{H} \to \mathcal{T}$*;*
- *for each conflation* $\epsilon : X \overset{i}{\rightarrowtail} Y \overset{p}{\twoheadrightarrow} Z$ *of* $\mathcal{H}$*, a morphism* $\partial\epsilon : IZ \to SIX$ *functorial in* ϵ *such that* $IX \xrightarrow{Ii} IY \xrightarrow{Ip} IZ \xrightarrow{\partial\epsilon} SIX$ *is a triangle of* $\mathcal{T}$*.*

For each exact category $\mathcal{H}$, the inclusion $I : \mathcal{H} \to \mathcal{D}^b(\mathcal{H})$ can be completed to a ∂-functor (I, ∂) in a unique way. Let $\mathcal{T}$ and $\mathcal{T}'$ be triangulated categories. If $(F, \varphi) : \mathcal{T} \to \mathcal{T}'$ is an S-functor and $(I, \partial) : \mathcal{H} \to \mathcal{T}$ is a ∂-functor, we say that F *respects* ∂ if $(F \circ I, \varphi(F\partial)) : \mathcal{H} \to \mathcal{T}'$ is a ∂-functor. Obviously each triangle functor respects ∂.

Proposition 7.1.1. *Let $\mathcal{H}$ be a k-linear hereditary abelian category and let $(I, \partial) : \mathcal{H} \to \mathcal{T}$ be a ∂-functor. Then there exists a unique (up to isomorphism) k-linear S-functor $F : \mathcal{D}^b(\mathcal{H}) \to \mathcal{T}$ which respects ∂.*

Proof. On $\mathcal{H}$ (which can be seen as a full subcategory of $\mathcal{D}^b(\mathcal{H})$), the functor F is uniquely determined. We want F to be an S-functor, so F is uniquely determined on $S^n\mathcal{H}$ for $n \in \mathbb{Z}$ too. Since $\mathcal{H}$ is hereditary, each object of $\mathcal{D}^b(\mathcal{H})$ is isomorphic to a direct sum of stalk complexes, *i.e.* complexes concentrated in a single degree. Thus, the functor F is uniquely determined on the objects. Now, let X and Y be stalk complexes of $\mathcal{D}^b(\mathcal{H})$ and $f : X \to Y$ a non-zero morphism. We can suppose that X is in $\mathcal{H}$ and Y is in $S^n\mathcal{H}$. If $n \neq 0, 1$, f is necessarily zero. If $n = 0$, then f is a morphism in $\mathcal{H}$ and Ff is uniquely determined. If $n = 1$, f is an element of $\mathsf{Ext}^1_{\mathcal{H}}(X, S^{-1}Y)$, so gives us a conflation $\epsilon : S^{-1}Y \overset{i}{\rightarrowtail} E \overset{p}{\twoheadrightarrow} X$ in $\mathcal{H}$. The functor F respects ∂, thus Ff has to be equal to $\varphi \circ \partial\epsilon$ where φ is the natural isomorphism between $SFS^{-1}Y$ and FY. Since ∂ is functorial, F is a functor. The result follows. □

A priori this functor is not a triangle functor. We recall a theorem proved by B. Keller [18, cor 2.7].

Theorem 7.1.2. *Let $\mathcal{H}$ be a k-linear exact category, and $\mathcal{T}$ be the base of a tower of triangulated categories* [18]*. Let $(I, \partial) : \mathcal{H} \to \mathcal{T}$ be a ∂-functor such that for each $n < 0$, and all objects X and Y of $\mathcal{H}$, the space $\mathsf{Hom}_{\mathcal{T}}(IX, S^nIY)$ vanishes. Then there exists a triangle functor $F : \mathcal{D}^b(\mathcal{H}) \to \mathcal{T}$ such that the following diagram commutes up to isomorphism of ∂-functors:*

$$\begin{tikzcd} \mathcal{H} \arrow[rr, hook] \arrow[dr, "(I,\partial)"'] & & \mathcal{D}^b(\mathcal{H}) \arrow[dl, "F"] \\ & \mathcal{T} & \end{tikzcd}$$

From theorem 7.1.2, and the proposition above we deduce the following corollary:

Corollary 7.1.3. *(compare to* [29]*) Let $\mathcal{T}$, $\mathcal{H}$ and $(I, \partial) : \mathcal{H} \to \mathcal{T}$ be as in theorem 7.1.2. If $\mathcal{H}$ is hereditary, then the unique functor $F : \mathcal{D}^b(\mathcal{H}) \to \mathcal{T}$ which respects ∂ is a triangle functor.*

7.2. **Proof of theorem 7.0.4.** Let F be the k-linear equivalence constructed in theorem 5.1.1 between an algebraic triangulated category $\mathcal{T}$ and $\mathcal{D}^b(\mathcal{H})$ where $\mathcal{H} = \mathsf{mod}\, k\Delta$ and Δ is a simply laced Dynkin graph. As we saw in section 6, the covering functor is an S-functor.

The category $\mathcal{H}$ is the heart of the standard t-structure on $\mathcal{D}^b(\mathcal{H})$. The image of this t-structure through F is a t-structure on $\mathcal{T}$. Indeed, F is an S-equivalence, so the conditions (i) and (ii) from [3, Def 1.3.1] hold obviously. And since $\mathcal{H}$ is hereditary, for an object X of $\mathcal{D}^b(\mathcal{H})$, the morphism $X \to S\tau_{\leq 0}X$ of the triangle

$$\tau_{\leq 0}X \longrightarrow X \longrightarrow \tau_{\geq 0}X \longrightarrow S\tau_{\leq 0}X$$

vanishes. Thus the image of this triangle through F is a triangle of $\mathcal{T}$ and condition (iii) of [3, Def 1.3.1] holds. Then we get a t-structure on $\mathcal{T}$ whose heart is $\mathcal{H}$.

It results from [3, Prop 1.2.4] that the inclusion of the heart of a t-structure can be uniquely completed to a ∂-functor. Thus we obtain a ∂-functor $(F_0, \partial) : \mathcal{H} \to \mathcal{T}$ with $F_0 = F_{|\mathcal{H}}$.

The functor F is an S-equivalence. Thus for each $n < 0$, and all objects X and Y of $\mathcal{H}$, the space $\mathsf{Hom}_{\mathcal{T}}(FX, S^nFY)$ vanishes. Now we can apply theorem 7.1.2

and we get the following commutative diagram:

$$\begin{tikzcd} \mathcal{H} \arrow[rr, hook] \arrow[dr, "(F_0,\partial)"'] & & \mathcal{D}^b(\mathcal{H}), \arrow[dl, shift left, "G"] \arrow[dl, shift right, "F"'] \\ & \mathcal{T} & \end{tikzcd}$$

where F is the S-equivalence and G is a triangle functor. Note that a priori F is an S-functor which does not respect ∂. The functors $F_{|\mathcal{H}}$ and $G_{|\mathcal{H}}$ are isomorphic. The functor F is an S-functor thus we have an isomorphism $F_{|S^n\mathcal{H}} \simeq G_{|S^n\mathcal{H}}$ for each $n \in \mathbb{Z}$. Thus the functor G is essentially surjective. Since $\mathcal{H}$ is the category $\mathsf{mod}\, k\Delta$, to show that G is fully faithful, we have just to show that for each $p \in \mathbb{Z}$, there is an isomorphism induced by G

$$\mathsf{Hom}_{\mathcal{D}^b(\mathcal{H})}(A, S^pA) \xrightarrow{\sim} \mathsf{Hom}_{\mathcal{T}}(GA, S^pGA)$$

where A is the free module $k\Delta$. For $p = 0$, this is clear because A is in $\mathcal{H}$. And for $p \neq 0$ both sides vanish.

Thus G is a triangle equivalence between $\mathcal{D}^b(\mathcal{H})$ and $\mathcal{T}$.

7.3. **Finite algebraic standard case.** For a small dg category $\mathcal{A}$, we denote by $\mathcal{CA}$ the category of dg $\mathcal{A}$-modules, by $\mathcal{DA}$ the derived category of $\mathcal{A}$ and by $\mathsf{per}\,\mathcal{A}$ the *perfect derived category* of $\mathcal{A}$, *i.e.* the smallest triangulated subcategory of $\mathcal{DA}$ which is stable under passage to direct factors and contains the free $\mathcal{A}$-modules $\mathcal{A}(?, A)$, where A runs through the objects of $\mathcal{A}$. Recall that a small triangulated category is *algebraic* if it is triangle equivalent to $\mathsf{per}\,\mathcal{A}$ for a dg category $\mathcal{A}$. For two small dg categories $\mathcal{A}$ and $\mathcal{B}$, a triangle functor $\mathsf{per}\,\mathcal{A} \to \mathsf{per}\,\mathcal{B}$ is *algebraic* if it is isomorphic to the functor

$$F_X = ? \overset{L}{\otimes}_{\mathcal{A}} X$$

associated with a dg bimodule X, *i.e.* an object of the derived category $\mathcal{D}(\mathcal{A}^{op} \otimes \mathcal{B})$.

Let Φ be an algebraic autoequivalence of $\mathcal{D}^b(\mathsf{mod}\, k\Delta)$ such that the orbit category $\mathcal{D}^b(\mathsf{mod}\, k\Delta)/\Phi$ is triangulated. Let Y be a dg $k\Delta$-$k\Delta$-bimodule such that $\Phi = F_Y$. In section 9.3 of [19], it was shown that there is a canonical triangle equivalence between this orbit category and the perfect derived category of a certain small dg category. Thus, the orbit category is algebraic, and endowed with a canonical triangle equivalence to the perfect derived category of a small dg category. Moreover, by the construction in [loc. cit.], the projection functor

$$\pi : \mathcal{D}^b(\mathsf{mod}\, k\Delta) \to \mathcal{D}^b(\mathsf{mod}\, k\Delta)/\Phi$$

is algebraic.

The proof of theorem 7.0.5 is based on the following universal property of the triangulated orbit category $\mathcal{D}^b(\mathsf{mod}\, k\Delta)/\Phi$. For the proof, we refer to section 9.3 of [19].

Proposition 7.3.1. *Let $\mathcal{B}$ be a small dg category and*

$$F_X = ? \overset{L}{\otimes}_{k\Delta} X : \mathcal{D}^b(\mathsf{mod}\, k\Delta) \to \mathsf{per}\,\mathcal{B}$$

an algebraic triangle functor given by a dg $k\Delta$-$\mathcal{A}$-bimodule X. Suppose that there is an isomorphism between $Y \overset{L}{\otimes}_{k\Delta} X$ and X in the derived bimodule category $\mathcal{D}(k\Delta^{op} \otimes \mathcal{B})$. Then the functor F_X factors, up to isomorphism of triangle functors, through the projection

$$\pi : \mathcal{D}^b(\mathsf{mod}\, k\Delta) \to \mathcal{D}^b(\mathsf{mod}\, k\Delta)/\Phi.$$

Moreover, the induced triangle functor is algebraic.

Let us recall a lemma of Van den Bergh [21]:

Lemma 7.3.2. *Let Q be a quiver without oriented cycles and $\mathcal{A}$ be a dg category. We denote by $k(Q)$ the category of paths of Q and by $Can : \mathcal{CA} \to \mathcal{DA}$ the canonical functor. Then we have the following properties:*

a) Each functor $F : k(Q) \to \mathcal{DA}$ lifts, up to isomorphism, to a functor $\tilde{F} : k(Q) \to \mathcal{CA}$ which verifies the following property: For each vertex j of Q, the induced morphism

$$\bigoplus_i \tilde{F}i \to \tilde{F}j,$$

where i runs through the immediate predecessors of j, is a monomorphism which splits as a morphism of graded $\mathcal{A}$-modules.

b) Let F and G be functors from $k(Q)$ to $\mathcal{CA}$, and suppose that F satisfies the property of a). Then any morphism of functors $\varphi : Can \circ F \to Can \circ G$ lifts to a morphism $\tilde{\varphi} : F \to G$.

Proof. *a)* For each vertex i of Q, the object Fi is isomorphic in $\mathcal{DA}$ to its cofibrant resolution X_i. Thus for each arrow $\alpha : i \to j$, F induces a morphism $f_\alpha : X_i \to X_j$ which can be lifted to $\mathcal{CA}$ since the X_i are cofibrant. Since Q has no oriented cycle, it is easy to choose the f_α such that the property is satisfied.

b) For each vertex i of Q, we may assume that Fi is cofibrant. Then we can lift $\varphi_i : Can \circ Fi \to Can \circ Gi$ to $\psi_i : Fi \to Gi$. For each arrow α of Q, the square

$$\begin{tikzcd} Fi \arrow[r, "F_\alpha"] \arrow[d, "\psi_i"] & Fj \arrow[d, "\psi_j"] \\ Gi \arrow[r, "G_\alpha"] & Gj \end{tikzcd}$$

is commutative in $\mathcal{DA}$. Thus the square

$$\begin{tikzcd} \bigoplus_i Fi \arrow[r, "F_\alpha"] \arrow[d, "(\psi_i)"] & Fj \arrow[d, "\psi_j"] \\ \bigoplus_i Gi \arrow[r, "G_\alpha"] & Gj \end{tikzcd}$$

is commutative up to nullhomotopic morphism $h : \bigoplus_i Fi \to Gj$. Since the morphism $f : \bigoplus_i Fi \to Fj$ is split mono in the category of graded $\mathcal{A}$-modules, h extends along f and we can modify Ψ_j so that the square becomes commutative in $\mathcal{CA}$. The quiver Q does not have oriented cycles, so we can construct $\tilde{\varphi}$ by induction. □

Proof. (of theorem 7.0.5) The category $\mathcal{T}$ is small and algebraic, thus we may assume that $\mathcal{T} = \mathsf{per}\,\mathcal{A}$ for some small dg category $\mathcal{A}$. Let $F : \mathcal{D}^b(\mathsf{mod}\, k\Delta) \to \mathcal{T}$ be the covering functor of theorem 5.1.1. Let Φ be an auto-equivalence of $\mathcal{D}^b(\mathsf{mod}\, k\Delta)$ such that the AR-quiver of the orbit category $\mathcal{D}^b(\mathsf{mod}\, k\Delta)/\Phi$ is isomorphic (as translation quiver) to the AR-quiver of $\mathcal{T}$. We may assume that $\Phi = - \overset{L}{\otimes}_{k\Delta} Y$ for an object Y of $\mathcal{D}(k\Delta^{op} \otimes k\Delta)$. The orbit category $\mathcal{D}^b(\mathsf{mod}\, k\Delta)/\Phi$ is algebraic, thus it is $\mathsf{per}\,\mathcal{B}$ for some dg category $\mathcal{B}$.

The functor $F_{|_{k(\Delta)}}$ lifts by lemma 7.3.2 to a functor $\tilde{F}$ from $k(\Delta)$ to $\mathcal{CA}$. This means that the object $X = \tilde{F}(k\Delta)$ has a structure of dg $k\Delta^{op} \otimes \mathcal{A}$-module. We denote by X the image of this object in $\mathcal{D}(k\Delta^{op} \otimes \mathcal{A})$.

The functors F and $- \overset{L}{\otimes}_{k\Delta} X$ become isomorphic when restricted to $k(\Delta)$. Moreover $- \overset{L}{\otimes}_{k\Delta} X$ satisfies the AR-property since it is a triangulated functor. Thus by lemma 5.3.1, they are isomorphic as k-linear functors. So we have the following

diagram:

$$\begin{tikzcd} \mathcal{D}^b(\mathsf{mod}\, k\Delta) \arrow[r, "-\overset{L}{\otimes}_{k\Delta} X"] \arrow[loop below, "-\overset{L}{\otimes}_{k\Delta} Y"] & \mathsf{per}\, \mathcal{A} = \mathcal{T} \end{tikzcd}$$

The category $\mathcal{T}$ is standard, thus there exists an isomorphism of k-linear functors:

$$c : - \overset{L}{\otimes}_{k\Delta} X \longrightarrow - \overset{L}{\otimes}_{k\Delta} Y \overset{L}{\otimes}_{k\Delta} X.$$

The functor $- \overset{L}{\otimes}_{k\Delta} X$ restricted to the category $k(\Delta)$ satisfies the property of $a)$ of lemma 7.3.2. Thus we can apply $b)$ and lift $c_{|_{k(\Delta)}}$ to an isomorphism $\tilde{c}$ between X and $Y \overset{L}{\otimes}_{k\Delta} X$ as dg-$k\Delta^{op} \otimes \mathcal{A}$-modules.

By the universal property of the orbit category, the bimodule X endowed with the isomorphism $\tilde{c}$ yields a triangle functor from $\mathcal{D}^b(\mathsf{mod}\, k\Delta)/\Phi$ to $\mathcal{T}$ which comes from a bimodule Z in $\mathcal{D}(\mathcal{B}^{op} \otimes \mathcal{A})$.

$$\begin{tikzcd} \mathcal{D}^b(\mathsf{mod}\, k\Delta) \arrow[r, "-\overset{L}{\otimes}_{k\Delta} X"] \arrow[loop above, "-\overset{L}{\otimes}_{k\Delta} Y"] \arrow[d, "\pi"'] & \mathsf{per}\, \mathcal{A} = \mathcal{T} \\ \mathcal{D}^b(\mathsf{mod}\, k\Delta)/\Phi = \mathsf{per}\, \mathcal{B} \arrow[ur, dashed, "-\overset{L}{\otimes}_{k\Delta} Z"'] & \end{tikzcd}$$

The functor $- \overset{L}{\otimes}_{k\Delta} Z$ is essentially surjective. Let us show that it is fully faithful. For M and N objects of $\mathcal{D}^b(\mathsf{mod}\, k\Delta)$ we have the following commutative diagram:

$$\begin{tikzcd} & \bigoplus_{n\in\mathbb{Z}} \mathsf{Hom}_{\mathcal{D}}(M, \Phi^n N) \arrow[dl, "\pi"'] \arrow[dr, "-\overset{L}{\otimes}_{k\Delta} X = F"] & \\ \mathsf{Hom}_{\mathcal{D}/\Phi}(\pi M, \pi N) \arrow[rr, "-\overset{L}{\otimes}_{k\Delta} Z"] & & \mathsf{Hom}_{\mathcal{T}}(FM, FN), \end{tikzcd}$$

where $\mathcal{D}$ means $\mathcal{D}^b(\mathsf{mod}\, k\Delta)$. The two diagonal morphisms are isomorphisms, thus so is the horizontal morphism. This proves that $- \overset{L}{\otimes}_{k\Delta} Z$ is a triangle equivalence between the orbit category $\mathcal{D}^b(\mathsf{mod}\, k\Delta)/\Phi$ and $\mathcal{T}$. □

8. Triangulated structure on the category of projectives

Let k be a algebraically closed field and $\mathcal{P}$ a k-linear category with split idempotents. The category $\mathsf{mod}\, \mathcal{P}$ of contravariant finitely presented functors from $\mathcal{P}$ to $\mathsf{mod}\, k$ is exact. As the idempotents split, the projectives of $\mathsf{mod}\, \mathcal{P}$ coincide with the representables. Thus the Yoneda functor gives a natural equivalence between $\mathcal{P}$ and $\mathsf{proj}\, \mathcal{P}$. Assume besides that $\mathsf{mod}\, \mathcal{P}$ has a structure of Frobenius category. The stable category $\underline{\mathsf{mod}}\, \mathcal{P}$ is a triangulated category, we write Σ for the suspension functor.

Let S be an auto-equivalence of $\mathcal{P}$. It can be extended to an exact functor from $\mathsf{mod}\, \mathcal{P}$ to $\mathsf{mod}\, \mathcal{P}$ and thus to a triangle functor of $\underline{\mathsf{mod}}\, \mathcal{P}$. The aim of this part is to find a necessary condition on the functor S such that the category $(\mathcal{P}, S)$ has a triangulated structure. Heller already showed [14, thm 16.4] that if there exists an isomorphism of triangle functors between S and Σ^3, then $\mathcal{P}$ has a pretriangulated structure. But he did not succeed in proving the octahedral axiom. We are going to impose a stronger condition on the functor S and prove the following theorem:

Theorem 8.0.3. *Assume there exists an exact sequence of exact functors from* $\mathsf{mod}\,\mathcal{P}$ *to* $\mathsf{mod}\,\mathcal{P}$*:*

$$0 \longrightarrow Id \longrightarrow X^0 \longrightarrow X^1 \longrightarrow X^2 \longrightarrow S \longrightarrow 0\ ,$$

where the X^i*,* $i = 0, 1, 2$*, take values in* $\mathsf{proj}\mathcal{P}$*. Then the category* $\mathcal{P}$ *has a structure of triangulated category with suspension functor* S*.*

For an M in $\mathsf{mod}\,\mathcal{P}$, denote $T_M : X^0M \longrightarrow X^1M \longrightarrow X^2M \longrightarrow SX^0M$ a *standard triangle.* A triangle of $\mathcal{P}$ will be a sequence $X : P \xrightarrow{u} Q \xrightarrow{v} R \xrightarrow{w} SP$ which is isomorphic to a standard triangle T_M for an M in $\mathsf{mod}\,\mathcal{P}$.

8.1. **S-complexes, Φ-S-complexes and standard triangles.** Let $\mathcal{A}cp(\mathsf{mod}\,\mathcal{P})$ be the category of acyclic complexes with projective components. It is a Frobenius category whose projective-injectives are the contractible complexes, *i.e.* the complexes homotopic to zero. The functor $Z^0 : \mathcal{A}cp(\mathsf{mod}\,\mathcal{P}) \to \mathsf{mod}\,\mathcal{P}$ which sends a complex

$$\cdots \longrightarrow X^{-1} \xrightarrow{x^{-1}} X^0 \xrightarrow{x^0} X^1 \xrightarrow{x^1} \cdots$$

to the kernel of x^0 is an exact functor. It sends the projective-injectives to projective-injectives and induces a triangle equivalence between $\underline{\mathcal{A}cp(\mathsf{mod}\,\mathcal{P})}$ and $\underline{\mathsf{mod}}\,\mathcal{P}$.

Definition 8.1.1. *An object of* $\mathcal{A}cp(\mathsf{mod}\,\mathcal{P})$ *is called an* S-complex *if it is* S*-periodic,* i.e. *if it has the following form:*

$$\cdots \longrightarrow P \xrightarrow{u} Q \xrightarrow{v} R \xrightarrow{w} SP \xrightarrow{Su} SQ \longrightarrow \cdots .$$

The category S-comp of S-complexes with S-periodic morphisms is a non full subcategory of $\mathcal{A}cp(\mathsf{mod}\,\mathcal{P})$. It is a Frobenius category. The projective-injectives are the S-contractibles, *i.e.* the complexes homotopic to zero with an S-periodic homotopy. Using the functor Z^0, we get an exact functor from S-comp to $\mathsf{mod}\,\mathcal{P}$ which induces a triangle functor:

$$\underline{Z^0} : \underline{S\text{-}\mathsf{comp}} \longrightarrow \underline{\mathsf{mod}}\,\mathcal{P}.$$

Fix a sequence as in theorem 8.0.3. Clearly, it induces for each object M of $\underline{\mathsf{mod}}\,\mathcal{P}$, a functorial isomorphism in $\underline{\mathsf{mod}}\,\mathcal{P}$, $\Phi_M : \Sigma^3 M \longrightarrow SM$.

Let Y be an S-complex,

$$Y : \cdots \longrightarrow P \xrightarrow{u} Q \xrightarrow{v} R \xrightarrow{w} SP \xrightarrow{Su} SQ \longrightarrow \cdots .$$

Let M be the kernel of u. Then Y induces an isomorphism θ (in $\underline{\mathsf{mod}}\,\mathcal{P}$) between $\Sigma^3 M$ and SM. If θ is equal to Φ_M, we will say that X is a Φ-S-*complex.*

Let M be an object of $\mathsf{mod}\,\mathcal{P}$. The standard triangle T_M can be see as a Φ-S-complex:

$$\cdots \longrightarrow X^0M \longrightarrow X^1M \longrightarrow X^2M \longrightarrow SX^0M \longrightarrow SX^1M \longrightarrow \cdots .$$

The functor T which sends an object M of $\mathsf{mod}\,\mathcal{P}$ to the S-complex T_M is exact since the X^i are exact. It satisfies the relation $Z^0 \circ T \simeq Id_{\mathsf{mod}\,\mathcal{P}}$. Moreover, as it preserves the projective-injectives, it induces a triangle functor:

$$T : \underline{\mathsf{mod}}\,\mathcal{P} \to \underline{S\text{-}\mathsf{comp}}\ .$$

8.2. **Properties of the functors Z^0 and T.**

Lemma 8.2.1. *An S-complex which is homotopy-equivalent to a Φ-S-complex is a Φ-S-complex.*

Proof. Let $X : P \xrightarrow{u} Q \xrightarrow{v} R \xrightarrow{w} SP$ be an S-complex homotopy-equivalent to the Φ-S-complex $X' : P' \xrightarrow{u'} Q' \xrightarrow{v'} R' \xrightarrow{w'} SP'$. Let M be the kernel of u and M' the kernel of u'. By assumption, there exists a S-periodic homotopy equivalence f from X to X', which induces a morphism $g = Z^0 f : M \to M'$. Thus, we get the following commutative diagram:

$$\begin{tikzcd}
 & P \arrow[r] \arrow[dd, "f^0"] & Q \arrow[r] \arrow[dd, "f^1"] & R \arrow[rr] \arrow[dd, "f^2"] \arrow[dr] & & & SP \arrow[dd, "Sf^0"] \\
M \arrow[ur] \arrow[dd, "g"] & & & & \Sigma^3 M \arrow[r, "\theta"] \arrow[dd, "\Sigma^3 g"] & SM \arrow[ur] \arrow[dd, "Sg"] & \\
 & P' \arrow[r] & Q' \arrow[r] & R' \arrow[rr] \arrow[dr] & & & SP'. \\
M' \arrow[ur] & & & & \Sigma^3 M' \arrow[r, "\Phi_{M'}"'] & SM' \arrow[ur] &
\end{tikzcd}$$

The morphism g is an isomorphism of $\underline{\mathsf{mod}}\,\mathcal{P}$ since f is an isomorphism of $\underline{S\text{-}\mathsf{comp}}$. Thus the morphisms $\Sigma^3 g$ and Sg are isomorphisms of $\underline{\mathsf{mod}}\,\mathcal{P}$. The following equality in $\underline{\mathsf{mod}}\,\mathcal{P}$

$$\theta = (Sg)^{-1}\Phi_{M'}\Sigma^3 g = \Phi_M$$

shows that the complex X is a Φ-S-complex. □

Lemma 8.2.2. *Let*

$$X : P \xrightarrow{u} Q \xrightarrow{v} R \xrightarrow{w} SP \quad \textit{and} \quad X' : P' \xrightarrow{u'} Q' \xrightarrow{v'} R' \xrightarrow{w'} SP'$$

be two Φ-S-complexes. Suppose that we have a commutative square:

$$\begin{tikzcd}
P \arrow[r, "u"] \arrow[d, "f^0"'] & Q \arrow[d, "f^1"] \\
P' \arrow[r, "u'"] & Q'.
\end{tikzcd}$$

Then, there exists a morphism $f^2 : R \to R'$ such that (f^0, f^1, f^2) extends to an S-periodic morphism from X to X'.

Proof. Let M be the kernel of u, M' be the kernel of u' and $f : M \to M'$ be the morphism induced by the commutative square. As R and R' are projective-injective objects, we can find a morphism $g^2 : R \to R'$ such that the following square commutes:

$$\begin{tikzcd}
Q \arrow[r, "v"] \arrow[d, "f^1"'] & R \arrow[d, "g^2"] \\
Q' \arrow[r, "v'"] & R'.
\end{tikzcd}$$

The morphism g^2 induces a morphism $g : SM \to SM'$ such that the following square is commutative in $\underline{\mathsf{mod}}\,\mathcal{P}$:

$$\begin{tikzcd} \Sigma^3 M \arrow[r, "\Phi_M"] \arrow[d, "\Sigma^3 f"'] & SM \arrow[d, "g"] \\ \Sigma^3 M' \arrow[r, "\Phi_{M'}"] & SM'. \end{tikzcd}$$

Thus the morphisms Sf and g are equal in $\underline{\mathsf{mod}}\,\mathcal{P}$, *i.e.* there exists a projective-injective I of $\mathsf{mod}\,\mathcal{P}$ and morphisms $\alpha : SM \to I$ and $\beta : I \to SM'$ such that $g - Sf = \beta\alpha$. Let p (resp. p') be the epimorphism from R onto SM (resp. from R' onto SM'). Then, as I is projective, β factors through p'.

$$\begin{tikzcd} Q \arrow[r, "v"] \arrow[dd, "f^1"'] & R \arrow[rr, "w"] \arrow[dr, "p"] \arrow[dd, "g^2"'] & & SP \arrow[dd, "Sf^0"] \\ & & SM \arrow[ur] \arrow[d, "\alpha"] \arrow[dd, bend right, "g"'] \arrow[dd, bend left, "Sf"] & \\ & & I \arrow[dl, dotted, "\gamma"'] \arrow[dd, "\beta"] & \\ Q' \arrow[r, "v'"] & R' \arrow[rr] \arrow[dr, "p'"'] & & SP' \\ & & SM' \arrow[ur] & \end{tikzcd}$$

We put $f^2 = g^2 - \gamma\alpha p$. Then obviously, we have the equalities $f^2 v = v' f^1$ and $w' f^2 = S f^0 w$. Thus the morphism (f^0, f^1, f^2) extends to a morphism of S-comp. $\square$

Proposition 8.2.3. *The functor* $\underline{Z^0} : \underline{\Phi\text{-}S\text{-}\mathsf{comp}} \longrightarrow \underline{\mathsf{mod}}\,\mathcal{P}$ *is full and essentially surjective. Its kernel is an ideal whose square vanishes.*

Proof. The functor $\underline{Z^0}$ is essentially surjective since we have the relation $\underline{Z^0} \circ \underline{T} = Id_{\underline{\mathsf{mod}}\,\mathcal{P}}$.

Let us show that $\underline{Z^0}$ is full. Let

$$X : P \xrightarrow{u} Q \xrightarrow{v} R \xrightarrow{w} SP \quad \text{and} \quad X' : P' \xrightarrow{u'} Q' \xrightarrow{v'} R' \xrightarrow{w'} SP'$$

be two Φ-S-complexes. Let M (resp. M') be the kernel of u (resp. u'). As P, Q, P' and Q' are projective-injective, there exist morphisms $f^0 : P \to P'$ and $f^1 : Q \to Q'$ such that the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, tail] \arrow[d, "f"'] & P \arrow[r, "u"] \arrow[d, "f^0"] & Q \arrow[d, "f^1"] \\ M' \arrow[r, tail] & P' \arrow[r, "u'"] & Q'. \end{tikzcd}$$

Now the result follows from lemma 8.2.2.

Now let $\underline{f} : X \to X'$ be a morphism in the kernel of $\underline{Z^0}$. Up to homotopy, we can suppose that $\underline{f}$ has the following form:

$$\begin{tikzcd}
P \arrow[r, "u"] \arrow[d, "0"'] & Q \arrow[r, "v"] \arrow[d, "0"'] & R \arrow[r, "w"] \arrow[d, "f^2"'] & SP \arrow[d, "0"] \\
P' \arrow[r, "u'"] & Q' \arrow[r, "v'"] & R' \arrow[r, "w'"] & SP'.
\end{tikzcd}$$

As the composition $w'f^2$ vanishes and as Q' is projective-injective, f^2 factors through v'. For the same argument, f^2 factors through w. If $\underline{f}$ and $\underline{f'}$ are composable morphisms of the kernel of $\underline{Z^0}$, we get the following diagram:

$$\begin{tikzcd}
P \arrow[r, "u"] \arrow[d, "0"'] & Q \arrow[r, "v"] \arrow[d, "0"'] & R \arrow[r, "w"] \arrow[d, "f^2"'] \arrow[dl, dotted, "h^2"'] & SP \arrow[d, "0"] \\
P' \arrow[r, "u'"] \arrow[d, "0"'] & Q' \arrow[r, "v'"] \arrow[d, "0"'] & R' \arrow[r, "w'"] \arrow[d, "f'^2"'] & SP' \arrow[d, "0"] \arrow[dl, dotted, "h'^3"'] \\
P'' \arrow[r, "u''"] & Q'' \arrow[r, "v''"] & R'' \arrow[r, "w''"] & SP''.
\end{tikzcd}$$

The composition $\underline{f'f}$ vanishes obviously.

□

Corollary 8.2.4. *A Φ-S-complex morphism f which induces an isomorphism $\underline{Z^0}(f)$ in $\underline{\mathsf{mod}}\,\mathcal{P}$ is an homotopy-equivalence.*

This corollary comes from the previous theorem and from the following lemma.

Lemma 8.2.5. *Let $F : \mathcal{C} \to \mathcal{C}'$ be a full functor between two additive categories. If the kernel of F is an ideal whose square vanishes, then F detects isomorphisms.*

Proof. Let $u \in \mathsf{Hom}_{\mathcal{C}}(A, B)$ be a morphism in $\mathcal{C}$ such that Fu is an isomorphism. Since the functor F is full, there exists v in $\mathsf{Hom}_{\mathcal{C}}(B, A)$ such that $Fv = (Fu)^{-1}$. The morphism $w = uv - Id_B$ is in the kernel of F, thus w^2 vanishes. Then the morphism $v(Id_B - w)$ is a right inverse of u. In the same way we show that u has a left inverse, so u is an isomorphism. □

Proposition 8.2.6. *The category of Φ-S-complexes is equivalent to the category of S-complexes which are homotopy-equivalent to standard triangles.*

Proof. Since standard triangles are ϕ-S-complexes, each S-complex that is homotopy equivalent to a standard triangle is a Φ-S-complex (lemma 8.2.1).

Let $X : P \xrightarrow{u} Q \xrightarrow{v} R \xrightarrow{w} SP$ be a Φ-S-complex. Let M be the kernel of u. Then there exist morphisms $f^1 : P \to X^0M$ and $f^1 : Q \to X^1M$ such that the following diagram is commutative:

$$\begin{tikzcd}
M \arrow[r, tail] \arrow[d, equal] & P \arrow[r, "u"] \arrow[d, "f^0"] & Q \arrow[d, "f^1"] \\
M \arrow[r, tail] & X^0M \arrow[r] & X^1M.
\end{tikzcd}$$

We can complete (lemma 8.2.2) f into an S-periodic morphism from X in T_M. The morphism f satisfies $Z^0f = Id_M$, so $\underline{Z}^0(T_M)$ and $Z^0(X)$ are equal in $\underline{\mathsf{mod}}\,\mathcal{P}$. By the corollary, T_M and X are homotopy-equivalent. Thus the inclusion functor T is essentially surjective.

□

These two diagrams summarize the results of this section:

$$\begin{tikzcd}
\Phi\text{-S-comp} \arrow[r, hook, "\text{full}"] & \underset{\text{(Frobenius)}}{\text{S-comp}} \arrow[r] & \underset{\text{(Frobenius)}}{\mathcal{A}cp(\mathsf{mod}\,\mathcal{P})} \arrow[dl, "Z^0 \text{ (exact)}"] \\
 & \underset{\text{(Frobenius)}}{\mathsf{mod}\,\mathcal{P}} \arrow[ul, "T"] \arrow[u, shift right, "T \text{ (exact)}"'] \arrow[u, shift left, "Z^0 \text{ (exact)}"] &
\end{tikzcd}$$

$$\begin{tikzcd}
\underline{\Phi\text{-S-comp}} \arrow[r, hook] & \underset{\text{(triang.)}}{\underline{\text{S-comp}}} \arrow[r] & \underset{\text{(triang.)}}{\underline{\mathcal{A}cp(\mathsf{mod}\,\mathcal{P})}} \arrow[dl, "\underline{Z^0} \text{ (triangle equivalence)}"] \\
 & \underset{\text{(triang.)}}{\underline{\mathsf{mod}}\,\mathcal{P}} \arrow[ul, "\underline{T} \text{ (ess.surj., pr. 8.2.6)}"] \arrow[u, shift right, "\underline{T} \text{ (triang.)}"'] \arrow[u, shift left, "\underline{Z^0} \text{ (exact, full, ess. surj., pr.8.2.3)}"] &
\end{tikzcd}$$

8.3. **Proof of theorem 8.0.3.** We are going to show that the Φ-S-complexes form a system of triangles of the category $\mathcal{P}$. We use triangle axioms as in [22].

TR0: For each object M of $\mathcal{P}$, the S-complex $M \xrightarrow{=} M \longrightarrow 0 \longrightarrow SM$ is homotopy-equivalent to the zero complex, so is a Φ-S-complex.

TR1: Let $u : P \to Q$ be a morphism of $\mathcal{P}$, and let M be its kernel. We can find morphisms f^0 and f^1 so as to obtain a commutative square:

$$\begin{tikzcd}
 & X^0M \arrow[r, "a"] \arrow[dd, "f^0"] & X^1M \arrow[rr, "b"] \arrow[dr] \arrow[dd, "f^1"] & & X^2M \\
M \arrow[ur, tail] \arrow[dd, equal] & & & \mathsf{Coker}a \arrow[ur, tail] \arrow[dd, "\gamma"] & \\
 & P \arrow[r, "u"] & Q \arrow[dr] & & \\
M \arrow[ur, tail] & & & \mathsf{Coker}u. &
\end{tikzcd}$$

We form the following push-out:

$$\begin{tikzcd}
0 \arrow[r] & \mathsf{Coker}a \arrow[r] \arrow[d, "\gamma"] \arrow[dr, phantom, "\text{PO}"] & X^2M \arrow[r] \arrow[d] & SM \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathsf{Coker}u \arrow[r] & R \arrow[r] & SM \arrow[r] & 0.
\end{tikzcd}$$

It induces a triangle morphism of the triangular category $\underline{\mathsf{mod}}\,\mathcal{P}$:

$$\begin{tikzcd}
\mathsf{Coker}\ a \arrow[r] \arrow[d, "\gamma"] & X^2M \arrow[r] \arrow[d] & SM \arrow[r] \arrow[d, equal] & \Sigma\mathsf{Coker}\ a \arrow[d, "\Sigma\gamma"] \\
\mathsf{Coker}\ u \arrow[r] & R \arrow[r] & SM \arrow[r] & \Sigma\mathsf{Coker}\ u.
\end{tikzcd}$$

The morphism γ is an isomorphism in $\underline{\mathsf{mod}}\,\mathcal{P}$ since $\mathsf{Coker}\ a$ and $\mathsf{Coker}\ u$ are canonically isomorphic to $\Sigma^2 M$ in $\underline{\mathsf{mod}}\,\mathcal{P}$. By the five lemma, $X^2M \to R$ is an isomorphism in $\underline{\mathsf{mod}}\,\mathcal{P}$. Since X^2M is projective-injective, so is R. Thus the complex $P \xrightarrow{u} Q \longrightarrow R \longrightarrow SP$ is an S-complex. Then we have to see that it is a Φ-S-complex. Let θ be the isomorphism between SM and $\Sigma^3 M$ induced by this complex. We write α (resp. β) for the canonical isomorphism in $\underline{\mathsf{mod}}\,\mathcal{P}$ between $\Sigma^2 M$ and $\mathsf{Coker}\ a$ (resp. $\mathsf{Coker}\ u$). From the commutative diagram:

$$\begin{tikzcd}
 & \mathsf{Coker}\ a \arrow[r] \arrow[dd, "\gamma"] & X^2M \arrow[r] \arrow[dd] & SM \arrow[rr] \arrow[dr, "\Phi_M"] \arrow[dd, equal] & & \Sigma\mathsf{Coker}\ a \arrow[dd, "\Sigma\gamma"] \\
\Sigma^2 M \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & & & \Sigma^3 M \arrow[ur, "\Sigma\alpha"] & \\
 & \mathsf{Coker}\ u \arrow[r] & R \arrow[r] & SM \arrow[rr] \arrow[dr, "\theta"] & & \Sigma\mathsf{Coker}\ u \\
 & & & & \Sigma^3 M \arrow[ur, "\Sigma\beta"] &
\end{tikzcd}$$

we deduce the equality $\theta = (\Sigma\beta)^{-1}\gamma\Sigma\alpha\Phi_M = \Phi_M$ in $\underline{\mathsf{mod}}\,\mathcal{P}$. The constructed S-complex is a Φ-S-complex.

TR2: Let $X : P \xrightarrow{u} Q \xrightarrow{v} R \xrightarrow{w} SP$ be a Φ-S-complex. It is homotopy-equivalent to a standard triangle T_M. Thus the S-complex

$$X' : Q \xrightarrow{-v} R \xrightarrow{-w} SP \xrightarrow{-Su} SQ$$

is homotopy-equivalent to $T_M[1]$. Since $\underline{T}$ is a triangle functor, the objects $T_{\Sigma M}$ and $T_M[1]$ are isomorphic in the stable category $\underline{S\text{-}\mathsf{comp}}$, *i.e.* they are homotopy-equivalent. Thus, by lemma 8.2.1, $T_M[1]$ is a Φ-S-complex and then so is X'.

TR3: This axiom is a direct consequence of lemma 8.2.2.

TR4: Let X and X' be two Φ-S-complexes and suppose we have a commutative diagram:

$$\begin{tikzcd}
X : P \arrow[r, "u"] \arrow[d, "f^0"] & Q \arrow[r, "v"] \arrow[d, "f^1"] & R \arrow[r, "w"] & SP \arrow[d, "Sf^0"] \\
X' : P' \arrow[r, "u'"] & Q' \arrow[r, "v'"] & R' \arrow[r, "w'"] & SP'.
\end{tikzcd}$$

Let M (resp. M') be the kernel of u (resp. u'), and $g : M \to M'$ the induced morphism. The morphism $Tg : T_M \to T_{M'}$ induces a S-complex morphism $\tilde{g} = (g^0, g^1, g^2)$ between X and X'.

We are going to show that we can find a morphism $f^2 : R \to R'$ such that (f^0, f^1, f^2) can be extended in an S-complex morphism that is homotopic to $\tilde{g}$. As (g^0, g^1) and (f^0, f^1) induce the same morphism g in the kernels, we have some morphisms $h^1 : Q \to P'$ and $h^2 : R \to Q'$ such that $f^0 - g^0 = h^1 u$ and $f^1 - g^1 = u'h^1 + h^2 v$. We put $f^2 = g^2 + v'h^2$. We have the following equalities:

$$\begin{array}{rcl} f^2 v & = & g^2 v + v'h^2 v \\ & = & v'(g^1 + h^2 v) \\ & = & v'(f^1 - u'h^1) \\ & = & v'f^1 \end{array} \qquad \text{and} \qquad \begin{array}{rcl} w'f^2 & = & w'g^2 \\ & = & (Sg^0)w \\ & = & (Sf^0 - Sh^1Su)w \\ & = & (Sf^0)w \end{array}$$

Thus (f^0, f^1, f^2) can be extended to an S-periodic morphism $\tilde{f}$ which is S-homotopic to $\tilde{g}$. Their respective cones $C(\tilde{f})$ and $C(\tilde{g})$ are isomorphic as S-complexes. Moreover, since $\tilde{g}$ is a composition of $Tg : T_M \to T_{M'}$ with homotopy-equivalences, the cones $C(\tilde{g})$ and $C(Tg)$ are homotopy-equivalent.

In $\underline{\mathsf{mod}}\,\mathcal{P}$, we have a triangle

$$M \xrightarrow{g} M' \longrightarrow C(g) \longrightarrow \Sigma M.$$

Since $\underline{T}$ is a triangle functor, the sequence

$$T_M \xrightarrow{T_g} T_{M'} \longrightarrow T_{C(g)} \longrightarrow T_{\Sigma M}$$

is a triangle in $\underline{S\text{-}\mathsf{comp}}$. But we know that

$$T_M \xrightarrow{Tg} T_{M'} \longrightarrow C(Tg) \longrightarrow T_M[1]$$

is a triangle in $\underline{S\text{-}\mathsf{comp}}$. Thus the objects $C(Tg)$ and $T_{C(g)}$ are isomorphic in $\underline{S\text{-}\mathsf{comp}}$, *i.e.* homotopy-equivalent. Thus, the cone $C(\tilde{f})$ of $\tilde{f}$ is a Φ-S-complex by lemma 8.2.1.

9. Application to the deformed preprojective algebras

In this section, we apply the theorem 8.0.3 to show that the category of finite dimensional projective modules over a deformed preprojective algebra of generalized Dynkin type (see [4]) is triangulated. This will give us some examples of non standard triangulated categories with finitely many indecomposables.

9.1. **Preprojective algebra of generalized Dynkin type.** Recall the notations of [4]. Let Δ be a generalized Dynkin graph of type $\mathbb{A}_n$, $\mathbb{D}_n$ $(n \geq 4)$, $\mathbb{E}_n$ $(n = 6, 7, 8)$, or $\mathbb{L}_n$. Let Q_Δ be the following associated quiver:

$\Delta = \mathbb{A}_n$ $(n \geq 1)$:

$$\begin{tikzcd} 0 \arrow[r, shift left, "a_0"] & 1 \arrow[l, shift left, "\overline{a}_0"] \arrow[r, shift left, "a_1"] & 2 \arrow[l, shift left, "\overline{a}_1"] \arrow[r, dotted] & n-2 \arrow[r, shift left, "a_{n-2}"] & n-1 \arrow[l, shift left, "\overline{a}_{n-2}"] \end{tikzcd}$$

$\Delta = \mathbb{D}_n$ $(n \geq 4)$:

$$\begin{tikzcd} 0 \arrow[dr, shift left, "a_0"] & & & & \\ & 2 \arrow[ul, shift left, "\overline{a}_0"] \arrow[r, shift left, "a_2"] & 3 \arrow[l, shift left, "\overline{a}_2"] \arrow[r, dotted] & n-2 \arrow[r, shift left, "a_{n-2}"] & n-1 \arrow[l, shift left, "\overline{a}_{n-2}"] \\ 1 \arrow[ur, shift left, "a_1"] & & & & \end{tikzcd}$$

$\Delta = \mathbb{E}_n$ $(n = 6, 7, 8)$:

$$\begin{tikzcd} & & 0 \arrow[d, shift left, "a_0"] & & & \\ 1 \arrow[r, shift left, "a_1"] & 2 \arrow[l, shift left, "\overline{a}_1"] \arrow[r, shift left, "a_2"] & 3 \arrow[l, shift left, "\overline{a}_2"] \arrow[u, shift left, "\overline{a}_0"] \arrow[r, shift left, "a_3"] & 4 \arrow[l, shift left, "\overline{a}_3"] \arrow[r, dotted] & n-2 \arrow[r, shift left, "a_{n-2}"] & n-1 \arrow[l, shift left, "\overline{a}_{n-2}"] \end{tikzcd}$$

$\Delta = \mathbb{L}_n$ $(n \geq 1)$:

$$\begin{tikzcd} \epsilon = \overline{\epsilon} \arrow[loop left] & 0 \arrow[r, shift left, "a_0"] & 1 \arrow[l, shift left, "\overline{a}_0"] \arrow[r, shift left, "a_1"] & 2 \arrow[l, shift left, "\overline{a}_1"] \arrow[r, dotted] & n-2 \arrow[r, shift left, "a_{n-2}"] & n-1 \arrow[l, shift left, "\overline{a}_{n-2}"] \end{tikzcd} .$$

The *preprojective algebra* $P(\Delta)$ associated to the graph Δ is the quotient of the path algebra kQ_Δ by the relations:

$$\sum_{sa=i} a\overline{a}, \quad \text{for each vertex } i \text{ of } Q_\Delta.$$

The following proposition is classical [4, prop 2.1].

Proposition 9.1.1. *The preprojective algebra $P(\Delta)$ is finite dimensional and self-injective. Its Nakayama permutation ν is the identity for $\Delta = \mathbb{A}_1$, $\mathbb{D}_{2n}$, $\mathbb{E}_7$, $\mathbb{E}_8$ and $\mathbb{L}_n$, and is of order 2 in all other cases.*

9.2. **Deformed preprojective algebras of generalized Dynkin type.** Let us recall the definition of deformed preprojective algebra introduced by [4]. Let Δ be a graph of generalized Dynkin type. We define an associated algebra $R(\Delta)$ as follows:

$$\begin{array}{rcl} R(\mathbb{A}_n) & = & k; \\ R(\mathbb{D}_n) & = & k\langle x,y\rangle/(x^2,y^2,(x+y)^{n-2}); \\ R(\mathbb{E}_n) & = & k\langle x,y\rangle/(x^2,y^3,(x+y)^{n-3}); \\ R(\mathbb{L}_n) & = & k[x]/(x^{2n}). \end{array}$$

Further, we fix an exceptional vertex in each graph as follows (with the notations of the previous section):

$$\begin{array}{lll} 0 & \text{for} & \Delta = \mathbb{A}_n \text{ or } \mathbb{L}_n, \\ 2 & \text{for} & \Delta = \mathbb{D}_n, \\ 3 & \text{for} & \Delta = \mathbb{E}_n. \end{array}$$

Let f be an element of the square $rad^2 R(\Delta)$ of the radical of $R(\Delta)$. The *deformed preprojective algebra* $P^f(\Delta)$ is the quotient of the path algebra kQ_Δ by the relations:

$$\sum_{sa=i} a\overline{a}, \quad \text{for each non exceptional vertex } i \text{ of } Q,$$

and

$$\begin{array}{lll} a_0\overline{a}_0 & & \text{for} \quad \Delta = \mathbb{A}_n; \\ \overline{a}_0a_0 + \overline{a}_1a_1 + a_2\overline{a}_2 + f(\overline{a}_0a_0, \overline{a}_1a_1), \text{ and } (\overline{a}_0a_0 + \overline{a}_1a_1)^{n-2} & & \text{for} \quad \Delta = \mathbb{D}_n; \\ \overline{a}_0a_0 + \overline{a}_2a_2 + a_3\overline{a}_3 + f(\overline{a}_0a_0, \overline{a}_2a_2), \text{ and } (\overline{a}_0a_0 + \overline{a}_2a_2)^{n-3} & & \text{for} \quad \Delta = \mathbb{E}_n; \\ \epsilon^2 + a_0\overline{a}_0 + \epsilon f(\epsilon), \text{ and } \epsilon^{2n} & & \text{for} \quad \Delta = \mathbb{L}_n. \end{array}$$

Note that if f is zero, we get the preprojective algebra $P(\Delta)$.

9.3. **Corollaries of** [4]**.** The following proposition [4, prop 3.4] shows that the category $\mathsf{proj}\, P^f(\Delta)$ of finite-dimensional projective modules over a deformed preprojective algebra satisfies the hypothesis of theorem 8.0.3.

Proposition 9.3.1. *Let $A = P^f(\Delta)$ be a deformed preprojective algebra. Then there exists an exact sequence of A-A-bimodules*

$$0 \longrightarrow {}_1A_{\Phi^{-1}} \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0,$$

where Φ is an automorphism of A and where the P_i's are projective as bimodules. Moreover, for each idempotent e_i of A, we have $\Phi(e_i) = e_{\nu(i)}$.

So we can easily deduce the corollary:

Corollary 9.3.2. *Let $P^f(\Delta)$ be a deformed preprojective algebra of generalized Dynkin type. Then the category $\mathsf{proj}\, P^f(\Delta)$ of finite dimensional projective modules is triangulated. The suspension is the Nakayama functor.*

Indeed, if $P_i = e_iA$ is a projective indecomposable, then $P_i \otimes_A A_\Phi$ is equal to $\Phi(e_i)A = e_{\nu(i)}A$ thus to $\nu(P_i)$.

Now we are able to answer to the question of the previous part and find a triangulated category with finitely many indecomposables which is not standard. The proof of the following theorem comes essentially from the theorem [4, thm 1.3].

Theorem 9.3.3. *Let k be an algebraically closed field of characteristic 2. Then there exist k-linear triangulated categories with finitely many indecomposables which are not standard.*

Proof. By theorem [4, thm 1.3], we know that there exist basic deformed preprojective algebras of generalized Dynkin type $P^f(\Delta)$ which are not isomorphic to $P(\Delta)$. Thus the categories $\mathsf{proj}\, P^f(\Delta)$ and $\mathsf{proj}\, P(\Delta)$ can not be equivalent. But both are triangulated by corollary 9.3.2 and have the same AR-quiver $\mathbb{Z}\Delta/\tau = Q_\Delta$. □

Conversely, we have the following theorem:

Theorem 9.3.4. *Let $\mathcal{T}$ be a finite 1-Calabi-Yau triangulated category. Then $\mathcal{T}$ is equivalent to $\mathsf{proj}\,\Lambda$ as k-category, where Λ is a deformed preprojective algebra of generalized Dynkin type.*

Proof. Let $M_1, \ldots, M_n$ be representatives of the isoclasses of indecomposable objects of $\mathcal{T}$. The k-algebra $\Lambda = \mathsf{End}(\bigoplus_{i=1}^n M_i)$ is basic, finite-dimensional and selfinjective since $\mathcal{T}$ has a Serre duality. It is easy to see that $\mathcal{T}$ and $\mathsf{proj}\,\Lambda$ are equivalent as k-categories.

Let $\mathsf{mod}\,\Lambda$ be the category of finitely presented Λ-modules. It is a Frobenius category. Denote by Σ the suspension functor of the triangulated category $\underline{\mathsf{mod}}\,\Lambda$. The category $\mathcal{T}$ is 1-Calabi-Yau, that is to say that the suspension functor S of the triangulated category $\mathcal{T}$ and the Serre functor ν are isomorphic. But in $\underline{\mathsf{mod}}\,\Lambda$, the functors S and Σ^3 are isomorphic. Thus, for each non projective simple Λ-module M we have an isomorphism $\Sigma^3 M \simeq \nu M$. By [4, thm 1.2], we get immediately the result. □

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