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OBSERVER-BASED APPROACH FOR SYNCHRONIZATION OF MODIFIED CHUA'S CIRCUIT

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Abstract: In their recent work, Wang *et al.* proposed an interesting technique for generating chaos in Chua's circuit via time-delay feedback (Wang *et al.*, 2001). For this class of nonlinear dynamical systems, we provide in this note, a specific solution to ensure synchronization. We show how to design an observer-based synchronization scheme, with a global convergence. The designed observer is tested on the example provided in (Wang *et al.*, 2001) in the deterministic and uncertain cases.

Keywords: Chaotic behavior, circuit models, observer, stability analysis, synchronization.

1. INTRODUCTION AND PROBLEM FORMULATION

Over the last decade, synchronization of chaotic systems has become a wide field of research activities. The pioneering papers (Pecora and Carroll, 1990), (Carroll and Pecora, 1991) have been the starting point of several synchronization schemes. The main interest of synchronization lies in its application to secure communications (Kolumbán *et al.*, 1997), (Kolumbán *et al.*, 1998). Indeed, once synchronization is achieved between two chaotic systems, one can send a message from the transmitter to the receiver, using for this purpose one of the following schemes : chaotic shift keying, chaotic modulation ... (see (*Special Issue on Chaos Synchronization and control : Theory and Applications*, 1997), (*Special Issue on Control of Chaos and Synchronization*, 1997)).

Pecora and Carroll have shown that, in spite of an extreme sensitivity in the initial conditions, which was a major drawback of the chaotic systems since then, two chaotic systems could synchronize under some conditions detailed in (Pecora and Carroll, 1990), (Carroll and Pecora, 1991). They defined in these papers the drive-response concept. It consists of two chaotic systems : the first one, called the *driving system*, transmits a signal (usually some of its state variables) to the second one, called the *response system*. This driving signal enables the receiver to synchronize with the transmitter. The response system is chosen as a part of the drive system, and it is shown that, under some assumptions on the conditional Lyapunov exponents of these systems, synchronization occurs.

Since the receiver has to be designed so as to assure synchronization of its dynamics with that of the transmitter, from the driving signal, the

response system can be seen as an observer of the driving system. Morgül and Nijmeijer have shown respectively in (Morgul and Solak, 1997), (Nijmeijer and Mareels, 1997) that the principle of synchronization could be related to the observer problem discussed in (non)linear control theory. Some papers used this observer-based concept to design synchronization schemes with well-known chaotic systems (Grassi and Mascolo, 1993), (Feki, 2003*b*), (Liao and Huang, 1999), (Boutayeb *et al.*, 2002).

The aim of our paper is to build an observer-based synchronization scheme for the modified Chua's circuit given in (Wang *et al.*, 2001). Indeed, Chua's classical circuit has been widely used to study chaos (Chua *et al.*, 1993*a*), (Chua *et al.*, 1993*b*) and consequently to study several synchronization schemes (see (Feki, 2003*a*), (Ogorzalek, 1993), (Liao and Huang, 1999), (Boutayeb *et al.*, 2002)). In (Yang *et al.*, 1997), Chua's circuit is used to design a secure communications scheme, but it has been shown in (Parker and Short, 2001) that it is not secure enough. In (Wang *et al.*, 2001), the authors have shown that the behavior of the classical Chua's circuit could be made more complex by adding a small-amplitude time-delay feedback. This method is well-known to produce complex chaotic systems (see references [2]-[9] in (Wang *et al.*, 2001)). This work is the starting point of a new secure communication scheme, based on the modified Chua's circuit whose dynamic model is given by :

$$\dot{X}(t) = AX(t) + F(X(t)) + H(X(t-\tau)) \quad (1)$$

where

$$X = (X_1\ X_2\ X_3)^T \quad (2)$$

$$A = \begin{pmatrix} -\frac{G}{C_1} & \frac{G}{C_1} & 0 \\ \frac{G}{C_2} & -\frac{G}{C_2} & \frac{1}{C_2} \\ 0 & -\frac{1}{L} & -\frac{R_0}{L} \end{pmatrix} \quad (3)$$

$$F(X(t)) = \begin{pmatrix} -\frac{1}{C_1} f(X_1(t)) \\ 0 \\ 0 \end{pmatrix} \quad (4)$$

with f the nonlinear (piecewise-linear) characteristic of Chua's diode :

$$f(X_1) = G_b X_1 + \frac{1}{2}(G_a - G_b)(|X_1 + E| - |X_1 - E|) \quad (5)$$

$$H(X(t-\tau)) = \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{L} w(X_1(t-\tau)) \end{pmatrix} \quad (6)$$

with w the time-delay feedback

$$w(X_1(t-\tau)) = \varepsilon \sin(\sigma X_1(t-\tau)) \quad (7)$$

and C_1, C_2, G, R_0, L, G_a, G_b, E, ε, σ, τ are defined in (Wang *et al.*, 2001).

Remark 1. Note that the functions F and H satisfy the Lipschitz condition, with respective constants $k_F = max\{|G_a|, |G_b|\}/C_1$ and $k_H = \varepsilon/L$. We recall that a function ϕ satisfies the Lipschitz condition with constant k if, given any x, y

$$\|\phi(x) - \phi(y)\| \leq k\|x - y\| \quad (8)$$

The aim of this work is therefore to design an observer-based synchronization scheme for this class of systems. We provide a general solution for which we specify how to design the observer matrices in order to ensure synchronization. Simulation results on the example provided in (Wang *et al.*, 2001) show the effectiveness of the proposed approach.

The layout of this paper is as follows. Section 2 is concerned by the design of an observer for the system (1), using the analytic solution of a Riccati equation. Finally, Section 3 ends this work with some numerical simulations.

Notations : Throughout the paper, I_n corresponds to the identity matrix of dimension n, the symbol $(.)^T$ denotes the transpose, and the symbol $\|.\|$ stands for the Euclidian norm.

2. DESIGN OF A FULL-ORDER OBSERVER IN THE CASE OF A KNOWN TIME-DELAY

In this part we deal with the design of a full order observer synchronizing with Chua's modified circuit (1), where the time-delay term τ is assumed to be known.

To this end, we recall the dynamic model to synchronize :

$$\begin{cases} \dot{X}(t) = AX(t) + F(X(t)) + H(X(t-\tau)) \\ Y(t) = CX(t) \end{cases} \quad (9)$$

The standard form of an observer of the system (9) is :

$$\begin{aligned} \dot{\hat{X}}(t) = & A\hat{X}(t) + F(\hat{X}(t)) + H(\hat{X}(t-\tau))) \\ & + L(Y(t) - C\hat{X}(t)) \end{aligned} \quad (10)$$

We define the state estimation error vector as $e = X - \hat{X}$. The dynamic of the estimation error is given by :

$$\begin{aligned}\dot{e}(t) &= (A - LC)e(t) \\ &\quad +F(X(t)) - F(\hat{X}(t)) \\ &\quad +H(X(t-\tau)) - H(\hat{X}(t-\tau)) \\ &= (A - LC)e(t) \\ &\quad +F(X(t)) - F(X(t) - e(t)) \\ &\quad +H(X(t-\tau)) - H(X(t-\tau) - e(t-\tau))\end{aligned} \tag{11}$$

The following theorem presents a sufficient condition for the synchronization of the observer (10) with the system (9).

Theorem 2. If there exist two symmetric, positive-definite matrices P and Q such that :

$$\begin{aligned}(A - LC)^T P + P(A - LC) + (1 + k_F^2)P^2 \\ +(1 + k_H^2)I_n = -Q\end{aligned} \tag{12}$$

then the system (10) is an asymptotic observer for the system (9).

PROOF. For the time-delay systems, it is classical to define a Lyapunov-Krasovskii functional of the form :

$$V(t, e) = e(t)^T P e(t) + \xi \int_{-\tau}^{0} e(t+\theta)^T e(t+\theta)\, \mathrm{d}\theta \tag{13}$$

where P is a symmetric, positive definite matrix, and ξ is a positive scalar. It is easy to show that V is positive definite and upper bounded.
We compute the derivative of V along the trajectories of (11) :

$$\begin{aligned}\dot{V}(t, e) &= e(t)^T[\ (A - LC)^T P + P(A - LC) \\ &\quad +\xi I_n\ ]e(t) \\ &\quad +2e(t)^T P[\ F(X(t)) - F(X(t) - e(t))\ ] \\ &\quad +2e(t)^T P[\ H(X(t-\tau)) - H(X(t-\tau) \\ &\quad -e(t-\tau))\ ] - \xi e(t-\tau)^T e(t-\tau)\end{aligned} \tag{14}$$

The Cauchy-Schwarz inequality and the Lipschitz property of f give :

$$\begin{aligned}&2e(t)^T P(F(X(t)) - F(X(t) - e(t))) \\ &\leq 2\|e(t)^T P\|\|F(X(t)) - F(X(t) - e(t))\| \\ &\leq 2k_F\|e(t)^T P\|\|e(t)\|\end{aligned} \tag{15}$$

Then the Young's inequality leads to :

$$2k_F\|e(t)^T P\|\|e(t)\| \leq k_F^2 e(t)^T P P e(t) + \|e(t)\|^2 \tag{16}$$

Proceeding in the same manner, it follows that :

$$\begin{aligned}&2e(t)^T P[\ H(X(t-\tau)) - H(X(t-\tau) - e(t-\tau))\ ] \\ &\leq e(t)^T P P e(t) + k_H^2\|e(t-\tau)\|^2\end{aligned} \tag{17}$$

By replacing (15), (16) and (17) in (14), this yields to :

$$\begin{aligned}&\dot{V}(t, e) \\ &\leq e(t)^T[(A - LC)^T P + P(A - LC) + \xi I_n]e(t) \\ &\quad +k_F^2 e(t)^T P P e(t) + \|e(t)\|^2 \\ &\quad +e(t)^T P P e(t) + k_H^2\|e(t-\tau)\|^2 - \xi\|e(t-\tau)\|^2 \\ &\leq e(t)^T[(A - LC)^T P + P(A - LC) \\ &\quad +(\xi + 1)I_n + (1 + k_F^2)P^2]e(t) \\ &\quad +(k_H^2 - \xi)\|e(t-\tau)\|^2\end{aligned} \tag{18}$$

ξ must be chosen so that : $k_H^2 - \xi \leq 0$. We choose $\xi = k_H^2$. Then (18) becomes :

$$\begin{aligned}\dot{V}(t, e) \leq\ & e(t)^T[(A - LC)^T P + P(A - LC) \\ & +(1 + k_F^2)P^2 + (1 + k_H^2)I_n]e(t)\end{aligned} \tag{19}$$

Using equation (12), it implies

$$\dot{V}(t, e) \leq -e(t)^T Q e(t) \tag{20}$$

□

Remark 3. Equation (12) implies that

$$(A - LC)^T P + P(A - LC) < 0 \tag{21}$$

so the matrix $A - LC$ is stable, which is equivalent to : the pair $(A,\ C)$ is detectable.

The problem now is to solve the Riccati-like equation (12). This represents the main contribution of this paper, since we give an analytical and simple solution of (12).

To give a particular solution of the Riccati equation for the modified Chua's circuit, and with a view to simplify the search of this solution, we take $C = (0\ 1\ 0)$ in the dynamic model (9), and we assume that P is a positive diagonal matrix :

$$P = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \text{ with } a, b, c > 0.$$

The positive-definite matrix Q is chosen of the form :

$$\begin{pmatrix} q_1 & q_4 & 0 \\ q_4 & q_2 & q_5 \\ 0 & q_5 & q_3 \end{pmatrix} \tag{22}$$

We note

$$L = \begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix} \tag{23}$$

and $\gamma_F = 1 + k_F^2$, $\gamma_H = 1 + k_H^2$.

The Riccati-like equation (12) becomes then :

$$(A - LC)^T P + P(A - LC) + \gamma_F P^2 = -Q - \gamma_H I_3 \tag{24}$$

We introduce the following notations for simplicity of the presentation : $a_1 = \frac{G}{C_1}$, $a_2 = \frac{G}{C_2}$, and $c_2 = \frac{1}{C_2}$.

$$\begin{pmatrix} -2a_1a+\gamma_F a^2 & a_2b+aa_1-al_1 & 0 \\ a_2b+aa_1-al_1 & -2b(a_2+l_2)+\gamma_F b^2 & bc_2-(\frac{1}{L}+l_3)c \\ 0 & bc_2-(\frac{1}{L}+l_3)c & -2\frac{R_0}{L}c+\gamma_F c^2 \end{pmatrix} = \begin{pmatrix} -\gamma_H-q_1 & -q_4 & 0 \\ -q_4 & -\gamma_H-q_2 & -q_5 \\ 0 & -q_5 & -\gamma_H-q_3 \end{pmatrix} \quad (25)$$

Hence, using this notations in the Riccati equation (12), we obtain the equation (25).
This relation leads to the conditions :

$$-2a_1a+\gamma_F a^2 = -\gamma_H - q_1 \quad (26a)$$
$$a_2b+aa_1-al_1 = -q_4 \quad (26b)$$
$$-2b(a_2+l_2)+\gamma_F b^2 = -\gamma_H - q_2 \quad (26c)$$
$$bc_2-(\frac{1}{L}+l_3)c = -q_5 \quad (26d)$$
$$-2\frac{R_0}{L}c+\gamma_F c^2 = -\gamma_H - q_3 \quad (26e)$$

We recall that the unknowns of this five-equation system (26) are : l_1, l_2, l_3 , q_i $(i = 1, 5)$ and the positive scalars a, b, c.

The first condition (26a) is a second-degree equation, with unknown a :

$$\gamma_F a^2 - 2a_1a + \gamma_H + q_1 = 0 \quad (27)$$

The discriminant $\Delta = 4a_1^2 - 4\gamma_F(\gamma_H + q_1)$ is superior or equal to zero if

$$q_1 \leq \frac{a_1^2}{\gamma_F} - \gamma_H \quad (28)$$

In this case, the solution is (a must be positive) :

$$a = \frac{a_1 + \sqrt{a_1^2 - \gamma_F(\gamma_H + q_1)}}{\gamma_F} \quad (29)$$

The second condition (26b) leads to the equality :

$$l_1 = a_1 + \frac{a_2b + q_4}{a} \quad (30)$$

The third equation (26c) gives the relation :

$$l_2 = \frac{b\gamma_F}{2} - a_2 + \frac{\gamma_H + q_2}{2b} \quad (31)$$

The fourth condition (26d) leads to :

$$l_3 = -\frac{1}{L} + \frac{bc_2 + q_5}{c} \quad (32)$$

The last equation (26e) is also a second-degree equation, with unknown c, whose roots can be found in the same way than for (27).
Hence we get :

$$c = \frac{\frac{R_0}{L} + \sqrt{\frac{R_0^2}{L^2} - \gamma_F(\gamma_H + q_3)}}{\gamma_F} \quad (33)$$

under the condition

$$q_3 \leq \frac{R_0^2}{L^2\gamma_F} - \gamma_H \quad (34)$$

The procedure to construct P, L, Q can be summarized as follows :

(1) We choose q_i, $i = 1, 5$ such that $Q > 0$ and (28), (34) hold.

(2) We can find a and c respectively using the equations (29) and (33).

(3) We can arbitrarily choose $b > 0$ to determine the diagonal, positive-definite matrix P.

(4) We compute l_1, l_2, and l_3 using the conditions (30), (31), (32).

By applying the Theorem 2, the observer that will synchronize with the modified Chua's circuit (9) is given by the dynamic equations :

$$\begin{cases} \dot{\hat{X}}_1 = -\frac{G}{C_1}\hat{X}_1 + (\frac{G}{C_1} - l_1)\hat{X}_2 - \frac{1}{C_1}f(\hat{X}_1) \\ \qquad +l_1Y \\ \dot{\hat{X}}_2 = \frac{G}{C_2}\hat{X}_1 - (\frac{G}{C_2} + l_2)\hat{X}_2 + \frac{1}{C_2}\hat{X}_3 + l_2Y \\ \dot{\hat{X}}_3 = -(\frac{1}{L} + l_3)\hat{X}_2 - \frac{R_0}{L}\hat{X}_3 - \frac{1}{L}w(\hat{X}_1(t-\tau)) \\ \qquad +l_3Y \end{cases} \quad (35)$$

Now, we will show the efficiency of this result on numerical simulations.

3. NUMERICAL SIMULATIONS

3.1 Synchronization of the observer with Chua's modified circuit

We take the same values as in (Wang *et al.*, 2001) :

$R = 1950,\ G = 1/R,\ C_1 = 10^{-8},\ C_2 = 10^{-7},$
$L = 18.68\ .10^{-3},\ R_0 = 16,\ E = 1,$
$G_a = -0.75\ .10^{-3},\ G_b = -0.41\ .10^{-3},$
$\tau = 0.001,\ \varepsilon = 0.2,\ \sigma = 0.5$

The following initial conditions are chosen :

$$X_0 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad (36)$$

and

$$\hat{X}_0 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \tag{37}$$

The Riccati equation (12) is verified with the matrices (we give rounded values) :

$$P = 10^{-4} \begin{pmatrix} 1.4276 & 0 & 0 \\ 0 & 0.01 & 0 \\ 0 & 0 & 1.8871 \end{pmatrix} \tag{38}$$

and

$$Q = \begin{pmatrix} -215.63 & 10^{-6} & 0 \\ 10^{-6} & -100 & 10^{-6} \\ 0 & 10^{-6} & -315.63 \end{pmatrix} \tag{39}$$

and the observer gain is

$$L = 10^{6} \begin{pmatrix} 0.051318 \\ 7.8137 \\ 0.052937 \end{pmatrix} \tag{40}$$

The figure 1. shows the curves representing the norm of the state estimation error.

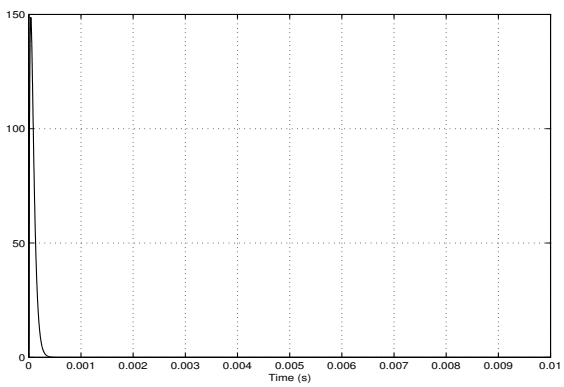

Fig. 1. Plot of the norm of the state estimation error vector

3.2 Robustness of the synchronization scheme

Synchronization takes an important place in some processes of secure communication and it should be avoided that anybody could decipher the message sent by the transmitter. Even if we chose not to deal with the problem of the robustness of the designed synchronization scheme, we present some numerical simulations by making variations on the parameters G and C_1 in the observer (35).
The figure 2. shows that when the parameters G and C_1 have a 1% mismatch in the observer, there is no more synchronization.

Hence, if the parameters of the observer synchronizing with Chua's modified circuit are not perfectly known (mismatch above 1%), the synchronization does not happen.

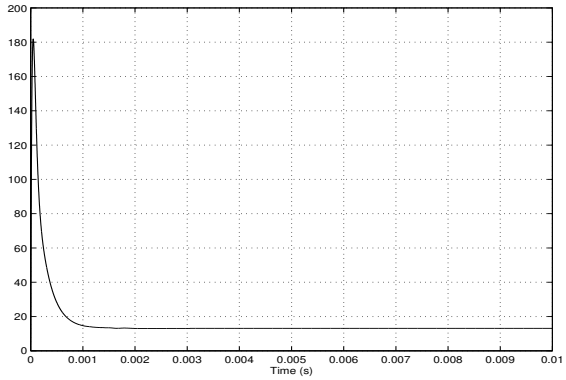

Fig. 2. Plot of the norm of the state estimation error vector with a 1% error on G and C_1

4. CONCLUSION

In this work, we addressed the problem of synchronization of a modified Chua's circuit, which belongs to a class of non-linear systems with a time delay. We have designed an observer-based scheme for synchronization and we have given a particular solution to a Riccati equation linked with the design of the observer. The synchronization of the observer with the modified Chua's circuit has been tested on numerical simulations in the deterministic and uncertain cases.

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