Observer-based approach for synchronization of modified Chua’s circuit

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To cite this version:

Estelle Cherrier, Mohamed Boutayeb, José Ragot. Observer-based approach for synchronization of modified Chua’s circuit. Aug 2004, pp.CDROM. hal-00118574
Abstract: In their recent work, Wang et al. proposed an interesting technique for generating chaos in Chua’s circuit via time-delay feedback (Wang et al., 2001). For this class of nonlinear dynamical systems, we provide in this note, a specific solution to ensure synchronization. We show how to design an observer-based synchronization scheme, with a global convergence. The designed observer is tested on the example provided in (Wang et al., 2001) in the deterministic and uncertain cases.

Keywords: Chaotic behavior, circuit models, observer, stability analysis, synchronization.

1. INTRODUCTION AND PROBLEM FORMULATION

Over the last decade, synchronization of chaotic systems has become a wide field of research activities. The pioneering papers (Pecora and Carroll, 1990), (Carroll and Pecora, 1991) have been the starting point of several synchronization schemes. The main interest of synchronization lies in its application to secure communications (Kolumbán et al., 1997), (Kolumbán et al., 1998). Indeed, once synchronization is achieved between two chaotic systems, one can send a message from the transmitter to the receiver, using for this purpose one of the following schemes: chaotic shift keying, chaotic modulation . . . (see (Special Issue on Chaos Synchronization and control : Theory and Applications, 1997), (Special Issue on Control of Chaos and Synchronization, 1997)).

Pecora and Carroll have shown that, in spite of an extreme sensitivity in the initial conditions, which was a major drawback of the chaotic systems since then, two chaotic systems could synchronize under some conditions detailed in (Pecora and Carroll, 1990), (Carroll and Pecora, 1991). They defined in these papers the drive-response concept. It consists of two chaotic systems: the first one, called the driving system, transmits a signal (usually some of its state variables) to the second one, called the response system. This driving signal enables the receiver to synchronize with the transmitter. The response system is chosen as a part of the drive system, and it is shown that, under some assumptions on the conditional Lyapunov exponents of these systems, synchronization occurs.

Since the receiver has to be designed so as to assure synchronization of its dynamics with that of the transmitter, from the driving signal, the
The layout of this paper is as follows. Section 2 is concerned by the design of an observer for the system (1), using the analytic solution of a Riccati equation. Finally, Section 3 ends this work with some numerical simulations.

**Notations:** Throughout the paper, \( I_n \) corresponds to the identity matrix of dimension \( n \), the symbol \((-)^T\) denotes the transpose, and the symbol \( || \cdot || \) stands for the Euclidian norm.

2. DESIGN OF A FULL-ORDER OBSERVER IN THE CASE OF A KNOWN TIME-DELAY

In this part we deal with the design of a full order observer synchronizing with Chua’s modified circuit (1), where the time-delay term \( \tau \) is assumed to be known.

To this end, we recall the dynamic model to synchronize:

\[
\begin{cases}
    \dot{X}(t) = AX(t) + F(X(t)) + H(X(t - \tau)) \\
    Y(t) = CX(t)
\end{cases}
\]  

(9)

The standard form of an observer of the system (9) is:

\[
\begin{align*}
\dot{\hat{X}}(t) & = A\hat{X}(t) + F(\hat{X}(t)) + H(\hat{X}(t - \tau)) \\
& \quad + L(Y(t) - CX(t))
\end{align*}
\]  

(10)

**Remark 1.** Note that the functions \( F \) and \( H \) satisfy the Lipschitz condition, with respective constants \( k_F = \max\{|G_a|, |G_b|\}/C_1 \) and \( k_H = \varepsilon/L \). We recall that a function \( \phi \) satisfies the Lipschitz condition with constant \( k \) if, given any \( x, y \):

\[
||\phi(x) - \phi(y)|| \leq k||x - y||
\]  

(8)
We define the state estimation error vector as $e = X - \hat{X}$. The dynamic of the estimation error is given by:

$$
\dot{e}(t) = (A - LC)e(t) + F(X(t)) - F(\hat{X}(t)) + H(X(t - \tau)) - H(\hat{X}(t - \tau))
$$

(11)

The following theorem presents a sufficient condition for the synchronization of the observer (10) with the system (9).

**Theorem 2.** If there exist two symmetric, positive-definite matrices $P$ and $Q$ such that:

$$(A - LC)^T P + P(A - LC) + (1 + k_F^2)P^2 + (1 + k_H^2)I_n = -Q
$$

(12)

then the system (10) is an asymptotic observer for the system (9).

**Proof.** For the time-delay systems, it is classical to define a Lyapunov-Krasovskii functional of the form:

$$
V(t, e) = e(t)^T Pe(t) + \xi \int_{-\tau}^{0} e(t + \theta)^T e(t + \theta) \, d\theta
$$

(13)

where $P$ is a symmetric, positive definite matrix, and $\xi$ is a positive scalar. It is easy to show that $V$ is positive definite and upper bounded. We compute the derivative of $V$ along the trajectories of (11):

$$
\dot{V}(t, e) = e(t)^T [(A - LC)^T P + P(A - LC) + \xi I_n] e(t) + 2e(t)^T P F(X(t)) - F(X(t) - e(t)) + 2e(t)^T P H(X(t - \tau)) - H(X(t - \tau) - e(t - \tau)) - \xi e(t - \tau)^T e(t - \tau)
$$

(14)

By replacing (15), (16) and (17) in (14), this yields to:

$$
\dot{V}(t, e) = e(t)^T [(A - LC)^T P + P(A - LC) + \xi I_n] e(t) + k_F^2 e(t)^T P Pe(t) + \|e(t)\|^2 + e(t)^T P Pe(t) + k_H^2 \|e(t - \tau)\|^2 - \xi \|e(t - \tau)\|^2
$$

$$
\leq e(t)^T [(A - LC)^T P + P(A - LC) + (1 + k_F^2)P^2 + (1 + k_H^2)I_n] e(t) + (1 + k_F^2) P^2 + (1 + k_H^2) I_n\|e(t)\|^2
$$

(18)

$\xi$ must be chosen so that $k_F^2 - \xi \leq 0$. We choose $\xi = k_H^2$. Then (18) becomes:

$$
\dot{V}(t, e) \leq -e(t)^T Q e(t)
$$

(19)

Using equation (12), it implies

$$
\dot{V}(t, e) \leq -e(t)^T Q e(t)
$$

(20)

**Remark 3.** Equation (12) implies that

$$(A - LC)^T P + P(A - LC) < 0
$$

(21)

so the matrix $A - LC$ is stable, which is equivalent to the pair $(A, C)$ is detectable.

The problem now is to solve the Riccati-like equation (12). This represents the main contribution of this paper, since we give an analytical and simple solution of (12).

To give a particular solution of the Riccati equation for the modified Chua’s circuit, and with a view to simplify the search of this solution, we take $C = (0 \ 1 \ 0)$ in the dynamic model (9), and we assume that $P$ is a positive diagonal matrix:

$$
P = \begin{pmatrix}
a & 0 & 0 \\
0 & b & 0 \\
0 & 0 & c
\end{pmatrix}
$$

(22)

with $a, b, c > 0$.

The positive-definite matrix $Q$ is chosen of the form:

$$
Q = \begin{pmatrix}
q_1 & q_4 & 0 \\
q_4 & q_2 & q_5 \\
0 & q_5 & q_3
\end{pmatrix}
$$

(23)

and $\gamma_F = 1 + k_F^2$, $\gamma_H = 1 + k_H^2$.

The Riccati-like equation (12) becomes then:

$$(A - LC)^T P + P(A - LC) + \gamma_F P^2 = -Q - \gamma_H I_3
$$

(24)

We introduce the following notations for simplicity of the presentation: $a_1 = \frac{q_1}{\gamma_F}$, $a_2 = \frac{q_2}{\gamma_F}$, and $c_2 = \frac{1}{\gamma_F}$. 

Hence, using this notations in the Riccati equation (12), we obtain the equation (25).

This relation leads to the conditions:

\[-2a_1a + \gamma Fa^2 - a_2b + aa_1 - al_1 = 0 \]  
\[a_2b + aa_1 - al_1 = -q_1 \]  
\[-2b(a_2 + l_2) + \gamma F b^2 = -\gamma H - q_2 \]  
\[bc_2 - \left(1 + l_3\right)c = -q_5 \]  
\[\frac{-2R_0c}{L} + \gamma F c^2 = -\gamma H - q_3 \]

We recall that the unknowns of this five-equation system (26) are: \(l_1, l_2, l_3, q_i (i = 1, 5)\) and the positive scalars \(a, b, c\).

The first condition (26a) is a second-degree equation, with unknown \(a\):

\[\gamma Fa^2 - 2a_1a + \gamma H + q_1 = 0 \] (27)

The discriminant \(\Delta = 4a_1^2 - 4\gamma F(\gamma H + q_1)\) is superior or equal to zero if

\[q_1 \leq \frac{a_1^2}{\gamma F} - \gamma H \] (28)

In this case, the solution is (\(a\) must be positive):

\[a = \frac{a_1 + \sqrt{a_1^2 - \gamma F(\gamma H + q_1)}}{\gamma F} \] (29)

The second condition (26b) leads to the equality:

\[l_1 = a_1 + \frac{a_2b + q_4}{a} \] (30)

The third equation (26c) gives the relation:

\[l_2 = \frac{b\gamma F}{2} - a_2 + \frac{\gamma H + q_2}{2b} \] (31)

The fourth condition (26d) leads to:

\[l_3 = \left(1 + l_3\right)c = -q_5 \] (32)

The last equation (26e) is also a second-degree equation, with unknown \(c\), whose roots can be found in the same way than for (27). Hence we get:

\[c = \frac{R_0}{L} + \sqrt{\frac{R_0^2}{L^2} - \gamma F(\gamma H + q_3)} \gamma F \] (33)

under the condition

\[q_3 \leq \frac{R_0^2}{L^2\gamma F} - \gamma H \] (34)

The procedure to construct \(P, L, Q\) can be summarized as follows:

1. We choose \(q_i, i = 1, 5\) such that \(Q > 0\) and (28), (34) hold.
2. We can find \(a\) and \(c\) respectively using the equations (29) and (33).
3. We can arbitrarily choose \(b > 0\) to determine the diagonal, positive-definite matrix \(P\).
4. We compute \(l_1, l_2, l_3\) using the conditions (30), (31), (32).

By applying the Theorem 2, the observer that will synchronize with the modified Chua’s circuit (9) is given by the dynamic equations:

\[
\begin{align*}
\dot{\hat{X}}_1 &= -\frac{G}{C_1}\hat{X}_1 + \left(\frac{G}{C_1} - l_1\right)\hat{X}_2 - \frac{1}{C_1}f(\hat{X}_1) + l_1 Y \\
\dot{\hat{X}}_2 &= \frac{G}{C_2}\hat{X}_1 - \left(\frac{G}{C_2} + l_2\right)\hat{X}_2 + \frac{1}{C_2}\hat{X}_3 + l_2 Y \\
\dot{\hat{X}}_3 &= -\left(1 + l_3\right)\hat{X}_2 - \frac{R_0}{L}\hat{X}_3 - \frac{1}{L}\frac{\mu}{(\hat{X}_1(t - \tau))} + l_3 Y
\end{align*}
\] (35)

Now, we will show the efficiency of this result on numerical simulations.

3. NUMERICAL SIMULATIONS

3.1 Synchronization of the observer with Chua’s modified circuit

We take the same values as in (Wang et al., 2001):

\(R = 1950, \ G = 1/R, \ C_1 = 10^{-8}, \ C_2 = 10^{-7}, \ L = 18.68 \cdot 10^{-3}, \ R_0 = 16, \ E = 1, \ G_a = -0.75 \cdot 10^{-3}, \ G_b = -0.41 \cdot 10^{-3}, \ \tau = 0.001, \ \varepsilon = 0.2, \ \sigma = 0.5\)

The following initial conditions are chosen:

\[X_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \] (36)
The Riccati equation (12) is verified with the matrices (we give rounded values):

\[
P = 10^{-4} \begin{pmatrix} 1.4276 & 0 & 0 \\ 0 & 0.01 & 0 \\ 0 & 0 & 1.8871 \end{pmatrix}
\] (38)

and

\[
Q = \begin{pmatrix} -215.63 & 10^{-6} & 0 \\ 10^{-6} & -100 & 10^{-6} \\ 0 & 10^{-6} & -315.63 \end{pmatrix}
\] (39)

and the observer gain is

\[
L = 10^6 \begin{pmatrix} 0.051318 \\ 7.8137 \\ 0.052937 \end{pmatrix}
\] (40)

The figure 1. shows the curves representing the norm of the state estimation error.

![Fig. 1. Plot of the norm of the state estimation error vector](image1)

**3.2 Robustness of the synchronization scheme**

Synchronization takes an important place in some processes of secure communication and it should be avoided that anybody could decipher the message sent by the transmitter. Even if we chose not to deal with the problem of the robustness of the designed synchronization scheme, we present some numerical simulations by making variations on the parameters \(G\) and \(C_1\) in the observer (35).

The figure 2. shows that when the parameters \(G\) and \(C_1\) have a 1% mismatch in the observer, there is no more synchronization.

Hence, if the parameters of the observer synchronizing with Chua’s modified circuit are not perfectly known (mismatch above 1%), the synchronization does not happen.

In this work, we addressed the problem of synchronization of a modified Chua’s circuit, which belongs to a class of non-linear systems with a time delay. We have designed an observer-based scheme for synchronization and we have given a particular solution to a Riccati equation linked with the design of the observer. The synchronization of the observer with the modified Chua’s circuit has been tested on numerical simulations in the deterministic and uncertain cases.

**REFERENCES**


