Markov loops, determinants and Gaussian fields.

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MARKOV LOOPS, DETERMINANTS AND GAUSSIAN FIELDS

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1 Introduction

The purpose of this article is to explore some simple relations between loop measures, spanning trees, determinants, and Gaussian Markov fields. These relations are related to Dynkin's isomorphism (cf [1], [11], [7]) . Their potential interest could be suggested by noting that loop measures were defined in [5] for planar Brownian motion and are related to SLE processes (see also [17]). It is also the case for the free field as shown in [13]. We present the results in the elementary framework of symmetric Markov chains on a finite space, and then indicate how they can be extended to more general Markov processes such as the two dimensional Brownian motion.

2 Symmetric Markov processes on finite spaces

Notations: Functions on finite (or countable) spaces are often denoted as vectors and measures as covectors in coordinates with respect to the canonical bases associated with points (the dual base being given by Dirac measures δ_x).

The multiplication operators defined by a function, f acting on functions or on measures are in general simply denoted by f, but sometimes multiplication operators by a function f or a measure λwill be denoted M_f or M_λ. The function obtained as the density of a measure μ with respect to some other measure ν is simply denoted $\frac{\mu}{\nu}$.

2.1 Energy and Markovian semigroups

Let us first consider for simplicity the case of a symmetric irreducible Markov chain with exponential holding times on a finite space X, with generator $L_y^x = q^x(P_y^x - \delta_y^x)$, $\lambda_x, x \in X$ being a positive measure and P a λ-symmetric stochastic transition matrix: $\lambda_x P_y^x = \lambda_y P_x^y$ with $P_x^x = 0$ for all x in X.

We denote P_t the semigroup $\exp(Lt) = \sum \frac{t^k}{k!} L^k$ and by m_x the measure $\frac{\lambda_x}{q^x}$. L and P_t are m-symmetric.

Recall that for any complex function $z^x, x \in X$, the "energy"

$$e(z) = \langle -Lz, \overline{z} \rangle_m = \sum_{x \in X} -(Lz)^x \overline{z}^x m_x$$

is nonnegative as it can be written

$$e(z) = \frac{1}{2} \sum_{x,y} C_{x,y}(z^x - z^y)(\overline{z}^x - \overline{z}^y) + \sum_x \kappa_x z^x \overline{z}^x = \sum_x \lambda_x z^x \overline{z}^x - \sum_{x,y} C_{x,y} z^x \overline{z}^y$$

with $C_{x,y} = C_{y,x} = \lambda_x P_y^x$ and $\kappa_x = \lambda_x(1 - \sum_y P_y^x)$, i.e. $\lambda_x = \kappa_x + \sum_y C_{x,y} = e(1_{\{x\}})$.

We say (x, y) is a link iff $C_{x,y} > 0$. An important exemple is the case of a graph: Conductances are equal to zero or one and the conductance matrix is the incidence matrix of the graph.

The (complex) Dirichlet space $\mathbb{H}$ is the space of complex functions equipped with the energy scalar product defined by polarisation of e. Note that the non negative symmetric "conductance matrix" C and the non negative equilibrium or "killing" measure κ are the free parameters of the model. (so is q but we will see it is irrelevant for our purpose and we will mostly take it equal to 1). The lowest eigenvector of $-L$ is nonnegative by the well known argument which shows that the modulus contraction $z \to |z|$ lowers the energy. We will assume (although it is not always necessary) the corresponding eigenvalue is positive which means there is a "mass gap": For some positive ε, the energy $e(z)$ dominates $\varepsilon \langle z, \overline{z} \rangle_m$ for all z.

We denote by V the associated potential operator $(-L)^{-1} = \int_0^\infty P_t dt$. They can be expressed in terms of the spectral resolution of L.

We denote by G the Green function defined on X^2 as $G^{x,y} = \frac{V_y^x}{m_y} = \frac{1}{\lambda_y}[(I - P)^{-1}]_y^x$ i.e. $G = (M_\lambda - C)^{-1}$. It verifies $e(f, G\mu) = \langle f, \mu \rangle$ for all function f and measure μ. In particular $G\kappa = 1$.

Different Markov chains associated to the same energy are equivalent under time change. If g is a positive function on X, in the new time scale $\int_0^t g_{\xi_s} ds$, we obtain a Markov chain with gm-symmetric generator $\frac{1}{g}L$. Objects invariant under time change are called *intrinsic*. The energy e, P and the Green function G are obviously intrinsic but L, V and P_t are not. We will be interested only in intrinsic objects. In this elementary framework, it is possible to define a natural canonical time scale by taking $q = 1$, but it will not be true on continuous spaces.

2.2 Recurrent chain

Assume for simplicity that $q = 1$. It will be convenient to add a cemetery point Δ to X, and extend C, λ and G to $X^\Delta = \{X \cup \Delta\}$by setting $C_{x,\Delta} = \kappa_x$, $\lambda_\Delta = \sum_{x\in X} \kappa_x$. and $G^{x,\Delta} = 0$. Note that $\lambda(X^\Delta) = \sum_{X\times X} C_{x,y} + 2\sum_X \kappa_x)$

One can consider the recurrent "resurrected" Markov chain defined by the extensions the conductances to X^Δ. An energy e^R is defined by the formula

$$e^R(z) = \frac{1}{2}\sum_{x,y} C_{x,y}(z^x - z^y)(\overline{z}^x - \overline{z}^y)$$

We denote by P^R the transition kernel on X^Δ defined by

$$e^R(z) = \left\langle z - P^R z, \overline{z}\right\rangle_\lambda$$

or equivalently by

$$[P^R]_y^x = \frac{C_{x,y}}{\sum_{y\in X^\Delta} C_{x,y}} = \frac{C_{x,y}}{\lambda_x}$$

Note that $P^R 1 = 1$ so that λ is now an invariant measure. Let $\lambda^\perp$ be the space of functions on X^Δ of zero λ measure and by V^R the inverse of the restriction of $I - P^R$ to $\lambda^\perp$.It vanishes on constants and has a mass gap on $\lambda^\perp$. Setting for any signed measure ν of total charge zero $G^R\nu = V^R\frac{\nu}{\lambda}$. we have for any function f, $\langle \nu, f\rangle = e^R(G^R\nu, f)$ and in particular $f^x - f^y = e^R(G^R(\delta_x - \delta_y), f)$.

Note that for $\mu \in \lambda^\perp$ and carried by X, for all $x \in X$, $\mu_x = e^R(G^R\mu, 1_x) = \lambda_x((I - P)G^R\mu)(x) - \kappa_x G^R\mu(\Delta)$. Hence, applying G , it follows that on X, $G^R\mu = G^R\mu(\Delta)G\kappa + G\mu = G^R\mu(\Delta) + G\mu$. Moreover, as $G^R\mu$ is in $\lambda^\perp$, $G^R\mu(\Delta)\lambda(X^\Delta) + \sum_{x\in X} \lambda_x(G\mu)_x = 0$.

Therefore, $G^R\mu(\Delta) = \frac{-\langle\lambda, G\mu\rangle}{\lambda(X^\Delta)}$ and $G^R\mu = \frac{-\langle\lambda, G\mu\rangle}{\lambda(X^\Delta)} + G\mu$

2.3 Transfer matrix

We can define a scalar product on the space $\mathbb{A}$ of antisymmetric functions on $X^\Delta \times X^\Delta$ as follows

$\langle \omega, \eta \rangle = \sum_{x,y} C_{x,y} \omega^{x,y} \eta^{x,y}$. Denoting as in [9] $df^{u,v} = f^u - f^v$, we note that $\langle df, dg \rangle = e^R(f,g)$ In particular

$$\left\langle df, dG^R(\delta_x - \delta_y) \right\rangle = df^{x,y}$$

As the antisymmetric functions df span the space of antisymmetric functions, it follows that the scalar product is positive definite.

The symmetric transfer matrix K, indexed by pairs of oriented links, is defined to be

$$K^{(x,y),(u,v)} = G^R(\delta_x - \delta_y)^u - G^R(\delta_x - \delta_y)^v =< dG^R(\delta^x - \delta^y), dG^R(\delta^u - \delta^v) >$$

for $x, y, u, v \in X^\Delta$, with $x \neq y, u \neq v$.

We see that for x and y in X, $G^R(\delta_x - \delta_y)^u - G^R(\delta_x - \delta_y)^v = G(\delta_x - \delta_y)^u - G(\delta_x - \delta_y)^v$.

We can see also that $G^R(\delta_x - \delta_\Delta) = G\delta_x - \frac{-\langle \lambda, G\delta_x \rangle}{\lambda(X^\Delta)}$. So the same identity holds in X^Δ.

Therefore, as $G^{x,\Delta} = 0$, in all cases,

$$K^{(x,y),(u,v)} = G^{x,u} + G^{y,v} - G^{x,v} - G^{y,u}$$

For every oriented link $\xi = (x,y)$ in X^Δ,set $K^\xi = dG^R(\delta^x - \delta^y) = dG(\delta^x - \delta^y)$.

We have $\left\langle K^\xi, K^\eta \right\rangle = K^{\xi,\eta}$. K will be viewed as a linear operator on $\mathbb{A}$, self adjoint with respect to $\langle \cdot, \cdot \rangle$. (It can also be viewed as symmetric with respect to the euclidean scalar product if we wish to use it Then it appears as the inverse of the operator defined by $\langle \cdot, \cdot \rangle$).

3 Loop measures

3.1 Definitions

For any integer k, let us define a based loop with p points in X as a couple $(\xi, \tau) = ((\xi_m, 1 \leq m \leq p), (\tau_m, 1 \leq m \leq p+1),)$ in $X^p \times \mathbb{R}_+^{p+1}$, and set $\xi_1 = \xi_{p+1}$. p will be denoted $p(\xi)$.

Based loops have a natural time parametrisation $\xi(t)$ and a time period $T(\xi) = \sum_{i=1}^{p(\xi)+1} \tau_i$. If we denote $\sum_{i=1}^{m} \tau_i$ by T_m: $\xi(t) = \xi_{m-1}$ on $[T_{m-1}, T_m)$ (with by convention $T_0 = 0$ and $\xi_0 = \xi_p$).

A σ-finite measure μ_0 is defined on based loops by

$$\mu_0 = \sum_{x \in X} \int_0^\infty \frac{1}{t} \mathbb{P}_t^{x,x} dt$$

where $\mathbb{P}_t^{x,x}$ denotes the (non normalized) "law" of a path from x to x of duration t : If $\sum_{i=1}^{h+1} t_i = t$,

$$\mathbb{P}_t^{x,x}(\xi(t_1) = x_1, ..., \xi(t_h) = x_h) = [P_{t_1}]_{x_1}^{x} [P_{t_2 - t_1}]_{x_2}^{x_1} ... [P_{t - t_h}]_{x}^{x_h}$$

Note also that

$$\begin{aligned}
&\mathbb{P}_t^{x,x}(p = k, \xi_2 = x_2, ..., \xi_k = x_k, T_1 \in dt_1, ..., T_k \in dt_k) \\
&\quad = [P]_{x_2}^{x} [P]_{x_3}^{x_2} ... [P]_{x}^{x_k} 1_{\{0 < t_1 < ... t_k < t\}} q^x e^{-q^x t_1} ... q_{x_k} e^{-q_{x_k}(t_k - t_{k-1})} e^{-q_x(t - t_k)} dt_1 ... dt_k
\end{aligned}$$

A loop is defined as an equivalence class of based loops for the $\mathbb{R}$-shift that acts naturally. μ_0 is shift invariant, It induces a *measure μ on loops*.

Note also that the measure $d\widetilde{\mu}_0 = \frac{T q_{\xi_1}}{\int_0^T q_{\xi(s)} ds} d\mu_0$ which is not shift invariant also induces μ on loops.

It writes

$$\begin{aligned}
&\widetilde{\mu}_0(p(\xi) = k, \xi_1 = x_1, ..., \xi_k = x_k, T_1 \in dt_1, ..., T_k \in dt_k, T \in dt) \\
&\quad = [P]_{x_2}^{x_1} [P]_{x_3}^{x_2} ... [P]_{x}^{x_k} \frac{1_{\{0 < t_1 < ... < t_k < t\}}}{\int_0^t q_{\xi(s)} ds} e^{-q^{x_1} t_1} e^{-q_{x_2}(t_2 - t_1)} ... e^{-q_x(t - t_k)} q_{x_1} dt_1 ... q_{x_k} dt_k q_{x_1} dt
\end{aligned}$$

for $k \geq 2$ and

$$\widetilde{\mu}_0\{p(\xi) = 1, \xi_1 = x, \tau_1 \in dt_1\} = \frac{e^{-q_x t_1}}{t_1} dt_1$$

It is clear, in that form, that a time change transforms the $\widetilde{\mu}_0$'s of Markov chains associated with the same energy one into each other, and therefore the same holds for μ: this is analogous to conformal invariance. Hence the restriction μ_I of μ to the σ-field of sets of loops invariant by time change (i.e. intrinsic sets) is intrinsic. It depends only on e. As we are interested in the restriction μ_I of μ to intrinsic sets, from now on we will denote simply μ_I by μ

Intrinsic sets are defined by the discrete loop ξ_i (in circular order, up to translation) and the associated intrinsic times $\frac{\tau_i}{m_i} = \tau_i^*$. Conditionally to the discrete loop, these are independent exponential variables with parameters λ_i.

$$\mu = \sum_{x \in X} e^{-\lambda_x \tau^*} \frac{d\tau^*}{\tau^*} + \sum_{p=2}^{\infty} \sum_{(\xi_i, i \in \mathbb{Z}/p\mathbb{Z}) \in X^p} \prod_{i \in \mathbb{Z}/p\mathbb{Z}} C_{\xi_i, \xi_{i+1}} e^{-\lambda_{\xi_i} \tau_i^*} d\tau_i^* \qquad (1)$$

Sets of discrete loop are the most important intrinsic sets, though we will see that to establish a connection with Gaussian fields it is important to consider occupation times. The simplest intrinsic variables are

$$N_{x,y} = \#\{i : \xi_i = x, \xi_{i+1} = y\}$$

and

$$N_x = \sum_y N_{x,y}$$

Note that $N_x = \#\{i \geq 1 : \xi_i = x\}$ except for trivial one point loops.

A *bridge measure* $\mu^{x,y}$ can be defined on paths γ from x to y: $\mu^{x,y}(d\gamma) = \frac{1}{m_y} \int_0^\infty \mathbb{P}_t^{x,y}(d\gamma) dt$ with

$$\mathbb{P}_t^{x,y}(\gamma(t_1) = x_1, ..., \gamma(t_h) = x_h) = P_{t_1}(x, x_1) P_{t_2 - t_1}(x_1, x_2) ... P_{t - t_h}(x_h, y)$$

Note that the mass of $\mu^{x,y}$ is $\frac{V_y^x}{m_y} = G^{x,y}$. We also have, with similar notations as the one defined for loops

$$\begin{aligned} &\mu^{x,y}(p(\gamma) = k, \gamma_2 = x_2, ..., \gamma_{k-1} = x_{k-1}, T_1 \in dt_1, ..., T_{k-1} \in dt_{k-1}, T \in dt) \\ &\qquad = \frac{C_{x,x_2} C_{x_2,x_3} ... C_{x_{k-1},y}}{\lambda_x \lambda_{x_2} ... \lambda_y} 1_{\{0 < t_1 < ... < t_k < t\}} e^{-q_x t_1} e^{-q_{x_2}(t_2 - t_1)} ... e^{-q_y (t - t_k)} q_x dt_1 ... q_{x_{k-1}} dt_k q_y dt \end{aligned}$$

so that the restriction of $\mu^{x,y}$ to intrinsic sets of paths is intrinsic.

Finally, we denote $\mathbb{P}^x$ the family of probability laws on paths defined by P_t.

$$\mathbb{P}^x(\gamma(t_1) = x_1, ..., \gamma(t_h) = x_h) = P_{t_1}(x, x_1) P_{t_2 - t_1}(x_1, x_2) ... P_{t_h - t_{h-1}}(x_{h-1}, x_h)$$

$$\mathbb{P}_x(p(\gamma)=k,\gamma_2=x_2,...,\gamma_k=x_k,T_1\in dt_1,...,T_k\in dt_k)$$
$$=\frac{C_{x,x_2}...C_{x_{k-1},x_k}\kappa_{x_k}}{\lambda_x\lambda_{x_2}...\lambda_{x_k}}1_{\{0<t_1<...<t_k\}}e^{-q_xt_1}...e^{-q_{x_k}(t_k-t_{k-1})}q_xdt_1...q_{x_k}dt_k$$

3.2 First properties

If D is a subset of X, the restriction of μ to loops contained in D, denoted μ^D is clearly the loop measure induced by the Markov chain killed at the exit of D. This can be called the *restriction property*.

Let us recall that this killed Markov chain is defined by the restriction of λ to D and the restriction P^D of P to D^2 (or equivalently by the restriction e_D of the Dirichlet norm e to functions vanishing outside D) and (for the time scale), by the restriction of q to D.

From now on in this section, we will take $q_x=1$ for all x. Then μ_0 takes a simpler form:

$$\mu_0(p(\xi)=k,\xi_1=x_1,...,\xi_k=x_k,T_1\in dt_1,...,T_k\in dt_k,T\in dt)$$
$$=P^{x_1}_{x_2}...P^{x_k}_{x_1}\frac{1_{\{0<t_1<...<t_k<t\}}}{t}e^{-t}dt_1...dt_kdt$$

for $k>1$ and $\mu_0\{p(\xi)=1,\xi_1=x_1,\tau_1\in dt_1\}=\frac{e^{-t_1}}{t_1}dt_1$

It follows that for $k>0$,

$$\mu_0(p(\xi)=k,\xi_1=x_1,...,\xi_k=x_k)=\frac{1}{k}P^{x_1}_{x_2}...P^{x_k}_{x_1}=\frac{1}{k}\prod_{x,y}C^{N_{x,y}}_{x,y}\prod_x\lambda_x^{-N_x}$$

as$\int\frac{t^{k-1}}{k!}e^{-t}dt=\frac{1}{k}$ and conditionally to $p(\xi)=k,\xi_1=x_1,...,\xi_k=x_k$, T is a gamma variable of density $\frac{t^{k-1}}{(k-1)!}e^{-t}$ on $\mathbb{R}_+$ and $(\frac{T_i}{T}1\le i\le k)$ an independent ordered $k-$sample of the uniform distribution on $(0,1)$.

In particular, we obtain that, for $k\ge 2$

$$\mu(p=k)=\mu_0(p=k)=\frac{1}{k}Tr(P^k)$$

and therefore, as $Tr(P)=0$,

$$\mu(p>0)=-\log(\det(I-P))=-\log(\frac{\det(G)}{\prod_x\lambda_x})$$

as denoting M_λ the diagonal matrix with entries λ_x, $\det(I-P) = \frac{\det(M_\lambda - C)}{\det(M_\lambda)}$. Moreover

$$\int p(l)\mu(dl) = Tr((I-P)^{-1}P)$$

Similarly, for any $x \neq y$ in X and $s \in [0,1]$, setting $P^{(s)}_{u,v} = P^u_v$ if $(u,v) \neq (x,y)$ and $P^{(s)}_{x,y} = sP^x_y$, we have:

$$\mu(s^{N_{x,y}} 1_{\{p>0\}}) = -\log(\det(I - P^{(s)}))$$

Differentiating in $s = 1$, it comes that

$$\mu(N_{x,y}) = [(I-P)^{-1}]^y_x P^x_y = G^{x,y} C_{x,y}$$

and $\mu(N_x) = \sum_y \mu(N_{x,y}) = \lambda_x G^{x,x} - 1$ (as $(M_\lambda - C)G = Id$).

4 Poisson process of loops and occupation field

4.1 Occupation field

To each loop l we associate an occupation field $\{\widehat{l_x}, x \in X\}$ defined by

$$\widehat{l^x} = \int_0^{T(l)} 1_{\{\xi(s)=x\}} \frac{q^{\xi_s}}{m_{\xi(s)}} ds = \sum_{i=1}^{p(l)} 1_{\{\xi_{i-1}=x\}} \frac{q^x \tau_i}{m_x} = \sum_{i=1}^{p(l)} 1_{\{\xi_{i-1}=x\}} \tau_i^*$$

for any representative (ξ, τ) of l. It is independent of the time scale (i.e."intrinsic").

For a path γ, $\widehat{\gamma}$ is defined in the same way.

From now on *we will take* $q = 1$.

Note that

$$\mu((1 - e^{-\alpha \widehat{l^x}}) 1_{\{p=1\}}) = \int_0^\infty (e^{-(\frac{\alpha}{\lambda_x}+1)t} - e^{-t}) \frac{dt}{t} = \log(\frac{\lambda_x}{\alpha + \lambda_x}) \qquad (2)$$

In particular, $\mu(\widehat{l^x} 1_{\{p=1\}}) = \frac{1}{\lambda_x}$.

From formula 1, we get easily that for any function Φ of the discrete loop and $k \geq 1$,

$$\mu((\widehat{l^x})^k 1_{\{p>1\}} \Phi) = \mu((N_x + k - 1)...(N_x + 1) N_x \Phi)$$

In particular, $\mu(\widehat{l^x}) = \frac{1}{\lambda_x}[\mu(N_x) + 1] = G^{x,x}$.

Note that functions of $\widehat{l}$ are not the only intrinsic functions. Other intrinsic variables of interest are, for $k \geq 2$

$\widehat{l}^{x_1,...,x_k} = \frac{1}{k}\sum_{j=0}^{k-1}\int_{0<t_1<...<t_k<T} 1_{\{\xi(t_1)=x_{1+j},....\xi(t_{k-j})=x_k,...\xi(t_k)=x_j\}}\prod\frac{1}{\lambda_{x_i}}dt_i$

$= \frac{1}{k}\sum_{j=0}^{k-1}\sum_{1\leq i_1<..<i_k\leq p(l)}\prod_{l=1}^{k} 1_{\{\xi_{i_l-1}=x_{l+j}\}}\tau^*_{i_l}$ and one can check that $\mu(\widehat{l}^{x_1,...,x_k}) = G^{x_1,x_2}G^{x_2,x_3}...G^{x_k,x_1}$. Note that in general $\widehat{l}^{x_1,...,x_k}$ cannot be expressed in terms of $\widehat{l}$ for $k > 3$.

For $x_1 = x_2 = ... = x_k$, we obtain self intersection local times $\widehat{l}^{x,k} = \sum_{1\leq i_1<..<i_k\leq p(l)}\prod_{l=1}^{k} 1_{\{\xi_{i_l-1}=x\}}\tau^*_{i_l}$

For any function Φ of the discrete loop, $\mu(\widehat{l}^{x,2}\Phi) = \mu(\frac{N_x(N_x-1)}{2}\Phi)$ since $\widehat{l}^{x,2} = \frac{1}{2}((\widehat{l}^x)^2 - \sum_{i=1}^{p(l)} 1_{\{\xi_{i-1}=x\}}(\tau^*_i)^2)$ and $\mu(\Phi\sum_{i=1}^{p(l)} 1_{\{\xi_{i-1}=x\}}(\tau^*_i)^2)) = 2\mu(\Phi N_x)$

More generally one proves in a similar way that $\mu(\widehat{l}^{x,k}\Phi) = \mu(\frac{N_x(N_x-1)...(N_x-k+1)}{k!}\Phi)$

From the Feynman-Kac formula, it comes easily that, denoting $M_{\frac{\chi}{\lambda}}$ the diagonal matrix with coefficients $\frac{\chi_x}{\lambda_x}$

$\mathbb{P}^t_{x,x}(e^{-\langle\widehat{l},\chi\rangle} - 1) = \exp(t(P - I - M_{\frac{\chi}{\lambda}}))_{x,x} - \exp(t(P-I))_{x,x}$. Integrating in t after expanding, we get from the definition of μ (first for χ small enough):

$$\int(e^{-\langle\widehat{l},\chi\rangle} - 1)d\mu(l) = \sum_{k=1}^{\infty}\frac{1}{k}[Tr((P - M_{\frac{\chi}{\lambda}})^k) - Tr((P)^k)]$$

Hence

$$\int(e^{-\langle\widehat{l},\chi\rangle} - 1)d\mu(l) = \log[\det(-L(-L + M_{\chi/\lambda})^{-1})] = -\log\det(I + VM_{\frac{\chi}{\lambda}})$$

which now holds for all non negative χ. Set $V_\chi = (-L + M_{\frac{\chi}{\lambda}})^{-1}$ and $G_\chi = V_\chi M_{\frac{1}{\lambda}}$. It is an intrinsic symmetric nonnegative function on $X \times X$. G_0 is the Green function G, and G_χ can be viewed as the Green function of the energy form $e_\chi = e + \|\|^2_{L^2(\chi)}$. Note that e_χ has the same conductances C as e, but χ is added to the killing measure. We have also the "resolvent" equation $V - V_\chi = VM_{\frac{\chi}{\lambda}}V_\chi = V_\chi M_{\frac{\chi}{\lambda}}V$. Then, $G - G_\chi = GM_\chi G_\chi = G_\chi M_\chi G$. Also:

$$\det(I + GM_\chi)^{-1} = \det(I - G_\chi M_\chi) = \frac{\det(G_\chi)}{\det(G)} \tag{3}$$

Finally we have the

Proposition 1 *i)*$\mu(e^{-\langle \widehat{l},\chi\rangle}-1) = -\log(\det(I+GM_\chi)) = \log(\det(I-G_\chi M_\chi)) = \log(\det(G_\chi G^{-1}))$

Note that in this calculation, the trace and the determinant are applied to matrices indexed by X. Note also that $\det(I+GM_\chi) = \det(I+M_{\sqrt{\chi}}GM_{\sqrt{\chi}})$ and $\det(I-G_\chi M_\chi) = \det(I-M_{\sqrt{\chi}}G_\chi M_{\sqrt{\chi}})$, so we can deal with symmetric matrices..

In view of generalizing them to continuous spaces in an intrinsic form (i.e. in a form invariant under time change), , G and G_χ will be interpreted as symmetric elements of $\mathbb{H}\otimes\mathbb{H}$, or as linear operators from $\mathbb{H}'$ into $\mathbb{H}$. G is a canonical bijection. $\frac{\det(G_\chi)}{\det(G)}$ can be viewed as the determinant of the operator $G_\chi G^{-1}$ acting on $\mathbb{H}$.

4.2 Poisson process of loops

Still following the idea of [5], define, for all positive α, the Poisson process of loops $\mathcal{L}_\alpha$ with intensity $\alpha\mu$. We denote by $\mathbb{P}$ or $\mathbb{P}_{\mathcal{L}_\alpha}$ its distribution. Note that by the restriction property, $\mathcal{L}_\alpha^D = \{l\in\mathcal{L}_\alpha, l\subseteq D\}$ is a Poisson process of loops with intensity μ^D, and that $\mathcal{L}_\alpha^D$ is independent of $\mathcal{L}_\alpha\backslash\mathcal{L}_\alpha^D$.

We denote by $\mathcal{L}_\alpha^d$ the set of non trivial discrete loops in $\mathcal{L}_\alpha$. Then, $\mathbb{P}(\mathcal{L}_\alpha^d = \{l_1,l_2,...l_k\}) = e^{-\alpha\mu(p>0)}\alpha^k\frac{\mu(l_1)...\mu(l_k)}{k!} = [\frac{\det(G)}{\prod_x\lambda_x}]^\alpha\prod_{x,y}C_{x,y}^{N_{x,y}^{(\alpha)}}\prod_x\lambda_x^{-N_x^{(\alpha)}}$ with $N_x^{(\alpha)} = \sum_{l\in\mathcal{L}_\alpha}N_x(l)$ and $N_{x,y}^{(\alpha)} = \sum_{l\in\mathcal{L}_\alpha}N_{x,y}(l)$.

Remark 2 *It follows that the probability of a discrete loop configuration depends only on the variables $N_{x,y}+N_{y,x}$, i.e. the total number of traversals of non oriented links. In particular, it does not depend on the orientation of the loops It should be noted that under loop or path measures, the conditional distributions of discrete loops or paths given the values of all $N_{x,y}+N_{y,x}$'s is uniform. The $N_{x,y}+N_{y,x}$ ($N_{x,y}$) configuration can be called the associated random (oriented) graph. Note however that any configuration of $N_{x,y}+N_{y,x}$ does not correspond to a loop configuration.*

We can associate to $\mathcal{L}_\alpha$ the σ-finite measure

$$\widehat{\mathcal{L}_\alpha} = \sum_{l\in\mathcal{L}_\alpha}\widehat{l}$$

Then, for any non-negative measure χ on X

$$\mathbb{E}(e^{-\langle\widehat{\mathcal{L}_\alpha},\chi\rangle}) = \exp(\alpha\int(e^{-\langle\widehat{l},\chi\rangle}-1)d\mu(l))$$

and

$$\mathbb{E}(e^{-\left\langle \widehat{\mathcal{L}_\alpha},\chi\right\rangle}) = [\det(-L(-L+M_{\chi/\lambda})^{-1})]^\alpha = \det(I+VM_{\frac{\chi}{\lambda}})^{-\alpha}$$

Finally we have the

Proposition 3 $\mathbb{E}(e^{-\left\langle \widehat{\mathcal{L}_\alpha},\chi\right\rangle}) = \det(I+GM_\chi)^{-\alpha} = \det(I-G_\chi M_\chi)^\alpha = \det(G_\chi G^{-1})^\alpha$

Many calculations follow from proposition 1.

It follows that $\mathbb{E}(\widehat{\mathcal{L}_\alpha}^x) = \alpha G_{x,x}$ and we recover that $\mu(\widehat{l}_x) = G_{x,x}$.

On loops and paths, we define the restricted intrinsic σ-field $\mathcal{I}_R$ as generated the variables $N_{x,y}$ with $y.$ possibly equal to Δ in the case of paths, with $N_{x,\Delta} = 0$ or 1. from (2),

$$\mathbb{E}(e^{-\sum \chi_i \left\langle \widehat{\mathcal{L}_\alpha},\delta_{x_i}\right\rangle}|\mathcal{I}_R) = \prod_{i=1}^k (\frac{\lambda_{x_i}}{\lambda_{x_i}+\chi_i})^{N_{x_i}^{(\alpha)}+1}$$

The distribution of $\{N_x^{(\alpha)}, x \in X\}$ follows easily, in terms of generating functions:

$$\mathbb{E}(\prod_{i=1}^k (s_i^{N_{x_i}^{(\alpha)}+1}) = \det(\delta_{i,j} + \sqrt{\frac{\lambda_{x_i}\lambda_j(1-s_i)(1-s_j)}{s_i s_j}} G_{x_i,x_j})^{-\alpha}$$

Note also that

$$\mathbb{E}((\widehat{\mathcal{L}_\alpha}^x)^k|\mathcal{I}_R) = \frac{(N_x^{(\alpha)}+k)(N_x^{(\alpha)}+k-1)...(N_x^{(\alpha)}+1)}{k!\lambda_x^k}$$

and if self intersection local times are defined as $\widehat{\mathcal{L}_\alpha}^{x,k} = \sum_{m=1}^k \sum_{k_1+...+k_m=k} \sum_{l_1\neq l_2...\neq l_m \in \mathcal{L}_\alpha^+} \prod_{j=1}^m \widehat{l}_j^{x,k_j}$, we get easily that

$$\mathbb{E}(\widehat{\mathcal{L}_\alpha}^{x,k}|\mathcal{I}_R) = \frac{1}{\lambda_x^k}(N_x^{(\alpha)}-k+1)...(N_x^{(\alpha)}-1)N_x^{(\alpha)}$$

Note also that since $G_\chi M_\chi$ is a contraction, from determinant expansions given in [15] and [16], we have

$$\mathbb{E}(\left\langle \widehat{\mathcal{L}_\alpha},\chi\right\rangle^k) = \sum \chi_{i_1}...\chi_{i_k} Per_\alpha(G_{i_l,i_m}, 1\leq l,m\leq k)$$

Here the α-permanent Per_a is defined as $\sum_{\sigma\in\mathcal{S}_k}\alpha^{m(\sigma)}G_{i_1,i_{\sigma(1)}}...G_{i_k,i_{\sigma(k)}}$ with $m(\sigma)$ denoting the number of cycles in σ.

Let $[H^F]^x_\cdot$ be the hitting distribution of F by the Markov chain starting at F. Set $D=F^c$ and denote e^D, $V^D=[(I-P)|_{D\times D}]^{-1}$ and $G^D=[(M_\lambda - C)|_{D\times D}]^{-1}$ the Dirichlet norm, the potential and the Green function of the process killed at the hitting of F. Recall that $V=V^D+H^FV$ and $G=G^D+H^FG$.

Taking $\chi=a1_F$ with F finite, and letting a increase to infinity, we get $\lim_{a\uparrow\infty}(G_\chi M_\chi)=H^F$ which is I on F. Therefore by proposition 1, one checks that $\mathbb{P}(\widehat{\mathcal{L}}_\alpha(F)=0)=\det(I-H^F)=0$ and $\mu(\widehat{l}(F)>0)=\infty$. But this is clearly due to trivial loops as it can be seen directly from the definition of μ that in this simple framework they cover the whole space X.

Note however that $\mu(\widehat{l}(F)>0,p>0)=\mu(p>0)-\mu(\widehat{l}(F)=0,p>0)$

$=\mu(p>0)-\mu^D(p>0)=-\log(\frac{\det(I-P)}{\det_{D\times D}(I-P)})=\log(\frac{\det(G^D)}{\prod_{x\in F}\lambda_x\det(G)})$

It follows that the probability no non trivial loop (i.e.a loop which is not reduced to a point) in $\mathcal{L}_\alpha$ intersects F equals $(\frac{\det(G^D)}{\prod_{x\in F}\lambda_x\det(G)})^\alpha$

Recall that for any $(n+p,n+p)$ invertible matrix A, $\det(A^{-1})\det(A_{ij}1\le i,j\le n)=\det(A^{-1})\det(Ae_1,...Ae_n,e_{n+1},...e_{n+p})$

$=\det(e_1,...e_n,A^{-1}e_{n+1},...A^{-1}e_{n+p})=\det((A^{-1})_{k,l},n\le k,l\le n+p)$.

In particular, $\det(G^D)=\frac{\det(G)}{\det(G|_{F\times F})}$, so we have the

Corollary 4 *The probability that no non trivial loop in $\mathcal{L}_\alpha$ intersects F equals $(\prod_{x\in F}\lambda_x\det_{F\times F}(G)^{-\alpha}$*

In particular, it follows that the probability a non trivial loop in $\mathcal{L}_\alpha$ visits x equals $1-(\frac{1}{\lambda_x G^{x,x}})^\alpha$

Also, if F_1 and F_2 are disjoint, $\mu(\prod\widehat{l}(F_i)>0)=\mu(p>0)+\mu(\sum\widehat{l}(F_i)=0,p>0)-\mu(\widehat{l}(F_1)=0,p>0)-\mu(\widehat{l}(F_2)=0,p>0)$

$=\log(\frac{\det(G)\det(G^{D_1\cap D_2})}{\det(G^{D_1})\det(G^{D_2})})$ and this formula is easily generalized to n disjoint sets.

$$\mu(\prod\widehat{l}(F_i)>0)=\log(\frac{\det(G)\prod_{i<j}\det(G^{D_i\cap D_j})...}{\prod\det(G^{D_i})\prod_{i<j<k}\det(G^{D_i\cap D_j\cap D_k})...}$$

The positivity yields an interesting determinant product inequality.

It follows in particular that the probability a non trivial loop in $\mathcal{L}_\alpha$ visits two distinct points x and y equals $1-(\frac{G^{x,x}G^{y,y}-(G^{x,y})^2}{G^{x,x}G^{y,y}})^\alpha$ and $\frac{(G^{x,y})^2}{G^{x,x}G^{y,y}}$ if $\alpha=1$.

Note finally that if χ has support in D, by the restriction property

$$\mu(1_{\{\widehat{l}(F)=0\}}(e^{-<\widehat{l},\chi>}-1)) = -\log(\det(I+G^D M_\chi)) = \log(\det(G^D_\chi)[G^D]^{-1})$$

Here the determinants are taken on matrices indexed by D. or equivalently on operators on $\mathbb{H}^D$.

For paths we have $\mathbb{P}_t^{x,y}(e^{-\langle \widehat{l},\chi\rangle}) = \exp(t(L-M_{\frac{\chi}{\chi}}))_{x,y}$.

Hence $\mu^{x,y}(e^{-\langle\widehat{\gamma},\chi\rangle}) = \frac{1}{\lambda_y}((I-P+M_{\chi/m})^{-1})_{x,y} = [G_\chi]^{x,y}$.

Also $\mathbb{E}^x(e^{-\langle\widehat{\gamma},\chi\rangle}) = \sum_y [G_\chi]^{x,y}\kappa_y$.

In the case of a lattice, one can consider a Poisson process of loops with intensity $\mu_{00}^{\#}$

5 Associated Gaussian field

By a well known calculation, if X is finite, for any $\chi \in \mathbb{R}_+^X$,

$$\frac{\det(M_\lambda - C)}{(2\pi)^{|X|}} \int (e^{-\frac{1}{2}<z\overline{z},\chi>} e^{-\frac{1}{2}e(z)} \Pi_{u\in X} \frac{i}{2} dz_u \wedge d\overline{z}_u = \frac{\det(G_\chi)}{\det(G)}$$

and

$$\frac{\det(M_\lambda + M_\chi - C)}{(2\pi)^{|X|}} \int z^x \overline{z}^y (e^{-\frac{1}{2}<z\overline{z},\chi>} e^{-\frac{1}{2}e(z)} \Pi_{u\in X} \frac{i}{2} dz^u \wedge d\overline{z}^u = (G_\chi)^{x,y}$$

This can be easily reformulated by introducing the complex Gaussian field ϕ defined by the covariance $\mathbb{E}_\phi(\phi^x\overline{\phi}^y) = 2G^{x,y}$ (this reformulation cannot be dispensed with when X becomes infinite)

So we have $\mathbb{E}((e^{-\frac{1}{2}<\phi\overline{\phi},\chi>}) = \det(I+GM_\chi)^{-1} = \det(G_\chi G^{-1})$ and

$\mathbb{E}((\phi^x\overline{\phi}^y e^{-\frac{1}{2}<\phi\overline{\phi},\chi>}) = (G_\chi)^{x,y}\det(G_\chi G^{-1})$ Then the following holds:

Theorem 5 *a) The fields $\widehat{\mathcal{L}_1}$ and $\frac{1}{2}\phi\overline{\phi}$ have the same distribution.*

b) $\mathbb{E}_\phi((\phi^x\overline{\phi}^y F(\phi\overline{\phi})) = \int \mathbb{E}(F(\widehat{\mathcal{L}_1}+\widehat{\gamma}))\mu^{x,y}(d\gamma)$ for any functional F of a non negative field.

This is a version of Dynkin's isomorphism (Cf [1]). It can be extended to non symmetric generators (Cf [10]).

Note it implies immediately that the process $\phi\overline{\phi}$ is infinitely divisible. See [2] and its references for a converse and earlier proofs of this last fact.

In fact an analogous result can be given when α is any positive half integer, by using a real scalar or vector valued Gaussian field.

Recall that for any $f \in \mathbb{H}$, the law of $f + \phi$ is absolutely continuous with respect to the law of ϕ, with density $\exp(< -Lf, \phi >_m -\frac{1}{2}e(f))$

Recall (it was observed by Nelson in the context of the free field) that the Gaussian field ϕ is Markovian: Given any subset F of X, denote $\mathcal{H}_F$ the Gaussian space spanned by $\{\phi^y, y \in F\}$. Then, for $x \in D = F^c$, the projection of ϕ^x on $\mathcal{H}_F$ is $\sum_{y\in F}[H^F]^x_y\phi^y$.

Moreover, $\phi^D = \phi - H^F\phi$ is the Gaussian field associated with the process killed at the exit of D.

Note also that if a function h is such that $Lh \leq 0$, the loop measure defined by the h^2m-symmetric generator $L_h = \frac{1}{h}LM_h$ is associated with the Gaussian field $h\phi$. The killing measure becomes $\frac{-Lh}{h}\lambda$

Remark finally that the transfer matrix K is the covariance matrix of the Gaussian field $d\phi^{x,y} = \phi^x - \phi^y$ indexed by oriented links.

6 Energy variation and currents

The loop measure μ depends on the energy e which is defined by the free parameters C, κ. It will sometimes be denoted μ_e. We shall denote $\mathcal{Z}_e$ the determinant $\det(G) = \det(M_\lambda - C)^{-1}$. Then $\mu(p > 0) = \log(\mathcal{Z}_e) + \sum \log(\lambda_x)$.

Other intrinsic variables of interest on the loop space are associated with real antisymmetric matrices $\omega_{x,y}$ indexed by X^Δ: $\omega_{x,y} = -\omega_{y,x}$.. Let us mention a few elementary results.

The operator $[P^\omega]^x_y = P^x_y \exp(i\omega_{x,y})$ is self adjoint in $L^2(\lambda)$.The associated loop variable writes $\sum_{j=1}^p \omega_{\xi_j,\xi_{j+1}}$ or $\sum_{x,y} \omega_{x,y}N_{x,y}(l)$. We will denote it $\int_l \omega$. This notation will be used even when ω is not antisymmetric. Note it is invariant if $\omega_{x,y}$ is replaced by $\omega_{x,y} + g(x) - g(y)$ for some g. Set $[G^\omega]^{x,y} = \frac{[(I-P^\omega)^{-1}]^x_y}{\lambda_y}$ and denote $\mathcal{Z}_{e,\omega}$ the determinant $\det(G^\omega)$. By an argument similar to the one given above for the occupation field, we have:

$\mathbb{P}^t_{x,x}(e^{i\int_l \omega} - 1) = \exp(t(P^\omega - I))_{x,x} - \exp(t(P - I))_{x,x}$. Integrating in t after expanding, we get from the definition of μ :

$$\int (e^{i\int_l \omega} - 1)d\mu(l) = \sum_{k=1}^{\infty} \frac{1}{k}[Tr((P^\omega)^k) - Tr((P)^k)]$$

Hence

$$\int (e^{i\int_l \omega} - 1) d\mu(l) = \log[\det(-L(I - P^\omega)^{-1}]$$

and

$$\mu(\exp(\sum_{l \in \mathcal{L}_\alpha} i \int_l \omega) - 1) = \log(\det(G^\omega G^{-1})) = \log(\frac{\mathcal{Z}_{e,\omega}}{\mathcal{Z}_e}) \tag{4}$$

The following result is suggested by an analogy with quantum field theory (Cf [3]).

Proposition 6 *i)* $\frac{\partial \mu}{\partial \kappa_x} = \widehat{l^x}\mu$

ii) $\frac{\partial \mu}{\partial \log C_{x,y}} = -T_{x,y}\mu$

with $T_{x,y}(l) = C_{x,y}(\widehat{l^x} + \widehat{l^y}) - N_{x,y}(l) - N_{y,x}(l)$

Note that the formula i) would a direct consequence of the Dynkin isomorphism if we considered only sets defined by the occupation field.

Recall that $\mu = \sum_{x \in X} e^{-\lambda_x \tau^*} \frac{d\tau^*}{\tau^*} + \sum_{p=2}^{\infty} \sum_{(\xi_i, i \in \mathbb{Z}/p\mathbb{Z}) \in X^p} \prod_{i \in \mathbb{Z}/p\mathbb{Z}} C_{\xi_i, \xi_{i+1}} e^{-\lambda_{\xi_i} \tau_i^*} d\tau_i^*$

$C_{x,y} = C_{y,x} = \lambda_x P_y^x$ and $\lambda_x = \kappa_x + \sum_y C_{x,y}$

The formulas follow by elementary calculation.

Recall that $\mu(\widehat{l^x}) = G^{x,x}$.and $\mu(N_{x,y}) = G^{x,y} C_{x,y}$

So we have $\mu(T_{x,y}) = C_{x,y}(G^{x,x} + G^{y,y} - 2G^{x,y})$

Then, the above proposition allows to compute all moments of T and $\widehat{l}$ relative to μ_e (Schwinger functions)

Consider now another energy form e' defining an equivalent norm on $\mathbb{H}$. Then we have the following identity:

$$\frac{\partial \mu_{e'}}{\partial \mu_e} = e^{\sum N_{x,y} \log(\frac{C'_{x,y}}{C_{x,y}}) - \sum (\lambda'_x - \lambda_x) \widehat{l^x}}$$

The above proposition is the infinitesimal form of this formula. Note that from the above expression of μ (**??**),

$$\mu_e((e^{\sum N_{x,y} \log(\frac{C'_{x,y}}{C_{x,y}}) - \sum (\lambda'_x - \lambda_x) \widehat{l^x}} - 1)) = \log(\frac{\mathcal{Z}_{e'}}{\mathcal{Z}_e})$$

(the proof goes by evaluating separately the contribution of trivial loops, which equals $\sum_x \log(\frac{\lambda_x}{\lambda'_x})$).

Note that if $C'_{x,y} = h^x h^y C_{x,y}$ et $\kappa'_x = \frac{-Lh}{h}\lambda$ for some positive function h on E such that $Lh \leq 0$, $\frac{\mathcal{Z}_{e'}}{\mathcal{Z}_e} = \frac{1}{\prod (h^x)^2}$.

Note also that $\frac{\mathcal{Z}_{e'}}{\mathcal{Z}_e} = \mathbb{E}(e^{-\frac{1}{2}[e'-e](\phi)})$

Equivalently

$$\mu_e(\prod_{(x,y)}[\frac{C'_{x,y}}{C_{x,y}}]^{N_{x,y}} \prod_x [\frac{\lambda_x}{\lambda'_x}]^{N_x+1} - 1) = \mu_e(\prod_{x,y}[\frac{P'^x_y}{P^x_y}]^{N_{x,y}} \prod_x [\frac{\lambda_x}{\lambda'_x}] - 1) = \log(\frac{\mathcal{Z}_{e'}}{\mathcal{Z}_e}) \tag{5}$$

and therefore

$$\mathbb{E}_{\mathbb{E}_{\mathcal{L}_\alpha}}(\prod_{(x,y)}[\frac{C'_{x,y}}{C_{x,y}}]^{N^{(\alpha)}_{x,y}} \prod_x [\frac{\lambda_x}{\lambda'_x}]^{N^{(\alpha)}_x+1}) = (\frac{\mathcal{Z}_{e'}}{\mathcal{Z}_e})^\alpha$$

Note also that $\prod_{(x,y)}[\frac{C'_{x,y}}{C_{x,y}}]^{N_{x,y}} = \prod_{\{x,y\}}[\frac{C'_{x,y}}{C_{x,y}}]^{N_{x,y}+N_{y,x}}$

N.B.: These $\frac{\mathcal{Z}_{e'}}{\mathcal{Z}_e}$ determine, when e' varies with $\frac{C'}{C} \leq 1$ and $\frac{\lambda'}{\lambda} = 1$, the Laplace transform of the distribution of the traversal numbers of non oriented links $N_{x,y} + N_{y,x}$, hence the loop distribution μ_e.

More generally

$$\mu_e(e^{-\sum N_{x,y}\log(\frac{C'_{x,y}}{C_{x,y}}) - \sum(\lambda'_x - \lambda_x)\widehat{l}_x + i\int_l \omega} - 1) = \log(\frac{\mathcal{Z}_{e',\omega}}{\mathcal{Z}_e}) \tag{6}$$

or

$$\mu_e(\prod_{x,y}[\frac{C'_{x,y}}{C_{x,y}} e^{i\omega_{x,y}}]^{N_{x,y}} \prod_x [\frac{\lambda_x}{\lambda'_x}]^{N_x+1} - 1) = \log(\frac{\mathcal{Z}_{e',\omega}}{\mathcal{Z}_e})$$

Note also that this last formula applies to the calculation of loop indices if we have for exemple a simple random walk on an oriented two dimensional lattice. In such cases, $\omega_{z'}$ can be chosen such that $\int_l \omega_{z'}$ is the winding number of the loop around a given point z' of the dual lattice[1] X'. Then $e^{i\pi\sum_l} \in \mathcal{L}_\alpha \int_l \omega'_z$ is a spin system of interest.

We then get for exemple that

$$\mu(\int_l \omega) \neq 0) = -\frac{1}{2\pi}\int_0^{2\pi} \log(\det(G^{2\pi u\omega}G^{-1}))du$$

[1]The construction of ω can be done as follows: Let P' be the uniform Markov transition probability on neighbouring points of the dual lattice and let h be a function such that $P'h = h$ except in z'. Then if the link xy in X intersects $x'y'$ in X', with $\det(x-y, x'-y') > 0$, set $\omega_{x,y} = h(y') - h(x')$

and hence

$$\mathbb{P}(\sum_{l\in\mathcal{L}_\alpha} | \int_l \omega'_z|) = 0) = e^{\frac{\alpha}{2\pi}\int_0^{2\pi}\log(\det(G^{2\pi u\omega}G^{-1}))du}$$

Conditional distributions of the occupation field with respect to values of the winding number can also be obtained.

We can apply the formula 5 to calculations concerning the links visited by the loops (similar to those done in section 4 for sites).

For exemple, R is a set of links, denote $e^{]R[}$ the energy form defined from e by setting all conductances in R to zero and increasing κ in such a way that λ is unchanged..

Then $\mu_e(\sum_{(x,y)\in R} N_{x,y} + N_{y,x} > 0) = -\log(\frac{\det(G^{]R[})}{\det(G)})$ and therefore, the probability no loop in $\mathcal{L}_\alpha$ visits R equals $\frac{\det(G^{]R[})}{\det(G)} = (\frac{\mathcal{Z}_{e^{]R[}}}{\mathcal{Z}_e})^\alpha$.

7 Self-avoiding paths and spanning trees.

Recall that link f is a pair of points (f^+, f^-) such that $C_f = C_{f^+,f^-} \neq 0$. Define $-f = (f^-, f^+)$.

Let $\mu^{\neq}_{x,y}$ be the measure induced by C on discrete self-avoiding paths.between x and y: $\mu^{x,y}_{\neq}(x, x_2, ..., x_{n-1}, y) = C_{x,x_2}C_{x_1,x_3}...C_{x_{n-1},y}$.

Another way to defined a measure on discrete self avoiding paths from x to y is loop erasure (see for exemple [4]). One checks easily the following:

Proposition 7 *the image of $\mu^{x,y}$ by the loop erasure map $\gamma \to \gamma^{BE}$ is $\mu^{x,y}_{BE}$ defined on self avoiding paths by $\mu^{x,y}_{BE}(\eta) = \mu^{x,y}_{\neq}(\eta)\frac{\det(G)}{\det(G^{\{\eta\}^c})} = \mu^{x,y}_{\neq}(\eta)\det(G_{|\{\eta\}\times\{\eta\}})$ (Here $\{\eta\}$ denotes the set of points in the path η)*

Proof: If $\eta = (x_1 = x, x_2, ...x_n = y)$,and $\eta_m = (x, ...x_m)$, $\mu^{x,y}(\gamma^{BE} = \eta) = V^x_x P^x_{x_2}[V^{\{x\}^c}]^{x_2}_{x_2}...[V^{\{\eta_{n-1}\}^c}]^{x_{n-1}}_{x_{n-1}} P^{x_{n-1}}_y [V^{\{\eta\}^c}]^y_y \lambda_y^{-1} = \mu^{x,y}_{\neq}(\eta)\frac{\det(G)}{\det(G^{\{\eta\}^c})}$ as $[V^{\{\eta_{m-1}\}^c}]^{x_m}_{x_m} = \frac{\det([(I-P]|_{\{\eta_m\}^c\times\{\eta_m\}^c})}{\det([(I-P]|_{\{\eta_{m-1}\}^c\times\{\eta_{m-1}\}^c})} = \frac{\det(V^{\{\eta_{m-1}\}^c})}{\det(V^{\{\eta_m\}^c})} = \frac{\det(G^{\{\eta_{m-1}\}^c})}{\det(G^{\{\eta_m\}^c})}\lambda^{x_m\cdot}$ for all $m \leq n-1$.

Also:$\int e^{-<\widehat{\gamma},\chi>}1_{\{\gamma^{BE}=\eta\}}\mu^{x,y}(d\gamma) = \frac{\det(G_\chi)}{\det(G_\chi^{\{\eta\}^c})}e^{-<\widehat{\eta},\chi>}\mu^{x,y}_{\neq}(\eta)$

$= \det(G_\chi)_{|\{\eta\}\times\{\eta\}}e^{-<\widehat{\eta},\chi>}\mu^{x,y}_{\neq}(\eta) = \frac{\det(G_\chi)_{|\{\eta\}\times\{\eta\}}}{\det(G_{|\{\eta\}\times\{\eta\}})}e^{-<\widehat{\eta},\chi>}\mu^{x,y}_{BE}(\eta)$ for any self-avoiding path η.

Therefore, under $\mu^{x,y}$, the conditional distribution of $\widehat{\gamma}-\widehat{\eta}$ given $\gamma^{BE}=\eta$ is the distribution of $\widehat{\mathcal{L}_1}-\widehat{\mathcal{L}_1^{\{\eta\}^c}}$ i.e. the occupation field of the loops of $\mathcal{L}_1$ which intersect η.

More generally, it can be shown that

Proposition 8 *the conditional distribution of the set $\mathcal{L}_\gamma$ of loops of γ given $\gamma^{BE}=\eta$ is the distribution of $\mathcal{L}_1/\mathcal{L}_1^{\{\eta\}^c}$ i.e. the loops of $\mathcal{L}_1$ which intersect η.*

Proof: First an elementary calculation shows that
$\mu_{e'}^{x,y}(\gamma^{BE}=\eta)=\frac{C'_{x,x_2}C'_{x_1,x_3}...C'_{x_{n-1},y}}{C_{x,x_2}C_{x_1,x_3}...C_{x_{n-1},y}}\mu_e^{x,y}(\prod_{u\neq v}[\frac{C'_{u,v}}{C_{u,v}}]^{N_{x,y}(\mathcal{L}_\gamma)+N_{y,x}(\mathcal{L}_\gamma)}\prod_u[\frac{\lambda_u}{\lambda'_u}]^{N_u(\mathcal{L}_\gamma)}1_{\{\gamma^{BE}=\eta\}})$
Therefore, by the previous proposition,
$\mu_e^{x,y}(\prod_{u\neq v}[\frac{C'_{u,v}}{C_{u,v}}]^{N_{x,y}(\mathcal{L})+N_{y,x}(\mathcal{L})}\prod_u[\frac{\lambda_u}{\lambda'_u}]^{N_u(\mathcal{L})}|\gamma^{BE}=\eta)=\frac{\mathcal{Z}_e\mathcal{Z}_{[e']\{\eta\}^c}}{\mathcal{Z}_{e\{\eta\}^c}\mathcal{Z}_{e'}}$.
Moreover, by 5 and the properties of the Poisson processes,
$\mathbb{E}(\prod_{u\neq v}[\frac{C'_{u,v}}{C_{u,v}}]^{N_{x,y}(\mathcal{L}_1/\mathcal{L}_1^{\{\eta\}^c})+N_{y,x}(\mathcal{L}_1/\mathcal{L}_1^{\{\eta\}^c})}\prod_u[\frac{\lambda_u}{\lambda'_u}]^{N_u(\mathcal{L}_1/\mathcal{L}_1^{\{\eta\}^c})}=\frac{\mathcal{Z}_e\mathcal{Z}_{[e']\{\eta\}^c}}{\mathcal{Z}_{e\{\eta\}^c}\mathcal{Z}_{e'}}$
It follows that the distributions of the $N_{x,y}+N_{y,x}$'s are identical for the set of erased loops and $\mathcal{L}_1/\mathcal{L}_1^{\{\eta\}^c}$. Moreover, remark 2 allows to conclude, since the same conditional equidistribution property holds for the configurations of erased loops.

Similarly one can define the image of $\mathbb{P}^x$ by BE which is given by
$\mathbb{P}^x_{BE}(\eta)=C_{x_1,x_2}...C_{x_{n-1},x_n}\kappa_{x_n}\det(G_{|\{\eta\}\times\{\eta\}})$, for $\eta=(x_1,...,x_n)$, and get the same results.

Wilson's algorithm (see [9]) iterates this construction, starting with $x's$ in arbitrary order. Each step of the algorithm reproduces the first step except it stops when it hits the already constructed tree of self avoiding paths. It provides a construction of the probability measure $\mathbb{P}^e_{ST}$ on the set $ST_{X,\Delta}$ of spanning trees of X rooted at the cemetery point Δ defined by the energy e. The weight attached to each oriented link $\xi=(x,y)$ of $X\times X$ is the conductance and the weight attached to the link (x,Δ) is κ_x. As the determinants simplify, the probability of a tree Υ is given by the simple formula

$$\mathbb{P}^e_{ST}(\Upsilon)=\mathcal{Z}_e\prod_{\xi\in\Upsilon}C_\xi$$

Proposition 9 *The random set of discrete loops $\mathcal{L}_W$ constructed in this algorithm is independent of the random spanning tree, and independent of the ordering. It has the same Poisson distribution as the non trivial discrete loops of $\mathcal{L}_1$.*

It follows easily from proposition 8 .

Together with the spanning tree these discrete loops define an interesting random graph.

First note that, since we get a probability

$$\mathcal{Z}_e \sum_{\Upsilon \in ST_{X,\Delta}} \prod_{(x,y)\in\Upsilon} C_{x,y} \prod_{x,(x,\Delta)\in\Upsilon} \kappa_x = 1$$

or equivalently

$$\sum_{\Upsilon \in ST_{X,\Delta}} \prod_{(x,y)\in\Upsilon} P^x_y \prod_{x,(x,\Delta)\in\Upsilon} P^x_\Delta = \frac{1}{\prod_{x\in X}\lambda_x \mathcal{Z}_e}$$

so that

$$\mathbb{P}^e_{ST}(\Upsilon) = \mathcal{Z}_e \prod_{x\in X}\lambda_x \prod_{(x,y)\in\Upsilon} P^x_y \prod_{x,(x,\Delta)\in\Upsilon} P^x_\Delta$$

Then, it comes that, for any e',

$$\mathbb{E}^e_{ST}(\prod_{(x,y)\in\Upsilon} \frac{P'^x_y}{P^x_y} \prod_{x,(x,\Delta)\in\Upsilon} \frac{P'^x_\Delta}{P^x_\Delta}) = \frac{\prod_{x\in X}\lambda_x}{\prod_{x\in X}\lambda'_x}\frac{\mathcal{Z}_e}{\mathcal{Z}_{e'}}$$

and

$$\mathbb{E}^e_{ST}(\prod_{(x,y)\in\Upsilon} \frac{C'_{x,y}}{C_{x,y}} \prod_{x,(x,\Delta)\in\Upsilon} \frac{\kappa'_x}{\kappa_x}) = \frac{\mathcal{Z}_e}{\mathcal{Z}_{e'}} \tag{7}$$

We also have $\mathbb{P}^e_{ST}((x,y)\in\Upsilon) = \mathbb{P}^{BE}_x(\eta_1 = y) = V^x_x P^x_y \mathbb{P}^y(T_x = \infty) = C_{x,y}G^{x,x}(1-\frac{G^{x,y}}{G^{x,x}})$

From the results exposed in [8] and [9], or directly from the above, we recover Kirchhoff's theorem:

$\mathbb{P}^e_{ST}(\pm(x,y)\in\Upsilon) = C_{x,y}[G^{x,x}(1-\frac{G^{x,y}}{G^{x,x}}) + G^{y,y}(1-\frac{G_{x,y}}{G_{y,y}})] = C_{x,y}(G^{x,x} + G^{y,y} - 2G^{x,y}) = C_{x,y}K^{x,y),(x,y)}$ and more generally Pemantle's transfer current theorem:

$$\mathbb{P}^e_{ST}(\pm\xi_1,...\pm\xi_k\in\Upsilon)=(\prod_1^k C_{\xi_i})\det(K^{\xi_i,\xi_j}\ 1\le i,j\le k)$$

Note this determinant does not depend on the orientation of the links.

Proof: We use recurrence on k. Let M denote the smallest subset of X^Δ containing the links $\pm\xi_1,...\pm\xi_k$ and denote $E-M$ by D Let V be the subspace of $\mathbb{A}$ spanned by all $K^{(x,y)}$ with x and y in M_k. Note that the orthogonal of V in $\mathbb{A}$ is spanned by $dG^D(\delta_x)$ and that for any $\eta=(u,v)$ the projection of K^η on $V^\perp$ is $dG^D(\delta_v-\delta_u)$, and $\left\langle dG^D(\delta_v-\delta_u), dG^D(\delta_v-\delta_u)\right\rangle = [G^D]^{u,u}+[G^D]^{v,v}-2[G^D]^{u,v}$

Moreover $\det(K^{\xi_i,\xi_j}\ 1\le i,j\le k)=\left\|K^{\xi_1}\wedge...\wedge K^{\xi_k}\right\|^2_{\wedge^k\mathbb{A}}$. Therefore, if $\xi_{k+1}=\eta$ $\det(K^{\xi_i,\xi_j}\ 1\le i,j\le k+1)=\det(K^{\xi_i,\xi_j}\ 1\le i,j\le k)([G^D]^{u,u}+[G^D]^{v,v}-2[G^D]^{u,v})$

But the argument given for $k=1$ shows also that $\mathbb{P}^e_{ST}(\pm\eta\in\Upsilon|\pm\xi_1,...\pm\xi_k\in\Upsilon)=C_{u,v}([G^D]^{u,u}+[G^D]^{v,v}-2[G^D]^{u,v})$ so we can conclude.

Therefore, given any function g on non oriented links, $\mathbb{E}^e_{ST}(e^{-\sum_{\xi\in\Upsilon}g(\xi)})=\mathbb{E}^e_{ST}(\prod_\xi(1+(e^{-g(\xi)}-1)1_{\xi\in\Upsilon})=\sum Tr((M_{C(e^{-g}-1)}K)^{\wedge k})$ and we have

$$\mathbb{E}^e_{ST}(e^{-\sum_{\xi\in\Upsilon}g(\xi)})=\det(I+KM_{C(e^{-g}-1)})=\det(I-M_{\sqrt{C(1-e^{-g})}}KM_{\sqrt{C(1-e^{-g})}})$$

This is an exemple of the Fermi point processes discussed in [14].

But, by (7) and (5), it comes that

$$\log(\mathbb{E}^e_{ST}(\prod_{(x,y)\in\Upsilon}\frac{C'_{x,y}}{C_{x,y}}\prod_{x,(x,\Delta)\in\Upsilon}\frac{\kappa'_x}{\kappa_x})=-\mu_e(\prod_{x,y}[\frac{C'_{x,y}}{C_{x,y}}]^{N_{x,y}}\prod_x[\frac{\lambda_x}{\lambda'_x}]^{N_x+1}-1)=\log(\frac{\mathcal{Z}_e}{\mathcal{Z}_{e'}})$$

The first identity could also be derived from proposition.9.

As $\frac{\lambda_x}{\lambda'_x}=\frac{1}{\sum P^x_y\frac{C'_{x,y}}{C_{x,y}}+P^x_\Delta\frac{\kappa'_x}{\kappa_x}}$, (with the convention $\frac{P^x_\Delta}{\kappa_x}=\lambda_x^{-1}$ if $\kappa_x=0$) we obtain

For any function g on non oriented link of X_Δ, non negative on links of X

$$\mathbb{E}_{\mathcal{L}_\alpha}(e^{-\sum_{x,y}g(\{x,y\})N_{x,y}}\prod_x[\sum_{z\in X^\Delta}P^x_z e^{-g\{x,z\}}]^{-N_x-1})=\det(I-KM_{C(1-e^{-g})})^{-\alpha}$$

We can check that this formula allows to recover the identity $\mathbb{E}_{\mathcal{L}_\alpha}(\frac{N_x+N_y+2}{\lambda_x} - N_{x,y} - N_{y,x}) = \alpha C_{x,y}(G^{x,x} + G^{y,y} - 2G^{x,y})$ It also gives back prpositin 3 for $g = 0$ on $X \times X$.

If κ is positive everywhere, we can adjust $g(\{x, \Delta\})$ to make $\sum_{z\in X^\Delta} P^x_z e^{-g\{x,z\}} = 1$ This means we have to choose $\kappa_x(1-e^{-g(\{x,\Delta\})}) = \kappa_x(1-\frac{\lambda_x}{\kappa_x}(1-\sum_{z\in X} P^x_z e^{-g(\{x,z\})}))$

$= (\kappa_x - \lambda_x + . \sum_{z\in X} C_{xz} e^{-g(\{x,z\})}) = \sum_{z\in X} C_{xz}(e^{-g(\{x,z\})} - 1)$

We check also also that by 5

$\mathbb{E}_{\mathcal{L}_\alpha}(e^{-\sum_{x,y} g(\{x,y\})N_{x,y}}) = (\frac{\mathcal{Z}_{e'}}{\mathcal{Z}_e})^\alpha = (\frac{\det(M_\lambda - Ce^{-g})}{\det(M_\lambda - C)})^{-a}$ Finally, the restriction on κ can be removed by taking a limit and we obtain:

Proposition 10 *For any function g on non oriented link of X_Δ, non negative on links of X, set $Tg(\xi) = C_\xi(1 - e^{-g(\xi)})$ if ξ is a link of X and $Tg(\{x, \Delta\}) = \sum_{z\in X} C_{xz}(e^{-g(\{x,z\})} - 1)$ for all x. Then*

$$\det(I + C(I - [e^{-g}]))^{-\alpha} = \mathbb{E}_{\mathcal{L}_\alpha}(e^{-\sum_{x,y} g(\{x,y\})N_{x,y}}) = \det(I - KM(g))^{-\alpha}$$

We see that the Poisson measure on loops $\mathcal{L}_\alpha$ induces a point process N on the space of non oriented links defined by the pair (α, K) which reminds the point processes discussed in [14]. Note however a difference of sign in the right hand side determinant, which is not a Laplace transform for positive α.

8 Fock spaces and Wick product

Recall that the Gaussian space $\mathcal{H}$ spanned by $\{\phi^x, x \in X\}$ is isomorphic to $\mathbb{H}$ by the linear map mapping $\mathrm{Re}(\phi_x)$ on $G_{x,\cdot}$ which extends into an isomorphism between the space of square integrable functionals of the Gaussian fields and the symmetric Fock space obtained as the closure of the sum of all symmetric tensor powers of $\mathbb{H}$ (Bose second quantization). We have seen that L^2 functionals of $\widehat{\mathcal{L}_1}$ can be represented in this symmetric Fock space.

In order to prepare the extension of these isomorphisms to a more interesting framework (including especially the planar Brownian motion considered in [5]) we shall introduce the renormalized (or Wick) powers of $\phi\overline{\phi}$.

The Laguerre polynomials $L^0_n(x)$ are defined by their generating function $\frac{e^{-\frac{xt}{1-t}}}{1-t} = \sum t^n L^0_n(x)$

Then one defines the polynomial $P_n(\cdot) = (-1)^n n! L^0_n(\cdot)$

Setting $\sigma_x = G_{x,x}$, it comes that $\sigma^n P_n(\frac{\phi\overline{\phi}}{\sigma})$ is the inverse image of a $2n$-th tensor in the Fock space denoted $:(\phi\overline{\phi})^n:$.Note that $:(\phi\overline{\phi}):=(\phi\overline{\phi})-2\sigma$ These variables are orthogonal in L^2. Set $\widetilde{l^x} = \widehat{l^x} - \sigma_x$ be the centered occupation field. Note that an equivalent formulation of proposition 5 is that the fields $\frac{1}{2} :\phi\overline{\phi}:$ and $\widetilde{\mathcal{L}_1}$ have the same law.

Let us now consider the relation of higher Wick powers with self intersection local times.

9 Decompositions

If $D \subset X$ and we set $F = D^c$, the orthogonal decomposition of the Dirichlet norm $e(f)$ into $e^D(f - H^F f) + e(H^F f)$ (cf [6] and references) leads to the decomposition of the Gaussian field mentionned above and also to a decomposition of the Markov chain into the Markov chain killed at the exit of D and the trace of the Markov chain on F.

Proposition 11 *The trace of the Markov chain on F is defined by the Dirichlet norm $e^{\{F\}}(f) = e(H^F f)$, for which*

$$C^{\{F\}}_{x,y} = C_{x,y} + \sum_{a,b\in D} C_{x,a}C_{b,y}[G^D]^{a,b}$$

$$\lambda^{\{F\}}_x = \lambda_x - \sum_{a,b\in D} C_{x,a}C_{b,x}[G^D]^{a,b}$$

and

$$\mathcal{Z}_e = \mathcal{Z}_{e^D}\mathcal{Z}_{e^{\{F\}}}$$

Proof: The first assertion is well known. For the second, note first that for any $y \in F$, $[H^F]^x_y = 1_{x=y} + 1_D(x)\sum_{b\in D}[G^D]^{x,b}C_{b,y}$. Moreover, $e(H^F f) = \left\langle f, H^F f\right\rangle_e$ and therefore

$\lambda^{\{F\}}_x = e^{\{F\}}(1_{\{x\}}) = e(1_{\{x\}}, H^F 1_{\{x\}}) = \lambda_x - \sum_{a\in D} C_{x,a}[H^F]^a_x = \lambda_x - \sum_{a,b\in D} C_{x,a}C_{b,x}[G^D]^{a,b}$.

Then for distinct x and y in F,

$C^{\{F\}}_{x,y} = -\left\langle 1_{\{x\}}, 1_{\{y\}}\right\rangle_{e^{\{F\}}} = -\left\langle 1_{\{x\}}, H^F 1_{\{y\}}\right\rangle_e = C_{x,y} + \sum_a C_{x,a}[H^F]^a_y = C_{x,y} + \sum_{a,b\in D} C_{x,a}C_{b,y}[G^D]^{a,b}$.

Finally, note also that $G^{\{F\}}$ is the restriction of G to F. as for all $x, y \in F$, $\left\langle G^{\cdot,y}, 1_{\{x\}}\right\rangle_{e^{\{F\}}} = \left\langle G^{\cdot,y}, [H^F]^{\cdot}_x\right\rangle_e = 1_{\{x=y\}}$. Hence the determinant decomposition already used in yields the final formula.

The cases where F has one point was already treated in section 3-2.

The transition matrix $[P^{\{F\}}]_y^x$ can also be computed directly and equals $P_y^x + \sum_{a,b\in D} P_a^x P_y^b V^{D\cup\{x\}}]_b^a = P_y^x + \sum_{a,b\in D} P_a^x C_{b,y}[G^{D\cup\{x\}}]^{a,b}$. The calculation of $\frac{C_{x,y}^{\{F\}}}{\lambda_x^{\{F\}}}$ yields a decomposition in two parts according whether the jump to y occurs from x or from D.

If we set $e_\chi = e + \|\|_{L^2(\chi)}$ and denote $[e_\chi]^{\{F\}}$ by $e^{\{F,\chi\}}$ we have

$$C_{x,y}^{\{F,\chi\}} = C_{x,y} + \sum_{a,b} C_{x,a}C_{b,y}[G_\chi^D]^{a,b}$$

and

$$\lambda_x^{\{F,\chi\}} = \lambda_x - \sum_{a,b} C_{x,a}C_{b,x}[G_\chi^D]^{a,b})$$

More generally, if $e^\#$ is such that $C^\# = C$ on $F\times F$, and $\lambda = \lambda^\#$ on F we have:

$$C_{x,y}^{\#\{F\}} = C_{x,y} + \sum_{a,b} C'_{x,a}C'_{b,y}[G_\chi'^D]^{a,b}$$

and

$$\lambda_x^{\#\{F\}} = \lambda_x - \sum_{a,b} C'_{x,a}C'_{b,x}[G_\chi'^D]^{a,b})$$

A loop in X which hits F can be decomposed into a loop $l^{\{F\}}$ in F and its excursions in D which may come back to their starting point.

Set $\nu_{x,y}^D = C_{x,y}\delta_\emptyset + \sum_{a,b\in D} C_{x,a}C_{b,y}\mu_D^{a,b}$ and $\nu_D^x = \lambda_x^{-1}\delta_\emptyset + \sum_{n=1}^\infty \lambda_x^{-n}(\sum_{a,b\in D} C_{x,a}C_{b,x}\mu_D^{a,b})^{\otimes n}$. Here $\mu_D^{a,b}$ denotes the bridge measure (with mass $[G^D]^{a,b}$ associated with e^D. Note that $\nu_{x,y}^D(1) = C_{x,y}^{\{F\}}$ and $\nu_x^D(1) = \frac{1}{\lambda_x^{\{F\}}}$. We get a decomposition of μ into its restriction μ^D to loops in D (associated to the process killed at the exit of D), a loop measure $\mu^{\{F\}}$ defined on loops of F by the trace of the Markov chain on F, measures $\nu_{x,y}^D$ on excursions in D indexed by pairs of points in F and measures ν_x^D on finite sequences of excursions in D indexed by points of F.

Conversely, a loop $l^{\{F\}}$ of points ξ_i in F (possibly reduced to a point), a family of excursions $\gamma_{\xi_i,\xi_{i+1}}$ attached to the jumps of $l^{\{F\}}$ and systems of i.i.d. excursions , γ_{ξ_i} attached to the points of $l^{\{F\}}$ defines a loop $\Lambda(l^{\{F\}}, (\gamma_{\xi_i,\xi_{i+1}}), (\gamma_{\xi_i}))$.

Note excursions can be empty. Then $\mu - \mu^D$ is the image measure by Λ of $\mu^{\{F\}}(dl^{\{F\}}) \prod(\widetilde{\nu}^D_{\xi_i,\xi_{i+1}})(d\gamma_{\xi_i,\xi_{i+1}}) \prod \widetilde{\nu}^D_{\xi_i}(d\gamma_{\xi_i})$, denoting $\widetilde{\nu}$ the normalised measures $\frac{\nu}{\nu(1)}$.

The Poisson process $\mathcal{L}^{\{F\}}_\alpha = \{l^{\{F\}}, l \in \mathcal{L}_\alpha\}$ has intensity $\mu^{\{F\}}$ and is independent of $\mathcal{L}^D_\alpha$.

In particular, if χ is a measure carried by D, we have:

$$E(e^{-\langle \widehat{\mathcal{L}_\alpha}, \chi\rangle} | \mathcal{L}^{\{F\}}_\alpha) = [\frac{\mathcal{Z}_{e^D_\chi}}{\mathcal{Z}_{e^D}}]^\alpha \prod_{l^{\{F\}} \in \mathcal{L}^{\{F\}}_\alpha} (\prod_{x,y \in F} [\frac{C^{\{F,\chi\}}_{x,y}}{C^{\{F\}}_{x,y}}]^{N_{x,y}} \prod_{x \in F} [\frac{\lambda^{\{F\}}_x}{\lambda^{\{F,\chi\}}_x}]^{N_x+1})$$

More generally

Proposition 12 *if $C^\# = C$ on $F \times F$, and $\lambda = \lambda^\#$ on F*
$\mathbb{E}_{\mathcal{L}_\alpha}(\prod_{x,y \notin F \times F} [\frac{C^\#_{x,y}}{C_{x,y}}]^{N_{x,y}} e^{-\sum_{x \in D} \widehat{l}_x(\lambda^\#_x - \lambda_x)} | \mathcal{L}^{\{F\}}_\alpha) =$
$[\frac{\mathcal{Z}_{e^{\#D}}}{\mathcal{Z}_{e^D}}]^\alpha \prod_{l^{\{F\}} \in \mathcal{L}^{\{F\}}_\alpha} (\prod_{x,y \in F} [\frac{C^{\#\{F\}}_{x,y}}{C^{\{F\}}_{x,y}}]^{N_{x,y}} \prod_{x \in F} [\frac{\lambda^{\{F\}}_x}{\lambda'^{\{F\}}_x}]^{N_x+1})$

The proof can be done by decomposing all e' into $e^\# + (e' - e^\#)$, with $(e' - e^\#)$ carried by$F$

These decomposition formulas extend to include a current ω provided it is closed (i.e. vanish on every loop) in D. In particular, it allows to define ω^F such that:

$$\mathcal{Z}_{e,\omega} = \mathcal{Z}_{e^D} \mathcal{Z}_{e^{\{F\}},\omega^F}$$

10 Reflection positivity and Hilbert space

Let us fix α. In view of physical applications, it is appropriate to assume that X is the union of two parts $X^\pm$ exchanged by an involution ρ under which e is invariant. Each configuration $\mathcal{L}_\alpha$ of loops induces a configuration Λ of loops in $X^0 = X^+ \cap X^-$. Given a function F on loops configuration in X^+, it follows from the previous proposition the following

Corollary 13 $E(F(\mathcal{L}_\alpha|_{X^+})|\Lambda) = E(F \circ \rho(\mathcal{L}_\alpha|_{X^-})|\Lambda)$

so that the reflection positivity (also called physical positivity) property holds:

$$\|F\| = E(F(\overline{F} \circ \rho)) \geq 0.$$

The physical space is the quotient space modulo functionals of zero norm. It identifies with L^2 functionals of Λ. Osterwalder-Schrader-type construction can be used to produce non commuting field observables. More precisely, after extending the framework to infinite spaces (see section below), one can assume for exemple X has a product structure $S \times \mathbb{Z}$ and that the time translation τ and the time reversal ρ leave e invariant. Then τ induces a self adjoint contraction T of the physical space, hence a Hamiltonian $\log(T)$ and by complex exponentiation, a unitary dynamic U. Non commuting observables are obtained by conjugation of an observable by the opeators U^n. This extends the construction of the relativistic non commuting quantum free field observables out of the Euclidean Gaussian field.

11 The case of general Markov processes

We now explain briefly how some of the above results will be extended to a symmetric Markov process on an infinite space X. The construction of the loop measure as well as a lot of computations can be performed quite generally, using Dirichlet space theory Let us consider more closely the occupation field $\widehat{l}$. The extension is rather straightforward when points are not polar. We can start with a Dirichlet space of continuous functions and a measure m such that there is a mass gap. Let P_t the associated Feller semigroup. Then the Green function is well defined as the mutual energy of the Dirac measures δ_x and δ_y which have finite energy. It is the covariance function of a Gaussian Markov field ϕ^x, which will be associated to the field $\widehat{l}$ of local times.of the Poisson process of random loops whose intensity is given by the loop measure defined by the semigroup P_t. More precisely, Propositions 1 and 5 still hold (χ being defined as a Radon measure with compact support on X) as long as the continuous Green function G will be locally trace class. This will apply to exemples related to one dimensional Brownian motion or to Markov chains on countable spaces.

When points are polar, one needs to be more careful. We will consider only the case of the two and three dimensional Brownian motion in a bounded domain killed at the boundary, i.e. associated with the classical energy with Dirichlet boundary condition. The Green function is not locally

trace class but it is still Hilbert-Schmidt which allows to define renormalized determinants $\det_2$ (Cf [12]) and to extend the statement of proposition 5 to the centered occupation field and the Wick square $:\phi\overline{\phi}:$ of the generalized Gaussian Markov field ϕ. These three generalized fields are not defined pointwise but have to be smeared by measures of finite energy χ such that $\int G^{x,y}\chi(dx)\chi(dy) < \infty$. The centered occupation field $\widetilde{l}$ is defined as follows: Let A_t^χ be the additive functional associated with χ of finite energy. Then $\left\langle \widetilde{l}, \chi \right\rangle$ is defined as $\lim_{\varepsilon\downarrow 0} \int_0^{(T-\varepsilon)^+} dA_t^\chi - \mu_0(\int_0^{(T-\varepsilon)^+} dA_t^\chi)$ which converges in $L^2(\mu_0)$. It is an intrinsic quantity.. We then have

Proposition 14 *a) The centered occupation field $\widetilde{\mathcal{L}_1}$ and the Wick square $\frac{1}{2}:\phi\overline{\phi}:$ have the same distribution.*

b) $E(e^{-\left\langle \widetilde{\mathcal{L}_\alpha},\chi\right\rangle}) = \det_2(G_\chi G^{-1})^\alpha$

To justify the use of $\det_2$, note that in the finite case $\det(I + GM_\chi) = \det_2(I + GM_\chi)e^{\chi(\sigma)}$ where we recall that σ_x^{-1} is the capacity of x, which vanishes now since x is polar.

In two dimensions, higher Wick powers of $\phi\overline{\phi}$ are associated with self intersection local times of the loops.

Let us now consider currents. We will restrict our attention to the two dimensional Brownian case, X being an open subset of the plane. Currents can be defined by divergence free vector fields, with compact support. Then $\int_l \omega$ and $\int_X(\overline{\phi}\partial_\omega\phi - \phi\partial_\omega\overline{\phi})dx$ are well defined square integrable variables (it can be checked easily in the case of the square by Fourier series). The distribution of the centered occupation field of the loop process "twisted" by the complex exponential $\exp(\sum_{l\in\mathcal{L}_\alpha}\int_l i\omega + \frac{1}{2}\widehat{l}(\|\omega\|^2))$ appears to be the same as the distribution of $:\phi\overline{\phi}:$ "twisted" by the complex exponential $\exp(\int_X(\overline{\phi}\partial_\omega\phi - \phi\partial_\omega\overline{\phi})dx)$ (Cf[10]).

These points, among others, will be developped in a forthcoming article.

References

[1] Dynkin, E.B. Local times and Quantum fields. Seminar on Stochastic processes, Gainesville 1982. 69-84 Progr. Prob. Statist. 7 Birkhauser. (1984).

[2] Eisenbaum, N. Kaspi, H. A characterization of the infinitely divisible squared Gaussian processes. Ann. Prob. 34 728-742 (2006).

[3] Gawedzki, K. Conformal field theory. Lecture notes. I.A.S. Princeton.

[4] Lawler, G. Loop erased random walks. H. Kesten Festshrift: Perplexing problems in probability. Progr.Prob. 44 197-217 Birkhaüser (1999)

[5] Lawler, G. , Werner, W. The Brownian loop soup. PTRF 128 565-588 (2004)

[6] Le Jan, Y. Mesures associées à une forme de Dirichlet. Applications. Bull. Soc. Math. Fr. 106 61-112 (1978).

[7] Le Jan, Y. On the Fock space representation of functionals of the occupation field and their renormalization. J;F.A. 80, 88-108 (1988)

[8] Lyons, R. Determinantal Probability Measures. Publ. Math. Inst. Hautes Etudes Sci. 98, 167-212 (2003)

[9] Lyons, R., Peres Y. Probability on trees and networks. Prepublication.

[10] Le Jan, Y. Dynkin's isomorphism without symmetry. Math. PR/O610571

[11] Marcus M.B. , Rosen J. Sample path properties of the local times of strongly symmetric Markov processes via Gaussian processes. Ann. Prob. 20, 1603-1684 (1992)

[12] Simon,B. Trace ideals and their applications. London Math Soc Lect. Notes 35 Cambridge (1979)

[13] Schramm, O. , Sheffield, S. Contour lines of the two dimensional discrete Gaussian free field. Math. PR/0605337

[14] Shirai, T. , Takahashi, Y. Random point fields associated with certain Fredholm determinants I: fermion, Poisson ans boson point processes. J. Functional Analysis 205 414-463 (2003)

[15] Vere Jones, D. A generalization of permanents and determinants. Linear Algebra and Appl. 111 (1988)

[16] Vere Jones, D. Alpha permanents and their applications. New Zeland J. Math. 26 125-149 (1997)

[17] Werner, W. The conformally invariant measure on self-avoiding loops. Math. PR/0511605