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MARKOV LOOPS, DETERMINANTS AND GAUSSIAN FIELDS

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1 Introduction

The purpose of this note is to explore some simple relations between loop measures, determinants, and Gaussian Markov fields. These relations are related to Dynkin's isomorphism (cf[1], [8], [5]) . Their potential interest could be suggested by noting that loop measures were defined in [3] for planar Brownian motion and are related to SLE processes (see also [11]). It is also the case for the free field as shown in [10]. We present the results first in the elementary framework of symmetric Markov chains on a finite space, and then indicate how they can be extended to more general Markov processes such as the two dimensional Brownian motion.

2 Symmetric Markov processes on finite spaces

Let us first consider for simplicity the case of a m -symmetric Markov chain with exponential holding times on a finite space X , with generator $L = q(P - I)$, P being a mq -symmetric stochastic transition matrix: $m_x q_x P_{x,y} = m_y q_y P_{y,x}$ with $P_{x,x} = 0$ for all x in X .

We denote P_t the semigroup $\exp(Lt) = \sum e^{-t \frac{t^k}{k!}} L^k$.

Recall that for any complex function z on X , the "energy"

$$e(z) = \langle -Lz, \bar{z} \rangle_m = \sum_{x \in X} -(Lz)_x \bar{z}_x m_x$$

. is nonnegative as it can be written

$$\frac{1}{2} \sum_{x,y} C_{x,y} (z_x - z_y) (\bar{z}_x - \bar{z}_y) + \sum_x \kappa_x z_x \bar{z}_x$$

with $C_{x,y} = C_{y,x} = m_x q_x P_{x,y}$ and $\kappa_x = m_x q_x (1 - \sum_y P_{x,y})$. The (complex) Dirichlet space \mathbb{H} is the space of complex functions equipped with the energy scalar product. The lowest eigenvector of $-L$ is nonnegative by the well known argument which shows that the module contraction $z \rightarrow |z|$ lowers the energy. We will assume (although it is not always necessary) the corresponding eigenvalue is positive which means there is a "mass gap": For some positive ε , the energy $e(z)$ dominates $\varepsilon \langle z, \bar{z} \rangle_m$ for all z . We denote by V the associated potential $(-L)^{-1} = \int_0^\infty P_t dt$ and by G the Green function $G_{x,y} = \frac{V_{x,y}}{m_y}$. It verifies $\langle f, G\mu \rangle_{\mathbb{H}} = \langle f, \mu \rangle$ for all function f and measure μ .

Different Markov chains associated to the same energy are equivalent under time change. If g is a positive function on X , in the new time scale $\int_0^t g_{\xi_s} ds$, we obtain a Markov chain with gm -symmetric generator $\frac{1}{g}L$. Objects invariant under time change are called *intrinsic*. The energy e , P and the Green function G are obviously intrinsic but L , V and P_t are not.

3 Loop measures

For any integer k , let us define a based loop with k jumps in X as a couple $(\xi, \tau) = ((\xi_m, 0 \leq m \leq k), (\tau_m, 1 \leq m \leq k))$ in $X^{k+1} \times \mathbb{R}_+^k$, such that $\xi_0 = \xi_k$. k will be denoted $p(\xi)$.

Based loops have a natural time parametrisation ξ_t and a time period $T(\xi) = \sum_{i=1}^{p(\xi)} \tau_i$. If we denote $\sum_{i=1}^m \tau_i$ by T_m : $\xi_t = \xi_{m-1}$ on $[T_{m-1}, T_m)$ (with $T_0 = 0$).

A σ -finite measure μ_0 is defined on based loops by

$$\mu_0 = \sum_{x \in X} \int_0^\infty \frac{1}{t} \mathbb{P}_{x,x}^t dt$$

where \mathbb{P}_{xx}^t denotes the (non normalized) "law" of an excursion from x of duration t : If $\sum_{i=1}^{h+1} t_i = t$,

$$\mathbb{P}_{xx}^t(\xi_{t_1} = x_1, \dots, \xi_{t_h} = x_h) = P_{t_1}(x, x_1) P_{t_2}(x_1, x_2) \dots P_{t_{h+1}}(x_h, x)$$

Note also that

$$\begin{aligned} \mathbb{P}_{xx}^t(p = k, \xi_1 = x_1, \dots, \xi_{k-1} = x_{k-1}, T_1 \in dt_1, \dots, T_k \in dt_k) \\ = P_{x, x_1} P_{x_1, x_2} \dots P_{x_{k-1}, x} \mathbf{1}_{\{0 < t_1 < \dots < t_k < t\}} q_x e^{-q_x t_1} \dots q_{x_{k-1}} e^{-q_{x_{k-1}}(t_k - t_{k-1})} e^{-q_x(t - t_k)} dt_1 \dots dt_k \end{aligned}$$

A loop is defined as an equivalence class of based loops for the \mathbb{R} -shift that acts naturally. μ_0 is shift invariant, It induces a measure μ on loops.

Note also that the measure $d\tilde{\mu}_0 = \frac{T q_{\xi_0}}{\int_0^T q_{\xi_s} ds} d\mu_0$ which is not shift invariant also induces μ on loops.

It writes

$$\begin{aligned} \tilde{\mu}_0(p(\xi) = k, \xi_0 = x_0, \dots, \xi_{k-1} = x_{k-1}, T_1 \in dt_1, \dots, T_k \in dt_k, T \in dt) \\ = P_{x_0, x_1} P_{x_1, x_2} \dots P_{x_{k-1}, x_0} \frac{\mathbf{1}_{\{0 < t_1 < \dots < t_k < t\}}}{\int_0^t q_{\xi_s} ds} e^{-q_{x_0} t_1} e^{-q_{x_1}(t_2 - t_1)} \dots e^{-q_x(t - t_k)} q_{x_0} dt_1 \dots q_{x_{k-1}} dt_k q_{x_0} dt \end{aligned}$$

for $k \geq 1$ and

$$\tilde{\mu}_0\{p(0) = 1, \xi_0 = x, \tau_1 \in dt_1\} = \frac{e^{-q_x t_1}}{t_1} dt_1$$

It is clear, in that form, that a time change transforms the $\tilde{\mu}_0$'s of Markov chains associated with the same energy one into each other, and therefore the same holds for μ : this is analogous to conformal invariance. Hence the restriction μ_I of μ to the σ -field of sets of loops invariant by time change (i.e. intrinsic sets) is intrinsic. It depends only on e .

To each loop l we associate an occupation field $\{\hat{l}_x, x \in X\}$ defined by

$$\hat{l}_x = \int_0^{T(l)} \mathbf{1}_{\{\xi_s = x\}} \frac{1}{m_{\xi_s}} ds = \sum_{i=1}^{p(l)} \mathbf{1}_{\{\xi_{i-1} = x\}} \frac{\tau_i}{m_x}$$

for any representative (ξ, τ) of l . It is independent of the time scale (we say "intrinsic"). But functions of \hat{l} are not the only intrinsic functions. They will be studied more completely in a forthcoming paper. Intrinsic objects will appear more important when X is not discrete since then there is in general no privileged parametrisation.

Other intrinsic variables of interest are

$\hat{l}_{x_1, \dots, x_k}^{(k)} = \frac{1}{k} \sum_{j=0}^{k-1} \int_{0 < t_1 < \dots < t_k < T} \mathbf{1}_{\{\xi_{t_1} = x_{1+j}, \dots, \xi_{t_{k-j}} = x_k, \dots, \xi_{t_k} = x_j\}} \prod \frac{1}{m_{x_i}} dt_i$ and one can check that $\mu(\hat{l}_{x_1, \dots, x_k}^{(k)}) = G_{x_1, x_2} G_{x_2, x_3} \dots G_{x_k, x_1}$. Note that $\hat{l}_{x_1, \dots, x_k}^{(k)}$ cannot be expressed in terms of \hat{l} for $k > 3$.

If D is a subset of X , the restriction of μ to loops contained in D , denoted μ^D is clearly the loop measure induced by the Markov chain killed at the exit of D . This is called the *restriction property*.

When $q_x = 1$ for all x , μ_0 takes a simpler form:

$$\begin{aligned} \mu_0(p(\xi) = k, \xi_0 = x_0, \dots, \xi_{k-1} = x_{k-1}, T_1 \in dt_1, \dots, T_k \in dt_k, T \in dt) \\ = P_{x_0, x_1} P_{x_1, x_2} \dots P_{x_{k-1}, x_0} \frac{1_{\{0 < t_1 < \dots < t_k < t\}}}{t} e^{-t} dt_1 \dots dt_k dt \end{aligned}$$

for $k \geq 1$ and $\mu_0\{p(\xi) = 1, \xi_0 = x_0, \tau_1 \in dt_1\} = \frac{e^{-t_1}}{t_1} dt_1$

It follows that for $k > 0$,

$$\mu_0(p(\xi) = k, \xi_0 = x_0, \dots, \xi_{k-1} = x_{k-1}) = \frac{1}{k} P_{x_0, x_1} P_{x_1, x_2} \dots P_{x_k, x_0}$$

as $\int \frac{t^{k-1}}{k!} e^{-t} dt = \frac{1}{k}$ and conditionally to $p(\xi) = k, \xi_0 = x_0, \dots, \xi_{k-1} = x_{k-1}$, T is a gamma variable of density $\frac{t^{k-1}}{(k-1)!} e^{-t}$ on \mathbb{R}_+ and $(\frac{T_i}{T} \mathbf{1} \leq i \leq k)$ an independent ordered k -sample of the uniform distribution on $(0, 1)$.

4 Loops Poisson process

Still following the idea of [3], define, for all positive α , the Poisson process of loops \mathcal{L}_α with intensity $\alpha\mu$. We can associate to \mathcal{L}_α the σ -finite measure

$$\widehat{\mathcal{L}}_\alpha = \sum_{l \in \mathcal{L}_\alpha} \widehat{l}$$

Then, for any non-negative measure χ on X

$$E(e^{-\langle \widehat{\mathcal{L}}_\alpha, \chi \rangle}) = \exp(\alpha \int (e^{-\langle \widehat{l}, \chi \rangle} - 1) d\mu(l))$$

Moreover, from the definition of μ it comes easily that, denoting $M_{\frac{\chi}{m}}$ the diagonal matrix with coefficients $\frac{\chi_x}{m_x}$

$\mathbb{P}_{x,x}^t(e^{-\langle \widehat{l}, \chi \rangle} - 1) = \exp(t(L - M_{\frac{\chi}{m}})) - \exp(tL)$. As we are interested in μ_I , we can take $q = 1$. Integrating in t after introducing by force a factor e^{-t} and expanding, we get (first for χ small enough):

$$\int (e^{-\langle \widehat{l}, \chi \rangle} - 1) d\mu(l) = \sum_{k=1}^{\infty} \frac{1}{k} [Tr((P - M_{\frac{\chi}{m}})^k) - Tr((P)^k)]$$

Hence

$$E(e^{-\langle \widehat{\mathcal{L}}_{\alpha}, \chi \rangle}) = [\det(-L(-L + M_{\chi/m})^{-1})]^{\alpha} = \det(I + (-L)^{-1} M_{\frac{\chi}{m}})^{-\alpha} = \det(I + V M_{\frac{\chi}{m}})^{-\alpha}$$

which now holds for all non negative χ . Set $G_{\chi} = (-L + M_{\frac{\chi}{m}})^{-1} M_{\frac{\chi}{m}}$. It is an intrinsic nonnegative function on $X \times X$. G_0 is the Green function G . We have also the "resolvent" equation $G - G_{\chi} = G M_{\chi} G_{\chi} = G_{\chi} M_{\chi} G$. Then

$$E(e^{-\langle \widehat{\mathcal{L}}_{\alpha}, \chi \rangle}) = \det(I + G M_{\chi})^{-\alpha} = \det(I - G_{\chi} M_{\chi})^{\alpha}$$

as $(I + G M_{\chi})(I - G_{\chi} M_{\chi}) = I$ by the resolvent equation.

In this calculation, the trace and the determinant are applied to matrices indexed by X . One could write equivalent forms in $L^2(m)$ but in view of generalizing them in an intrinsic form (i.e. in a form invariant under time change), it is useful to write an equivalent formula for operators in \mathbb{H} in the following:

Proposition 1 $E(e^{-\langle \widehat{\mathcal{L}}_{\alpha}, \chi \rangle}) = \det_{\mathbb{H}}(G_{\chi})^{\alpha}$ and $\mu(e^{-\langle \widehat{l}, \chi \rangle} - 1) = \log(\det_{\mathbb{H}}(G_{\chi}))$

Indeed, it follows from the calculation above and the identities:

$$\langle f, G_{\chi} \mu \rangle_{\mathbb{H}} = \langle f, G \mu - G(\chi G_{\chi} \mu) \rangle_{\mathbb{H}} = \langle f - G_{\chi}(f \chi), \mu \rangle$$

Note finally it follows easily that $E(\langle \widehat{\mathcal{L}}_1, \chi \rangle) = \mu(\langle \widehat{l}, \chi \rangle) = \sum_x \chi_x G_{x,x}$

Let $H^F(x, \cdot)$ be the hitting distribution of F . Denote \mathbb{H}^{F^c} , V^{F^c} and G^{F^c} the Dirichlet space, the potential and the Green function of the process killed at the hitting of F . Recall that $V = V^{F^c} + H^F V$ and $G = G^{F^c} + H^F G$.

Taking $\chi = a 1_F$ with F finite, and letting a increase to infinity, we get $\lim_{a \uparrow \infty} (G_{\chi} M_{\chi}) = H^F$ which is I on F . Therefore by proposition 1: $\mathbb{P}(\widehat{\mathcal{L}}_{\alpha}(F) = 0) = \det(I - H^F) = 0$ and $\mu(\widehat{l}(F) > 0) = \infty$.

More generally, if χ has support in F^c , by the restriction property

$$\mu(1_{\{\widehat{l}(F)=0\}}(e^{-\langle \widehat{l}, \chi \rangle} - 1)) = -\log(\det(I + G^{F^c} M_{\chi})) = \log(\det(I - G_{\chi}^{F^c} M_{\chi})) = \log(\det_{\mathbb{H}^{F^c}}(G_{\chi}^{F^c}))$$

Where the two first determinants are taken on matrices indexed by F^c .

5 Associated Gaussian field

By a well known calculation, if X is finite, for any $\chi \in \mathbb{R}_+^X$,

$$\frac{\det(-L)}{(2\pi)^{|X|}} \int (e^{-\frac{1}{2}\langle z\bar{z}, \chi \rangle} e^{\frac{1}{2}\langle Lz, \bar{z} \rangle_m} \prod_{u \in X} \frac{i}{2} dz_u \wedge d\bar{z}_u = \det(I + VM_{\frac{\chi}{m}})^{-1}$$

This can be easily reformulated by introducing the complex Gaussian field ϕ defined by the covariance $\mathbb{E}(\phi_x \bar{\phi}_y) = 2G_{x,y}$ (this reformulation cannot be dispensed with when X becomes infinite)

So we have $\mathbb{E}(e^{-\frac{1}{2}\langle \phi\bar{\phi}, \chi \rangle}) = \det(I + VM_{\frac{\chi}{m}})^{-1} = \det_{\mathbb{H}}(G_{\chi})$ and then the following holds:

Proposition 2 : *The fields $\widehat{\mathcal{L}}_1$ and $\frac{1}{2}\phi\bar{\phi}$ have the same distribution.*

This is a version of Dynkin's isomorphism (Cf [1]). It can be extended to non symmetric generators (Cf [7]).

Note it implies immediately that the process $\phi\bar{\phi}$ is infinitely divisible. See [2] and its references for a converse and earlier proofs of this last fact.

Recall that for any $f \in \mathbb{H}$, the law of $f + \phi$ is absolutely continuous with respect to the law of ϕ , with density $\exp(\langle -Lf, \phi \rangle_m - \frac{1}{2}e(f))$

Recall (it was observed by Nelson in the context of the free field) that the Gaussian field ϕ is Markovian: Given any subset F of X , denote \mathcal{H}_F the Gaussian space spanned by $\{\phi_x, x \in F\}$. Then, for $y \in D = F^c$, the projection of ϕ_y on \mathcal{H}_F is $\sum_{x \in F} H^F(y, x)\phi_x$.

Moreover, $\phi^D = \phi - H^F\phi$ is the Gaussian field associated with the process killed at the exit of D .

Given any f in the restriction of \mathbb{H} to F , the conditioning by $\phi = f$ on F is well defined.

The decomposition of the Dirichlet space (cf [4]) leads to a decomposition of the Gaussian field and should lead to a decomposition of μ into its restriction to loops in D (associated to the process killed at the exit of D), a loop measure induced on F , and measures on excursions in D indexed by pairs of points in F .

Note also that if a function h is such that $Lh \leq 0$, the loop measure defined by the h^2m -symmetric generator $L_h = \frac{1}{h}LM_h$ is associated with the Gaussian field $h\phi$.

6 Fock space and Wick powers

Recall that the Gaussian space \mathcal{H} spanned by $\{\phi_x, x \in X\}$ is isomorphic to \mathbb{H} by the linear map mapping $\text{Re}(\phi_x)$ on G_x , which extends into an isomorphism between the space of square integrable functionals of the Gaussian fields and the Fock space obtained as the closure of the sum of all symmetric tensor powers of \mathbb{H} (Bose second quantization).

In order to prepare the extension of the isomorphism to a more interesting framework (including especially the planar Brownian motion considered in [3]) we shall introduce the renormalized (or Wick) powers of $\phi\bar{\phi}$.

The Laguerre polynomials $L_n^0(x)$ are defined by their generating function $\frac{e^{-xt}}{1-t} = \sum t^n L_n^0(x)$

Then one defines the polynomial $P_n^\sigma(\cdot) = (-\sigma)^n n! L_n^0(\frac{\cdot}{\sigma})$

Setting $\sigma_x = G_{x,x}$, it comes that $P_n^\sigma(\frac{\phi\bar{\phi}}{2} - \sigma)$ is the inverse image of a $2n$ -th tensor in the Fock space denoted $:(\phi\bar{\phi})^n:$. These variables are orthogonal in L^2 . Set $\tilde{l}_x = \widehat{l}_x - \sigma_x$ be the centered occupation field. Note that an equivalent formulation of proposition 2 is that the fields $\frac{1}{2} : \phi\bar{\phi} :$ and $\tilde{\mathcal{L}}_1$ have the same law.

7 Currents

Other intrinsic variables of interest on the loop space are associated with currents. In this setting, we define them as antisymmetric matrices $\omega_{x,y}$ indexed by X : $\omega_{x,y} = -\omega_{y,x}$.

The operator $P_{x,y}^\omega = P_{x,y} \exp(i\omega_{x,y})$ is self adjoint in $L^2(mq)$. Set $L_{x,y}^\omega = q(P^\omega - I)$ and $V^\omega = (-L^\omega)^{-1}$. Set $G_{x,y}^\omega = \frac{V_{x,y}^\omega}{m_y}$.

The associated loop variable writes $\sum_{j=1}^p \omega_{\xi_j, \xi_{j+1}}$. We will denote it $\int_l \omega$. Note it is invariant if $\omega_{x,y}$ is replaced by $\omega_{x,y} + g(x) - g(y)$ for some g . By an argument similar to the one given above for the occupation field, we have:

$$E(\exp(i \sum_{l \in \mathcal{L}_\alpha} \int_l \omega)) = [\det(-L(-L^\omega)^{-1})]^\alpha = \det_{\mathbb{H}}(G^\omega)^\alpha$$

and also, in a similar way: $E(\exp(\sum_{l \in \mathcal{L}_\alpha} i \int_l \omega + \langle \widehat{l}, \chi \rangle)) = \det_{\mathbb{H}}(G_\chi^\omega)^\alpha$
For $\alpha = 1$,

$$E(\exp(i \sum_{l \in \mathcal{L}_1} (\int_l \omega + \langle \hat{l}, \chi \rangle))) = \frac{\det(-L)}{(2\pi)^{|\mathcal{X}|}} \int (e^{-\frac{1}{2} \langle z \bar{z}, \chi \rangle} e^{\frac{1}{2} \langle L^\omega z, \bar{z} \rangle} \prod_{u \in X} \frac{i}{2} dz_u \wedge d\bar{z}_u)$$

which gives an extension of Dynkin's isomorphism .

Note also that this formula applies to the calculation of loop indices if we have for exemple a simple random walk on an oriented two dimensional lattice. In such cases, ω can be chosen to be \mathbb{Z} -valued and such that Z_ω is the winding number of the loop around a given point x'_0 of the dual lattice X'^1 .

We then get for exemple that

$$\mu(Z_\omega(l) \neq 0) = \frac{-1}{2\pi} \int_0^{2\pi} \log(\det_{\mathbb{H}}(G^{2\pi u \omega})) du$$

Conditional distributions of the occupation field with respect to values of the winding number can also be obtained.

8 The case of general Markov processes

We now explain briefly how the above will be extended to a symmetric Markov process on an uncountable space X . The construction of the loop measure as well as a lot of computations can be performed quite generally, using Dirichlet space theory. Let us consider more closely the occupation field \hat{l} . The extension is rather straightforward when points are not polar. We can start with a Dirichlet space of continuous functions and a measure m such that there is a mass gap. Then the Green function is well defined as the Gaussian Markov field ϕ_x . \hat{l} is a field of local times. Propositions 1 and 2 still hold (χ being defined as a positive bounded measure on X) as the continuous Green function G will be locally trace class. This will apply to exemples related to one dimensional Brownian motion or to Markov chains on countable spaces.

When points are polar, one needs to be more careful. We will consider only the case of the two and three dimensional Brownian motion in a bounded

¹The construction of ω can be done as follows: Let P' be the uniform Markov transition probability on neighbouring points of the dual lattice and let h be a function such that $P'h = h$ except in x'_0 . Then if the link xy in X intersects $x'y'$ in X' , with $\det(x-y, x'-y') > 0$, set $\omega_{x,y} = h(y') - h(x')$

domain. The Green function is not locally trace class but it is still Hilbert-Schmidt which allows to define renormalized determinants \det_2 (Cf [9]) and to extend the statement of proposition 1 to the centered occupation field and the Wick square : $\phi\bar{\phi}$: of the generalized Gaussian Markov field ϕ . These three generalized fields are not defined pointwise but have to be smeared by measures of finite energy χ . The centered occupation field \tilde{l} is defined as follows: Let A_t^χ be the additive functional associated with χ such that $\int G_{x,y}\chi(dx)\chi(dy) < \infty$. Then $\langle \tilde{l}, \chi \rangle$ is defined as $\lim_{\varepsilon \downarrow 0} \int_0^{(T-\varepsilon)^+} dA_t^\chi - \mu_0(\int_0^{(T-\varepsilon)^+} dA_t^\chi)$ which converges in $L^2(\mu_0)$. It is an intrinsic quantity. We then have

Proposition 3 a) *The centered occupation field $\tilde{\mathcal{L}}_1$ and the Wick square $\frac{1}{2} : \phi\bar{\phi} :$ have the same distribution.*

b) $E(e^{-\langle \tilde{\mathcal{L}}_\alpha, \chi \rangle}) = \det_{2, \mathbb{H}}(G_\chi)^\alpha$

To justify the use of \det_2 , note that in the finite case $\det(I + GM_\chi) = \det_2(I + GM_\chi)e^{\chi(\sigma)}$ where we recall that σ_x^{-1} is the capacity of x , which vanishes now since x is polar.

In two dimensions, higher Wick powers of $\phi\bar{\phi}$ are associated with self intersection local times of the loops.

Let us now consider currents. We will restrict our attention to the two dimensional Brownian case, X being an open subset of the plane. Currents can be defined by curl free vector fields, with compact support. Then $\int_l \omega$ and $\int_X (\bar{\phi}\partial_\omega\phi - \phi\partial_\omega\bar{\phi})dx$ are well defined square integrable variables. The distribution of the centered occupation field of the loop process "twisted" by the complex exponential $\exp(i \sum_{l \in \mathcal{L}_\alpha} \int_l \omega)$ appears again to be the same as the distribution of $: \phi\bar{\phi} :$ "twisted" by the complex exponential $\exp(\int_X (\bar{\phi}\partial_\omega\phi - \phi\partial_\omega\bar{\phi})dx)$ (Cf[7]). It implies the joint distribution of the centered occupation field with $\sum_{l \in \mathcal{L}_\alpha} \int_l \omega$ coincides with the joint distribution of $: \phi\bar{\phi} :$ with $i \int_X (\phi\partial_\omega\bar{\phi} - \bar{\phi}\partial_\omega\phi)dx$. These points, among others, should be developed in a forthcoming article.

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