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# On Decidability Properties of Local Sentences

Olivier Finkel<sup>1</sup>

*Equipe de Logique Mathématique  
U.F.R. de Mathématiques, Université Paris 7  
2 Place Jussieu, 75251 Paris cedex 05, France*

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## Abstract

Local (first order) sentences, introduced by Ressayre, enjoy very nice decidability properties, following from some stretching theorems stating some remarkable links between the finite and the infinite model theory of these sentences [Res88]. Another stretching theorem of Finkel and Ressayre implies that one can decide, for a given local sentence  $\varphi$  and an ordinal  $\alpha < \omega^\omega$ , whether  $\varphi$  has a model of order type  $\alpha$ . This result is very similar to Büchi's one who proved that the monadic second order theory of the structure  $(\alpha, <)$ , for a countable ordinal  $\alpha$ , is decidable. It is in fact an extension of that result, as shown in [Fin01] by considering the expressive power of monadic sentences and of local sentences over languages of words of length  $\alpha$ . The aim of this paper is twofold. We wish first to attract the reader's attention on these powerful decidability results proved using methods of model theory and which should find some applications in computer science and we prove also here two additional results on local sentences.

The first one is a new decidability result in the case of local sentences whose function symbols are at most unary: one can decide, for every *regular cardinal*  $\omega_\alpha$  (the  $\alpha$ -th cardinal), whether a local sentence  $\varphi$  has a model of order type  $\omega_\alpha$ .

Secondly we show that this result can not be extended to the general case. Assuming the consistency of an inaccessible cardinal we prove that the set of local sentences having a model of order type  $\omega_2$  is not determined by the axiomatic system  $ZFC + GCH$ , where  $GCH$  is the generalized continuum hypothesis.

*Key words:* local sentences, decidability properties, model of ordinal order type  $\alpha$ , monadic theory of an ordinal,  $\omega_2$ -model, Kurepa tree, independence result.

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## 1 Introduction

A local sentence is a first order sentence which is equivalent to a universal sentence and satisfies some semantic restrictions: closure in its models takes

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<sup>1</sup> Email: [finkel@logique.jussieu.fr](mailto:finkel@logique.jussieu.fr)

a finite number of steps. Ressayre introduced local sentences in [Res88] and established some remarkable links between the finite and the infinite model theory of these sentences given by some stretching theorems. Assuming that a binary relation symbol belongs to the signature of a local sentence  $\varphi$  and is interpreted by a linear order in every model of  $\varphi$ , the stretching theorems state that the existence of some well ordered models of  $\varphi$  is equivalent to the existence of some finite model of  $\varphi$ , generated by some particular kind of indiscernibles, like special, remarkable or monotonic ones. Another stretching theorem of Finkel and Ressayre establishes the equivalence between the existence of a model of order type  $\alpha$  (where  $\alpha$  is an infinite ordinal  $< \omega^\omega$ ) and the existence of a finite model (of another local sentence  $\varphi_\alpha$ ) generated by  $N_{\varphi_\alpha}$  semi-monotonic indiscernibles (where  $N_{\varphi_\alpha}$  is a positive integer depending on  $\varphi_\alpha$ ) [FR96].

This theorem provides some decision algorithms which show the decidability of the following problem: (P) “For a given local sentence  $\varphi$  and an ordinal  $\alpha < \omega^\omega$ , has  $\varphi$  a model of order type  $\alpha$  ?”

This last result is very similar to Büchi’s one who proved that the monadic second order theory of the structure  $(\alpha, <)$ , for a countable ordinal  $\alpha$ , is decidable, [Büc62] [Tho90] [BS73]. Büchi obtained some decision algorithms by proving firstly that, for  $\alpha$ -languages (languages of words of length  $\alpha$ ) over a finite alphabet, definability by monadic second order sentences is equivalent to acceptance by finite automata where a transition relation is added for limit steps.

We can compare the expressive power of monadic sentences and of local sentences, considering languages defined by these sentences. For each ordinal  $\alpha < \omega^\omega$ , an  $\alpha$ -language is called local in [Res88] [FR96] (or also locally finite in [Fin01] [Fin04] [Fin02]) iff it is defined by a second order sentence in the form  $\exists R_1 \dots \exists R_k \varphi$ , where  $\varphi$  is local in the signature  $S(\varphi) = \{<, R_1, \dots, R_k\}$  and  $R_1, \dots, R_k$  are relation or function symbols.

The class  $LOC_\alpha$  of local  $\alpha$ -languages, for  $\omega \leq \alpha < \omega^\omega$ , is a strict extension of the class  $REG_\alpha$  of regular  $\alpha$ -languages, defined by monadic second order sentences [Fin01]. Moreover this extension is very large. This can be seen by considering the topological complexity of  $\alpha$ -languages and firstly of  $\omega$ -languages. It is well known that all regular  $\omega$ -languages are boolean combinations of  $\Sigma_2^0$  Borel sets hence  $\Delta_3^0$  Borel sets, [Tho90] [PP04]. On the other hand the class  $LOC_\omega$  meets all finite levels of the Borel hierarchy, contains some Borel sets of infinite rank and even some analytic but non Borel sets, [Fin02].

Thus the decision algorithm for local sentences provides in fact a very large extension, for  $\alpha < \omega^\omega$ , of Büchi’s result about the decidability of the monadic second order theory of  $(\alpha, <)$ . Moreover, at least for  $\alpha = \omega$ , the algorithm for local sentences is of much lower complexity than the corresponding algorithm for monadic second order sentences [Fin02].

We think that these powerful decidability results proved using methods of

model theory should find some applications in computer science and that the study of local sentences could become an interdisciplinary subject for both model theory and computer science communities.

So the aim of this paper is twofold: firstly to attract the reader's attention on these good properties of local sentences and their possible further applications; secondly to prove two new results on local sentences, described below.

Büchi showed that for every ordinal  $\alpha < \omega_2$ , where  $\omega_2$  is the second uncountable cardinal, the monadic theory of  $(\alpha, <)$  is decidable. This result cannot be extended to  $\omega_2$ . Assuming the existence of a weakly compact cardinal (a kind of large cardinal) Gurevich, Magidor and Shelah proved that the monadic theory of  $(\omega_2, <)$  is not determined by the set theory axiomatic system  $ZFC$ . They proved even much more: for any given  $S \subseteq \omega$  there is a model of ZFC where the monadic theory of  $(\omega_2, <)$  has the Turing degree of  $S$ ; in particular it can be non-recursive [GMS83].

Ressayre asked similarly for which ordinals  $\alpha$  it is decidable whether a given local sentence  $\varphi$  has a model of order type  $\alpha$ . The question is solved in [FR96] for  $\alpha < \omega^\omega$  but for larger ordinals the problem was still open.

We firstly consider local sentences whose function symbols are at most unary. We show that these sentences satisfy an extension of the stretching theorem implying new decidability properties. In particular, for each *regular* cardinal  $\omega_\alpha$  (hence in particular for each  $\omega_n$  where  $n$  is a positive integer), it is decidable whether a local sentence  $\varphi$  has a model of order type  $\omega_\alpha$ . To know that this restricted class  $LOCAL(1)$  of local sentences has more decidability properties is of interest because it has already a great expressive power.

Sentences in  $LOCAL(1)$  can define all regular finitary languages [Res88], and all the quasirational languages forming a large class of context free languages containing all linear languages [Fin01].

If we consider their expressive power over infinite words, sentences in  $LOCAL(1)$  can define all regular  $\omega$ -languages [Fin01], but also some  $\Sigma_n^0$ -complete and some  $\Pi_n^0$ -complete Borel sets for every integer  $n \geq 1$ , [Fin02].

Next we show that this decidability result can not be extended to local sentences having  $n$ -ary function symbols for  $n \geq 2$ . Assuming the consistency of an inaccessible cardinal, we prove that the local theory of  $\omega_2$  (the set of local sentences having a model of order type  $\omega_2$ ) is not determined by the system  $ZFC + GCH$ , where  $GCH$  is the generalized continuum hypothesis. This is also extended to many larger ordinals.

We could still expect, as Shelah did in [She75] about the monadic theory of  $\omega_2$ , that there are only finitely many possible local theories of  $\omega_2$ , and that each of them is decidable. But it seems more plausible that the situation is much more complicated, as it is the case for the monadic theory of  $\omega_2$  [GMS83].

This result is obtained by showing that there is a local sentence which has a model of order type  $\omega_2$  if and only if there is a Kurepa tree, i.e. a tree of height  $\omega_1$  whose levels are countable and which has more than  $\omega_1$  branches of length  $\omega_1$ . Kurepa trees have been much studied in set theory and their existence

has been shown to be independent of  $ZFC + GCH$ , via the consistency of  $ZFC +$  “there is an inaccessible cardinal”.

It is remarkable that our proof needs only the consistency of an inaccessible cardinal which is still a large cardinal but a very smaller cardinal than a weakly compact cardinal. This gives another indication of the great expressive power of local sentences with regard to that of monadic sentences.

This new result is seemingly far from problems arising in concrete applications studied in computer science. However it is obtained by encoding (Kurepa) trees in models of a local sentence and methods used here for such coding might be very useful for problems arising in computer science where (finite or infinite) trees are a widely used tool.

## 2 Review of previous results

In this paper the (first order) signatures are finite, always contain one binary predicate symbol  $=$  for equality, and can contain both functional and relational symbols.

When  $M$  is a structure in a signature  $\Lambda$  and  $X \subseteq |M|$ , we define:

$$\begin{aligned} cl^1(X, M) &= X \cup \bigcup_{\{f \text{ n-ary function of } \Lambda\}} f^M(X^n) \cup \bigcup_{\{a \text{ constant of } \Lambda\}} a^M \\ cl^{n+1}(X, M) &= cl^1(cl^n(X, M), M) \quad \text{for an integer } n \geq 1 \\ \text{and } cl(X, M) &= \bigcup_{n \geq 1} cl^n(X, M) \text{ is the closure of } X \text{ in } M. \end{aligned}$$

The signature of a first order sentence  $\varphi$ , i.e. the set of non logical symbols appearing in  $\varphi$ , is denoted  $S(\varphi)$ .

**Definition 2.1** *A first order sentence  $\varphi$  is local if and only if:*

- (a)  $M \models \varphi$  and  $X \subseteq |M|$  imply  $cl(X, M) \models \varphi$
- (b)  $\exists n \in \mathbb{N}$  such that  $\forall M$ , if  $M \models \varphi$  and  $X \subseteq |M|$ , then  $cl(X, M) = cl^n(X, M)$ , (closure in models of  $\varphi$  takes less than  $n$  steps).

For a local sentence  $\varphi$ ,  $n_\varphi$  is the smallest integer  $n \geq 1$  satisfying (b) of the above definition. In this definition, (a) implies that a local sentence  $\varphi$  is always equivalent to a universal sentence, so we may assume that this is always the case.

**Example 2.2** *Let  $\varphi$  be the sentence in the signature  $S(\varphi) = \{<, P, i, a\}$ , where  $<$  is a binary relation symbol,  $P$  is a unary relation symbol,  $i$  is a unary function symbol, and  $a$  is a constant symbol, which is the conjunction of:*

- (1)  $\forall xyz[(x \leq y \vee y \leq x) \wedge ((x \leq y \wedge y \leq x) \leftrightarrow x = y) \wedge ((x \leq y \wedge y \leq z) \rightarrow x \leq z)]$ ,
- (2)  $\forall xy[(P(x) \wedge \neg P(y)) \rightarrow x < y]$ ,
- (3)  $\forall xy[(P(x) \rightarrow i(x) = x) \wedge (\neg P(y) \rightarrow P(i(y)))]$ ,
- (4)  $\forall xy[(\neg P(x) \wedge \neg P(y) \wedge x \neq y) \rightarrow i(x) \neq i(y)]$ ,

(5)  $\neg P(a)$ .

We now explain the meaning of the above sentences (1)-(5).

Assume that  $M$  is a model of  $\varphi$ . The sentence (1) expresses that  $<$  is interpreted in  $M$  by a linear order; (2) expresses that  $P^M$  is an initial segment of the model  $M$ ; (3) expresses that the function  $i^M$  is trivially defined by  $i^M(x) = x$  on  $P^M$  and is defined from  $\neg P^M$  into  $P^M$ . (4) says that  $i^M$  is an injection from  $\neg P^M$  into  $P^M$  and (5) ensures that the element  $a^M$  is in  $\neg P^M$ .

The sentence  $\varphi$  is a conjunction of universal sentences thus it is equivalent to a universal one, and closure in its models takes at most two steps (one adds the constant  $a$  in one step then takes the closure under the function  $i$ ). Thus  $\varphi$  is a local sentence.

If we consider only the order types of well ordered models of  $\varphi$ , we can easily see that  $\varphi$  has a model of order type  $\alpha$ , for every finite ordinal  $\alpha \geq 2$  and for every infinite ordinal  $\alpha$  which is not a cardinal.

Because of lack of space for this paper we cannot give here many more examples of local sentences. Some examples will be given later in Section 4. The reader may also find many other ones in the papers [Res88] [FR96] [Fin01] [Fin89] [Fin02] [Fin04].

The set of local sentences is recursively enumerable but not recursive [Fin01]. However there exists a “recursive presentation” up to logical equivalence of all local sentences.

**Theorem 2.3 (Ressayre, see [Fin01])** *There exists a recursive set  $\mathbf{L}$  of local sentences and a recursive function  $\mathbf{F}$  such that:*

- 1)  $\psi$  local  $\longleftrightarrow \exists \psi' \in \mathbf{L}$  such that  $\psi \equiv \psi'$ .
- 2)  $\psi' \in \mathbf{L} \longrightarrow n_{\psi'} = \mathbf{F}(\psi')$ .

The elements of  $\mathbf{L}$  are the  $\psi \wedge C_n$ , where  $\psi$  run over the universal formulas and  $C_n$  run over the universal formulas in the signature  $S(\psi)$  which express that closure in a model takes at most  $n$  steps.

$\psi \wedge C_n$  is local and  $n_{\psi \wedge C_n} \leq n$ . Then we can compute  $n_{\psi \wedge C_n}$ , considering only finite models of cardinal  $\leq m$ , where  $m$  is an integer depending on  $n$ . And each local sentence  $\psi$  is equivalent to a universal formula  $\theta$ , hence  $\psi \equiv \theta \wedge C_{n_\psi}$ .

From now on we shall assume that the signature of local sentences contain a binary predicate  $<$  which is interpreted by a linear ordering in all of their models.

We recall now the stretching theorem for local sentences. Below, semi-monotonic, special, and monotonic indiscernibles are particular kinds of indiscernibles which are precisely defined in [FR96].

**Theorem 2.4 ([FR96])** *For each local sentence  $\varphi$  there exists a positive integer  $N_\varphi$  such that*

- (A)  $\varphi$  has arbitrarily large finite models if and only if  $\varphi$  has an infinite model if and only if  $\varphi$  has a finite model generated by  $N_\varphi$  indiscernibles.
- (B)  $\varphi$  has an infinite well ordered model if and only if  $\varphi$  has a finite model generated by  $N_\varphi$  semi-monotonic indiscernibles.
- (C)  $\varphi$  has a model of order type  $\omega$  if and only if  $\varphi$  has a finite model generated by  $N_\varphi$  special indiscernibles.
- (D)  $\varphi$  has well ordered models of unbounded order types in the ordinals if and only if  $\varphi$  has a finite model generated by  $N_\varphi$  monotonic indiscernibles.

To every local sentence  $\varphi$  and every ordinal  $\alpha$  such that  $\omega \leq \alpha < \omega^\omega$  one can associate by an effective procedure a local sentence  $\varphi_\alpha$ , a unary predicate symbol  $P$  being in the signature  $S(\varphi_\alpha)$ , such that:

- (C $_\alpha$ )  $\varphi$  has a well ordered model of order type  $\alpha$  if and only if  $\varphi_\alpha$  has a finite model  $M$  generated by  $N_{\varphi_\alpha}$  semi-monotonic indiscernibles into  $P^M$ .

The integer  $N_\varphi$  can be effectively computed from  $n_\varphi$  and  $q$  where  $\varphi = \forall x_1 \dots \forall x_q \theta(x_1, \dots, x_q)$  and  $\theta$  is an open formula, i.e. a formula without quantifiers. If  $v(\varphi)$  is the maximum number of variables of terms of complexity  $\leq n_\varphi + 1$  (resulting by at most  $n_\varphi + 1$  applications of function symbols) and  $v'(\varphi)$  is the maximum number of variables of an atomic formula involving terms of complexity  $\leq n_\varphi + 1$  then  $N_\varphi = \max\{3v(\varphi); v'(\varphi) + v(\varphi); q.v'(\varphi)\}$ .

From Theorem 2.4 we can prove the decidability of several problems about local sentences. For instance (C) states that a local sentence  $\varphi$  has an infinite well ordered model iff it has a *finite* model generated by  $N_\varphi$  semi-monotonic indiscernibles. Therefore in order to check the existence of an infinite well ordered model of  $\varphi$  one can only consider models whose cardinals are bounded by an integer depending on  $n_\varphi$  and  $N_\varphi$ , because closure in models of  $\varphi$  takes at most  $n_\varphi$  steps. This can be done in a finite amount of time.

Notice that the set of local sentences is not recursive so the algorithms given by the following theorem are applied to local sentences in the recursive set  $\mathbf{L}$  given by Proposition 2.3. In particular  $\varphi$  is given with the integer  $n_\varphi$ .

**Theorem 2.5** ([FR96]) *It is decidable, for a given local sentence  $\varphi$ , whether*

- (1)  $\varphi$  has arbitrarily large finite models.
- (2)  $\varphi$  has an infinite model.
- (3)  $\varphi$  has an infinite well ordered model.
- (4)  $\varphi$  has well ordered models of unbounded order types in the ordinals.
- (5)  $\varphi$  has a model of order type  $\alpha$ , where  $\alpha < \omega^\omega$  is a given ordinal.

These decidable problems (1) – (4) and (5) (at least for  $\alpha = \omega$ ) are in the class  $\mathbf{NTIME}(2^{\mathbf{O}(n \cdot \log(n))})$ , (and even probably of lower complexity):

Using non determinism a Turing machine may guess a finite structure  $M$  of

signature  $S(\varphi)$  generated by  $N_\varphi$  elements  $y_1, \dots, y_{N_\varphi}$  in at most  $n_\varphi$  steps. Then, assuming  $\varphi = \forall x_1 \dots \forall x_q \theta(x_1, \dots, x_q)$  where  $\theta$  is an open formula, the Turing machine checks that  $\theta(x_1, \dots, x_q)$  holds for all  $x_1 \dots x_q$  in  $M$ , and that the elements  $y_1, \dots, y_{N_\varphi}$  are indiscernibles (respectively, semi-monotonic, special, monotonic, indiscernibles) in  $M$ .

On the other hand Büchi's procedure to decide whether a monadic second order formula of size  $n$  of  $S1S$  is true in the structure  $(\omega, <)$  might run in time  $\underbrace{2^{2^{2^n}}}_{O(n)}$ , [Büc62] [Saf89]. Moreover Meyer proved that one cannot essentially improve this result: the monadic second order theory of  $(\omega, <)$  is not elementary recursive, [Mey75].

We know that the expressive power of local sentences is much greater than that of monadic second order sentences hence this is a remarkable fact that decision algorithms for local sentences given by Theorem 2.5 are of much lower complexity than the algorithm for decidability of the monadic second order theory  $S1S$  of one successor over the integers.

### 3 More decidability results

We assume in this section that the function symbols of a local sentence  $\varphi$  are at most unary. We shall prove in this case some more decidability results which rely on an extension of the stretching Theorem 2.4.

The cardinal of a set  $X$  will be denoted by  $card(X)$ .

We recall that the infinite cardinals are usually denoted by  $\aleph_0, \aleph_1, \aleph_2, \dots, \aleph_\alpha, \dots$ . We refer for instance to [Dev84] [Jec02] for the notion of regular cardinal. The cardinal  $\aleph_\alpha$  is also denoted by  $\omega_\alpha$ , as usual when it is considered as an ordinal.

We recall now the notion of special indiscernibles, [FR96], in that particular case where all function symbols hence all terms of  $S(\varphi)$  are unary.

A set  $X$  included in a structure  $M$ , having a linear ordering  $<$  in its signature, is a set of indiscernibles iff whenever  $\bar{x}$  and  $\bar{y}$  are order isomorphic sequences from  $X$  they satisfy in  $M$  the same atomic sentences. The indiscernibles of  $X$  are special iff they satisfy (i) and (ii):

- (i) for all  $x < y$  in  $X$  and all terms  $t$ :  $t(x) < y$ .
- (ii) for all  $x < y$  in  $X$  and all terms  $t$ :  $t(y) < x \rightarrow t(y) = t(z)$  for all elements  $z > x$  of  $X$  (i.e.  $t$  is constant on  $\{z \in X \mid z > x\}$ ).

**Theorem 3.1** *For each local sentence  $\varphi$  whose function symbols are at most unary, there is a positive integer  $N_\varphi$  such that, for each regular cardinal  $\omega_\alpha$ , the following statements are equivalent:*



- (a)  $\varphi$  has an  $\omega$ -model.
- (b)  $\varphi$  has a finite model generated by  $N_\varphi$  special indiscernibles.
- (c)  $\varphi$  has a  $\beta$ -model, for all limit ordinals  $\beta$ .
- (d)  $\varphi$  has an  $\omega_\alpha$ -model.

**Proof.** It is proved in [FR96] that for each local sentence  $\varphi$  there is a positive integer  $N_\varphi$  such that (a) is equivalent to (b).

To prove (a)  $\rightarrow$  (c) assume that  $\varphi$  has an  $\omega$ -model  $M$ . Then it is proved in [FR96] that there exists an infinite set  $X$  of special indiscernibles in  $M$ . Recall that every linear order  $Y$  can be extended to a model  $M(Y)$  of  $\varphi$ , called the stretching of  $M$  along  $Y$ , so that:

- (1)  $M(X)$  is the submodel of  $M$  generated by the set  $X$ .
- (2)  $Y \subseteq Z$  implies  $M(Y) \subseteq M(Z)$ .
- (3) Every order embedding  $f : Y \rightarrow Z$  has an extension  $M(f)$  which is an embedding of  $M(Y)$  into  $M(Z)$ .

Let then  $\beta$  be a limit ordinal and  $M(\beta)$  be the stretching of  $M$  along  $\beta$ . We are going to show that  $M(\beta)$  is of order type  $\beta$ . The model  $M(\beta)$  is generated by the set  $\beta$  in a finite number of steps so there is a finite set  $T_\varphi$  of (unary) terms of the signature  $S(\varphi)$  such that the domain of  $M(\beta)$  is  $\beta \cup \bigcup_{t \in T_\varphi} \bigcup_{\gamma < \beta} t(\gamma)$ . The indiscernibles are special thus for each term  $t \in T_\varphi$ , either  $t$  is constant on  $\beta$  or for all indiscernibles  $x < y < z$  in  $\beta$  we have  $x < t(y) < z$ . It is then easy to see that  $M(\beta)$  is of order type  $\beta$ .

(c)  $\rightarrow$  (d) is trivial so it remains to prove (d)  $\rightarrow$  (a).

We assume that  $\alpha$  is an ordinal and that  $M$  is a model of  $\varphi$  of order type  $\omega_\alpha$  where  $\omega_\alpha$  is a *regular* cardinal. We are going to show that there exists in  $M$  an infinite set of special indiscernibles. These indiscernibles have to satisfy (i) and (ii) only for terms of complexity  $\leq n_\varphi$  because for each term  $t$  of complexity greater than  $n_\varphi$  there will be another term  $t'$  of complexity  $\leq n_\varphi$  such that  $t(x) = t'(x)$  for all indiscernibles  $x$ . This finite set of terms of complexity  $\leq n_\varphi$  will be denoted by  $T = \{t_1, t_2, \dots, t_N\}$ .

Using the fact that  $\omega_\alpha$  is a *regular* cardinal, we can firstly construct by induction a strictly increasing sequence  $(x_\delta)_{\delta < \omega_\alpha}$  of elements of  $M$  such that for each ordinal  $\delta < \omega_\alpha$  and each term  $t \in T$  it holds that  $t(x_\delta) < x_{\delta+1}$ . We denote  $X_0 = \{x_\delta \mid \delta < \omega_\alpha\}$ ; this set has cardinal  $\aleph_\alpha$ .

We consider now the three following cases:

**First case.** The set  $\{x_\delta \in X_0 - \{x_0\} \mid t_1(x_\delta) = x_0\}$  has cardinal  $\aleph_\alpha$ . Then we denote this set by  $X_0^1$ .

**Second case.** The set  $\{x_\delta \in X_0 - \{x_0\} \mid t_1(x_\delta) < x_0\}$  has cardinal  $\aleph_\alpha$  and the first case does not hold. The initial segment  $\{x \in M \mid x < x_0\}$  of  $M$  has cardinal smaller than  $\aleph_\alpha$  thus there is a subset of  $\{x_\delta \in X_0 - \{x_0\} \mid t_1(x_\delta) < x_0\}$  which has cardinal  $\aleph_\alpha$  and on which  $t_1$  is constant. Then we denote this set by  $X_0^1$ .

**Third case.** The set  $\{x_\delta \in X_0 - \{x_0\} \mid t_1(x_\delta) > x_0\}$  has cardinal  $\aleph_\alpha$  and the

two first cases do not hold. Then we call this set  $X_0^1$ .

We can repeat now this process, replacing  $X_0$  by  $X_0^1$  and the term  $t_1$  by the term  $t_2$ , so we obtain a new set  $X_0^2 \subseteq X_0^1$  having still cardinal  $\aleph_\alpha$ . Next we repeat the process replacing  $X_0^1$  by  $X_0^2$  and the term  $t_2$  by the term  $t_3$ , so we obtain a new set  $X_0^3 \subseteq X_0^2$  having still cardinal  $\aleph_\alpha$ .

After having considered all terms  $t_1, t_2, \dots, t_N$  we have got a set  $X_0^N \subseteq X_0^{N-1} \subseteq \dots \subseteq X_0$ . We denote  $X_1 = X_0^N$ .

Let  $x_{\delta_1}$  be the first element of  $X_1$ . We can repeat all the above process replacing  $X_0$  by  $X_1$  and  $x_0$  by  $x_{\delta_1}$ . This way, considering successively each of the terms  $t_1, t_2, \dots, t_N$ , we construct new sets  $X_1^N \subseteq X_1^{N-1} \subseteq \dots \subseteq X_1^1 \subseteq X_1$ , each of them having cardinal  $\aleph_\alpha$ , and we set  $X_2 = X_1^N$ .

Assume now that we have applied this process  $K$  times for some integer  $K \geq 2$ . Then we have constructed successively some sets  $X_1, X_2, \dots, X_K$  of cardinal  $\aleph_\alpha$ . Let now  $x_{\delta_K}$  be the first element of  $X_K$ . We can repeat the above process replacing  $X_0$  by  $X_K$  and  $x_0$  by  $x_{\delta_K}$ . This way we construct a new set  $X_{K+1} = X_K^N$  of cardinal  $\aleph_\alpha$ .

Then we can construct by induction the sets  $X_K$  for all integers  $K \geq 1$ . We set  $X = \{x_{\delta_i} \mid 0 \leq i < \omega\}$  where for all  $i$ ,  $x_{\delta_i}$  is the first element of  $X_i$ .

Let now  $X^{[n]}$  be the set of strictly increasing  $n$ -sequences of elements of  $X$ . Let  $\sim$  be the equivalence relation defined on  $X^{[v'(\varphi)]}$  by:  $x \sim y$  if and only if  $x$  and  $y$  satisfy in  $M$  the same atomic formulas of complexity  $\leq n_\varphi + 1$  (i.e. whose terms are of complexity  $\leq n_\varphi + 1$ ). Applying the Infinite Ramsey Theorem, we can now get an infinite set  $Y \subseteq X$  such that  $Y^{[v'(\varphi)]}$  is contained in a single equivalence class of  $\sim$ .

$Y$  is a set of indiscernibles in  $M$  because if  $z$  and  $z'$  are two elements of  $Y^{[n]}$  for  $n \geq v'(\varphi)$ , then they satisfy in  $M$  the same atomic sentences of complexity  $\leq n_\varphi + 1$  hence of any complexity by Fact 1 of [FR96, page 568].

By the above construction of the set  $X$ , the indiscernibles of  $Y$  are special. Thus the submodel  $M(Y)$  of  $M$  generated by  $Y$  is a model of  $\varphi$  of order type  $\omega$ .  $\square$

Notice that one cannot omit the hypothesis of the *regularity* of the cardinal  $\omega_\alpha$  in the above theorem, as it will be shown in the full version of this paper.

By Theorem 2.5 it is decidable whether a local sentence  $\varphi$  has an  $\omega$ -model so we have got the following decidability result.

**Theorem 3.2** *It is decidable, for a given local sentence  $\varphi$  whose function symbols are at most unary, and a given regular cardinal  $\omega_\alpha$ , whether:*

- (1)  $\varphi$  has an  $\omega_\alpha$ -model
- (2)  $\varphi$  has a  $\beta$ -model for all limit ordinals  $\beta$ .

So in particular one can decide, for a given local sentence  $\varphi$  whose function symbols are at most unary, whether  $\varphi$  has a model of order type  $\omega_1$ , (respectively,  $\omega_2$ ,  $\omega_n$  where  $n$  is a positive integer).

## 4 The local theory of $\omega_2$

It was proved in [FR96] that there exists a local sentence  $\psi$  (whose signature contains binary function symbols) having well ordered models of order type  $\alpha$  for every ordinal  $\alpha$  in the segment  $[\omega; 2^{\aleph_0}]$  but not any well ordered model of order type  $\alpha$  for  $\text{card}(\alpha) > 2^{\aleph_0}$ . On the other hand it is well known that the continuum hypothesis  $CH$  is independent of the axiomatic system  $ZFC$ . This means that there are some models of  $ZFC$  in which  $2^{\aleph_0} = \aleph_1$  and some others in which  $2^{\aleph_0} \geq \aleph_2$ . Therefore the statement “ $\psi$  has a model of order type  $\omega_2$ ” is independent of  $ZFC$ .

However if we assume the continuum hypothesis and even the generalized continuum hypothesis  $GCH$  saying that, for every cardinal  $\aleph_\alpha$ ,  $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ , then the above result of [FR96] does not imply a similar independence result.

Nevertheless we are going to prove the existence of a local sentence  $\Phi$  such that “ $\Phi$  has a model of order type  $\omega_2$ ” is independent of  $ZFC + GCH$ .

For that purpose we shall use results about Kurepa trees which we now recall.

A partially ordered set  $(T, \prec_T)$  is called a tree if for every  $t \in T$  the set  $\{s \in T \mid s \prec_T t\}$  is well ordered under  $\prec_T$ . Then the order type of the set  $\{s \in T \mid s \prec_T t\}$  is called the height of  $t$  in  $T$  and is denoted by  $ht(t)$ . We shall not distinguish a tree from its base set.

For every ordinal  $\alpha$  the  $\alpha$ -th level of  $T$  is  $T_\alpha = \{t \in T \mid ht(t) = \alpha\}$ .

The height of  $T$ , denoted by  $ht(T)$ , is the smallest ordinal  $\alpha$  such that  $T_\alpha = \emptyset$ . A branch of  $T$  will be a linearly ordered subset of  $T$  intersecting every non-empty level of  $T$ . The set of all branches of  $T$  will be denoted  $\mathcal{B}(T)$ .

A tree  $T$  is called an  $\omega_1$ -tree if  $\text{card}(T) = \aleph_1$  and  $ht(T) = \omega_1$ . An  $\omega_1$ -tree  $T$  is called a Kurepa tree if  $\text{card}(\mathcal{B}(T)) > \aleph_1$  and for every ordinal  $\alpha < \omega_1$ ,  $\text{card}(T_\alpha) < \aleph_1$ .

Recall now the well known results about Kurepa trees, [Dev84]:

### Theorem 4.1

- (i) *If  $ZF$  is consistent so too is the theory:  $ZFC + GCH +$  “there is a Kurepa tree”.*
- (ii) *If the theory  $ZFC +$  “there is an inaccessible cardinal” is consistent so too is the theory  $ZFC + GCH +$  “there are no Kurepa trees”.*
- (iii) *If the theory  $ZFC +$  “there are no Kurepa trees” is consistent so too is the theory  $ZFC +$  “there is an inaccessible cardinal”.*

In order to use the above result in the context of local sentences we state now

the main technical result of this section.

**Theorem 4.2** *There exists a local sentence  $\Phi$  such that:*

$$[\Phi \text{ has an } \omega_2\text{-model}] \iff [\text{there is a Kurepa tree}].$$

To prove this theorem we shall firstly state the two following lemmas.

**Lemma 4.3** *There exists a local sentence  $\varphi_0$  such that  $\varphi_0$  has a well ordered model of order type  $\omega$  but has no well ordered model of order type  $> \omega$ .*

**Proof.** Such a sentence is given in [FR96] in the signature  $S(\varphi_0) = \{<, P, f, p_1, p_2\}$ , where  $P$  is a unary predicate,  $f$  is a binary function, and  $p_1, p_2$  are unary functions.  $\square$

**Lemma 4.4** *There exists a local sentence  $\varphi_1$  such that  $\varphi_1$  has well ordered models of order type  $\alpha$ , for every ordinal  $\alpha \in [\omega, \omega_1]$ , but has no well ordered model of order type  $> \omega_1$ .*

**Proof.** We give below the sentence  $\varphi_1$  in the signature  $S(\varphi_1) = S(\varphi_0) \cup \{Q, g\} = \{<, P, f, p_1, p_2, Q, g\}$ , where  $Q$  is a unary predicate and  $g$  is a binary function.  $\varphi_1$  is the conjunction of the following sentences (1)-(10) whose meaning is explained below:

- (1)  $\forall xyz[(x \leq y \vee y \leq x) \wedge ((x \leq y \wedge y \leq x) \leftrightarrow x = y) \wedge ((x \leq y \wedge y \leq z) \rightarrow x \leq z)],$
- (2)  $\forall xy[(Q(x) \wedge \neg Q(y)) \rightarrow x < y],$
- (3)  $\forall xy[(Q(x) \wedge Q(y)) \rightarrow f(x, y) \in Q],$
- (4)  $\forall x[Q(x) \rightarrow Q(p_i(x))],$  for each  $i \in [1, 2],$
- (5)  $\forall xy[(\neg Q(x) \vee \neg Q(y)) \rightarrow f(x, y) = x],$
- (6)  $\forall x[\neg Q(x) \rightarrow p_i(x) = x],$  for each  $i \in [1, 2],$
- (7)  $\forall x_1 \dots x_j \in Q[\varphi'_0(x_1, \dots, x_j)],$  where  $\varphi_0 = \forall x_1 \dots x_j \varphi'_0(x_1, \dots, x_j)$  with  $\varphi'_0$  an open formula,
- (8)  $\forall xy[(\neg Q(x) \wedge \neg Q(y) \wedge y < x) \rightarrow Q(g(x, y))],$
- (9)  $\forall xyz[(\neg Q(x) \wedge \neg Q(y) \wedge \neg Q(z) \wedge y < z < x) \rightarrow g(x, y) \neq g(x, z)],$
- (10)  $\forall xy[(Q(x) \vee Q(y) \vee \neg(y < x)) \rightarrow g(x, y) = x].$

We now explain the meaning of the above sentences (1)-(10).

Assume that  $M$  is a model of  $\varphi_1$ . The sentence (1) expresses that  $<$  is interpreted in  $M$  by a linear order; (2) expresses that  $Q^M$  is an initial segment of the model  $M$ ; (3) and (4) state that  $Q^M$  is closed under the functions of  $S(\varphi_0)$  while (5) and (6) state that these functions are trivially defined elsewhere; (7) means that the restriction of the model  $M$  to the domain  $Q^M$  and to the signature of  $S(\varphi_0)$  is a model of  $\varphi_0$ ; Finally (8) and (9) ensure that, for each  $x \in \neg Q$ , the binary function  $g$  realizes an injection from the segment  $\{y \in \neg Q \mid y < x\}$  into  $Q$  and (10) states that the function  $g$  is trivially defined where it is not useful for that purpose.

The sentence  $\varphi_1$  is a conjunction of universal sentences thus it is equivalent to a universal one, and closure in its models takes at most  $n_\varphi + 1$  steps: one applies first the function  $g$  and then the functions of  $S(\varphi_0)$ . Thus the sentence  $\varphi_1$  is local.

Consider now a well ordered model  $M$  of  $\varphi_1$ . The restriction of  $M$  to the domain  $Q^M$  and to the signature of  $S(\varphi_0)$  is a well ordered model of  $\varphi_0$  hence it is of order type  $\leq \omega$ . But the function  $g$  defines an injection from each initial segment of  $\neg Q$  into  $Q$  thus each initial segment of  $\neg Q$  is countable and this implies that the order type of  $\neg Q^M$  is smaller than or equal to  $\omega_1$ . Finally we have proved that the order type of  $M$  is  $\leq \omega_1$ .

Conversely it is easy to see that every ordinal  $\alpha \in [\omega, \omega_1]$  is the order type of some model of  $\varphi_1$ .  $\square$

Return now to the construction of the sentence  $\Phi$  given by Theorem 4.2. We are going to explain this construction by several successive steps.

A model  $M$  of  $\Phi$  will be totally ordered by  $<$  and will be the disjoint union of four successive segments. This will be expressed by the following sentence  $\Phi_1$  in the signature  $S(\Phi_1) = \{P_0, P_1, P_2, P_3\}$ , where  $P_0, P_1, P_2, P_3$ , are unary predicate symbols.  $\Phi_1$  is the conjunction of:

- (1)  $\forall xyz[(x \leq y \vee y \leq x) \wedge ((x \leq y \wedge y \leq x) \leftrightarrow x = y) \wedge ((x \leq y \wedge y \leq z) \rightarrow x \leq z)]$ ,
- (2)  $\forall xy \bigwedge_{0 \leq i < j \leq 3} [(P_i(x) \wedge P_j(y)) \rightarrow x < y]$ .

We want now to ensure that, if  $M$  is a well ordered model of  $\Phi$ , then  $P_0^M$  is of order type  $\leq \omega$  and  $P_1^M$  is of order type  $\leq \omega_1$ . For that purpose, the signature of  $\Phi$  will contain the signature  $S(\varphi_1) = S(\varphi_0) \cup \{Q, g\} = \{<, P, f, p_1, p_2, Q, g\}$  and  $\Phi$  will express that if  $M$  is a model of  $\Phi$ , then  $P_0^M = Q^M$  and the restriction of the model  $M$  to  $(P_0^M \cup P_1^M)$  and to the signature of  $\varphi_1$  is a model of  $\varphi_1$ .

This is expressed by the following sentence  $\Phi_2$  which is the conjunction of:

- (1)  $\forall x [Q(x) \leftrightarrow P_0(x)]$ ,
- (2)  $\forall xy [(x \in P_0 \cup P_1 \wedge y \in P_0 \cup P_1) \rightarrow f(x, y) \in P_0 \cup P_1]$ ,
- (3)  $\forall xy [(x \in P_0 \cup P_1 \wedge y \in P_0 \cup P_1) \rightarrow g(x, y) \in P_0 \cup P_1]$ ,
- (4)  $\forall x [(x \in P_0 \cup P_1) \rightarrow p_i(x) \in P_0 \cup P_1]$ , for each  $i \in [1, 2]$ ,
- (5)  $\forall xy [(x \notin P_0 \cup P_1 \vee y \notin P_0 \cup P_1) \rightarrow f(x, y) = x]$ ,
- (6)  $\forall xy [(x \notin P_0 \cup P_1 \vee y \notin P_0 \cup P_1) \rightarrow g(x, y) = x]$ ,
- (7)  $\forall x [x \notin P_0 \cup P_1 \rightarrow p_i(x) = x]$ , for each  $i \in [1, 2]$ ,
- (8)  $\forall x_1 \dots x_k \in (P_0 \cup P_1) [\varphi'_1(x_1, \dots, x_k)]$ , where  $\varphi_1 = \forall x_1 \dots x_k \varphi'_1(x_1, \dots, x_k)$  with  $\varphi'_1$  an open formula.

Above sentences (2)-(4) state that in a model  $M$  the set  $(P_0 \cup P_1)^M$  is closed under the functions of  $S(\varphi_1)$  while (5)-(7) state that these functions are triv-

ially defined elsewhere; (8) means that the restriction of the model  $M$  to the domain  $(P_0 \cup P_1)^M$  and to the signature of  $S(\varphi_1)$  is a model of  $\varphi_1$ .

We want now that, in a model  $M$  of  $\Phi$ , the set  $P_2^M$  represents the base set of a tree  $(T, \prec)$ . We shall use a binary relation symbol  $\prec$ . The following sentence  $\Phi_3$  is the conjunction of:

- (1)  $\forall xy[x \prec y \rightarrow P_2(x) \wedge P_2(y)]$ ,
- (2)  $\forall xyz[((x \prec y \wedge y \prec x) \leftrightarrow x = y) \wedge ((x \prec y \wedge y \prec z) \rightarrow x \prec z)]$ .
- (3)  $\forall xy[x \prec y \rightarrow x < y]$ .

Above sentences (1)-(2) express that  $\prec$  is a partial order on  $P_2$  and the sentence (3) ensures that, in a well ordered (for  $<$ ) model  $M$  of  $\Phi_3$ , for every  $t \in P_2$ , the set  $\{s \in P_2 \mid s \prec t\}$  is well ordered under  $\prec$  because  $M$  itself is well ordered under  $<$ .

Moreover we want now that in an  $\omega_2$ -model  $M$  of  $\Phi$ , the set  $P_2^M$  represents the base set of an  $\omega_1$ -tree  $T$  whose levels are countable.

We have firstly to distinguish the different levels of the tree  $T$ . We shall use for that purpose unary functions  $I$  and  $p$  and the following sentence  $\Phi_4$  conjunction of:

- (1)  $\forall xy \in P_2[(I(y) \leq y) \wedge (y \leq x \rightarrow I(y) \leq I(x)) \wedge (I(y) \leq x \leq y \rightarrow I(x) = I(y))]$ .
- (2)  $\forall xy \in P_2[x \prec y \rightarrow I(x) < I(y)]$ ,
- (3)  $\forall xyz \in P_2[(x \prec y \wedge z \prec y \wedge I(x) = I(z)) \rightarrow x = z]$ ,
- (4)  $\forall xy \in P_2[I(x) < I(y) \rightarrow (I(p(I(x), y)) = I(x) \wedge p(I(x), y) \prec y)]$ ,
- (5)  $\forall xy[(\neg P_2(x) \vee \neg P_2(y) \vee I(x) \neq x \vee I(x) \geq I(y)) \rightarrow p(x, y) = x]$ ,
- (6)  $\forall x[\neg P_2(x) \rightarrow I(x) = x]$ .

Above the sentence (1) is used to divide the segment  $P_2$  of a model of  $\Phi_4$  into successive segments. The function  $I$  is constant on each of these segments and the image  $I(x)$  of an element  $x \in P_2$  is the first element of the segment containing  $x$ .

Sentences (2)-(3) ensure that if  $y \in P_2$  then every element  $x \in P_2$  such that  $x \prec y$  belongs to some segment  $I_z = \{w \in P_2 \mid I(w) = I(z)\}$  for some  $z < I(y)$ . Moreover for each  $z < I(y)$ , the segment  $I_z$  contains at most one element of  $\{x \in P_2 \mid x \prec y\}$ .

The function  $p$  is used to ensure that, for each  $z < I(y)$ , the segment  $I_z$  contains in fact exactly one element  $x \in P_2$  such that  $x \prec y$ : the element  $p(I(z), y)$ . This is implied by the sentence (4).

Thus  $\Phi_4$  will imply that each segment  $I_z$  is really a level of the tree  $T$ .

If  $y \in P_2$  is at level  $\alpha$  of the tree  $T$  and if  $x \in P_2$  and  $I_x$  represents the  $\beta$ -th level  $T_\beta$  of the tree  $T$  for some  $\beta < \alpha$  (so  $I(x) < I(y)$ ), then the element  $p(I(x), y)$  is the unique element  $t \in T_\beta$  such that  $t \prec y$ .

Finally sentences (5)-(6) are used to trivially define the functions  $p$  and  $I$  where they are not useful as explained above.

The following sentence  $\Phi_5$  will imply that all levels of the tree  $T$  are countable and that  $ht(T) \leq \omega_1$  hence also  $card(T) \leq \aleph_1$ . The signature of  $\Phi_5$  is  $\{<, P_0, P_1, P_2, I, i, j\}$ , where  $i$  and  $j$  are two new unary function symbols, and  $\Phi_5$  is the conjunction of:

- (1)  $\forall x[P_2(x) \rightarrow P_0(i(x))]$ ,
- (2)  $\forall xy[(P_2(x) \wedge P_2(y) \wedge I(x) = I(y) \wedge x \neq y) \rightarrow i(x) \neq i(y)]$ ,
- (3)  $\forall x[P_2(x) \rightarrow P_1(j(x))]$ ,
- (4)  $\forall xy[(P_2(x) \wedge P_2(y) \wedge x < y) \rightarrow j(x) < j(y)]$ ,
- (5)  $\forall x[\neg P_2(x) \rightarrow i(x) = x]$ ,
- (6)  $\forall x[\neg P_2(x) \rightarrow j(x) = x]$ .

Above sentences (1)-(2) say that the function  $i$  is defined from  $P_2$  into  $P_0$  and that it is an injection from any level of the tree  $T$  into  $P_0$ . We have seen that in a well ordered model  $M$  of  $\Phi$  the set  $P_0^M$  will be of order type  $\leq \omega$  thus each level of the tree will be countable.

Sentences (3)-(4) say that the function  $j$  is strictly increasing from  $P_2$  into  $P_1$  thus in a well ordered model  $M$  of  $\Phi$  the set  $P_1^M$  hence also  $P_2^M$  will be of order type  $\leq \omega_1$ . So we shall have  $ht(T) \leq \omega_1$  and  $card(T) \leq \aleph_1$ .

Finally sentences (5)-(6) are used to trivially define the functions  $i$  and  $j$  on  $\neg P_2 = P_0 \cup P_1 \cup P_3$ .

In a well ordered model  $M$  of  $\Phi$  of order type  $\omega_2$ , the set  $P_2^M$  will be the base set of an  $\omega_1$ -tree  $T$  and the set  $P_3^M$  will be identified to a set of branches of  $T$ . For that purpose we use two new binary function symbols  $h$  and  $k$  and the following sentence  $\Phi_6$ , conjunction of:

- (1)  $\forall xy[(P_2(x) \wedge P_3(y)) \rightarrow (P_2(h(I(x), y)) \wedge I(h(I(x), y)) = I(x))]$ ,
- (2)  $\forall xyz[(P_2(x) \wedge P_2(y) \wedge P_3(z) \wedge I(x) < I(y)) \rightarrow h(I(x), z) \prec h(I(y), z)]$ ,
- (3)  $\forall xy[(\neg P_2(x) \vee \neg P_3(y) \vee x \neq I(x)) \rightarrow h(x, y) = x]$ ,
- (4)  $\forall xy[(P_3(x) \wedge P_3(y) \wedge x \neq y) \rightarrow (I(k(x, y)) = k(x, y) \wedge P_2(k(x, y)))]$ ,
- (5)  $\forall xy[(P_3(x) \wedge P_3(y) \wedge x \neq y) \rightarrow h(k(x, y), x) \neq h(k(x, y), y)]$ ,
- (6)  $\forall xy[(\neg P_3(x) \vee \neg P_3(y) \vee x = y) \rightarrow k(x, y) = x]$ .

Above sentences (1)-(2) are used to associate a branch  $b(z)$  of  $T$  to an element  $z \in P_3$ . For each level  $T_\alpha$  of the tree which is represented by the segment of  $P_2$  whose first element is  $I(x)$ , the sentence (1) says that  $h(I(x), z)$  is an element at the same level  $T_\alpha$  and (2) says that the elements  $h(I(x), z)$ , for  $x \in P_2$ , are linearly ordered for  $\prec$  hence they form a branch  $b(z)$  of the tree  $T$ .

The function  $k$  is used to associate to two different elements  $x$  and  $y$  of  $P_3$  a level of the tree  $T$ , which is represented by the element  $k(x, y)$ : the first

element of the segment of  $P_2$  representing this level. This is expressed by the sentence (4).

The sentence (5) says that, for two distinct elements  $x$  and  $y$  of  $P_3$ , the branches  $b(x)$  and  $b(y)$  differ at the level represented by  $k(x, y)$ .

Finally sentences (3) and (6) are used to trivially define the functions  $h$  and  $k$  in other cases.

We have seen that in a well ordered model  $M$  of  $\Phi$ ,  $P_1^M$  and  $P_2^M$  will be of order type  $\leq \omega_1$ . The following sentence  $\Phi_7$  will then imply that  $P_3^M$  is of order type  $\leq \omega_2$ . Its signature is  $\{<, P_1, P_3, l\}$ , where  $l$  is a binary function symbol, and  $\Phi_7$  is the conjunction of:

- (1)  $\forall xy[(P_3(x) \wedge P_3(y) \wedge y < x) \rightarrow P_2(l(x, y))]$ ,
- (2)  $\forall xyz[(P_3(x) \wedge P_3(y) \wedge P_3(z) \wedge y < z < x) \rightarrow l(x, y) \neq l(x, z)]$ ,
- (3)  $\forall xy[(\neg P_3(x) \vee \neg P_3(y) \vee \neg(y < x)) \rightarrow l(x, y) = x]$ .

Above sentences (1)-(3) are in fact very similar to sentences (8)-(10) used in the construction of the sentence  $\varphi_1$ .

(1) and (2) ensure that, for each  $x \in P_3$ , the binary function  $l$  realizes an injection from the segment  $\{y \in P_3 \mid y < x\}$  into  $P_2$  and (3) states that the function  $l$  is trivially defined where it is not useful for that purpose.

We can now define the sentence

$$\Phi = \bigwedge_{1 \leq i \leq 7} \Phi_i$$

in the signature

$$S(\Phi) = \bigwedge_{1 \leq i \leq 7} S(\Phi_i) = \{<, P_0, P_1, P_2, P_3, Q, p_1, p_2, f, g, \prec, p, I, i, j, h, k, l\}.$$

$\Phi$  is a conjunction of universal sentences thus it is equivalent to a universal sentence and closure in its models takes at most 7 steps: one takes firstly closure under the function  $l$  then under the functions  $I$  and  $k$ , then under the functions  $h$  and  $p$ , then under  $i$  and  $j$ , then under the function  $g$ , then under the functions  $p_1$  and  $p_2$ , and finally under the function  $f$ . Notice that the two last steps are due to the construction of  $\varphi_0$  and the fact that  $n_{\varphi_0} = 2$  (see [FR96]).

Assume now that  $M$  is a well ordered model of  $\Phi$ . By construction  $P_0^M$  is of order type  $\leq \omega$ ,  $P_1^M$  and  $P_2^M$  are of order types  $\leq \omega_1$ , and  $P_2^M$  is the base set of a tree  $T$  whose levels are countable. Moreover every strict initial segment of  $P_3^M$  is of cardinal  $\leq \aleph_1$ , so  $P_3^M$  is of order type  $\leq \omega_2$ . Finally we have got that  $M$  itself is of order type  $\leq \omega_2$ .

Suppose now that  $M$  is of order type  $\omega_2$ . Then  $P_3^M$  also is of order type  $\omega_2$



and for every strict initial segment  $J$  of  $P_3^M$  there is an injection from  $J$  into  $P_2^M$  thus  $P_2^M$  is of cardinal  $\aleph_1$ . But its order type is  $\leq \omega_1$ , hence it is in fact equal to  $\omega_1$ .

The tree  $T$  is then really an  $\omega_1$ -tree and all its levels are countable. Moreover the set  $P_3^M$  can be identified to a set of branches of  $T$  thus  $\text{card}(\mathcal{B}(T)) > \aleph_1$  and  $T$  is a Kurepa tree.

Conversely if there exists a Kurepa tree, we can easily see that  $\Phi$  has an  $\omega_2$ -model.  $\square$

We can now infer from Theorems 4.1 and 4.2 the following result which shows that the local theory of  $\omega_2$  is not determined by the axiomatic system  $ZFC + GCH$ .

**Theorem 4.5** *If the theory  $ZFC +$  “there is an inaccessible cardinal” is consistent then “ $\Phi$  has an  $\omega_2$ -model” is independent of  $ZFC + GCH$ .*

Notice that this result can be extended easily to ordinals larger than  $\omega_2$ . For instance reasoning as in the construction of the local sentence  $\varphi_1$  from the local sentence  $\varphi_0$  (see Lemma 4.4 above), we can construct by induction, for each integer  $n \geq 2$ , a local sentence  $\Phi_n$  such that: for all ordinals  $\alpha \in ]\omega_n, \omega_{n+1}]$ ,  
(  $\Phi_n$  has an  $\alpha$ -model ) iff (  $\Phi$  has an  $\omega_2$ -model ) iff ( there is a Kurepa tree ).  
This implies the following extension of Theorem 4.5.

**Theorem 4.6** *If the theory  $ZFC +$  “there is an inaccessible cardinal” is consistent then for each integer  $n \geq 2$  and each ordinal  $\alpha \in ]\omega_n, \omega_{n+1}]$ , “ $\Phi_n$  has an  $\alpha$ -model” is independent of  $ZFC + GCH$ .*

A similar result can be obtained for larger ordinals, for example for ordinals of cofinality  $\omega_n$ , for an integer  $n \geq 2$ , using some methods of [FR96].

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