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Probability invariance and entropy as a measure of uncertainty

Qiuping A. Wang

Institut Supérieur des Matériaux et Mécaniques Avancées du Mans, 44 Av. Bartholdi,
72000 Le Mans, France

Abstract

Starting from a variational definition of entropy as a measure of dynamical uncertainty, we discuss possible entropy forms for the probability distributions in exponential law, power law and Zipf-Mandelbrot law by consideration of their invariant properties. In this approach, other entropy forms different from the Shannon formula are possible for nonexponential probability distribution.

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1) Introduction

The relationship between probability distribution and the associated entropy as a measure of the dynamical uncertainty has always been a question in statistical and informational science. There are many relationships established from postulated properties of entropy. The reader can refer to the references [1] to [9] to see several examples of entropies proposed in the past decades. Among all these entropies, the most famous one is the Shannon informational entropy ($S = -\sum_i p_i \ln p_i$) [2] which was almost the only one widely used in equilibrium thermodynamics (Boltzmann-Gibbs entropy) and in nonequilibrium dynamics (Kolmogorov-Sinai entropy for example). Recently, a nonextensive statistics [6][8][9] (NES) proposed to use other entropies for thermodynamics and stochastic dynamics of certain nonextensive systems. NES has given rise to a large number of papers in the last decade with very different viewpoints dealing with equilibrium and nonequilibrium systems, which have incited more and more debates [10][11] within the statistical physics community. In the debates, we can distinguish two kinds of questions. The first one is about whether or not NES can be used for some given systems. The discussions are rather related to technical aspects of numerical simulation and of the definition of physical quantities such as temperature and pressure¹. The second one is more fundamental and about the physical foundation of the new entropies: whether or not it is necessary to replace Boltzmann-Gibbs-Shannon entropy with other ones in different physical situation? what should be the possible and useful forms of entropy? The present work is an investigation of these questions under a different angle without postulating mathematical properties of entropy as usual. We use entropy as a measure of uncertainty of given probability distribution and introduce a variational definition of entropy inspired by a thermodynamic relationship between entropy and energy. Then from the consideration of the invariant properties of certain observed probability distributions, we derive the entropy functionals. One of the objectives of this work is to show that, within the variational approach to stochastic dynamics, Shannon entropy may be replaced by other ones having different invariant properties and capable of yielding non-exponential probability distribution by their optimization with appropriate constraints.

¹ We would like to indicate here that the criticisms raised by R. Balian, M. Nauenberg and R. Luzzi *et al* about the violation of thermodynamic laws by NS due to the absence of temperature and pressure within the NS formalisms and about its failure for systems having different q [10] are flawed and misleading. A review of the results on these points can be found in [12].

2) Invariance and probability distribution laws

In this section, by some trivial calculations, we want to recall the invariant properties of some probability distributions and to relate these properties to the corresponding entropies.

a) Translation invariance and exponential law

Suppose that $f(x)$ is invariant by a translation of $x \rightarrow x+b$, i.e.

$$f(x+b) = g(b)f(x) \quad (1)$$

where $g(b)$ depending on b renormalizes $f(x+b)$ if the latter is a probability. We have

$$\frac{df(x+b)}{db} = \frac{df(x+b)}{d(x+b)} = g'(b)f(x) \quad (2)$$

Let $b=0$, we get

$$\frac{df(x)}{dx} = g'(0)f(x) \text{ or } \frac{df(x)}{f} = g'(0)dx \quad (3)$$

which means

$$\ln f(x) = g'(0)x + c \text{ or } f(x) = ce^{g'(0)x} \quad (4)$$

b) Scale invariance and power law

Suppose that $f(x)$ is scale invariant, we must have

$$f(bx) = g(b)f(x) \quad (5)$$

Where b some factor, $g(b)$ is a function of b . We make following calculation

$$\frac{df(bx)}{db} = \frac{df(bx)}{d(bx)} x = g'(b)f(x). \quad (6)$$

Let $b=1$, we get

$$\frac{df(x)}{dx} x = g'(1)f(x) \quad (7)$$

which means

$$f(x) = c x^{g'(1)}. \quad (8)$$

The Levy flight is a good example of power law with $g'(1) = -1 - \alpha$ where $0 < \alpha < 2$.

c) The q -exponential and its invariance properties

Here we would like to mention a probability functional which has attracted a lot of attention in the last years:

$$f(x) = c[1 - a\beta x]_a^{-1} \quad (9)$$

where a and β are some constants. The Zipf-Mandelbrot law $f(x) = c[1 + x]^{-\alpha}$ can be considered as a kind of q -exponential law. Another example of this law is the equilibrium thermodynamic distribution for finite systems in equilibrium with a finite heat bath, where a can be related to the number of elements N of the heat bath and tends to zero if N is very large[13].

This distribution is not a power law in the sense of Eq.(8). It has neither the scale invariance nor the translation invariance mentioned above. The operator on x that keeps $f(x)$ unchanged is a generalized addition $x+_a b = x + b + axb$.

3) A definition of entropy as a measure of dynamical uncertainty

Suppose we have a random (discrete) variable x with a probability distribution $p_i = f(x_i)$ where i is the state index. The average of x is given by $\bar{x} = \sum_i x_i p_i$ and the normalization is $\sum_i p_i = 1$. The uncertainty in this probability distribution of x can be measured by many quantities. For example, the standard deviation σ or the variance $\sigma^2 = \overline{x^2} - \bar{x}^2$ can surely be used if they exist. A disadvantage of σ^2 is that it may not exist for many probability distributions. Here we propose another measure which seems much general. This is a variational definition of entropy as a measure of uncertainty given by following relationship

$$dI = d\bar{x} - \bar{dx} = \sum_i x_i d p_i \quad (10)$$

This choice of uncertainty measure has been in a way inspired by the first and second laws of thermodynamics in equilibrium statistical thermodynamics. Considering the definition of internal energy $\bar{E} = \sum_i p_i E_i$ where E_i is the energy of the state i with probability p_i , we can

write $\delta\bar{E} = \sum_i \delta p_i E_i + \sum_i p_i \delta E_i = \sum_i \delta p_i E_i + \overline{\delta E_i}$. It can be proved that

$\overline{\delta E_i} = \sum_i p_i \delta E_i = \sum_j \left(\sum_i p_i \frac{\partial E_i}{\partial q_j} \right) \delta q_j$ is the work done to the system by external forces

$F_j = (\sum_i p_i \frac{\partial E_i}{\partial q_j})$ where q_j is extensive variables such as volume, distance or electrical polarization. According to the first law of thermodynamics, the quantity $\sum_i \delta p_i E_i = \delta \bar{E} - \overline{\delta E_i}$ must be the heat change in the system, that is, $\sum_i \delta p_i E_i = \delta Q = T \delta S$ for a reversible process, where S is the thermodynamic entropy and T the absolute temperature. Hence the thermodynamic entropy must satisfy the following variational relation

$$\delta S = \frac{1}{T} (\delta \bar{E} - \overline{\delta E}). \quad (11)$$

This relationship is extended in eq.(10) to arbitrary random variables x . By this definition, it is obvious that if the distribution is not exponential, the entropy functional may not be logarithmic.

4) Scale invariance and entropy functional

In this case, as shown above, we have power law probability distribution p_i :

$$p_i = \frac{1}{Z} x_i^{-a}. \quad (12)$$

So that:

$$dI = \sum_i (p_i Z)^{-1/a} d p_i = Z^{-1/a} \frac{1}{1-1/a} d \sum_i p_i^{1-1/a} = -Z^{-1/a} d [\sum_i p_i^{1-1/a} / (1-1/a) + c] \quad (13)$$

where c is an arbitrary constant. We finally get

$$I \propto \sum_i p_i^{1-1/a} / (1-1/a) + c \quad (14)$$

In order to determine c , we imagine a system with two states $i=1$ and 2 with $p_1 = 0$ and $p_2 = 1$. In this case, $I=0$ so that

$$\left(\frac{0+1}{1-1/a} \right) + c = 0 \quad (15)$$

i.e.,

$$c = -\frac{1}{1-1/a} \quad (16)$$

We finally get

$$I = \frac{\sum_i p_i^{1-1/a} - 1}{1-1/a} \quad (17)$$

Let $q = \frac{1}{a}$, we can write

$$I = -\frac{1 - \sum_i p_i^{1-q}}{1-q} = -\sum_i \frac{p_i - p_i^{1-q}}{1-q} \quad (18)$$

Notice that this functional is different from the Tsallis entropy in q -logarithm (see below). It does not yield Shannon entropy when $q \rightarrow 1$. As a matter of fact, q must be positive and smaller than unity. I may be negative if q is greater than unity, which is not possible. For Lévy flight for example, $1 < a < 3$, so $\frac{1}{3} < q < 1$.

It can be easily shown that the entropy in Eq.(18) is scale invariant. Its variation with probability value is shown in Figure 1. Its maximization with a Lagrange multiplier β such as $\delta(I - \beta \bar{x}) = 0$ yields the power law of Eq.(12) with $\beta = Z^q = Z^{1/a}$.

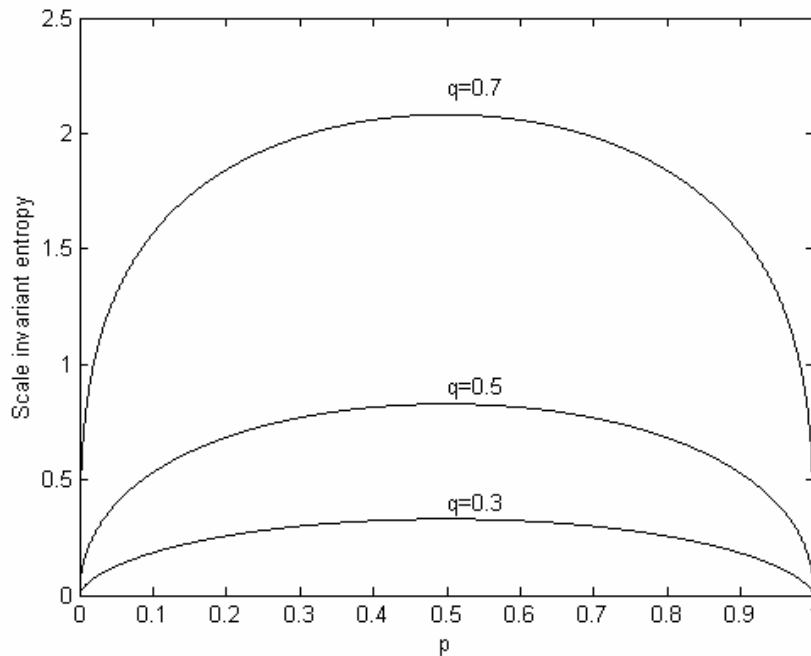


Figure 1, The variation of the scale invariant entropy $I = -\sum_{i=1}^2 \frac{p_i - p_i^{1-q}}{1-q}$ with $p_1=p$ and $p_2=1-p$ for different q values.

5) The entropy for q-exponential probability

We have seen above that the probability $p_i = c[1 - a\beta x_i]^{\frac{1}{a}}$ had a special invariant property.

Let us express x as a function of p_i and put it into Eq.(10) to get

$$dI = \sum_i \frac{1 - (p_i/c)^a}{a\beta} d p_i = -\frac{1}{a\beta c^a} \sum_i p_i^a d p_i = -\frac{1}{a\beta c^a (1+a)} d(\sum_i p_i^{1+a} + c) \quad (19)$$

By the same tricks for determining c in the above section, we get $c=-1$. So we can write

$$I = -\frac{\sum_i p_i^{1+a} - 1}{a} = -\sum_i \frac{p_i - p_i^q}{1-q} \quad (20)$$

Where $q=1+a$ and we have used the normalization $\sum_i p_i=1$. This is the Tsallis entropy which

tends to the Shannon entropy in the case of $q \rightarrow 1$ or $a \rightarrow 0$ where $p_i = c[1 - a\beta x_i]^{\frac{1}{a}}$ tends to an exponential distribution.

6) Concluding remarks

We have proposed an entropy as a measure of dynamical uncertainty with a single random variable x . This entropy has a variational definition such as $dI = \overline{dx} - \overline{dx}$ which can stem from the relation between energy and entropy in equilibrium thermodynamics and can be used for probability distributions of x as long as it has finite expectation value. On this basis, we have discussed the possible entropy forms for the probability distributions with three kinds of invariance: two translation invariances and scale invariance. The translation invariance of probability is associated with Shannon entropy, scale invariance with an entropy like

$$I = -\sum_i \frac{p_i^{1-q}}{1-q} \quad \text{where } 0 < q < 1, \text{ and the invariance for a generalized translation}$$

$$x_{+1-q} b = x + b + (1-q)xb \quad \text{with Tsallis entropy } I = -\sum_i \frac{p_i - p_i^q}{1-q} \quad \text{where } q \text{ is positive.}$$

We would like to mention that the present definition of entropy as a measure of dynamical uncertainty is in accordance with the variational approach in statistical and information theory, i.e., the determination of probability distribution from the optimization of entropy or uncertainty. It is easy to verify this with the three entropies above. Therefore they make sense and are particularly important from the point of view of variational approach. However, they are by no means the only forms of uncertainty measure even in these particular cases. We

would like also to mention that, in the present formalism, the entropy variational method using Lagrange multipliers can be written in a more concise form like $\overline{\delta x}=0$ for a single random variable, as shown in the context of an extended least action principle where we have $\overline{\delta A}=0$ for Lagrange action A on the random paths of stochastic dynamics[14].

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