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Estimation of bivariate excess probabilities for elliptical models

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Abstract

Let (X, Y) be a random vector whose conditional excess probability $\theta(x, y) := P(Y \leq y \mid X > x)$ is of interest. Estimating this kind of probability is a delicate problem as soon as x tends to be large, since the conditioning event becomes an extreme set. Assume that (X, Y) is elliptically distributed, with a rapidly varying radial component. In this paper, three statistical procedures are proposed to estimate $\theta(x, y)$, for fixed x, y , with x large. They respectively make use of an approximation result of Abdous *et al.* (cf. Abdous *et al.* (2005, Theorem 1)), a new second-order refinement of Abdous *et al.*'s Theorem 1, and a non-approximating method. The estimation of the conditional quantile function $\theta(x, \cdot)^{-}$ for large fixed x is also addressed, and these methods are compared via simulations.

Key words: Asymptotic independence; Conditional excess probability; Elliptic law; Rapidly varying tails.

1 Introduction

Let (X, Y) be a random pair whose conditional excess probability $\theta(x, y) := P(Y \leq y \mid X > x)$ is of interest. Estimating this kind of probability is a delicate problem as soon as x tends to be large, since the conditioning event becomes an extreme set. The conditional empirical distribution function then fails to be of any use, even if the probability $\theta(x, y)$ in itself is *not* a small probability, nor close to 1. Alternative methods have to be considered.

A classical approach is to call upon multivariate extreme value theory. Many refined inference procedures have been developed, making use of the structure of multivariate max-stable distributions introduced by de Haan and Resnick (1977), Pickands (1981), de Haan (1985). These procedures are successful in the rather general situation where (X, Y) are asymptotically dependent (for the maxima), which means heuristically that X and Y can be simultaneously large (see e.g. Resnick (1987) or Beirlant et al. (2004) for more details).

Efforts have recently been made to the problem in the opposite case of asymptotic independence. In some papers, attempts are made to provide models for joint tails, see e.g. Ledford and Tawn (1996, 1997, 1998), Draisma et al. (2004), Resnick (2002), Maulik and Resnick (2004). In a parallel way, Heffernan and Tawn (2004) explored modeling for multivariate tail regions which are not necessarily joint tails, and Heffernan and Resnick (2007) provided a complementary mathematical framework in the bivariate case.

In these papers, the key assumption is that there exists a limit for the conditional distribution of Y suitably centered and renormalized, given that X tends to infinity. This assumption was first checked for bivariate spherical distributions by Berman (1992, Theorem 12.4.1). Abdous et al. (2005) obtained it for bivariate elliptical distributions; Hashorva (2006) for multivariate elliptical distributions and Balkema and Embrechts (2007) for generalized multivariate elliptical distributions.

Elliptical distributions form a large family of multivariate laws, which have received considerable attention, especially in the financial risk context. See Artzner et al. (1999), Embrechts et al. (2003), Hult and Lindskog (2002), Demarta and McNeil (2005), among others. Assume from now on that (X, Y) is elliptically distributed; Theorem 1 of Abdous et al. (2005) exhibits the asymptotic behavior of $\theta(x, y)$ when $x \rightarrow \infty$ for such an elliptical pair. The appropriate rate $y = y(x)$ is explicated to get a non degenerate behavior of $\lim_{x \rightarrow \infty} \theta(x, y)$. This rate depends on the tail behavior of the radial random variable R defined by the relation $R^2 = (X^2 - 2\rho XY + Y^2)/(1 - \rho^2)$, where ρ is the Pearson correlation coefficient between X and Y . The only parameters involved in $y(x)$ and in the limiting distribution are: the Pearson correlation coefficient ρ , and the index of regular variation of R (say α) or an auxiliary function of R (say ψ), depending on whether R has a regularly or rapidly varying upper tail. Abdous et al. (2005) provided a simulation study in the specific case where R has a regularly varying tail. Existing estimators of ρ and α were used therein to obtain a practical way to estimate excess probabilities.

The aim of this paper is to focus on the case where the radial component R associated with the elliptical pair (X, Y) has a rapidly varying upper tail. A second-order approximation result is obtained, which refines Theorem 1 of Abdous *et al.* in the case of rapid variation of R . In the situation mentioned

above, the associated auxiliary function ψ has to be estimated. To the best of our knowledge, estimating the auxiliary function has never been considered in the literature. We propose three statistical procedures for the estimation of $\theta(x, y)$, for fixed x, y , with x large. They respectively make use of Abdous *et al.*'s approximation result, its second-order refinement, or an alternative method which does not rely on an analytic approximation result.

A related problem encountered in practice is to estimate the conditional quantile function $\theta(x, \cdot)^{\leftarrow}$ when x is fixed and large. This problem is also addressed in detail in the present paper, and the simulation study performed is presented in both terms of estimation of θ and $\theta(x, \cdot)^{\leftarrow}$.

The paper is organized as follows: The second-order approximation result is presented in Section 2, as well as some remarks and examples illustrating the theorem. The statistical procedures are described in Section 3, where a semi-parametric estimator of ψ is proposed. Section 4 deals with a comparative simulation study. Some concluding comments are provided in Section 5. Proofs are relegated to the Appendix.

2 Asymptotic approximation

Consider a bivariate elliptical random vector (X, Y) . General background on elliptical distributions can be found e.g. in Fang *et al.* (1990). One can focus without loss of theoretical generality on the standard case where $\mathbb{E} X = \mathbb{E} Y = 0$ and $\text{Var} X = \text{Var} Y = 1$. A convenient representation is then the following (see for example Hult and Lindskog (2002)): (X, Y) has a standard elliptical distribution with radial positive distribution function H and Pearson correlation coefficient ρ if it can be expressed as

$$(X, Y) = R \left(\cos U, \rho \cos U + \sqrt{1 - \rho^2} \sin U \right) ,$$

where R and U are independent, R has distribution function H with $\mathbb{E} R^2 = 2$ and U is uniformly distributed on $[-\pi/2, 3\pi/2]$. Hereafter, to avoid trivialities we assume $|\rho| < 1$.

Let Φ denote the normal distribution function and φ its density, i.e.

$$\varphi(t) = \frac{e^{-t^2/2}}{\sqrt{2\pi}} , \quad \Phi(x) = \int_{-\infty}^x \varphi(t) dt .$$

This paper deals with elliptical distributions with rapidly varying marginal upper tails, or equivalently with a rapidly varying radial component. More precisely, the radial component R associated with (X, Y) is assumed to be

such that there exists an *auxiliary function* ψ for which one gets, for any positive t

$$\lim_{x \rightarrow \infty} \frac{P\{R > x + t\psi(x)\}}{P(R > x)} = e^{-t}. \quad (1)$$

Such a condition implies that R belongs to the max-domain of attraction of the Gumbel distribution, see Resnick (1987, p.26) for more details. de Haan (1970) introduced this class of distributions as the Γ -*varying class*. The function ψ is positive, absolutely continuous and satisfies $\lim_{t \rightarrow \infty} \psi'(t) = 0$, $\lim_{t \rightarrow \infty} \psi(t)/t = 0$ and $\lim_{t \rightarrow \infty} \psi\{t + x\psi(t)\}/\psi(t) = 1$ for each positive x . It is only unique up to asymptotic equivalence.

Let us recall Abdous et al. (2005)'s result in this rapidly varying context (see Theorem 1, (ii)):

Theorem 1 *Let (X, Y) be a bivariate standardized elliptical random variable with Pearson correlation coefficient ρ and radial component satisfying (1). Then for each $z \in \mathbb{R}$ one has*

$$\lim_{x \rightarrow \infty} P(Y \leq \rho x + z\sqrt{1 - \rho^2}\sqrt{x\psi(x)} \mid X > x) = \Phi(z).$$

In the following subsection, a rate of convergence is provided for the approximation result stated in Theorem 1, as well as a second order correction. Note that if (X, Y) has an elliptic distribution with correlation coefficient ρ , then the couple $(X, -Y)$ has an elliptic distribution with correlation coefficient $-\rho$. Therefore, one can focus on non-negative ρ . From now on, assume that $\rho \geq 0$. As a consequence, one can assume that both $x > 0$ and $y > 0$.

2.1 Main result

For each distribution function H , denote by \bar{H} the survival function $\bar{H} = 1 - H$. The following assumption will be sufficient to obtain the main result. It is a strengthening of (1).

Hypothesis 2 *Let H be a rapidly varying distribution function such that*

$$\left| \frac{\bar{H}\{x + t\psi(x)\}}{\bar{H}(x)} - e^{-t} \right| \leq \chi(x)\Theta(t), \quad (2)$$

for all $t \geq 0$ and x large enough, where $\lim_{x \rightarrow \infty} \chi(x) = 0$, ψ satisfies

$$\lim_{x \rightarrow \infty} \frac{\psi(x)}{x} = 0; \quad (3)$$

and Θ is locally bounded and integrable over $[0, \infty)$.

The following result is a second-order approximation for conditional excess probabilities in the elliptical case with rapidly varying radial component.

Theorem 3 *Let (X, Y) be a bivariate elliptical vector with Pearson correlation coefficient $\rho \in [0, 1)$ and radial distribution H that satisfies Hypothesis 2. Then for all $x > 0$ and $z \in \mathbb{R}$,*

$$\begin{aligned} & \mathbb{P}(Y \leq \rho x + z\sqrt{1-\rho^2}\sqrt{x\psi(x)} \mid X > x) \\ &= \Phi(z) - \sqrt{\frac{\psi(x)}{x}} \frac{\rho \varphi(z)}{\sqrt{1-\rho^2}} + O\left(\chi(x) + \frac{\psi(x)}{x}\right), \end{aligned} \quad (4)$$

$$\mathbb{P}(Y \leq \rho x + z\sqrt{1-\rho^2}\sqrt{x\psi(x)} + \rho\psi(x) \mid X > x) \quad (5)$$

$$= \Phi(z) + O\left(\chi(x) + \frac{\psi(x)}{x}\right). \quad (6)$$

All the terms $O(\cdot)$ are locally uniform with respect to z .

Remark 4 *This result provides a rate of convergence in the approximation result of Abdous et al. (2005, Theorem 1) and a second-order correction. This correction is useful only if $\chi(x) = o(\sqrt{\psi(x)/x})$.*

Remark 5 *Theorem 3 and the formula*

$$\begin{aligned} & \mathbb{P}(X \leq x' ; Y \leq y \mid X > x) \\ &= \mathbb{P}(Y \leq y \mid X > x) - \mathbb{P}(X > x' \mid X > x)\mathbb{P}(Y \leq y \mid X > x') \end{aligned}$$

yield (with some extra calculations) the asymptotic joint distribution (and the rate of convergence) of (X, Y) given that $X > x$ when x is large: for all $x > 0$ and $z \in \mathbb{R}$,

$$\begin{aligned} & \mathbb{P}(X \leq x + t\psi(x) ; Y \leq \rho x + z\sqrt{1-\rho^2}\sqrt{x\psi(x)} \mid X > x) \\ &= (1 - e^{-t})\Phi(z) + O\left(\chi(x) + \sqrt{\frac{\psi(x)}{x}}\right), \end{aligned} \quad (7)$$

where all the terms $O(\cdot)$ are locally uniform with respect to z .

Remark 6 *Hashorva (2006) obtained that the conditional limit distribution of Y given that $X = x$ for multivariate elliptical vectors with rapidly varying radial component is also the Gaussian distribution. The equality of these two asymptotic distributions is not true in general. In the elliptical context, this is only true if the radial variable is rapidly varying. The conditional distribution of Y given that $X = x$ is of course related to the joint distribution of (X, Y)*

given that $X > x$ via the formula

$$\mathbb{P}(X \leq x' ; Y \leq y \mid X > x) = \int_x^{x'} \mathbb{P}(Y \leq y \mid X = u) P_X(du),$$

where P_X is the distribution of X . However, the limiting behavior of the integrand is not sufficient to obtain the limit of the integral, so that (7) is not a straightforward consequence of Hashorva's result.

2.2 Remarks and examples

Hypothesis 2 gives a rate of convergence in the conditional excess probability approximation. To the best of our knowledge, the literature deals more classically with second order conditions providing limits (see for instance Beirlant et al. (2004, Section 3.3)) or with pointwise or uniform rates of convergence (cf. e.g. de Haan and Stadtmüller (1996); Raoult and Worms (2003); Beirlant et al. (2003)). The need here is to have a non uniform bound that can be used for dominated convergence arguments. However, in the examples given below, the non uniform rates $\chi(x)$ that we exhibit are the same as the optimal uniform rates provided by de Haan and Stadtmüller (1996).

One can however check that Hypothesis 2 holds for usual rapidly varying functions, in particular for most of the so-called *Von Mises distribution functions*, which satisfy (see e.g. Resnick (1987, p.40))

$$\bar{H}(x) = d \exp \left\{ - \int_{x_0}^x \frac{ds}{\psi_H(s)} \right\}, \quad (8)$$

for x greater than some $x_0 \geq 0$, where $d > 0$ and ψ_H is positive, absolutely continuous and $\lim_{x \rightarrow \infty} \psi_H'(x) = 0$. Note that under this assumption, $\psi_H = \bar{H}/H'$, and ψ_H is an auxiliary function in the sense of (1). In the sequel, auxiliary functions ψ_H satisfying (8) will be called *Von Mises auxiliary functions*.

The following lemma provides sufficient conditions for Hypothesis 2, which could be weakened at the price of additional technicalities.

Lemma 7 *Let H be a Von Mises distribution function, and assume that,*

- (i) ψ_H is ultimately monotone, differentiable and $|\psi_H'|$ is ultimately decreasing;
- (ii) if ψ_H is decreasing, then either $\lim_{x \rightarrow \infty} \psi_H(x) > 0$ or there exist positive constants c_1 and c_2 such that for all $x \geq 0$ and $u \geq 0$,

$$\frac{\psi_H(x)}{\psi_H(x+u)} \leq c_1 e^{c_2 u}. \quad (9)$$

Then Hypothesis 2 holds for ψ_H :

$$\left| \frac{\bar{H}\{x + t\psi_H(x)\}}{\bar{H}(x)} - e^{-t} \right| \leq \chi(x)\Theta(t),$$

with $\chi(x) = O(|\psi'_H(x)|)$ and $\Theta(t) = O((1+t)^{-\kappa})$ for an arbitrary $\kappa > 0$.

Remark 8 Assumption (9) holds if ψ_H is regularly varying with index $\gamma < 0$. It also holds if $\psi_H(x) = e^{-cx}$ for some $c > 0$.

Remark 9 If ψ_H is regularly varying with index $\gamma < 1$, and ψ'_H ultimately decreasing, then $\psi'_H(x) = o(\sqrt{\psi_H(x)/x})$ and hence the second-order correction is useful.

Remark 10 This bound corresponds to a worst case scenario. In many particular cases, a much faster rate of convergence can be obtained. For instance, if $\bar{H}(x) = e^{-x}$, then (2) holds with $\psi_H \equiv 1$, whence $\chi(x) = 1/x$, but for any positive x and t , $\bar{H}\{x + t\psi_H(x)\}/\bar{H}(x) = e^{-t}$. The rate of convergence is infinite here. If \bar{H} is the Gumbel distribution, then $\psi_H \equiv 1$ and the rate of convergence in (2) is exponential: $|\bar{H}(x+t)/\bar{H}(x) - e^{-t}| \leq 2e^{-x}e^{-t}$.

In the following, some examples of distributions satisfying Hypothesis 2 are given for illustration.

Example 11 (Standard normal distribution). For the standard normal distribution, $\psi_H = \bar{\Phi}/\varphi$ satisfies the previous assumptions. More precisely, it is well known that $\psi_H(x) \equiv x^{-1} + O(x^{-3})$. Thus $\chi(x) = O(x^{-2})$.

Example 12 (Lognormal distribution). Consider the lognormal distribution $H(x) = \Phi(\log(x))$ and density function $h(x) = x^{-1}\varphi(\log(x))$. The auxiliary function that satisfies (8) is $\psi_H(x) = x\bar{\Phi}(\log(x))/\varphi(\log(x)) = x/\log(x) + O(x/\log^3(x))$. Thus (2) holds with $\chi(x) = O(1/\log(x))$.

Example 13 Consider the survival function $\bar{H}(x) = 2e^{-x^2}/(1 + e^{-x^2})$. For this distribution, $\psi_H(x) = (1 + e^{-x^2})/(2x)$ (see Table 1). Thus (2) holds with $\chi(x) = O(x^{-2})$.

Example 14 (Weibull-type distributions). If H satisfies (8) with $\psi_H(x) = cx^\gamma + O(x^\delta)$ with $\delta < \gamma$, then the rate of convergence is $\chi(x) = O(x^{\gamma-1})$. Note that for the elliptical Kotz distribution with parameter β used in the simulations (see Table 1), the radial component has a strict Weibull distribution with $\psi_H(x) = x^{1-\beta}/\beta$.

Example 15 (Discrete distribution in the domain of attraction of the Gumbel law). Let ψ be a concave increasing function such that $\lim_{x \rightarrow \infty} \psi(x) = +\infty$ and

$\lim_{x \rightarrow \infty} \psi'(x) = 0$ and define

$$\bar{H}_{\#}(x) = \exp \left\{ - \int_{x_0}^x \frac{ds}{\psi(s)} \right\}, \quad \bar{H}(x) = \bar{H}_{\#}([x]),$$

where $[x]$ is the integer part of x . Then H is a discrete distribution function, \bar{H} belongs to the domain of attraction of the Gumbel distribution, but does not satisfy Condition (8). Nevertheless, Condition (2) holds with

$$\chi(x) = O(\psi'([x]) + 1/\psi(x)), \quad \Theta(t) = O((1+t)^{-\kappa}), \quad (10)$$

for any arbitrary $\kappa > 0$.

One can deduce from Lemma 7 that Theorem 3 holds for $\psi = \psi_H$. The following lemma says what happens if one uses an asymptotically equivalent auxiliary function ψ instead of ψ_H in Theorem 3.

Lemma 16 *Under the assumption of Lemma 7, let ψ be equivalent to ψ_H at infinity and define $\xi(x) = |\psi(x) - \psi_H(x)|/\psi_H(x)$. Then*

$$\left| \frac{\bar{H}(x + t\psi(x))}{\bar{H}(x)} - e^{-t} \right| \leq O \{ |\psi'_H(x)| + \xi(x) \} \Theta(t). \quad (11)$$

Remark 17 *A consequence of Lemma 16 is that if an auxiliary function ψ is used instead of ψ_H in Theorem 3, then the second-order correction is relevant provided that $\xi(x) = o(\sqrt{\psi_H(x)/x})$. This is the case in Examples 11 and 13 if one takes $\psi(x) = 1/x$ or $\psi(x) = 1/(2x)$ respectively; in Example 12 when using $\psi(x) = x/\log(x)$. As for Example 14, one can take $\psi(x) = cx^\gamma$ if $\delta < (3\gamma - 1)/2$.*

3 Statistical procedure

For given large positive x and y , consider the problems of estimating $\theta(x, y) = \mathbb{P}(Y \leq y \mid X > x)$ and the conditional quantile function $\theta(x, \cdot)^{\leftarrow}$. The empirical distribution function is useless, since there might be no observations in the considered range. We suggest to estimate these quantities by means of Theorem 3.

Assume that a sample $(X_1, Y_1), \dots, (X_n, Y_n)$ is available, drawn from an elliptical distribution with radial component satisfying Hypothesis 2. Note that in this section we do not assume that the distribution is standardized, so that a preliminary standardization is required.

3.1 Definition of the estimators

The estimation of $\theta(x, y)$ and the conditional quantile function requires estimates of $\mu_X, \mu_Y, \sigma_X, \sigma_Y, \rho$ and ψ . Let $\hat{\mu}_X, \hat{\mu}_Y, \hat{\sigma}_X, \hat{\sigma}_Y, \hat{\rho}_n$ and $\hat{\psi}_n$ denote such estimates. For fixed $x, y > 0$, define

$$\hat{\theta}_{n,1}(x, y) = \Phi \left(\frac{\hat{y} - \hat{\rho}_n \hat{x}}{\sqrt{1 - \hat{\rho}_n^2} \sqrt{\hat{x} \hat{\psi}_n(\hat{x})}} \right), \quad (12)$$

$$\hat{\theta}_{n,2}(x, y) = \Phi \left(\frac{\hat{y} - \hat{\rho}_n \hat{x} - \hat{\rho}_n \hat{\psi}_n(\hat{x})}{\sqrt{1 - \hat{\rho}_n^2} \sqrt{\hat{x} \hat{\psi}_n(\hat{x})}} \right), \quad (13)$$

where $\hat{x} = (x - \hat{\mu}_X)/\hat{\sigma}_X$ and $\hat{y} = (y - \hat{\mu}_Y)/\hat{\sigma}_Y$.

In order to estimate the conditional quantile function $\theta(x, \cdot)^{\leftarrow}$, define, for fixed $\theta \in (0, 1)$,

$$\hat{y}_{n,1} = \hat{\mu}_Y + \hat{\sigma}_Y \left\{ \hat{\rho}_n \hat{x} + \sqrt{1 - \hat{\rho}_n^2} \sqrt{\hat{x} \hat{\psi}_n(\hat{x})} \Phi^{-1}(\theta) \right\}, \quad (14)$$

$$\hat{y}_{n,2} = \hat{\mu}_Y + \hat{\sigma}_Y \left\{ \hat{\rho}_n \hat{x} + \hat{\rho}_n \hat{\psi}_n(\hat{x}) + \sqrt{1 - \hat{\rho}_n^2} \sqrt{\hat{x} \hat{\psi}_n(\hat{x})} \Phi^{-1}(\theta) \right\}. \quad (15)$$

Estimating $\mu_X, \mu_Y, \sigma_X, \sigma_Y$ and ρ is a classical topic, and the empirical version of each quantity can easily be used. Under the assumption of elliptical distributions, however, one can observe a better stability when the Pearson correlation coefficient is estimated by $\hat{\rho}_n = \sin(\pi \hat{\tau}_n/2)$, where $\hat{\tau}_n$ is the empirical Kendall's tau. See e.g. Hult and Lindskog (2002) for more details. This estimator is \sqrt{n} -consistent and asymptotically normal.

Consider now the problem of the estimation of ψ . Since auxiliary functions are defined up to asymptotic equivalence, a particular representant must be a priori chosen in order to define the estimator. We assume that an admissible auxiliary function is

$$\psi(x) = \frac{1}{c\beta} x^{1-\beta}, \quad (16)$$

for some constants $c > 0$ and $\beta > 0$. Under this assumption, estimation of ψ boils down to estimating c and β .

A wide literature exists on estimators of β , see e.g. Beirlant et al. (1999), Gardes and Girard (2006), Dierckx et al. (2006), among others. The method chosen here is the one proposed in Beirlant et al. (1996). Let k_n be a user chosen threshold and $R_{j,n}, 1 \leq j \leq n$ be the order statistics of the sample

R_1, \dots, R_n . The estimator of β is obtained as the slope of the Weibull quantile plot at the point $(\log \log(n/k_n), \log(R_{n-k_n, n}))$:

$$\hat{\beta}_n = \frac{k_n^{-1} \sum_{i=1}^{k_n} \log \log(n/i) - \log \log(n/k_n)}{k_n^{-1} \sum_{i=1}^{k_n} \log(R_{n-i+1, n}) - \log(R_{n-k_n, n})}. \quad (17)$$

An estimator of c is then naturally given by

$$\hat{c}_n = \frac{1}{k} \sum_{i=1}^{k_n} \frac{\log(n/i)}{R_{n-i+1, n}^{\hat{\beta}_n}}. \quad (18)$$

Actually, in our context, the radial component is not observed. We estimate the R_i s by

$$\hat{R}_i^2 = \hat{X}_i^2 + (\hat{Y}_i - \hat{\rho}_n \hat{X}_i)^2 / (1 - \hat{\rho}_n^2),$$

where $\hat{X}_i = (X_i - \hat{\mu}_X) / \hat{\sigma}_X$ and $\hat{Y}_i = (Y_i - \hat{\mu}_Y) / \hat{\sigma}_Y$, and we plug these values in (17) and (18). Define then $\hat{\psi}_n(x) = x^{1-\hat{\beta}_n} / (\hat{c}_n \hat{\beta}_n)$.

3.2 Discussion of the estimation error versus approximation error

Denote

$$\begin{aligned} \hat{z}_{n,1} &= \frac{y - \hat{\rho}_n x}{\sqrt{1 - \hat{\rho}_n^2} \sqrt{x \hat{\psi}_n(x)}}, & \hat{z}_{n,2} &= \frac{y - \hat{\rho}_n x - \hat{\rho}_n \hat{\psi}_n(x)}{\sqrt{1 - \hat{\rho}_n^2} \sqrt{x \hat{\psi}_n(x)}}, \\ z_1 &= \frac{y - \rho x}{\sqrt{(1 - \rho^2)} \sqrt{x^{2-\beta} / (c\beta)}}, & z_2 &= \frac{y - \rho x - \rho x^{1-\beta} / (c\beta)}{\sqrt{(1 - \rho^2)} \sqrt{x^{2-\beta} / (c\beta)}}. \end{aligned}$$

Then, for $i = 1, 2$,

$$\hat{\theta}_{n,i}(x, y) - \theta(x, y) = \Phi(\hat{z}_{n,i}) - \Phi(z_i) + \Phi(z_i) - \theta(x, y).$$

This shows that the estimators defined in (12) have two sources of error: the first one, $\Phi(\hat{z}_{n,i}) - \Phi(z_i)$, comes from the estimation of ρ , μ , σ and ψ , and the second one, $\Phi(z_i) - \theta(x, y)$, from the asymptotic nature of the approximations (4) and (6).

The order of magnitude of the estimation error can be measured by the rate of convergence of the estimators. In order to obtain a rate of convergence for the estimators $\hat{\beta}_n$ and \hat{c}_n , we assume that H is a Von Mises distribution function with

$$\psi_H(x) = \frac{1}{c\beta} x^{1-\beta} \{1 + t(x)\}, \quad (19)$$

where t is a regularly varying function¹ with index $\eta\beta$ for some $\eta < 0$. This implies that $\bar{H}(x) = \exp\{-cx^\beta[1 + s(x)]\}$, where s is also regularly varying with index $\eta\beta$. Under this assumption, the function ψ defined in (16) is an admissible auxiliary function and Girard (2004) has shown that $\hat{\beta}_n$ is $k_n^{1/2}$ -consistent, for any sequence k_n such that

$$k_n \rightarrow \infty, \quad k_n^{1/2} \log^{-1}(n/k_n) \rightarrow 0, \quad k_n^{1/2} b(\log(n/k_n)) \rightarrow 0,$$

where b is regularly varying with index η , see (Girard, 2004, Theorem 2) for details. Similarly, it can be shown that $k_n^{1/2}(\hat{c}_n - c) = O_P(1)$ under the same assumptions on the sequence k_n . Thus, for any x , $\hat{\psi}_n(x)$ is a $k_n^{1/2}$ -consistent estimator of $\psi(x)$. Besides, $\hat{\rho}_n$, $\hat{\mu}_n$ and $\hat{\sigma}_n$ are \sqrt{n} -consistent, so the estimation error $\Phi(\hat{z}_{n,i}) - \Phi(z_i)$ is of order $k_n^{-1/2}$ in probability.

If ψ_H satisfies (19), then Hypothesis 2 holds, and Theorem 3 and Lemma 16 provide a bound for the deterministic approximation error $\Phi(z_i) - \theta(x, y)$. By Example 14, we see that the second order correction is useful only if $\eta < -1/2$.

3.3 Discussion of an alternative method

The method described previously makes use of the asymptotic approximations of Theorem 3. It could be thought that a direct method, making use of Formula (A.1) and of an estimator of \bar{H} , would yield a better estimate of $\theta(x, y)$, since it would avoid this approximation step. Recall however that we specifically need to estimate the tail of the radial distribution, so that a nonparametric estimator of H cannot be considered. The traditional solution given by extreme value theory consists in fitting a parametric model for the tail. This will always induce an approximation error.

Nevertheless, there exists a situation in which the approximation error can be canceled: If the radial component is exactly Weibull distributed, i.e. $\bar{H}(x) = \exp\{-cx^\beta\}$, then ψ_H satisfies (16), so for any $x, u > 0$, (8) implies that

$$\frac{\bar{H}(xu)}{\bar{H}(x)} = \exp \left\{ \int_x^{xu} \frac{ds}{\psi(s)} \right\}.$$

Therefore, in this specific case, a consistent estimator of $\theta(x, y)$, say $\hat{\theta}_{n,3}(x, y)$, can be introduced, which does not make use of any asymptotic expansion as

¹ A function f is regularly varying at infinity with index α if for all $t > 0$, $\lim_{x \rightarrow \infty} f(tx)/f(x) = t^\alpha$.

in Theorem 3. Through formula (A.1), we get explicitly:

$$\hat{\theta}_{n,3}(x, y) = 1 - \frac{\int_{\arctan(\hat{t}_0)}^{\pi/2} \hat{K}(\hat{x}, \hat{x}, \cos(u)) du + \int_{-\hat{U}_0}^{\arctan(\hat{t}_0)} \hat{K}(\hat{x}, \hat{y}, \sin(u + \hat{U}_0)) du}{2 \int_0^{\pi/2} \hat{K}(\hat{x}, \hat{x}, \cos(u)) du}, \quad (20)$$

where

$$\begin{aligned} \hat{K}(x, y, v) &= \exp \left\{ \int_x^{y/v} \frac{ds}{\hat{\psi}_n(s)} \right\}, \\ \hat{t}_0 &= (\hat{y}/\hat{x} - \hat{\rho}_n) / \sqrt{1 - \hat{\rho}_n^2}, \quad \hat{U}_0 = \arctan(\hat{\rho}_n / \sqrt{1 - \hat{\rho}_n^2}), \\ \hat{x} &= (x - \hat{\mu}_X) / \hat{\sigma}_X, \quad \hat{y} = (y - \hat{\mu}_Y) / \hat{\sigma}_Y. \end{aligned}$$

We included this estimator in the simulation study as a benchmark when looking at elliptical Kotz distributed observations (see Table 1 or Example 14).

4 Simulation study

To assess the performance of the proposed estimators, we simulated several families of bivariate standard elliptical distributions. Recall that a standardized bivariate elliptical density function can be written as $f(x, y) = Cg\{(x^2 - 2\rho xy + y^2)/(1 - \rho^2)\}$, where g is called the generator, ρ is the Pearson correlation coefficient and C is a normalizing constant. The density of the radial component R is given by $H'(r) = Kr g(r^2)$, where K is a normalizing constant (see, e.g., Fang et al. (1990)).

The distributions used are presented in Table 1. The Pearson correlation coefficient will be either $\rho = 0.5$ or $\rho = 0.9$. Three of them (Normal, Kotz and Logis) are Von Mises distributions which satisfy both Hypothesis 2 and (19); in addition, the Von Mises auxiliary function of the Kotz distribution satisfies (16). The Lognor and the modified Kotz distributions satisfy Hypothesis 2 but not (16), and finally the bivariate Student distribution has a regularly varying radial component, so it does not satisfy any of the assumptions. These three distributions are used to explore the robustness of the proposed estimation method.

In each case, 200 samples of size 500 were simulated. Several values of x were chosen, corresponding to the $(1 - p)$ -quantile of the marginal distribution of X , with $p = 10^{-3}$, $p = 10^{-4}$ and $p = 10^{-5}$. For each value of x , we computed (by numerical integration) the theoretical values of y corresponding to the

Table 1

Bivariate elliptical distributions used for the simulations. The elliptical generator g is given, the Von Mises auxiliary function ψ_H (or an equivalent), and the values of the parameters used (in addition to $\rho \in \{0.5, 0.9\}$).

Bivariate law	Generator $g(u)$	$\psi_H(x)$	Parameters
Normal	$e^{-u/2}$	$1/x + O(1/x^3)$	
Kotz	$u^{\beta/2-1} \exp(-u^{\beta/2})$	$x^{1-\beta}/\beta$	$\beta \in \{1, 4\}$
Logis [†]	$e^{-u}/(1 + e^{-u})^2$	$(1 + e^{-x^2})/(2x)$	
Modified Kotz	$g_\star(u)^\ddagger$	$x^{1-\beta}/(1 + \beta \log x)$	$\beta = 3/2$
Lognor [†]	$e^{-(\log^2 u)/8}/u$	$x/\log x + O(x/\log^3 x)$	
Student	$(1 + u/\nu)^{-(\nu+2)/2}$	–	$\nu \in \{3, 20\}$

[†] “Logis” and “Lognor” refer to the elliptical distributions with logistic and lognormal generator, respectively.

[‡] $g_\star(u) = \{(3/8) \log u + 1/2\} u^{-1/4} \exp\{-(1/2)u^{3/4} \log u\}$.

probability $\theta(x, y) = .05, .1, .2, \dots, .8, .9, .95$. Then we estimated $\theta(x, y)$ via the three proposed methods (cf. Section 3). For the estimation of the auxiliary function ψ , the threshold chosen corresponds to the highest 10% of the estimated \hat{R}_i s. It must be noted that this choice is independent of x . For each fixed x we also estimated the conditional quantile function $\theta(x, \cdot)^-$ by both methods (14) and (15). We did not compute the estimated quantile function for Method 3 since it would involve the numerical inversion of the integrals which appear in (A.1). This actually is one advantage of Methods 1 and 2 over Method 3.

Some general features can be observed, which comfort the theoretical expectations. (i) First of all, in the given range of x and y , there were hardly any observations, so that the empirical conditional distribution function is useless. (ii) For a given probability θ , the variability of the estimators slightly increases with x for all underlying distributions. For a given x , the variability of the estimators is greater for medium values of θ . (iii) The results for the Student distribution are as expected: if the degree of freedom ν is large, the estimation shows a high variability but moderate bias, while if ν is small, then the estimation is clearly inconsistent. (iv) The results for the Logis and modified Kotz distributions are similar to the Gaussian distribution. (v) As described in Section 3.3, Method 3 is markedly better for the Kotz distribution only.

Hence we have chosen to report only the results for the largest value of x (corresponding to the 10^{-5} -quantile of the marginal distribution of X) and the Normal, Kotz (with parameter $\beta = 1$ and $\beta = 4$) and Lognor distributions, for $\rho = .5$ and $\rho = .9$. Figures A.1-A.4 illustrate the behavior of the estimators of the probability θ : Median, 2.5% and 97.5% quantiles of the estimation error $\hat{\theta}_{n,i}(x, y) - \theta(x, y)$ ($i = 1, 2, 3$) are shown as a function of the estimated proba-

bility. Figure A.5 shows the estimated conditional quantile functions $\hat{y}_{n,i}(x, y)$ ($i = 1, 2$) and the theoretical conditional quantile function $y = \theta(x, \cdot)^{\leftarrow}$ for only three distributions and $\rho = .9$, because the results are much more stable as the correlation or the distribution vary. Median, 2.5% and 97.5% quantiles of the estimated conditional quantile function $\hat{y}_{n,i}(x, y)$ ($i = 1, 2$) are given as a function of the probability.

From these simulation results, one can see that the estimator of θ by Method 1 presents a systematic positive bias, which of course induces an underestimation of the conditional quantile function. As expected, Method 2 corrects this systematic bias; the correction is better when ρ is large. This is also true for the Lognormal generator, though in a lesser extent.

As already mentioned, the Lognor and modified Kotz distributions do not satisfy Assumption (19). In both cases the radial component belongs to an extended Weibull type family, with auxiliary function ψ of the form

$$\psi(x) = cx^{1-\beta}(\log x)^{-\delta}\{1 + o(1)\},$$

with $c > 0$, $\beta \geq 0$ and $\delta > 0$ if $\beta = 0$. The modified Kotz distribution corresponds to $\beta = 3/2$ and $\delta = 1$ and Lognor corresponds to $\beta = 0$ and $\delta = 1$. The simulation results are much better for the modified Kotz than for the Lognor distribution. This tends to prove that the method is not severely affected by the logarithmic factor, as long as $\beta > 0$.

5 Concluding remarks

In this paper we focused attention on elliptically distributed random pairs (X, Y) having a rapidly varying radial component. Three methods have been proposed to estimate the conditional excess probability $\theta(x, y)$ for large x . Under this specific assumption of elliptical distributions, Methods 2 and 3 revealed comparable results and outperformed Method 1.

Methods 1 and 2 make use of an asymptotic approximation of the conditional excess distribution function by the Gaussian distribution function. As shown by Balkema and Embrechts (2007), this approximation remains valid outside the family of elliptical distributions, under geometric assumptions on the level curves of the joint density function of (X, Y) . This suggests that these methods may be robust to departure from the elliptical family. This is the subject of current research.

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A Appendix

Proof of Theorem 3:

Define $U_0 = \arctan(\rho/\sqrt{1-\rho^2})$. For $x > 0$ and $y \in (0, x)$, we have

$$\mathbb{P}(X > x, Y > y) = \mathbb{P}\left(R > \frac{x}{\cos U} \vee \frac{y}{\rho \cos U + \sqrt{1-\rho^2} \sin U}; -U_0 \leq U \leq \frac{\pi}{2}\right).$$

Set $t_0 = (y/x - \rho)/\sqrt{1-\rho^2}$. Then $-U_0 < \arctan(t_0)$, and $x/\cos u > y/(\rho \cos u + \sqrt{1-\rho^2} \sin u)$ if and only if $u > \arctan(t_0)$. Hence,

$$\begin{aligned} \mathbb{P}(X > x, Y > y) &= \int_{\arctan(t_0)}^{\pi/2} \bar{H}\left\{\frac{x}{\cos u}\right\} \frac{du}{2\pi} \\ &\quad + \int_{-U_0}^{\arctan(t_0)} \bar{H}\left\{\frac{y}{\sin(u+U_0)}\right\} \frac{du}{2\pi}, \end{aligned}$$

$$\begin{aligned} \mathbb{P}(Y > y | X > x) &= \frac{\int_{\arctan(t_0)}^{\pi/2} \bar{H}(x/\cos(u)) du + \int_{-U_0}^{\arctan(t_0)} \bar{H}(y/\sin(u+U_0)) du}{2 \int_0^{\pi/2} \bar{H}(x/\cos(u)) du}. \quad (\text{A.1}) \end{aligned}$$

If $t_0 \geq 0$, i.e. $y - \rho x \geq 0$, the changes of variables $v = 1/\cos(u)$ and $v = 1/\sin(u+U_0)$ yield:

$$\mathbb{P}(Y > y | X > x) = \frac{I_1 + I_2}{I_3}, \quad (\text{A.2})$$

with

$$\begin{aligned} I_1 &= \int_{w_1}^{\infty} \frac{\bar{H}(vx)}{\bar{H}(x)} \frac{dv}{v\sqrt{v^2-1}}, \\ I_2 &= \int_{w_2}^{\infty} \frac{\bar{H}(vy)}{\bar{H}(x)} \frac{dv}{v\sqrt{v^2-1}}, \\ I_3 &= 2 \int_1^{\infty} \frac{\bar{H}(vx)}{\bar{H}(x)} \frac{dv}{v\sqrt{v^2-1}}, \\ w_1 &= \sqrt{1+t_0^2} = \sqrt{1+(y/x-\rho)^2/(1-\rho^2)}, \\ w_2 &= xw_1/y. \end{aligned}$$

If $t_0 < 0$, then

$$\mathbb{P}(Y > y | X > x) = \frac{I_3 - I_1 + I_2}{I_3}. \quad (\text{A.3})$$

Denote $w_0 = x(w_1 - 1)/\psi(x)$. In I_1 and I_3 , the change of variable $v = 1 + \frac{\psi(x)}{x}t$ yields

$$I_1 = \sqrt{\frac{\psi(x)}{x}} \int_{w_0}^{\infty} \frac{\bar{H}(x + t\psi(x))}{\bar{H}(x)} \frac{dt}{(1 + t\frac{\psi(x)}{x})\sqrt{1 + \frac{t\psi(x)}{2x}}\sqrt{2t}},$$

$$I_3 = 2\sqrt{\frac{\psi(x)}{x}} \int_0^{\infty} \frac{\bar{H}(x + t\psi(x))}{\bar{H}(x)} \frac{dt}{(1 + t\frac{\psi(x)}{x})\sqrt{1 + \frac{t\psi(x)}{2x}}\sqrt{2t}}.$$

In I_2 , the change of variable $vy = x + t\psi(x)$ yields

$$I_2 = \frac{\psi(x)}{x} \int_{w_0}^{\infty} \frac{\bar{H}(x + t\psi(x))}{\bar{H}(x)} \frac{(y/x) dt}{(1 + t\frac{\psi(x)}{x})\sqrt{1 - (y/x)^2 + 2t\frac{\psi(x)}{x} + \frac{\psi^2(x)}{x^2}t^2}}.$$

Set $J_i = \sqrt{x/\psi(x)}I_i$, $i = 1, 3$ and $J_2 = (x/\psi(x))I_2$. We start with I_1 and I_3 . We will use the following bound, valid for all $B, C > 0$,

$$0 \leq 1 - \frac{1}{(1 + C)\sqrt{1 + B}} \leq B/2 + C, \quad (\text{A.4})$$

which follows from straightforward algebra and the concavity of the function $x \mapsto \sqrt{1 + x}$. Applying this bound with $B = \frac{\psi(x)t}{x}$ and $C = \frac{\psi(x)}{x}t$ yields

$$0 \leq 1 - \frac{1}{(1 + t\frac{\psi(x)}{x})\sqrt{1 + \frac{t\psi(x)}{2x}}} \leq \frac{5}{4} \frac{\psi(x)}{x} t.$$

We thus have

$$|J_1 - \sqrt{2\pi}\bar{\Phi}(\sqrt{2w_0})| + |J_3 - \sqrt{2\pi}| \leq 3\chi(x) \int_0^{\infty} \Theta(t) \frac{dt}{\sqrt{2t}} + \frac{15}{16}\sqrt{2\pi} \frac{\psi(x)}{x}.$$

Hence

$$\frac{I_1}{I_3} = \bar{\Phi}(\sqrt{2w_0}) + O\left(\chi(x) + \frac{\psi(x)}{x}\right). \quad (\text{A.5})$$

Consider now J_2 . Applying the bound (A.4) with $C = t\psi(x)/x$ and

$$B = \{2t\frac{\psi(x)}{x} + \frac{\psi^2(x)}{x^2}t^2\}/\sqrt{1 - (y/x)^2},$$

and making use of Hypothesis 2, we obtain

$$\left| J_2 - \frac{(y/x)\sqrt{2\pi}\varphi(\sqrt{2w_0})}{\sqrt{1 - (y/x)^2}} \right|$$

$$\leq \frac{y/x}{\sqrt{1 - (y/x)^2}} \chi(x) \int_0^{\infty} \Theta(t) dt + \frac{y/x}{1 - (y/x)^2} \frac{\psi(x)}{x} \left(\sqrt{1 - \left(\frac{y}{x}\right)^2} + 1 + \frac{\psi(x)}{x} \right).$$

Choose $y = \rho x + \sqrt{1 - \rho^2} \sqrt{x\psi(x)}z$ for some fixed $z \in \mathbb{R}$. Then for large enough x it does hold that $0 < y < x$ and

$$\frac{y/x}{\sqrt{1 - (y/x)^2}} = \frac{\rho}{\sqrt{1 - \rho^2}} + O\left(\sqrt{\psi(x)/x}\right).$$

Thus,

$$\frac{I_2}{I_3} = \frac{\rho}{\sqrt{1 - \rho^2}} \sqrt{\frac{\psi(x)}{x}} \varphi(\sqrt{2w_0}) + O\left(\chi(x) + \frac{\psi(x)}{x}\right). \quad (\text{A.6})$$

For $z \geq 0$ and large enough x , plugging (A.5) and (A.6) into (A.2) yields

$$\theta(x, y) = \Phi(\sqrt{2w_0}) - \frac{\rho}{\sqrt{1 - \rho^2}} \sqrt{\frac{\psi(x)}{x}} \varphi(\sqrt{2w_0}) + O\left(\chi(x) + \frac{\psi(x)}{x}\right).$$

For $z < 0$ and large enough x , plugging (A.5) and (A.6) into (A.3) yields

$$\theta(x, y) = \bar{\Phi}(\sqrt{2w_0}) - \frac{\rho}{\sqrt{1 - \rho^2}} \sqrt{\frac{\psi(x)}{x}} \varphi(\sqrt{2w_0}) + O\left(\chi(x) + \frac{\psi(x)}{x}\right).$$

Note now that $w_0 = z^2/2 + O(\psi(x)/x)$, hence $\sqrt{2w_0} = |z| + O(\psi(x)/x)$. Thus, in both cases $z \geq 0$ and $z < 0$, (4) holds. Set $z = z' + \rho\sqrt{\psi(x)/x}/\sqrt{1 - \rho^2}$. A Taylor expansion of Φ and φ around z' yields (6). □

Proof of Lemma 7:

$$\begin{aligned} \frac{\bar{H}\{x + t\psi(x)\}}{\bar{H}(x)} - e^{-t} &= \exp\left\{-\int_x^{x+t\psi(x)} \frac{ds}{\psi(s)}\right\} - e^{-t} \\ &= \exp\left[-\int_0^t \frac{\psi(x)}{\psi\{x + s\psi(x)\}} ds\right] - e^{-t}. \end{aligned}$$

Applying the inequality $|e^{-a} - e^{-b}| \leq |a - b|e^{-a \wedge b}$ valid for all $a, b \geq 0$ yields

$$\begin{aligned} &\left| \frac{\bar{H}\{x + t\psi(x)\}}{\bar{H}(x)} - e^{-t} \right| \\ &\leq \left| \int_0^t \frac{\psi\{x + s\psi(x)\} - \psi(x)}{\psi\{x + s\psi(x)\}} ds \right| \exp\left[-t \wedge \int_0^t \frac{\psi(x)}{\psi\{x + s\psi(x)\}} ds\right]. \end{aligned}$$

Denote

$$\int_0^t \frac{|\psi\{x + s\psi(x)\} - \psi(x)|}{\psi\{x + s\psi(x)\}} ds = I(x, t)$$

and

$$\exp\left[-t \wedge \int_0^t \frac{\psi(x)}{\psi\{x + s\psi(x)\}} ds\right] = E(x, t).$$

Case ψ increasing. If ψ is nondecreasing, then $\psi' \geq 0$ and ψ' is decreasing. Thus, for any $\delta > 0$ and large enough x , $\psi'(x) \leq \delta$ and

$$\begin{aligned} \int_0^t \frac{\psi(x)}{\psi\{x + s\psi(x)\}} ds &\geq \int_0^t \frac{\psi(x)}{\psi(x) + s\psi'(x)\psi(x)} ds \\ &= \int_0^t \frac{ds}{1 + s\psi'(x)} \geq \int_0^t \frac{ds}{1 + s\delta} = \frac{1}{\delta} \log(1 + \delta t). \end{aligned}$$

This implies that $E(x, t) \leq (1 + \delta t)^{-1/\delta}$ for large enough x . Since ψ is increasing and ψ' is decreasing, we also have

$$I(x, t) \leq \psi'(x) \int_0^t \frac{s\psi(x)}{\psi\{x + s\psi(x)\}} ds \leq \psi'(x) \frac{t^2}{2}.$$

Thus, for any $\delta > 0$, $I(x, t)E(x, t) = O(|\psi'(x)|t^2(1 + \delta t)^{-1/\delta})$.

Case ψ decreasing. If ψ is monotone non increasing, then

$$\int_0^t \frac{\psi(x)}{\psi\{x + s\psi(x)\}} ds \geq t$$

and $E(x, t) \leq e^{-t}$. Also, since $|\psi'|$ is decreasing,

$$I(x, t) \leq |\psi'(x)| \int_0^t \frac{s\psi(x)}{\psi\{x + s\psi(x)\}} ds.$$

If ψ has a positive limit at infinity, then $I(x, t) = O(\psi'(x)t^2)$. Otherwise, $\lim_{x \rightarrow \infty} \psi(x) = 0$ and (9) holds. This yields, for large enough x ,

$$\begin{aligned} I(x, t) &\leq |\psi'(x)| \int_0^t \frac{s\psi(x)}{\psi\{x + s\psi(x)\}} ds \\ &\leq c_1 |\psi'(x)| \int_0^t se^{c_2 s\psi(x)} ds \leq c_1 |\psi'(x)| \int_0^t se^{s/2} ds. \end{aligned}$$

Thus $I(x, t) = O(|\psi'(x)|t^2 e^{t/2})$ and $I(x, t)E(x, t) = O(|\psi'(x)|t^2 e^{-t/2})$. This concludes the proof. \square

Proof of Lemma 16: Write

$$\left| \frac{\bar{H}\{x + t\psi(x)\}}{\bar{H}(x)} - e^{-t} \right| \leq \left| \frac{\bar{H}\{x + t\psi(x)\}}{\bar{H}\{x + t\psi_H(x)\}} - 1 \right| e^{-t} \quad (\text{A.7})$$

$$+ \frac{\bar{H}\{x + t\psi(x)\}}{\bar{H}\{x + t\psi_H(x)\}} \left| \frac{\bar{H}\{x + t\psi_H(x)\}}{\bar{H}(x)} - e^{-t} \right|, \quad (\text{A.8})$$

$$\frac{\bar{H}\{x + t\psi(x)\}}{\bar{H}\{x + t\psi_H(x)\}} = \exp \left[- \int_t^{t\psi(x)/\psi_H(x)} \frac{\psi_H(x)}{\psi\{x + s\psi_H(x)\}} ds \right].$$

If ψ_H is increasing, then

$$\left| \int_t^{t\psi(x)/\psi_H(x)} \frac{\psi_H(x)}{\psi_H\{x + s\psi_H(x)\}} ds \right| \leq t\xi(x), \quad (\text{A.9})$$

$$\exp \left[- \int_t^{t\psi(x)/\psi_H(x)} \frac{\psi_H(x)}{\psi_H\{x + s\psi_H(x)\}} ds \right] \leq e^{t\xi(x)}. \quad (\text{A.10})$$

Since $\xi(x) \rightarrow 0$, gathering (A.9) and (A.10) yields, for large enough x ,

$$\left| \frac{\bar{H}\{x + t\psi(x)\}}{H\{x + t\psi_H(x)\}} - 1 \right| e^{-t} \leq t\xi(x)e^{-t/2}. \quad (\text{A.11})$$

If ψ_H is decreasing and $\lim_{x \rightarrow \infty} \psi_H(x) > 0$, the ratio $\psi_H(x)/\psi_H\{x + s\psi(x)\}$ is bounded above and away from 0, so that

$$\left| \int_t^{t\psi(x)/\psi_H(x)} \frac{\psi_H(x)}{\psi\{x + s\psi_H(x)\}} ds \right| \leq Ct\xi(x),$$

and (A.11) still holds.

If ψ_H is decreasing and $\lim_{x \rightarrow \infty} \psi_H(x) = 0$, then applying (9) gives that the left-hand side of the previous equation is bounded by $Ct\xi(x) \exp\{2c_2\psi(x)t\}$. Thus, for large enough x , (A.11) still holds.

This provides a bound for the right-hand side of (A.7). The term (A.8) is bounded by Lemma 16. \square

Proof of (10): We proceed as in the proof of Lemma 7.

$$\begin{aligned} \frac{\bar{H}\{x + t\psi(x)\}}{H(x)} &= \exp \left\{ - \int_{[x]}^{[x+t\psi(x)]} \frac{ds}{\psi(s)} \right\} \\ &= \exp \left[- \int_0^{([x+t\psi(x)]-[x])/ \psi(x)} \frac{\psi(x)}{\psi([x] + s\psi(x))} ds \right], \end{aligned}$$

$$\begin{aligned} &\int_0^{([x+t\psi(x)]-[x])/ \psi(x)} \frac{\psi(x)}{\psi([x] + s\psi(x))} ds - t \\ &= \int_0^{([x+t\psi(x)]-[x])/ \psi(x)} \frac{\psi(x) - \psi([x] + s\psi(x))}{\psi([x] + s\psi(x))} ds + \frac{[x + t\psi(x)] - [x]}{\psi(x)} - t. \end{aligned}$$

By concavity of ψ ,

$$\frac{|\psi(x) - \psi([x] + s\psi(x))|}{\psi([x] + s\psi(x))} = O((s \vee 1)\psi'([x])).$$

By definition of the integral part,

$$t - 1/\psi(x) \leq \frac{[x + t\psi(x)] - [x]}{\psi(x)} \leq t + 1/\psi(x) .$$

The previous bounds yield

$$\left| \int_0^{([x+t\psi(x)]-[x])/\psi(x)} \frac{\psi(x)}{\psi([x] + s\psi(x))} ds - t \right| = O\left((t \vee 1)^2 \psi'([x]) + 1/\psi(x)\right) .$$

We must also give a lower bound for the integral. Since ψ is concave increasing, $\psi([x] + s\psi(x)) \leq \psi(x) + s\psi'(x)\psi(x)$, hence

$$\begin{aligned} \int_0^{([x+t\psi(x)]-[x])/\psi(x)} \frac{\psi(x)}{\psi([x] + s\psi(x))} ds &\geq \int_0^{t-1/\psi(x)} \frac{ds}{1 + s\psi'(x)} \\ &= \frac{1}{\psi'(x)} \log(1 + t\psi'(x) - \psi'(x)/\psi(x)) . \end{aligned}$$

Since $\psi'(x) \rightarrow 0$ and $\psi(x) \rightarrow \infty$, the arguments of the proof of Lemma 7 can be applied again to conclude the proof of (10). \square

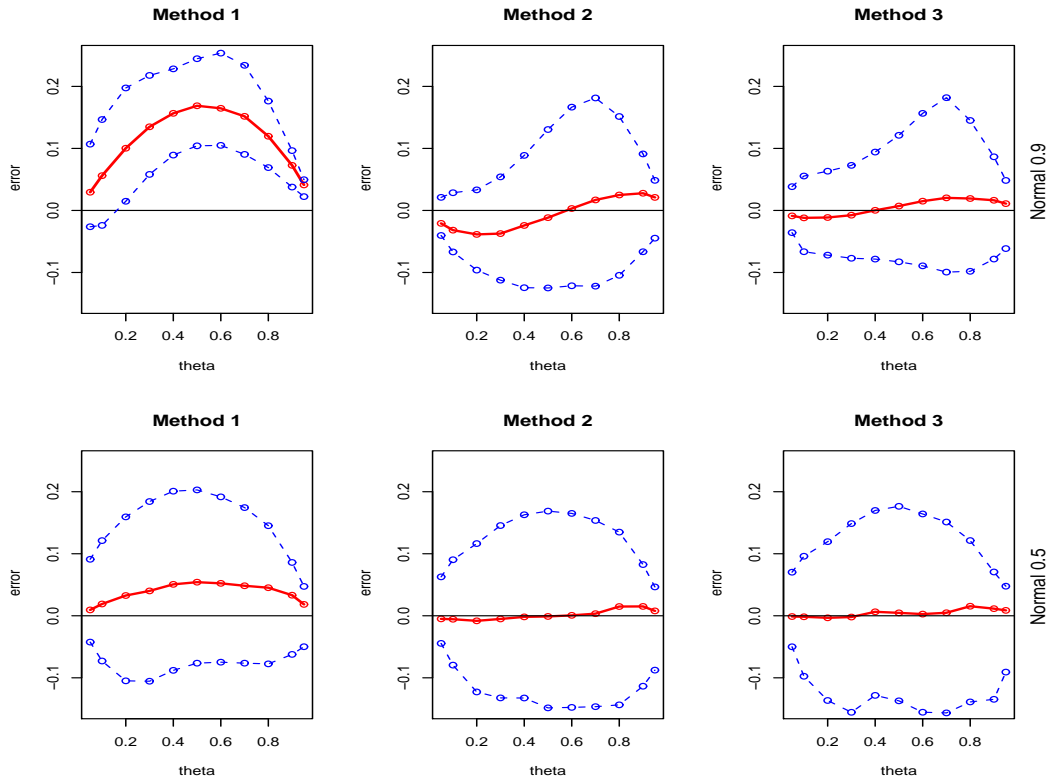


Fig. A.1. Median, 2.5% and 97.5% quantiles of the estimation error $\hat{\theta}_{n,i}(x, y) - \theta(x, y)$ ($i = 1, 2, 3$) as a function of the estimated probability. Gaussian distribution. First row: $\rho = .9$; second row: $\rho = .5$.

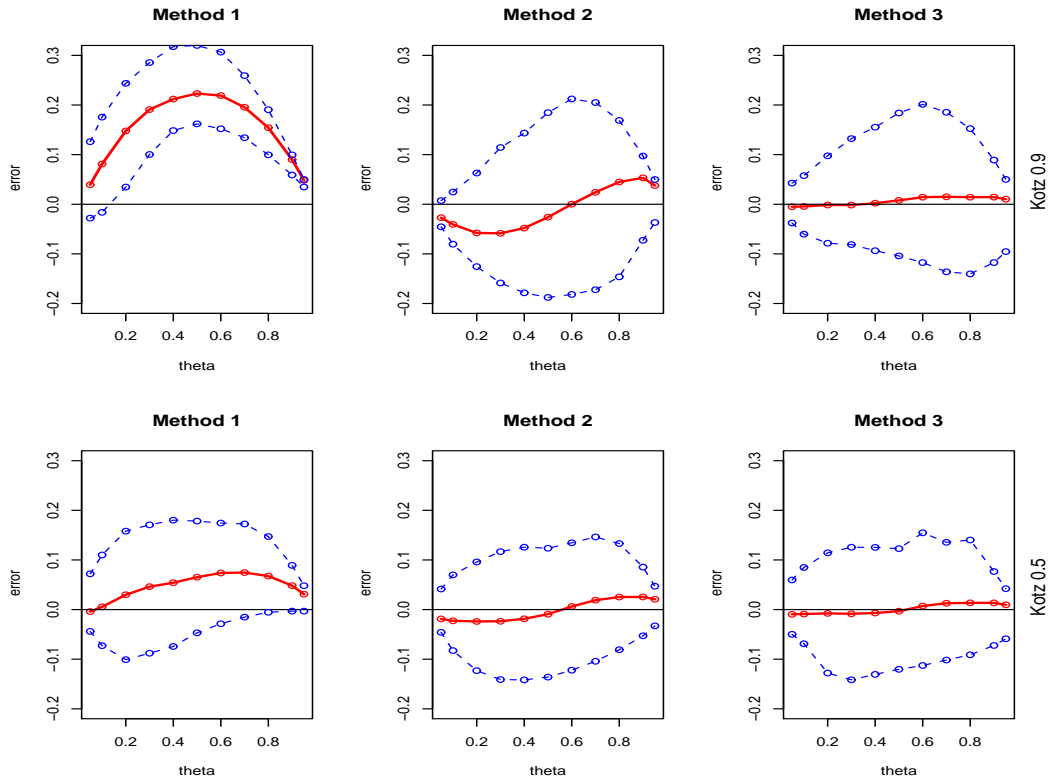


Fig. A.2. Median, 2.5% and 97.5% quantiles of the estimation error $\hat{\theta}_{n,i}(x, y) - \theta(x, y)$ ($i = 1, 2, 3$) as a function of the estimated probability. Kotz distribution, $\beta = 1$. First row: $\rho = .9$; second row: $\rho = .5$.

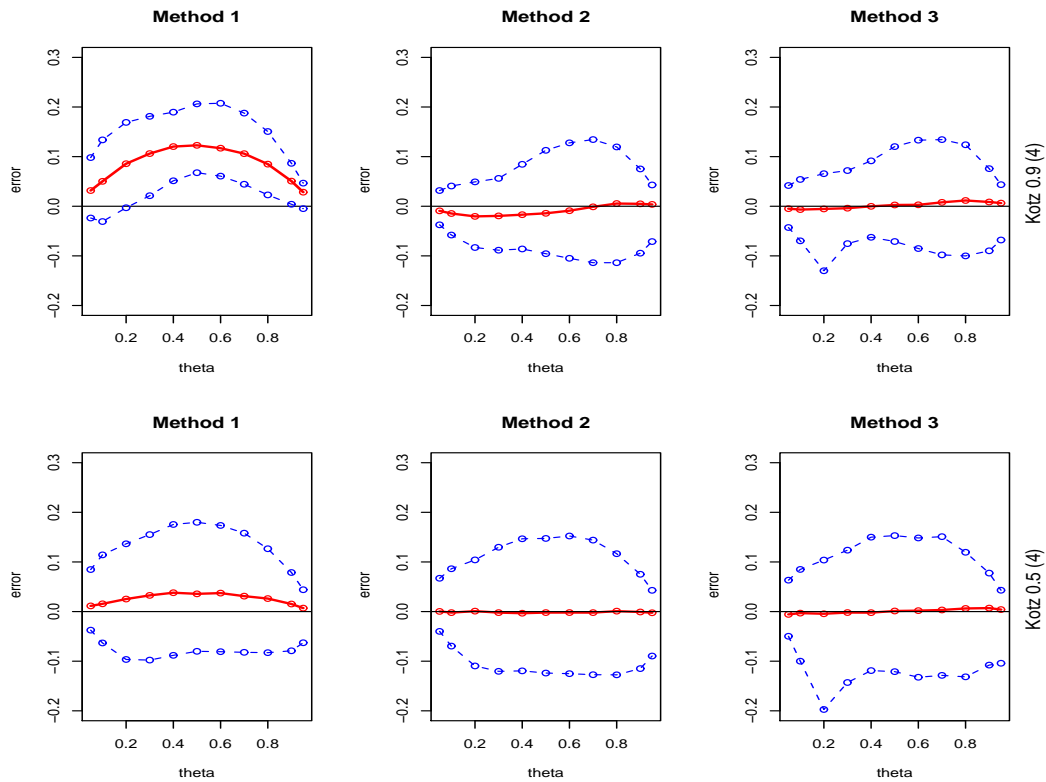


Fig. A.3. Median, 2.5% and 97.5% quantiles of the estimation error $\hat{\theta}_{n,i}(x,y) - \theta(x,y)$ ($i = 1,2,3$) as a function of the estimated probability. Kotz distribution, $\beta = 4$. First row: $\rho = .9$; second row: $\rho = .5$.

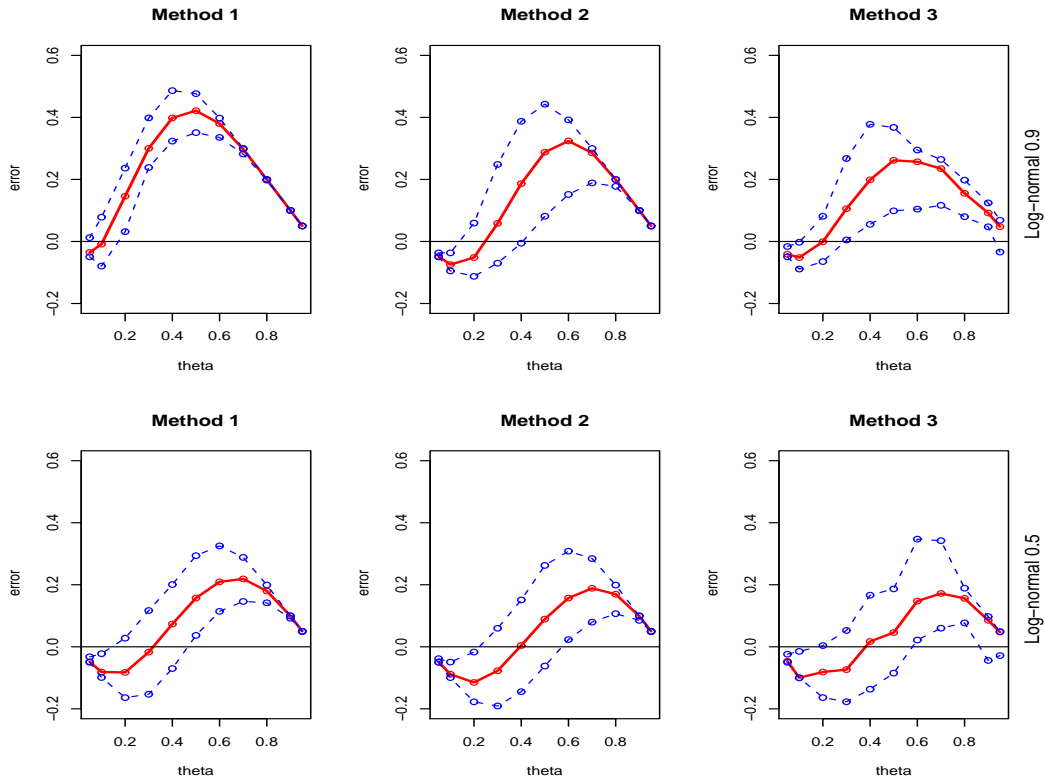


Fig. A.4. Median, 2.5% and 97.5% quantiles of the estimation error $\hat{\theta}_{n,i}(x,y) - \theta(x,y)$ ($i = 1,2,3$) as a function of the estimated probability. Lognormal distribution. First row: $\rho = .9$; second row: $\rho = .5$.

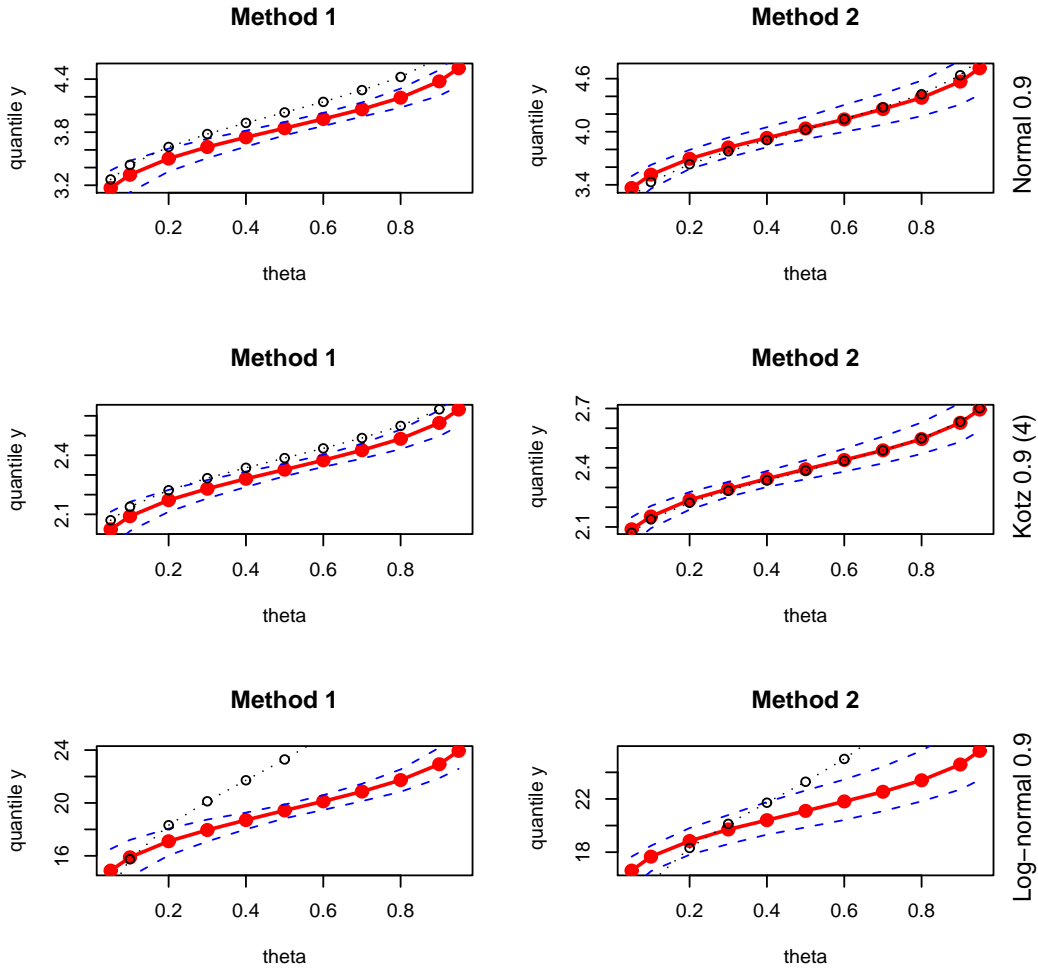


Fig. A.5. Median (solid line), 2.5% and 97.5% quantiles (dashed lines) of the estimated conditional quantile function $\hat{y}_{n,i}$ ($i = 1, 2$) defined in (14) and (15) and theoretical conditional quantile function y (dotted line) as a function of the probability θ . First row: Normal distribution; second row: Kotz distribution, $\beta = 4$; Lognormal distribution. For each of them $\rho = .9$.