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Estimation of bivariate excess probabilities for elliptical models

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Abstract

Let (X, Y) be a random vector whose conditional excess probability $\theta(x, y) := P(Y \leq y \mid X > x)$ is of interest. Estimating this kind of probability is a delicate problem as soon as x tends to be large, since the conditioning event becomes an extreme set. Assume that (X, Y) is elliptically distributed, with a rapidly varying radial component. In this paper, three statistical procedures are proposed to estimate $\theta(x, y)$, for fixed x, y , with x large. They respectively make use of an approximation result of Abdous *et al.* (cf. Abdous *et al.* (2005, Theorem 1)), a new second-order refinement of Abdous *et al.*'s Theorem 1, and a non-approximating method. The estimation of the conditional quantile function $\theta(x, \cdot)^+$ for large fixed x is also addressed, and these methods are compared via simulations.

Key words: Conditional excess probability, asymptotic independence, elliptic law.

1 Introduction

Let (X, Y) be a random pair whose conditional excess probability $\theta(x, y) := P(Y \leq y \mid X > x)$ is of interest. Estimating this kind of probability is a delicate problem as soon as x tends to be large, since the conditioning event becomes an extreme set. The conditional empirical distribution function then

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fails to be of any use, even if the probability $\theta(x, y)$ in itself is *not* a small probability, nor close to 1. Alternative methods have to be considered.

A classical approach is to call upon multivariate extreme value theory. Many refined inference procedures have been developed, making use of the structure of multivariate max-stable distributions introduced by de Haan and Resnick (1977), Pickands (1981), de Haan (1985). These procedures are successful in the rather general situation where (X, Y) are asymptotically dependent (for the maxima), which means heuristically that X and Y can be simultaneously large (see e.g. Resnick (1987) or Beirlant et al. (2004) for more definitions).

Efforts have recently been done to solve the problem in the opposite case of asymptotic independence. In some papers, attempts are made to provide models for joint tails, see e.g. Ledford and Tawn (1996, 1997, 1998), Draisma et al. (2004), Resnick (2002), Maulik and Resnick (2004). In a parallel way, Heffernan and Tawn (2004) explored modeling for multivariate tail regions which are not necessarily joint tails, and Heffernan and Resnick (2005) provided a complementary mathematical framework in the bivariate case.

In these papers, the key assumption is that there exists a limit for the conditional distribution of Y suitably centered and renormalized, given that X tends to infinity. Denote by \mathcal{A} this (informal) assumption. The earliest result is due to Berman (1992, Theorem 12.4.1), who proved that \mathcal{A} holds for bivariate spherical distributions. Abdous et al. (2005) obtained it for bivariate elliptical distributions; Hashorva (2006) for multivariate elliptical distributions and Balkema and Embrechts (2004) for generalized multivariate elliptical distributions.

Elliptical distributions form a large family of multivariate laws, which received an important attention, especially in the financial and risk contexts. See Artzner et al. (1999), Embrechts et al. (2003), Hult and Lindskog (2002), Demarta and McNeil (2005), among others. Assume from now on that (X, Y) is elliptically distributed; Theorem 1 of Abdous et al. (2005) exhibits the asymptotic behavior of $\theta(x, y)$ when $x \rightarrow \infty$ for such an elliptical pair. The appropriate rate $y = y(x)$ is explicited to get a non degenerate behavior of $\lim_{x \rightarrow \infty} \theta(x, y)$. This rate depends on the tail behavior of the radial random variable R defined by the relation $R^2 = (X^2 - 2\rho XY + Y^2)/(1 - \rho^2)$, where ρ is the Pearson correlation coefficient between X and Y . The only parameters involved in $y(x)$ and in the limiting distribution are: the Pearson correlation coefficient ρ , and the index of regular variation of R (say α) or an auxiliary function of R (say ψ), depending on whether R has a regularly or rapidly varying upper tail. Abdous et al. (2005) provided a simulation study in the specific case where R has a regularly varying tail. Existing estimators of ρ and α were used therein to obtain a practical way to estimate excess probabilities.

The aim of this paper is to focus on the case where the radial component R associated with the elliptical pair (X, Y) has a rapidly varying upper tail. A second-order approximation result is obtained, which refines Theorem 1 of Abdous *et al.* in the case of rapid variation of R . In the situation mentioned above, the associated auxiliary function ψ has to be estimated. To the best of our knowledge, this estimation problem has never been considered in the literature. We propose three statistical procedures for the estimation of $\theta(x, y)$, for fixed x, y , with x large. They respectively make use of Abdous *et al.*'s approximation result, its second-order refinement, or an alternative method which does not rely on an analytic approximation result.

A related problem encountered in practice is to estimate the conditional quantile function $\theta(x, \cdot)^{\leftarrow}$ when x is fixed and large. This problem is also addressed in detail in the present paper, and the simulation study performed is presented in both terms of estimation of θ and $\theta(x, \cdot)^{\leftarrow}$.

The rest of the paper is organized as follows: The second-order approximation result is presented in Section 2, as well as some remarks and examples illustrating the theorem. The statistical procedures are described in Section 3, where a semi-parametric estimator of ψ is proposed. Section 4 deals with a comparative simulation study. Some concluding comments are discussed in Section 5. Proofs are relegated to the Appendix.

2 Asymptotic approximation

Consider a bivariate elliptical random vector (X, Y) . General background on elliptical distributions can be found e.g. in Fang et al. (1990). One can focus without loss of generality on the standard case where $\mathbb{E}X = \mathbb{E}Y = 0$ and $\text{Var}X = \text{Var}Y = 1$. A convenient representation is then the following (see for example Hult and Lindskog (2002)): (X, Y) has a standard elliptical distribution with radial positive distribution function H and Pearson correlation coefficient ρ if it can be expressed as

$$(X, Y) = R \left(\cos U, \rho \cos U + \sqrt{1 - \rho^2} \sin U \right) ,$$

where R and U are independent, R has distribution function H and U is uniformly distributed on $[-\pi/2, 3\pi/2]$.

Let Φ denote the normal distribution function and φ its density, i.e.

$$\varphi(t) = \frac{e^{-t^2/2}}{\sqrt{2\pi}} , \quad \Phi(x) = \int_{-\infty}^x \varphi(t) dt .$$

This paper deals with elliptical distributions with rapidly varying marginal upper tails, or equivalently with a rapidly varying radial component. More precisely, the radial component R associated with (X, Y) is assumed to be such that there exists an *auxiliary function* ψ for which one gets, for any positive t

$$\lim_{x \rightarrow \infty} \frac{P\{R > x + t\psi(x)\}}{P(R > x)} = e^{-t}. \quad (1)$$

Such a condition is equivalent for R to belong to the max-domain of attraction of the Gumbel distribution, see Resnick (1987, p.26) for more details. de Haan (1970) introduced this class of distributions as the Γ -*varying class*. The function ψ is positive, absolutely continuous and satisfies $\lim_{t \rightarrow \infty} \psi'(t) = 0$, $\lim_{t \rightarrow \infty} \psi(t)/t = 0$ and $\lim_{t \rightarrow \infty} \psi\{t + x\psi(t)\}/\psi(t) = 1$. It is only unique up to asymptotic equivalence. Note that the elliptical distributions with such a rapidly varying radial component are precisely those belonging to the domain of attraction of the independence.

Let recall Abdous et al. (2005)'s result in this rapidly varying context (see Theorem 1, (ii)):

Theorem 1 *Let (X, Y) be a bivariate standardized elliptical random variable with Pearson correlation coefficient ρ and radial component satisfying (1). Then for each $z \in \mathbb{R}$ one has*

$$\lim_{x \rightarrow \infty} P(Y \leq \rho x + z\sqrt{1 - \rho^2}\sqrt{x\psi(x)} \mid X > x) = \Phi(z).$$

In the following subsection, a rate of convergence is provided for the approximation result stated in Theorem 1, as well as a second order correction. Note that if (X, Y) has an elliptic distribution with correlation coefficient ρ , then the couple $(X, -Y)$ has an elliptic distribution with correlation coefficient $-\rho$. Therefore, one can focus on non-negative ρ . From now on, assume that $\rho \geq 0$. As a consequence, one can assume that both $x > 0$ and $y > 0$.

2.1 Main result

For each distribution function H , denote by \bar{H} the survival function $\bar{H} = 1 - H$. The following assumption will be needed to get the main result. It is a strengthening of (1).

Hypothesis 2 *Let H be a rapidly varying distribution function such that*

$$\left| \frac{\bar{H}\{x + t\psi(x)\}}{\bar{H}(x)} - e^{-t} \right| \leq \chi(x)\theta(t), \quad (2)$$

for all $t \geq 0$ and x large enough, where $\lim_{x \rightarrow \infty} \chi(x) = 0$, ψ satisfies

$$\lim_{x \rightarrow \infty} \frac{\psi(x)}{x} = 0 ; \quad (3)$$

and θ satisfies

$$\int_0^\infty t^2 \theta(t) dt < \infty . \quad (4)$$

The following result is a second-order approximation for conditional excess probabilities in the elliptical case with rapidly varying radial component.

Theorem 3 *Let (X, Y) be a bivariate elliptical vector with Pearson correlation coefficient $\rho \in [0, 1)$ and radial distribution H that satisfies Hypothesis 2. Then for all $x > 0$ and $z \in \mathbb{R}$,*

$$\begin{aligned} \mathbb{P}(Y \leq \rho x + z\sqrt{1 - \rho^2}\sqrt{x\psi(x)} \mid X > x) \\ = \Phi(z) - \sqrt{\frac{\psi(x)}{x}} \frac{\rho \varphi(z)}{\sqrt{1 - \rho^2}} + O\left(\chi(x) + \frac{\psi(x)}{x}\right) , \end{aligned} \quad (5)$$

$$\mathbb{P}(Y \leq \rho x + z\sqrt{1 - \rho^2}\sqrt{x\psi(x)} + \rho\psi(x) \mid X > x) \quad (6)$$

$$= \Phi(z) + O\left(\chi(x) + \frac{\psi(x)}{x}\right) . \quad (7)$$

All the terms $O(\cdot)$ are locally uniform with respect to z .

Remark 4 *This result provides a rate of convergence in the approximation result of Abdous et al. (2005, Theorem 1) and a second-order correction. This correction is meaningful only if $\chi(x) = o(\sqrt{\psi(x)/x})$.*

2.2 Remarks and examples

Hypothesis 2 gives a rate of convergence in the conditional excess probability approximation. To the best of our knowledge, the literature deals more classically with second order conditions providing limits (see for instance Beirlant et al. (2004, Section 3.3)) or with pointwise or uniform rates of convergence (cf. e.g. Raoult and Worms (2003); Beirlant et al. (2003)). The need here is to have a non uniform bound that can be used for dominated convergence arguments.

One can however check that Hypothesis 2 holds for usual rapidly varying functions, in particular for most of the so-called *Von Mises distribution functions*,

which satisfy (see e.g. Resnick (1987, p.40))

$$\bar{H}(x) = c \exp \left\{ - \int_{x_0}^x \frac{ds}{\psi_H(s)} \right\} , \quad (8)$$

where $c > 0$ and ψ_H is positive, absolutely continuous and $\lim_{x \rightarrow \infty} \psi_H'(x) = 0$. Note that under this assumption, $\psi_H = \bar{H}/H'$, and ψ_H is an auxiliary function in the sense of (1).

The following lemma proves Hypothesis 2 under mild restrictions, which could be weakened at the price of additional technicalities.

Lemma 5 *Let H be a Von Mises distribution function, and assume that,*

- (i) ψ_H is ultimately monotone, differentiable and $|\psi_H'|$ is ultimately decreasing;
- (ii) if ψ_H is decreasing, then either $\lim_{x \rightarrow \infty} \psi_H(x) > 0$ or there exist positive constants c_1 and c_2 such that for all $x \geq 0$ and $u \geq 0$,

$$\frac{\psi_H(x)}{\psi_H(x+u)} \leq c_1 e^{c_2 u} . \quad (9)$$

Then Hypothesis 2 holds for ψ_H :

$$\left| \frac{\bar{H}\{x + t\psi_H(x)\}}{\bar{H}(x)} - e^{-t} \right| \leq \chi(x)\theta(t) ,$$

with $\chi(x) = O(|\psi_H'(x)|)$ and $\theta(t) = (1+t)^{-\kappa}$ for an arbitrary $\kappa > 0$.

Remark 6 *Assumption (9) holds if ψ_H is regularly varying with index $\gamma < 0$. It also holds if $\psi_H(x) = e^{-cx}$ for some $c > 0$.*

Remark 7 *If ψ_H is regularly varying with index $\gamma < 1$, and ψ' ultimately decreasing, then $\psi_H'(x) = o(\sqrt{\psi_H(x)/x})$ and hence the second-order correction is meaningful.*

Remark 8 *This bound is a worst case scenario. In many particular cases, a much faster rate of convergence can be obtained. For instance, if $\bar{H}(x) = e^{-t}$, then (2) holds with $\psi_H \equiv 1$, whence $\chi(x) = 1/x$, but for any positive x and t , $\bar{H}\{x + t\psi_H(x)\}/\bar{H}(x) = e^{-t}$. The rate of convergence is infinite here. If \bar{H} is the Gumbel distribution, then $\psi_H \equiv 1$ and the rate of convergence in (2) is exponential: $|\bar{H}(x+t)/\bar{H}(x) - e^{-t}| \leq 2e^{-x}$.*

In the following, some examples of distributions satisfying Hypothesis 2 are given for illustration.

Example 9 *(Standard normal distribution). For the standard normal distribution, $\psi_H = \bar{\Phi}/\varphi$ satisfies the previous assumptions. More precisely, it is well*

known that $\psi_H(x) \equiv x^{-1} + O(x^{-3})$. Thus $\chi(x) = O(x^{-2})$.

Example 10 (Lognormal distribution). Consider the lognormal distribution $H(x) = \Phi(\log(x))$ and density function $h(x) = x^{-1}\varphi(\log(x))$. The auxiliary function that satisfies (8) is $\psi_H(x) = x\bar{\Phi}(\log(x))/\varphi(\log(x)) = x/\log(x) + O(x/\log^3(x))$. Thus (2) holds with $\chi(x) = O(1/\log(x))$.

Example 11 Consider the survival function $\bar{H}(x) = 2e^{-x^2}/(1 + e^{-x^2})$. For this distribution, $\psi_H(x) = (1 + e^{-x^2})/(2x)$ (see Table 1). Thus (2) holds with $\chi(x) = O(x^{-2})$.

Example 12 (Weibull-type distributions). If H satisfies (8) with $\psi_H(x) = cx^\gamma + O(x^\delta)$ with $\delta < \gamma$, then the rate of convergence is $\chi(x) = O(x^{\gamma-1})$. Note that for the elliptical Kotz distribution with parameter β used in the simulations (see Table 1), the radial component has a strict Weibull distribution with $\psi_H(x) = x^{1-\beta}/\beta$.

Example 13 (Discrete distribution in the domain of attraction of the Gumbel law). Let ψ be a concave increasing function such that $\lim_{x \rightarrow \infty} \psi(x) = +\infty$ and $\lim_{x \rightarrow \infty} \psi'(x) = 0$ and define

$$\bar{H}_\#(x) = \exp \left\{ - \int_{x_0}^x \frac{ds}{\psi(s)} \right\}, \quad \bar{H}(x) = \bar{H}_\#([x]),$$

where $[x]$ is the integer part of x . Then H is a discrete distribution function, \bar{H} belongs to the domain of attraction of the Gumbel distribution, but does not satisfy Condition (8). Nevertheless, Condition (2) holds with

$$\chi(x) = O(\psi'([x]) + 1/\psi(x)), \quad \theta(t) = (1+t)^{-\kappa}, \quad (10)$$

for any arbitrary $\kappa > 0$.

One can deduce from Lemma 5 that Theorem 3 holds for $\psi = \psi_H$. The following lemma says what happens if one uses an asymptotically equivalent auxiliary function ψ instead of ψ_H in Theorem 3.

Lemma 14 Under the assumption of Lemma 5, let ψ be equivalent to ψ_H at infinity and define $\xi(x) = |\psi(x) - \psi_H(x)|/\psi_H(x)$. Then

$$\left| \frac{\bar{H}(x + t\psi(x))}{\bar{H}(x)} - e^{-t} \right| \leq O \{ |\psi'_H(x)| + \xi(x) \} \theta(t). \quad (11)$$

Remark 15 A consequence of Lemma 14 is that if an auxiliary function ψ is used instead of ψ_H in Theorem 3, then the second-order correction is relevant provided that $\xi(x) = o(\sqrt{\psi_H(x)/x})$. This is the case in Examples 9 and 11 if one takes $\psi(x) = 1/x$ or $\psi(x) = 1/(2x)$ respectively; in Example 10 when

using $\psi(x) = x/\log(x)$. As for Example 12, one can take $\psi(x) = cx^\gamma$ if $\delta < (3\gamma - 1)/2$.

3 Statistical procedure

Consider now the problems of estimating $\theta(x, y) = \mathbb{P}(Y \leq y \mid X > x)$ for given “large” positive x and y and the conditional quantile function $\theta(x, \cdot)^\leftarrow$ for large x . The empirical distribution function is useless, since there might be no observations in the considered range. We suggest to estimate these quantities by means of Theorem 3.

Assume that a sample $(X_1, Y_1), \dots, (X_n, Y_n)$ is available, coming from a standard elliptical distribution with radial component satisfying Hypothesis 2. Let $\hat{\rho}_n$ and $\hat{\psi}_n$ be estimates of ρ and ψ . For fixed $x, y > 0$, define

$$\hat{\theta}_{n,1}(x, y) = \Phi \left(\frac{y - \hat{\rho}_n x}{\sqrt{1 - \hat{\rho}_n^2} \sqrt{x \hat{\psi}_n(x)}} \right), \quad (12)$$

$$\hat{\theta}_{n,2}(x, y) = \Phi \left(\frac{y - \hat{\rho}_n x - \hat{\rho}_n \hat{\psi}_n(x)}{\sqrt{1 - \hat{\rho}_n^2} \sqrt{x \hat{\psi}_n(x)}} \right). \quad (13)$$

In order to estimate the conditional quantile function $\theta(x, \cdot)^\leftarrow$, define, for fixed $\theta \in (0, 1)$,

$$\hat{y}_{n,1} = \hat{\rho}_n x + \sqrt{1 - \hat{\rho}_n^2} \sqrt{x \hat{\psi}_n(x)} \Phi^{-1}(\theta), \quad (14)$$

$$\hat{y}_{n,2} = \hat{\rho}_n x + \hat{\rho}_n \hat{\psi}_n(x) + \sqrt{1 - \hat{\rho}_n^2} \sqrt{x \hat{\psi}_n(x)} \Phi^{-1}(\theta). \quad (15)$$

Estimating ρ is a classical topic, and the empirical version of ρ is generally used. Under the assumption of elliptical distributions, however, one can observe a better stability when the Pearson correlation coefficient is estimated by $\hat{\rho}_n = \sin(\pi \hat{\tau}_n/2)$, where $\hat{\tau}_n$ is the empirical Kendall’s tau. See e.g. Hult and Lindskog (2002) for more details. This estimator is \sqrt{n} -consistent and asymptotically normal.

Consider now the problem of the estimation of ψ . Assume that H is a Von Mises distribution. As recalled in Section 2, ψ can be taken as the inverse of the hazard rate function of the radial component R , $\psi = \psi_H = \bar{H}/H'$. A non parametric procedure could therefore be considered, using a Nelson-Aalen’s type estimator. See e.g. Uzunoğullari and Wang (1992), Wu and Wells (2003), Wang (2005). However, such procedures encounter some difficulties when dealing with the tail, which is precisely the domain one needs here to

focus on. To circumvent this problem, we adopt a semi-parametric approach, and estimate ψ under the assumption that the following expansion holds:

$$\psi(x) = \frac{1}{c\beta}x^{1-\beta} + o(x^{1-\beta}) , \quad (16)$$

for some constants $c > 0$ and $\beta > 0$. The distribution function H is in this case commonly called a "Weibull-type distribution", see e.g. Beirlant et al. (2004, p. 92). Under this assumption, estimating ψ boils down to estimating c and β . A wide literature exists on estimators of β , see e.g. Beirlant et al. (1999), Gardes and Girard (2006), Dierckx et al. (2006), among others.

The method chosen here is the one proposed in Beirlant et al. (1996). Let k be a user chosen threshold and $R_{j,n}$, $1 \leq j \leq n$ be the order statistics of the sample R_1, \dots, R_n . The estimator of β is obtained as the slope of the Weibull quantile plot at the point $(\log \log(n/k), \log(R_{n-k,n}))$:

$$\hat{\beta}_n = \frac{k^{-1} \sum_{i=1}^k \log \log(n/i) - \log \log(n/k)}{k^{-1} \sum_{i=1}^k \log(R_{n-i+1,n}) - \log(R_{n-k,n})} . \quad (17)$$

An estimator of c is then naturally given by

$$\hat{c}_n = \frac{1}{k} \sum_{i=1}^k \frac{\log(n/i)}{R_{n-i+1,n}^{\hat{\beta}_n}} . \quad (18)$$

Actually, in our context, the radial component is not observed. We estimate the R_i s by

$$\hat{R}_i^2 = X_i^2 + (Y_i - \hat{\rho}_n X_i)^2 / (1 - \hat{\rho}_n^2) ,$$

and we plug these values in (17) and (18). Beirlant et al. (1996) proved the asymptotic normality of these estimators in the case where the R_i s are observed. The rate of convergence is \sqrt{k} , where k is an intermediate sequence, i.e. $k \rightarrow \infty$ and $k/n \rightarrow 0$ (see also Dierckx et al. (2006) for more details). Since $\hat{\rho}_n$ is \sqrt{n} -consistent, it can be checked that there is asymptotically no effect of the estimation of ρ on this procedure.

Given any estimates \hat{c}_n and $\hat{\beta}_n$, define $\hat{\psi}_n(x) = x^{1-\hat{\beta}_n} / (\hat{c}_n \hat{\beta}_n)$. The estimators defined in (12) have two sources of error: the first one comes from the estimation of ρ and ψ , and the second one from the asymptotic nature of the approximations (5) and (7). Denote

$$\begin{aligned} \hat{z}_{n,1} &= \frac{y - \hat{\rho}_n x}{\sqrt{1 - \hat{\rho}_n^2} \sqrt{x \hat{\psi}_n(x)}} , & \hat{z}_{n,2} &= \frac{y - \hat{\rho}_n x - \hat{\rho}_n \hat{\psi}_n(x)}{\sqrt{1 - \hat{\rho}_n^2} \sqrt{x \hat{\psi}_n(x)}} , \\ z_1 &= \frac{y - \rho x}{\sqrt{(1 - \rho^2)} \sqrt{x^{2-\beta} / (c\beta)}} , & z_2 &= \frac{y - \rho x - \rho x^{1-\beta} / (c\beta)}{\sqrt{(1 - \rho^2)} \sqrt{x^{2-\beta} / (c\beta)}} . \end{aligned}$$

Then, for $i = 1, 2$,

$$\hat{\theta}_{n,i}(x, y) - \theta(x, y) = \Phi(\hat{z}_{n,i}) - \Phi(z_i) + \Phi(z_i) - \theta(x, y) .$$

Theorem 3 and Lemma 14 provide a bound for the deterministic approximation error $\Phi(z_i) - \theta(x, y)$. Note that the second order correction is meaningful only if the $o(\cdot)$ term in (16) is small enough.

For any reasonable estimators, there exists a rate of convergence v_n such that $v_n(\hat{c}_n - c, \hat{\beta}_n - \beta)$ is asymptotically normal. This rate of convergence is smaller than \sqrt{n} because of the semi-parametric nature of the problem. Besides, $\hat{\rho}_n$ is \sqrt{n} -consistent, so $\Phi(\hat{z}_{n,i})$ is a v_n -consistent estimator of $\Phi(z_i)$. Thus, for large n , one expects the deterministic approximation error to dominate the stochastic fluctuations. Slutsky's Theorem yields that $v_n\{\Phi(\hat{z}_{n,i}) - \Phi(z_i)\}$ is also asymptotically normal, and the limiting variance depends on x and y . Roughly, the computations show that, for a fixed probability θ , the variability increases logarithmically with x , and for a fixed x , it decreases as θ tends to 0 or 1.

A third estimator can finally be introduced here, which does not make use of any asymptotic expansion as in Theorem 3, and relies only on the assumption (16). For any $x, u > 0$, (8) implies that

$$\frac{\bar{H}(xu)}{\bar{H}(x)} = \exp \left\{ \int_x^{xu} \frac{ds}{\psi(s)} \right\} .$$

Therefore, an estimator of ψ yields an estimator of $\theta(x, y)$ through formula (A.1). Denote such an estimate by $\hat{\theta}_{n,3}(x, y)$.

4 Simulation study

To assess the performance of the proposed statistical procedures, we simulated several families of bivariate standard elliptical distributions. Recall that a standardized bivariate elliptical density function can be written as $f(x, y) = Cg\{(x^2 - 2\rho xy + y^2)/(1 - \rho^2)\}$, where g is called the generator, ρ is the Pearson correlation coefficient and C is a normalizing constant. The density of the radial component R is given by $H'(r) = Krg(r^2)$, where K is a normalizing constant (see, e.g., Fang et al. (1990)).

The distributions used are presented in Table 1. The Pearson correlation coefficient will be either $\rho = 0.5$ or $\rho = 0.9$. Three of them (Normal, Kotz and Logis) satisfy both Hypothesis 2 and (16), and three of them fail in at least one of these two assumptions: Lognor and the modified Kotz satisfy Hypothesis 2

but not (16), whereas the bivariate Student distribution has a regularly varying radial component, so does not satisfy any of the assumptions. These three distributions are used to explore the robustness of the proposed estimation method.

Table 1

Bivariate elliptical distributions used for the simulations. The elliptical generator g is given, the Von Mises auxiliary function ψ_H (or an equivalent), and the values of the parameters used (in addition to $\rho \in \{0.5, 0.9\}$).

Bivariate law	Generator $g(u)$	$\psi_H(x)$	Parameters
Normal	$e^{-u/2}$	$1/x + O(1/x^3)$	
Kotz	$u^{\beta/2-1} \exp(-u^{\beta/2})$	$x^{1-\beta}/\beta$	$\beta \in \{1, 4\}$
Logis [†]	$e^{-u}/(1 + e^{-u})^2$	$(1 + e^{-x^2})/(2x)$	
Modified Kotz	$g_\star(u)^\ddagger$	$x^{1-\beta}/(1 + \beta \log x)$	$\beta = 3/2$
Lognor [†]	$e^{-(\log^2 u)/8}/u$	$x/\log x + O(x/\log^3 x)$	
Student	$(1 + u/\nu)^{-(\nu+2)/2}$	—	$\nu \in \{3, 20\}$

[†] “Logis” and “Lognor” refer to the elliptical distributions with logistic and lognormal generator, respectively.

[‡] $g_\star(u) = \{(3/8) \log u + 1/2\} u^{-1/4} \exp\{-(1/2)u^{3/4} \log u\}$.

In each case, 200 samples of size 500 were simulated. Several values of x were chosen, corresponding to the $(1 - p)$ -quantile of the marginal distribution of X , with $p = 10^{-3}$, $p = 10^{-4}$ and $p = 10^{-5}$. For each value of x , we computed (by numerical integration) the theoretical values of y corresponding to the probability $\theta(x, y) = .05, .1, .2, \dots, .8, .9, .95$. Then we estimated $\theta(x, y)$ via the three proposed methods (cf. Section 3). For the estimation of the auxiliary function ψ , the threshold chosen corresponds to the highest 10% of the estimated \hat{R}_i s. It must be noted that this choice is independent of x . For each fixed x we also estimated the conditional quantile function $\theta(x, \cdot)^{-}$ by both methods (14) and (15). We did not compute the estimated quantile function for Method 3 since it would involve the numerical inversion of the integrals which appear in (A.1). This actually is one advantage of Methods 1 and 2 over Method 3.

Some general features can be observed, which comfort the theoretical expectations. (i) First of all, in the given range of x and y , there were hardly any observations, so that the empirical conditional distribution function is useless. (ii) For a given probability θ , the variability of the estimators slightly increases with x for all underlying distributions. For a given x , the variability of the estimators is greater for medium values of θ . (iii) The results for the Student distribution are as expected: if the degree of freedom ν is large, the estimation shows a high variability but moderate bias, while if ν is small, then the

estimation is clearly inconsistent. (iv) The results for the Logis and modified Kotz distributions are similar to the Gaussian distribution.

Hence we have chosen to report only the results for the largest value of x (corresponding to the 10^{-5} -quantile of the marginal distribution of X) and the Normal, Kotz (with parameter $\beta = 1$ and $\beta = 4$) and Lognor distributions, for $\rho = .5$ and $\rho = .9$. Figures A.1-A.4 illustrate the behavior of the estimators of the probability θ . Figure A.5 shows the estimated quantile functions for only three distributions and $\rho = .9$, because the results are much more stable.

From these simulation results, one can see that the estimator of θ by Method 1 presents a systematic positive bias, which of course induces an underestimation of the conditional quantile function. As expected, Method 2 corrects this systematic bias; the correction is better when ρ is large. This is also true for the Lognormal generator, though in a lesser extent.

As already mentioned, the Lognor and modified Kotz distributions do not satisfy Assumption (16). In both cases the radial component belongs to an extended Weibull type family, with auxiliary function ψ of the form

$$\psi(x) = cx^{1-\beta}(\log x)^{-\delta}\{1 + o(1)\} ,$$

with $c > 0$, $\beta \geq 0$ and $\delta > 0$ if $\beta = 0$. The modified Kotz distribution corresponds to $\beta = 3/2$ and $\delta = 1$ and Lognor corresponds to $\beta = 0$ and $\delta = 1$. The simulation results are much better for the modified Kotz than for the Lognor distribution. This tends to prove that the method is not severely affected by the logarithmic factor, as long as $\beta > 0$.

As noted in Section 3 and empirically checked (on plots not reported here), the bias of Methods 1 and 2 is mainly due to the approximation error. Nevertheless, in most cases, Methods 2 and 3 behave similarly both in terms of bias and variability. Only two cases in all our simulations showed a better behavior of Method 3.

5 Concluding remarks

In this paper we focused attention on elliptically distributed random pairs (X, Y) having a rapidly varying radial component. Three methods have been proposed to estimate the conditional excess probability $\theta(x, y)$ for large x . Under this specific assumption of elliptical distributions, Methods 2 and 3 revealed comparable results and outperformed Method 1.

Methods 1 and 2 make use of an asymptotic approximation of the conditional excess distribution function by the Gaussian distribution function. As shown

by Balkema and Embrechts (2004), this approximation remains valid outside the family of elliptical distributions, under geometric assumptions on the level curves of the joint density function of (X, Y) . This suggests that these methods may be robust to departure from the elliptical family. This is the subject of current research.

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A Appendix

Proof of Theorem 3: For $x > 0$ and $y \in (0, x)$, we have

$$\mathbb{P}(X > x, Y > y) = \mathbb{P}\left(R > \frac{x}{\cos U} \vee \frac{y}{\rho \cos U + \sqrt{1 - \rho^2} \sin U}\right).$$

Set $t_0 = (y/x - \rho)/\sqrt{1 - \rho^2}$. Then $x/\cos u > y/(\rho \cos u + \sqrt{1 - \rho^2} \sin u)$ if and only if $u > \arctan(t_0)$. Define $U_0 = \arctan(\rho/\sqrt{1 - \rho^2})$. Then $-U_0 < \arctan(t_0)$ and

$$\begin{aligned} \mathbb{P}(X > x, Y > y) &= \int_{\arctan(t_0)}^{\pi/2} \bar{H}\left\{\frac{x}{\cos u}\right\} \frac{du}{2\pi} \\ &\quad + \int_{-U_0}^{\arctan(t_0)} \bar{H}\left\{\frac{y}{\sin(u + U_0)}\right\} \frac{du}{2\pi}, \end{aligned}$$

$$\begin{aligned} \mathbb{P}(Y > y \mid X > x) &= \frac{\int_{\arctan(t_0)}^{\pi/2} \bar{H}(x/\cos(u)) du + \int_{-U_0}^{\arctan(t_0)} \bar{H}(y/\sin(u + U_0)) du}{2 \int_0^{\pi/2} \bar{H}(x/\cos(u)) du}. \quad (\text{A.1}) \end{aligned}$$

If $t_0 \geq 0$, i.e. $y - \rho x \geq 0$, the changes of variables $v = 1/\cos(u)$ and $v = 1/\sin(u + U_0)$ yield:

$$\mathbb{P}(Y > y \mid X > x) = \frac{I_1 + I_2}{I_3}, \quad (\text{A.2})$$

with

$$\begin{aligned} I_1 &= \int_{w_1}^{\infty} \frac{\bar{H}(vx)}{\bar{H}(x)} \frac{dv}{v\sqrt{v^2 - 1}}, \\ I_2 &= \int_{w_2}^{\infty} \frac{\bar{H}(vy)}{\bar{H}(x)} \frac{dv}{v\sqrt{v^2 - 1}}, \\ I_3 &= 2 \int_1^{\infty} \frac{\bar{H}(vx)}{\bar{H}(x)} \frac{dv}{v\sqrt{v^2 - 1}}, \\ w_1 &= \sqrt{1 + t_0^2} = \sqrt{1 + (y/x - \rho)^2/(1 - \rho^2)}, \\ w_2 &= xw_1/y. \end{aligned}$$

If $t_0 < 0$, then

$$\mathbb{P}(Y > y \mid X > x) = \frac{I_3 - I_1 + I_2}{I_3}. \quad (\text{A.3})$$

Denote $w_0 = x(w_1 - 1)/\psi(x)$. In I_1 and I_3 , the change of variable $v = 1 + \frac{\psi(x)}{x}t$ yields

$$I_1 = \sqrt{\frac{\psi(x)}{x}} \int_{w_0}^{\infty} \frac{\bar{H}(x + t\psi(x))}{\bar{H}(x)} \frac{dt}{(1 + t\frac{\psi(x)}{x})\sqrt{1 + \frac{t}{2}\frac{\psi(x)}{x}}\sqrt{2t}},$$

$$I_3 = 2\sqrt{\frac{\psi(x)}{x}} \int_0^{\infty} \frac{\bar{H}(x + t\psi(x))}{\bar{H}(x)} \frac{dt}{(1 + t\frac{\psi(x)}{x})\sqrt{1 + \frac{t}{2}\frac{\psi(x)}{x}}\sqrt{2t}}.$$

In I_2 , the change of variable $vy = x + t\psi(x)$ yields

$$I_2 = \frac{\psi(x)}{x} \int_{w_0}^{\infty} \frac{\bar{H}(x + t\psi(x))}{\bar{H}(x)} \frac{(y/x) dt}{(1 + t\frac{\psi(x)}{x})\sqrt{1 - (y/x)^2 + 2t\frac{\psi(x)}{x} + \frac{\psi^2(x)}{x^2}t^2}}.$$

Set $J_i = \sqrt{x/\psi(x)}I_i$, $i = 1, 3$ and $J_2 = (x/\psi(x))I_2$. We start with I_1 and I_3 . We will use the following bound, valid for all $B, C > 0$,

$$0 \leq 1 - \frac{1}{(1 + C)\sqrt{1 + B}} \leq B/2 + C, \quad (\text{A.4})$$

which follows from straightforward algebra and the concavity of the function $x \mapsto \sqrt{1 + x}$. Applying this bound with $B = \frac{\psi(x)}{x} \frac{t}{2}$ and $C = \frac{\psi(x)}{x}t$ yields

$$0 \leq 1 - \frac{1}{(1 + t\frac{\psi(x)}{x})\sqrt{1 + \frac{t}{2}\frac{\psi(x)}{x}}} \leq \frac{5}{4} \frac{\psi(x)}{x} t.$$

We thus have

$$\left| J_1 - \sqrt{2\pi}\bar{\Phi}(\sqrt{2w_0}) \right| + \left| J_3 - \sqrt{2\pi} \right| \leq \chi(x) \int_0^{\infty} \theta(t) \frac{dt}{\sqrt{2t}} + \frac{\psi(x)}{x} \int_0^{\infty} \sqrt{t} e^{-t} dt.$$

Hence

$$\frac{I_1}{I_3} = \bar{\Phi}(\sqrt{2w_0}) + O\left(\chi(x) + \frac{\psi(x)}{x}\right). \quad (\text{A.5})$$

Consider now J_2 . Applying the bound (A.4) with $C = t\psi(x)/x$ and

$$B = \{2t\frac{\psi(x)}{x} + \frac{\psi^2(x)}{x^2}t^2\}/\sqrt{1 - (y/x)^2},$$

Assumptions (2) and (4), we obtain

$$\left| J_2 - \frac{(y/x)\sqrt{2\pi}\varphi(\sqrt{2w_0})}{\sqrt{1 - (y/x)^2}} \right|$$

$$\leq \frac{y/x}{\sqrt{1 - (y/x)^2}} \chi(x) \int_0^{\infty} \theta(t) dt + \frac{y/x}{1 - (y/x)^2} \sqrt{\frac{\psi(x)}{x}} \int_0^{\infty} (\sqrt{2t} + 2t) e^{-t} dt.$$

Choose $y = \rho x + \sqrt{1 - \rho^2} \sqrt{x\psi(x)} z$ for some fixed $z \in \mathbb{R}$. Then for large enough x it does hold that $0 < y < x$ and

$$\frac{y/x}{\sqrt{1 - (y/x)^2}} = \frac{\rho}{\sqrt{1 - \rho^2}} + O\left(\sqrt{\psi(x)/x}\right).$$

Thus,

$$\frac{I_2}{I_3} = \frac{\rho}{\sqrt{1 - \rho^2}} \sqrt{\frac{\psi(x)}{x}} \varphi(\sqrt{2w_0}) + O\left(\chi(x) + \frac{\psi(x)}{x}\right). \quad (\text{A.6})$$

For $z \geq 0$ and large enough x , plugging (A.5) and (A.6) into (A.2) yields

$$\theta(x, y) = \Phi(\sqrt{2w_0}) - \frac{\rho}{\sqrt{1 - \rho^2}} \sqrt{\frac{\psi(x)}{x}} \varphi(\sqrt{2w_0}) + O\left(\chi(x) + \frac{\psi(x)}{x}\right).$$

For $z < 0$ and large enough x , plugging (A.5) and (A.6) into (A.3) yields

$$\theta(x, y) = \bar{\Phi}(\sqrt{2w_0}) - \frac{\rho}{\sqrt{1 - \rho^2}} \sqrt{\frac{\psi(x)}{x}} \varphi(\sqrt{2w_0}) + O\left(\chi(x) + \frac{\psi(x)}{x}\right).$$

Note now that $w_0 = z^2/2 + O(\psi(x)/x)$, hence $\sqrt{2w_0} = |z| + O(\psi(x)/x)$. Thus, in both cases $z \geq 0$ and $z < 0$, (5) holds. Set $z = z' + \rho\sqrt{\psi(x)/x}/\sqrt{1 - \rho^2}$. A Taylor expansion of Φ and φ around z' yields (7). \square

Proof of Lemma 5:

$$\begin{aligned} \frac{\bar{H}\{x + t\psi(x)\}}{\bar{H}(x)} - e^{-t} &= \exp\left\{-\int_x^{x+t\psi(x)} \frac{ds}{\psi(s)}\right\} - e^{-t} \\ &= \exp\left[-\int_0^t \frac{\psi(x)}{\psi\{x + s\psi(x)\}} ds\right] - e^{-t}. \end{aligned}$$

Applying the inequality $|e^{-a} - e^{-b}| \leq |a - b|e^{-a \wedge b}$ valid for all $a, b \geq 0$ yields

$$\begin{aligned} &\left|\frac{\bar{H}\{x + t\psi(x)\}}{\bar{H}(x)} - e^{-t}\right| \\ &\leq \left|\int_0^t \frac{\psi\{x + s\psi(x)\} - \psi(x)}{\psi\{x + s\psi(x)\}} ds\right| \exp\left[-t \wedge \int_0^t \frac{\psi(x)}{\psi\{x + s\psi(x)\}} ds\right]. \end{aligned}$$

Denote

$$\int_0^t \frac{|\psi\{x + s\psi(x)\} - \psi(x)|}{\psi\{x + s\psi(x)\}} ds = I(x, t)$$

and

$$\exp\left[-t \wedge \int_0^t \frac{\psi(x)}{\psi\{x + s\psi(x)\}} ds\right] = E(x, t).$$

Case ψ increasing. If ψ is nondecreasing, then $\psi' \geq 0$ and ψ' is decreasing. Thus, for any $\delta > 0$ and large enough x , $\psi'(x) \leq \delta$ and

$$\begin{aligned} \int_0^t \frac{\psi(x)}{\psi\{x + s\psi(x)\}} ds &\geq \int_0^t \frac{\psi(x)}{\psi(x) + s\psi'(x)\psi(x)} ds \\ &= \int_0^t \frac{ds}{1 + s\psi'(x)} \geq \int_0^t \frac{ds}{1 + s\delta} = \frac{1}{\delta} \log(1 + \delta t) . \end{aligned}$$

This implies that $E(x, t) \leq (1 + \delta t)^{-1/\delta}$ for large enough x . Since ψ is increasing and ψ' is decreasing, we also have

$$I(x, t) \leq \psi'(x) \int_0^t \frac{s\psi(x)}{\psi\{x + s\psi(x)\}} ds \leq \psi'(x) \frac{t^2}{2} .$$

Thus, for any $\delta > 0$, $I(x, t)E(x, t) = O(|\psi'(x)|t^2(1 + \delta t)^{-1/\delta})$.

Case ψ decreasing. If ψ is monotone non increasing, then

$$\int_0^t \frac{\psi(x)}{\psi\{x + s\psi(x)\}} ds \geq t$$

and $E(x, t) \leq e^{-t}$. Also, since $|\psi'|$ is decreasing,

$$I(x, t) \leq |\psi'(x)| \int_0^t \frac{s\psi(x)}{\psi\{x + s\psi(x)\}} ds .$$

If ψ has a positive limit at infinity, then $I(x, t) = O(\psi'(x)t^2)$. Otherwise, $\lim_{x \rightarrow \infty} \psi(x) = 0$ and (9) holds. This yields, for large enough x ,

$$\begin{aligned} I(x, t) &\leq |\psi'(x)| \int_0^t \frac{s\psi(x)}{\psi\{x + s\psi(x)\}} ds \\ &\leq c_1 |\psi'(x)| \int_0^t s e^{c_2 s \psi(x)} ds \leq c_1 |\psi'(x)| \int_0^t s e^{s/2} ds . \end{aligned}$$

Thus $I(x, t) = O(|\psi'(x)|t^2 e^{t/2})$ and $I(x, t)E(x, t) = O(|\psi'(x)|t^2 e^{-t/2})$. This concludes the proof. \square

Proof of Lemma 14: Write

$$\left| \frac{\bar{H}\{x + t\psi(x)\}}{\bar{H}(x)} - e^{-t} \right| \leq \left| \frac{\bar{H}\{x + t\psi(x)\}}{\bar{H}\{x + t\psi_H(x)\}} - 1 \right| e^{-t} \quad (\text{A.7})$$

$$+ \frac{\bar{H}\{x + t\psi(x)\}}{\bar{H}\{x + t\psi_H(x)\}} \left| \frac{\bar{H}\{x + t\psi_H(x)\}}{\bar{H}(x)} - e^{-t} \right| , \quad (\text{A.8})$$

$$\frac{\bar{H}\{x + t\psi(x)\}}{\bar{H}\{x + t\psi_H(x)\}} = \exp \left[- \int_t^{t\psi(x)/\psi_H(x)} \frac{\psi_H(x)}{\psi\{x + s\psi_H(x)\}} ds \right] .$$

If ψ_H is increasing, then

$$\left| \int_t^{t\psi(x)/\psi_H(x)} \frac{\psi_H(x)}{\psi\{x + s\psi_H(x)\}} ds \right| \leq t\xi(x), \quad (\text{A.9})$$

$$\exp \left[- \int_t^{t\psi(x)/\psi_H(x)} \frac{\psi_H(x)}{\psi_H\{x + s\psi_H(x)\}} ds \right] \leq e^{t\xi(x)}. \quad (\text{A.10})$$

Since $\xi(x) \rightarrow 0$, gathering (A.9) and (A.10) yields, for large enough x ,

$$\left| \frac{\bar{H}\{x + t\psi(x)\}}{H\{x + t\psi_H(x)\}} - 1 \right| e^{-t} \leq t\xi(x)e^{-t/2}. \quad (\text{A.11})$$

If ψ_H is decreasing and $\lim_{x \rightarrow \infty} \psi_H(x) > 0$, the ratio $\psi_H(x)/\psi_H\{x + s\psi(x)\}$ is bounded above and away from 0, so that

$$\left| \int_t^{t\psi(x)/\psi_H(x)} \frac{\psi_H(x)}{\psi\{x + s\psi_H(x)\}} ds \right| \leq Ct\xi(x),$$

and (A.11) still holds.

If ψ_H is decreasing and $\lim_{x \rightarrow \infty} \psi_H(x) = 0$, then applying (9) gives that the left-hand side of the previous equation is bounded by $Ct\xi(x) \exp\{2c_2\psi(x)t\}$. Thus, for large enough x , (A.11) still holds.

This provides a bound for the right-hand side of (A.7). The term in (A.8) is bounded by Lemma 14. \square

Proof of (10): We proceed as in the proof of Lemma 5.

$$\begin{aligned} \frac{\bar{H}\{x + t\psi(x)\}}{H(x)} &= \exp \left\{ - \int_{[x]}^{[x+t\psi(x)]} \frac{ds}{\psi(s)} \right\} \\ &= \exp \left[- \int_0^{([x+t\psi(x)]-[x])/ \psi(x)} \frac{\psi(x)}{\psi([x] + s\psi(x))} ds \right], \end{aligned}$$

$$\begin{aligned} &\int_0^{([x+t\psi(x)]-[x])/ \psi(x)} \frac{\psi(x)}{\psi([x] + s\psi(x))} ds - t \\ &= \int_0^{([x+t\psi(x)]-[x])/ \psi(x)} \frac{\psi(x) - \psi([x] + s\psi(x))}{\psi([x] + s\psi(x))} ds + \frac{[x + t\psi(x)] - [x]}{\psi(x)} - t. \end{aligned}$$

By concavity of ψ ,

$$\frac{|\psi(x) - \psi([x] + s\psi(x))|}{\psi([x] + s\psi(x))} = O((s \vee 1)\psi'([x])).$$

By definition of the integral part,

$$t - 1/\psi(x) \leq \frac{[x + t\psi(x)] - [x]}{\psi(x)} \leq t + 1/\psi(x) .$$

The previous bounds yield

$$\left| \int_0^{([x+t\psi(x)]-[x])/ \psi(x)} \frac{\psi(x)}{\psi([x] + s\psi(x))} ds - t \right| = O((t \vee 1)^2 \psi'([x])) .$$

We must also give a lower bound for the integral. Since ψ is concave increasing, $\psi([x] + s\psi(x)) \leq \psi(x) + s\psi'(x)\psi(x)$, hence

$$\begin{aligned} \int_0^{([x+t\psi(x)]-[x])/ \psi(x)} \frac{\psi(x)}{\psi([x] + s\psi(x))} ds &\geq \int_0^{t-1/\psi(x)} \frac{ds}{1 + s\psi'(x)} \\ &= \frac{1}{\psi'(x)} \log(1 + t\psi'(x) - \psi'(x)/\psi(x)) . \end{aligned}$$

Since $\psi'(x) \rightarrow 0$ and $\psi(x) \rightarrow \infty$, the arguments of the proof of Lemma 5 can be applied again to conclude the proof of (10). \square

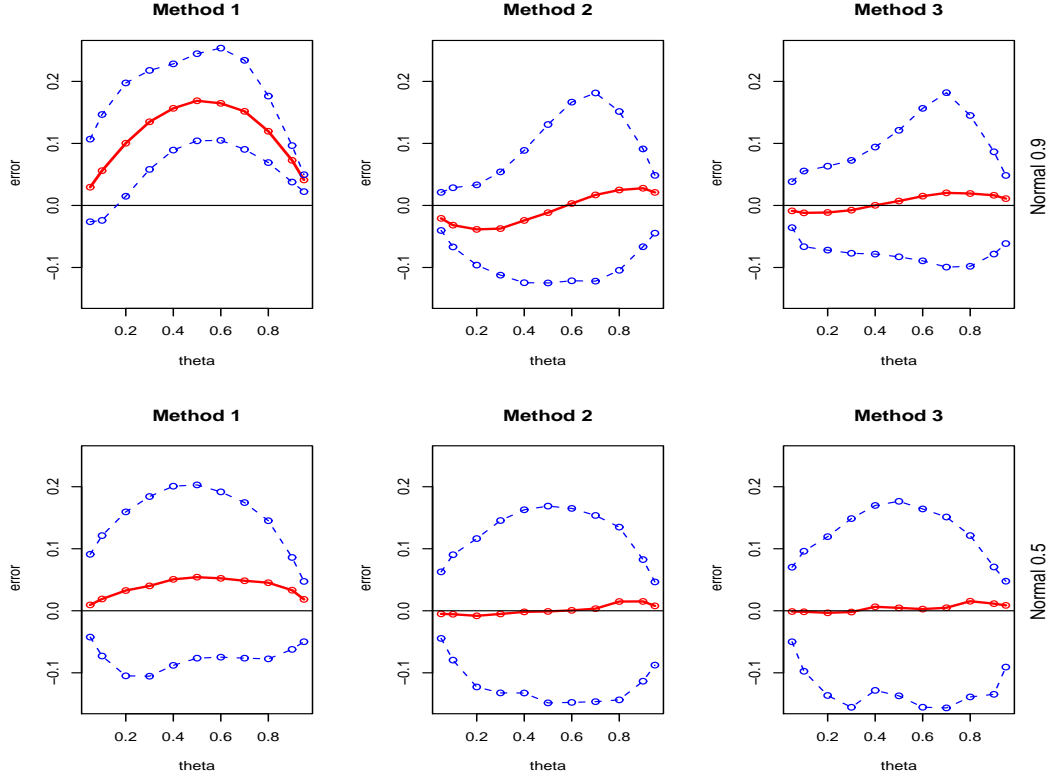


Fig. A.1. Median, 2.5% and 97.5% quantiles of the estimation error $\hat{\theta}_{n,i}(x, y) - \theta(x, y)$ ($i = 1, 2, 3$) as a function of the estimated probability. Gaussian distribution. First row: $\rho = .9$; second row: $\rho = .5$.

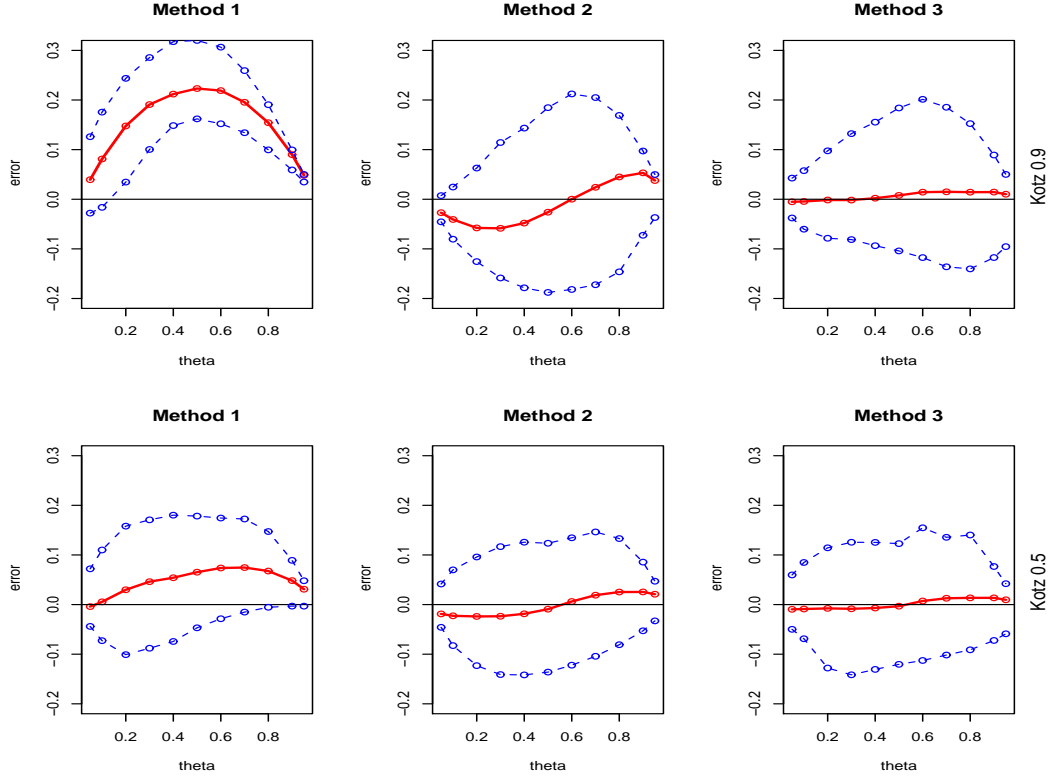


Fig. A.2. Median, 2.5% and 97.5% quantiles of the estimation error $\hat{\theta}_{n,i}(x, y) - \theta(x, y)$ ($i = 1, 2, 3$) as a function of the estimated probability. Kotz distribution, $\beta = 1$. First row: $\rho = .9$; second row: $\rho = .5$.

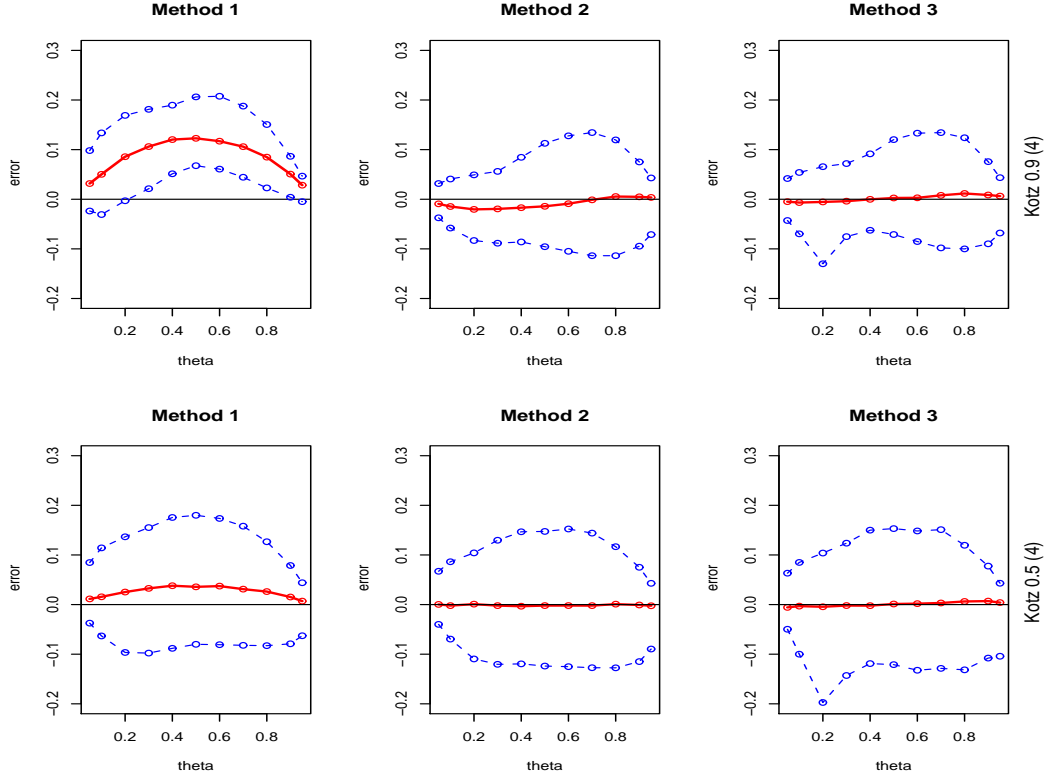


Fig. A.3. Median, 2.5% and 97.5% quantiles of the estimation error $\hat{\theta}_{n,i}(x, y) - \theta(x, y)$ ($i = 1, 2, 3$) as a function of the estimated probability. Kotz distribution, $\beta = 4$. First row: $\rho = .9$; second row: $\rho = .5$.

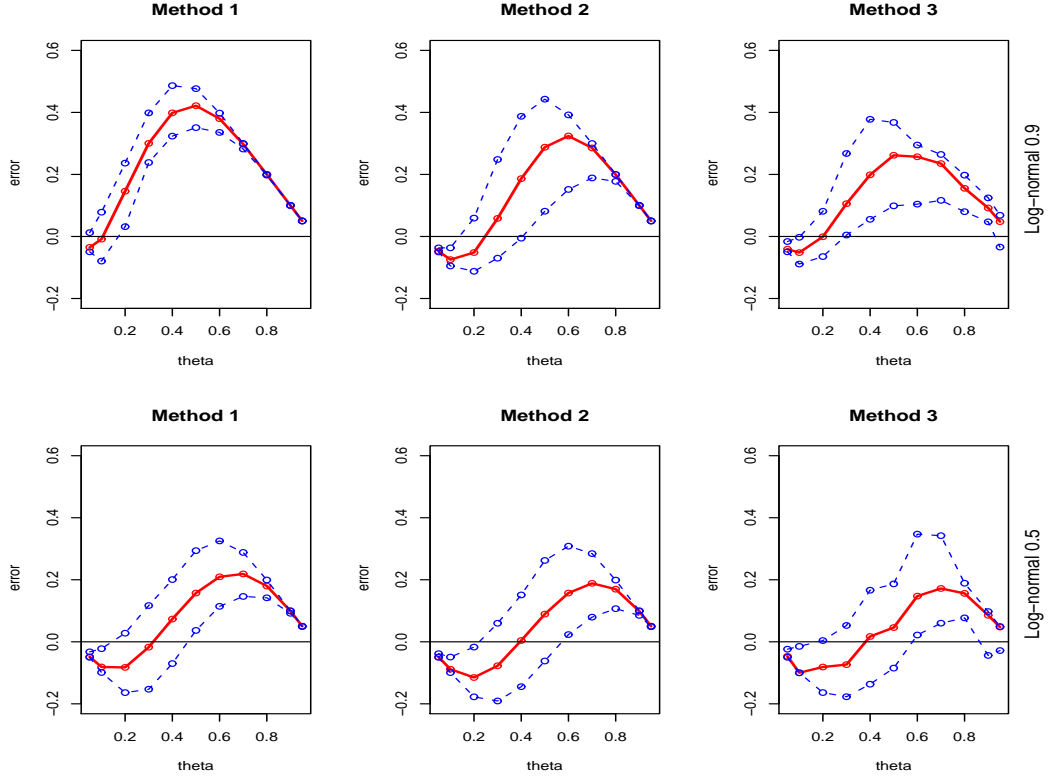


Fig. A.4. Median, 2.5% and 97.5% quantiles of the estimation error $\hat{\theta}_{n,i}(x,y) - \theta(x,y)$ ($i = 1,2,3$) as a function of the estimated probability. Lognormal distribution. First row: $\rho = .9$; second row: $\rho = .5$.

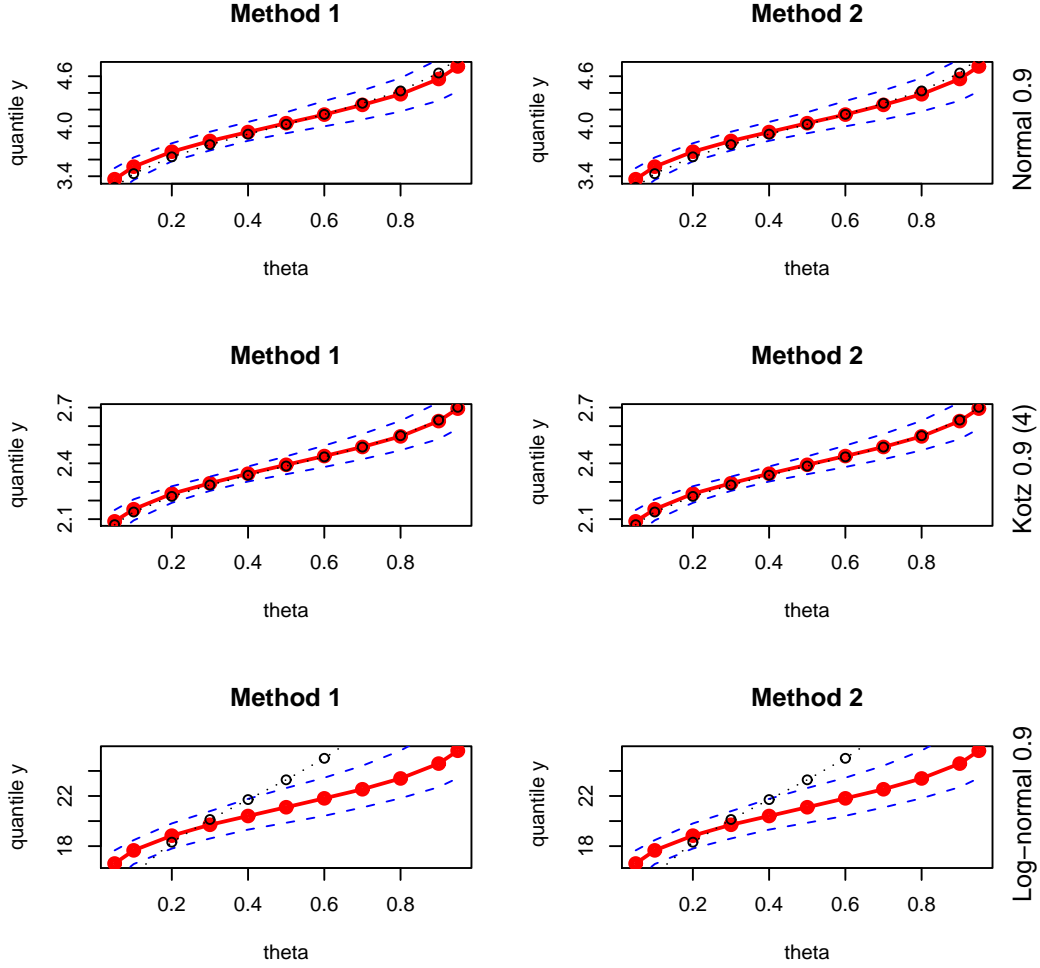


Fig. A.5. Median, 2.5% and 97.5% quantiles of the estimated conditional quantile function $\hat{y}_{n,i}(x, y)$ ($i = 1, 2$) as a function of the estimated probability. First row: Normal distribution; second row: Kotz distribution, $\beta = 4$; Lognor distribution. For each of them $\rho = .9$.