



**HAL**  
open science

# Stability of moving fronts under Griffith criterion: a computational approach using integral equations and domain derivatives

Marc Bonnet

► **To cite this version:**

Marc Bonnet. Stability of moving fronts under Griffith criterion: a computational approach using integral equations and domain derivatives. IUTAM symposium on variations of domains and free boundary problems in solid mechanics, Apr 1997, Paris, France. pp.269-276, 10.1007/978-94-011-4738-5\_32 . hal-00116518

**HAL Id: hal-00116518**

**<https://hal.science/hal-00116518v1>**

Submitted on 6 Nov 2024

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# STABILITY OF MOVING FRONTS UNDER GRIFFITH CRITERION: A COMPUTATIONAL APPROACH USING INTEGRAL EQUATIONS AND DOMAIN DERIVATIVES

MARC BONNET

*Laboratoire de Mécanique des Solides*

*Ecole Polytechnique, 91128 Palaiseau cedex, FRANCE*

PRESENTATION. Consider a linearly elastic body  $\Omega \subset \mathbf{R}^3$ . Its external boundary is divided into two complementary parts  $S_u$  (supporting prescribed displacements:  $\mathbf{u} = \bar{\mathbf{u}}$ ) and  $S_t$  (supporting prescribed tractions:  $\boldsymbol{\sigma} \cdot \mathbf{n} = \bar{\mathbf{t}}$ ). Besides, a crack (described by an open surface  $S$  across which the displacement is discontinuous:  $\boldsymbol{\varphi} = \mathbf{u}^+ - \mathbf{u}^-$  denotes the crack opening displacement (COD)) is embedded in  $\Omega$ .

In the classical Griffith approach, crack propagation may occur at points of the crack front  $\partial S = \Gamma$  where the energy release rate  $G(s)$  ( $s$ : arc length along the crack front  $\Gamma$ , for three-dimensional problems) reaches a certain threshold.  $G$  is mathematically defined (Eq. (5) below) as (minus) the kernel associated with the domain derivative of the equilibrium value  $W(\bar{\mathbf{u}}, \bar{\mathbf{t}}; S)$  of the potential energy for the current crack configuration. In the present context (linear elasticity, infinitesimal strains, no body forces), the latter is known in terms of the elastic variables on the external boundary:

$$W \equiv W(\bar{\mathbf{u}}, \bar{\mathbf{t}}; S) = \frac{1}{2} \int_{S_u} \bar{\mathbf{t}} \cdot \bar{\mathbf{u}} \, dS - \frac{1}{2} \int_{S_t} \bar{\mathbf{t}} \cdot \mathbf{u} \, dS \quad (1)$$

Then, the determination of the actual infinitesimal crack front perturbation involves the second-order domain derivative of  $W$  (Eqs. 7–9).

This paper investigates a computational approach which (i) uses analytical expressions for the first- and second-order domain derivatives of  $W$ , with no recourse to finite-difference evaluations, and (ii) is based on a Galerkin symmetric boundary integral equation (SGBIE)[8, 3] of the elastic equilibrium, in view of the major role played by geometry (the crack surface) and geometry perturbations in the problem. The latter governs the boundary unknowns  $\mathbf{t}|_{S_u}$ ,  $\mathbf{t}|_{S_t}$ ,  $\boldsymbol{\varphi}|_S$  and has the general form:

$$\forall(\tilde{\mathbf{u}}, \tilde{\mathbf{t}}, \tilde{\boldsymbol{\varphi}}) \quad \begin{cases} \mathcal{B}_{tt}(\tilde{\mathbf{t}}, \mathbf{t}) + \mathcal{B}_{tu}(\tilde{\mathbf{t}}, \mathbf{u}) + \mathcal{B}_{t\varphi}(\tilde{\mathbf{t}}, \boldsymbol{\varphi}) & = \mathcal{L}_t(\tilde{\mathbf{t}}) \\ \mathcal{B}_{ut}(\tilde{\mathbf{u}}, \mathbf{t}) + \mathcal{B}_{uu}(\tilde{\mathbf{u}}, \mathbf{u}) + \mathcal{B}_{u\varphi}(\tilde{\mathbf{u}}, \boldsymbol{\varphi}) & = \mathcal{L}_u(\tilde{\mathbf{u}}) \\ \mathcal{B}_{\varphi t}(\tilde{\boldsymbol{\varphi}}, \mathbf{t}) + \mathcal{B}_{\varphi u}(\tilde{\boldsymbol{\varphi}}, \mathbf{u}) + \mathcal{B}_{\varphi\varphi}(\tilde{\boldsymbol{\varphi}}, \boldsymbol{\varphi}) & = \mathcal{L}_\varphi(\tilde{\boldsymbol{\varphi}}) \end{cases} \quad (2)$$

where the  $B_{ij}$  are known in terms of double surface integrals (see [2] or [6] for detailed expressions) and such that the above formulation is symmetric, and the  $\tilde{\mathbf{u}}, \tilde{\mathbf{t}}, \tilde{\varphi}$  are trial functions of support  $S_u, S_t, S$ , respectively. After boundary element discretization, the above set of equations lead to a symmetric linear system of equations for the unknowns boundary DOFs.

As a result, a boundary-only formulation, based solely on a boundary element discretisation (i.e. no part of the domain needs discretization) is obtained for the instantaneous crack perturbation problem. Our approach is in fact an adaptation of the so-called  $\theta$ -method [5, 11], originally defined in the framework of weak formulations and implemented using finite elements, to boundary integral equation formulations. More specifically, Galerkin symmetric BIEs are used because their symmetry allows (like in the FEM context) to formulate  $W_{,\Omega\Omega}$  in terms of the elastic field variables and their first-order domain derivatives (i.e. no second-order field variable derivatives are needed). Also, like in the classical  $\theta$ -method, Lagrangian-type domain differentiation formulas are used, in order not to increase the crack front singular behaviour of the field variables through the differentiation process.

**ENERGY FORMULATION FOR THE CRACK EXTENSION PROBLEM** A quasistatic in-plane crack extension process, induced by a load increment, is considered. Using a small (non-physical) time  $t$ , a perturbed configuration of  $S$  is described by  $S(t) = S + \boldsymbol{\theta}t$ , where  $S \equiv S(0)$  is the initial crack surface and  $\boldsymbol{\theta}$ , the ‘‘initial velocity’’ of extension, must satisfy the requirement

$$\boldsymbol{\theta} \in \Theta = \{\boldsymbol{\theta} \in C^0(S) \mid \theta_n = 0 \text{ on } S, \theta_\nu = 0 \text{ on } \Gamma\} \quad (3)$$

where  $\boldsymbol{\nu}$  denotes the unit normal to  $\Gamma$  which lies in the tangent plane to  $S$  and points outwards to  $S$ . The variations of any field quantity induced by this domain perturbation are described in a Lagrangian manner, using  $S$  as the initial configuration. Denoting by  $\dot{f}^* = f_{,t} + \nabla f \cdot \boldsymbol{\theta}$  the lagrangian derivative of a field quantity  $f$ , then the material derivative of a surface integral at  $t = 0$  is given by:

$$I(f, S; t) = \int_{S(t)} f \, dS \quad \dot{I}^*(f, S) = \frac{dI}{dt} = \int_S \dot{f}^* \, dS + \int_S f \operatorname{div}_S \boldsymbol{\theta} \, dS \quad (4)$$

where  $\operatorname{div}_S(\cdot) \equiv \operatorname{div}(\cdot) - \mathbf{n} \cdot \nabla(\cdot) \cdot \mathbf{n}$  is the surface divergence of a vector field. Moreover, the crack front singularity of  $\nabla \dot{\varphi}^*$  equals that of  $\nabla \varphi$  instead of increasing it; this is the main motivation for our choice of the lagrangian framework, reminiscent of the ‘ $\theta$ -method’ (Mialon [7], Suo [11]).

The energy release rate  $G(s)$  associated to (virtual) tangent crack extensions is such that:

$$\int_\Gamma G(s) \theta_\nu(s) \, ds + \dot{W}^* = 0 \quad (\forall \boldsymbol{\theta} \in \Theta) \quad (5)$$

where  $W$  is given by Eq. (1) and  $(\dot{\cdot})^*$  denotes the lagrangian derivative with *fixed* load. Then, under the Griffith propagation criterion, the actual crack extension velocity  $\boldsymbol{\mu} \in \Theta$  verifies at any point  $s$  of the crack front  $\Gamma$ :

$$\boldsymbol{\mu} \in \Theta_c = \{\boldsymbol{\mu} \in \Theta \mid G(s) < G_c \Rightarrow \mu_\nu(s) = 0, G(s) = G_c \Rightarrow \mu_\nu(s) \geq 0\}$$

Following Nguyen [9], Pradeilles-Duval [10], define

$$Q(\boldsymbol{\theta}, \boldsymbol{\mu}) = \overset{\star\vee}{W} + \int_{\Gamma} G_c \theta_\nu \mu_\nu \kappa(s) ds \quad (6)$$

where  $(\dot{\cdot})^\vee$  denotes the lagrangian derivative in the transformation velocity  $\boldsymbol{\mu}$  under *fixed* load;  $\overset{\star\vee}{W}$  is the second-order domain derivative of  $W$ , i.e.  $\overset{\star\vee}{W} \equiv (\overset{\star}{W})^\vee$ , evaluated under the assumption  $\overset{\vee}{\boldsymbol{\theta}} = \mathbf{0}$ ;  $\kappa(s)$  is the algebraic curvature on  $G$ ). Then the three following statements hold [9, 10]: first,  $\boldsymbol{\mu}$  is governed by the variational inequality:

$$\text{Find } \boldsymbol{\mu} \in \Theta_c, \forall \boldsymbol{\theta} \in \Theta_c \quad Q(\boldsymbol{\theta} - \boldsymbol{\mu}, \boldsymbol{\mu}) + \overset{\star}{W}' \geq 0 \quad (7)$$

where  $(\cdot)'$  denotes the derivative with respect to load variations and with crack fixed. Second, a solution  $\boldsymbol{\mu}$  to (7) exists, i.e. the infinitesimal crack propagation is stable, if

$$\forall \boldsymbol{\theta} \in \Theta_c \quad (\boldsymbol{\theta} \neq \mathbf{0}) \Rightarrow Q(\boldsymbol{\theta}, \boldsymbol{\theta}) > 0 \quad (8)$$

Third, the solution  $\boldsymbol{\mu}$  to (7) is unique if (non-bifurcation criterion):

$$\forall (\boldsymbol{\theta}, \boldsymbol{\mu}) \in \Theta \quad (\boldsymbol{\theta} \neq \boldsymbol{\mu}) \Rightarrow Q(\boldsymbol{\theta} - \boldsymbol{\mu}, \boldsymbol{\theta} - \boldsymbol{\mu}) > 0 \quad (9)$$

Our objective is to propose a treatment of the statements (7–9) based on boundary-only expressions of the energy derivatives  $\overset{\star}{W}, \overset{\star\vee}{W}, \overset{\star}{W}'$ .

FIRST-ORDER DERIVATIVE OF  $W$ . Define the Lagrangian  $L$ :

$$L = W + \mathcal{B}(\tilde{\mathbf{u}}, \mathbf{u}) - \mathcal{L}(\tilde{\mathbf{u}}) \quad (10)$$

where  $\mathbf{u}, \tilde{\mathbf{u}}, \mathcal{B}, \mathcal{L}$  are compact notations for the set of unknowns, trial functions, bilinear forms and linear forms appearing in Eq. (2). Upon Lagrangian differentiation with  $(\tilde{\mathbf{u}})^* = \mathbf{0}$ , the derivative  $\overset{\star}{\mathcal{L}}$  takes the form

$$\overset{\star}{\mathcal{L}} = \frac{1}{2} \int_{S_u} \overset{\star}{\mathbf{t}} \cdot \overset{\star}{\tilde{\mathbf{u}}} dS - \frac{1}{2} \int_{S_t} \bar{\mathbf{t}} \cdot \overset{\star}{\tilde{\mathbf{u}}} dS + \mathcal{B}(\tilde{\mathbf{u}}, \overset{\star}{\mathbf{u}}) + \mathcal{B}^1(\tilde{\mathbf{u}}, \mathbf{u}; \boldsymbol{\theta}) - \mathcal{L}^1(\tilde{\mathbf{u}}; \boldsymbol{\theta}) \quad (11)$$

where  $\mathcal{B}^1, \mathcal{L}^1$ , respectively the Lagrangian derivatives of  $\mathcal{B}$  with  $\overset{\star}{\mathbf{u}} = \mathbf{0}$  and of  $\mathcal{L}$  with fixed load, are linear expressions of  $\boldsymbol{\theta}$ . Now define the adjoint state  $\mathbf{u}^A$  as the solution to the variational equation:

$$(\forall \overset{\star}{\mathbf{u}}) \quad \mathcal{B}(\mathbf{u}^A, \overset{\star}{\mathbf{u}}) = \frac{1}{2} \int_{S_t} \bar{\mathbf{t}} \cdot \overset{\star}{\tilde{\mathbf{u}}} dS - \frac{1}{2} \int_{S_u} \overset{\star}{\mathbf{t}} \cdot \overset{\star}{\tilde{\mathbf{u}}} dS \quad (12)$$

Then, using the symmetry of  $\mathcal{B}$ , one can eliminate  $\overset{\star}{\mathbf{u}}$  between Eqs. (11,12),

thus obtaining the following expression for  $\overset{\star}{W}$ :

$$\overset{\star}{W} = \mathcal{B}^1(\mathbf{u}^A, \mathbf{u}; \boldsymbol{\theta}) - \mathcal{L}^1(\mathbf{u}^A; \boldsymbol{\theta}) \quad (13)$$

SECOND-ORDER DERIVATIVES OF  $W$ . First, one has:

$$\overset{\star\vee}{W} = \mathcal{B}^1(\mathbf{u}^A, \overset{\vee}{\mathbf{u}}; \boldsymbol{\theta}) + \mathcal{B}^1(\overset{\vee}{\mathbf{u}}^A, \mathbf{u}; \boldsymbol{\theta}) + \mathcal{B}^2(\mathbf{u}^A, \mathbf{u}; \boldsymbol{\theta}, \boldsymbol{\mu}) - \mathcal{L}^2(\mathbf{u}^A; \boldsymbol{\theta}, \boldsymbol{\mu}) \quad (14)$$

whereas the first-order domain derivative states are governed by:

$$\mathcal{B}(\tilde{\mathbf{u}}, \overset{\star}{\mathbf{u}}) = \mathcal{L}^1(\tilde{\mathbf{u}}; \boldsymbol{\theta}) - \mathcal{B}^1(\tilde{\mathbf{u}}, \mathbf{u}; \boldsymbol{\theta}) \quad \mathcal{B}(\tilde{\mathbf{u}}, \overset{\star}{\mathbf{u}}^A) = -\mathcal{B}^1(\tilde{\mathbf{u}}, \mathbf{u}^A; \boldsymbol{\theta}) \quad (\forall \tilde{\mathbf{u}}) \quad (15)$$

upon choosing  $\tilde{\mathbf{u}} = \overset{\vee}{\mathbf{u}}^A$  in (15a) and  $\tilde{\mathbf{u}} = \overset{\vee}{\mathbf{u}}$  in (15b), one then obtains an expression for the second-order domain derivative of  $W$ :

$$\overset{\star\vee}{W} = \mathcal{B}^2(\mathbf{u}^A, \mathbf{u}; \boldsymbol{\theta}, \boldsymbol{\mu}) - \mathcal{L}^2(\mathbf{u}^A; \boldsymbol{\theta}, \boldsymbol{\mu}) - \mathcal{B}(\mathbf{u}^A, \overset{\vee}{\mathbf{u}}) - \mathcal{B}(\overset{\star}{\mathbf{u}}, \overset{\vee}{\mathbf{u}}^A) \quad (16)$$

Next, the second-order mixed derivative  $\overset{\star}{W}'$  is given by:

$$\overset{\star}{W}' = \mathcal{B}^1(\mathbf{u}^A, \mathbf{u}'; \boldsymbol{\theta}) + \mathcal{B}^1(\mathbf{u}^{A'}, \mathbf{u}; \boldsymbol{\theta}) - \frac{\partial}{\partial t} \mathcal{L}^1(\mathbf{u}^A; \boldsymbol{\theta}) \quad (17)$$

whereas the first-order load derivative states are governed by:

$$\mathcal{B}(\tilde{\mathbf{u}}, \mathbf{u}') = \frac{\partial}{\partial t} \mathcal{L}(\tilde{\mathbf{u}}) \quad (\forall \tilde{\mathbf{u}}) \quad (18)$$

$$\mathcal{B}(\tilde{\mathbf{u}}, (\mathbf{u}^A)') = \frac{1}{2} \int_{S_u} \mathbf{t} \cdot \tilde{\mathbf{u}}' dS - \frac{1}{2} \int_{S_t} \bar{\mathbf{t}}' \cdot \mathbf{u} dS \quad (19)$$

SPECIAL CASE: CRACK IN AN INFINITE BODY. For this particular configuration, assuming symmetrical loading of the crack faces (i.e.  $\boldsymbol{\sigma} \cdot \mathbf{n} = \pm \bar{\mathbf{t}}$  on  $S^\pm$ ), the only nonzero terms in the formulation (2) are (see [8], and also [4, 1] for plane cracks):

$$\mathcal{B}_{\varphi, \varphi}(\tilde{\varphi}, \varphi) = \int_S \int_S B_{ikqs}(\mathbf{y} - \mathbf{x}) R_q \tilde{\varphi}_i(\mathbf{x}) R_s \varphi_k(\mathbf{y}) dS_x dS_y \quad \mathcal{L}_\varphi = \int_S \bar{\mathbf{t}} \cdot \tilde{\varphi} dS$$

with  $R_a f \equiv e_{bca} n_b f_{,c}$  (a tangential differential operator) and:

$$B_{ikqs}(\mathbf{y} - \mathbf{x}) = -\frac{\mu}{8\pi} \left( \delta_{pr} \delta_{qs} + \delta_{ps} \delta_{qr} + \frac{2\nu}{1-\nu} \delta_{pq} \delta_{rs} \right) e_{iep} e_{kgr} r_{,eg}$$

$$r = \|\mathbf{y} - \mathbf{x}\| \quad r_{,i} = (\mathbf{y}_i - \mathbf{x}_i)/r \quad r_{,ij} = (\delta_{ij} - r_{,i} r_{,j})/r$$

This in turn leads to the following expressions:

$$\mathcal{B}^1(\tilde{\varphi}, \varphi; \boldsymbol{\theta}) = \int_S \int_S B_{ikqs}^1(\mathbf{x}, \mathbf{y}) R_q \tilde{\varphi}_i(\mathbf{x}) R_s \varphi_k(\mathbf{y}) dS_x dS_y \quad (20)$$

$$\mathcal{B}^2(\tilde{\varphi}, \varphi; \boldsymbol{\theta}, \boldsymbol{\mu}) = \int_S \int_S B_{ikqs}^2(\mathbf{x}, \mathbf{y}) R_q \tilde{\varphi}_i(\mathbf{x}) R_s \varphi_k(\mathbf{y}) dS_x dS_y \quad (21)$$

$$\mathcal{L}^1(\tilde{\varphi}; \theta) = - \int_S \bar{\mathbf{t}} \cdot \nabla \varphi \cdot \theta \, dS \quad (22)$$

$$\mathcal{L}^2(\tilde{\varphi}; \theta) = \int_S [\bar{\mathbf{t}} \cdot (\nabla \phi \cdot \nabla \mu \cdot \theta) + (\nabla \bar{\mathbf{t}} \cdot \mu + (\text{div}_S \mu) \bar{\mathbf{t}}) \cdot \nabla \varphi \cdot \theta] \, dS \quad (23)$$

with

$$\begin{aligned} B^1(\mathbf{x}, \mathbf{y}) &= [\theta_m(\mathbf{y}) - \theta_m(\mathbf{x})] B_{ikqs,m}(\mathbf{y} - \mathbf{x}) \\ &\quad + B_{ikqv}(\mathbf{y} - \mathbf{x}) \theta_{v,s}(\mathbf{y}) + B_{ikvs}(\mathbf{y} - \mathbf{x}) \theta_{v,q}(\mathbf{x}) \\ B^2(\mathbf{x}, \mathbf{y}) &= [\theta_m(\mathbf{y}) - \theta_m(\mathbf{x})] [\mu_n(\mathbf{y}) - \mu_n(\mathbf{x})] B_{ikqs,mn}(\mathbf{y} - \mathbf{x}) \\ &\quad + [\mu_n(\mathbf{y}) - \mu_n(\mathbf{x})] \{ \theta_{m,s}(\mathbf{y}) B_{ikqm,n}(\mathbf{y} - \mathbf{x}) + \theta_{m,q}(\mathbf{y}) B_{ikms,n}(\mathbf{y} - \mathbf{x}) \} \\ &\quad + [\theta_n(\mathbf{y}) - \theta_n(\mathbf{x})] \{ \mu_{m,s}(\mathbf{y}) B_{ikqm,n}(\mathbf{y} - \mathbf{x}) + \mu_{m,q}(\mathbf{y}) B_{ikms,n}(\mathbf{y} - \mathbf{x}) \} \\ &\quad + B_{ikmn}(\mathbf{y} - \mathbf{x}) \{ \mu_{m,s}(\mathbf{y}) \theta_{n,q}(\mathbf{x}) + \mu_{m,q}(\mathbf{x}) \theta_{n,s}(\mathbf{y}) \} \end{aligned}$$

NUMERICAL IMPLEMENTATION. The present approach has been numerically implemented for plane cracks of arbitrary shape and mixed-mode loading. The crack surface  $S$  and field variables  $\varphi, \dot{\varphi}, \varphi'$  are discretized using 9-noded quadrilateral boundary elements. It is worth noting that the bilinear forms  $\mathcal{B}, \mathcal{B}^1, \mathcal{B}^2$  are actually formulated in terms of *tangential* derivatives of boundary variables. Indeed, using a mapping  $(\xi_1, \xi_2) \in \Delta \rightarrow \mathbf{y}(\xi_1, \xi_2) \in S$  (where e.g.  $\Delta$  is a reference element) one can show that

$$e_{abc} n_a(\mathbf{y}) \phi_{b,c}(\mathbf{y}) e_c \, dS_y = (\phi_{,\xi_1} \mathbf{y}_{,\xi_2} - \phi_{,\xi_2} \mathbf{y}_{,\xi_1}) d\xi_1 d\xi_2 \quad (24)$$

This formula is also applied to the numerical evaluation of the element integrals which arise in the discretization process of  $\mathcal{B}, \mathcal{B}^1, \mathcal{B}^2$ .

Due to the present use of material-type differentiations, the known fact that ultimately the derivatives of  $P$  depend on the crack front velocities only through its normal component  $\theta_\nu |_{\partial\Gamma}, \mu_\nu |_{\partial\Gamma}$  [7], [10] is not apparent in Eqs. (16,17). It is then necessary to construct the velocity fields  $\theta, \mu$  such as to be entirely determined by  $\theta_\nu |_{\partial\Gamma}, \mu_\nu |_{\partial\Gamma}$ . At the BE level, this is achieved by introducing interpolations of the form:

$$\theta(\mathbf{y}) = \mathbf{B}^k(\mathbf{y}) \theta_k \quad \text{pour } \mathbf{y} \in E(\partial\Gamma) \quad (25)$$

where  $\theta_k$  ( $1 \leq k \leq \text{NC}$ ) are the values *at the NC crack front nodes*  $A_k$  of the normal velocity  $\theta_\nu$ . The local numbering of the elements adjacent to  $\Gamma$  is made so that the line  $\xi_2 = -1$  is on the approximate crack front. The vector shape functions  $\mathbf{B}^k$  ( $1 \leq k \leq \text{NC}$ ), which must realize:

$$\mathbf{B}^k |_{\xi_2=-1} = 0 \quad \mathbf{B}^k(A_l) = \delta_{kl} \boldsymbol{\nu}(A_k) \quad \mathbf{B}^k \cdot \mathbf{n} = 0 \quad (26)$$

are given (in local numbering) by

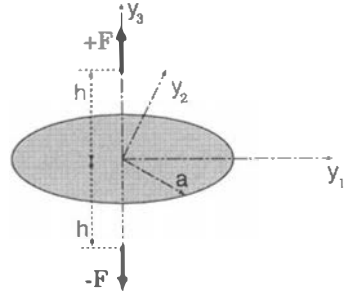
$$\mathbf{B}^j(\boldsymbol{\xi}) = \frac{1}{2}(1 - \xi_2) S_j(\xi_1) \mathbf{a} \wedge \mathbf{n} \quad (j = 1, 2, 3 \quad \mathbf{a} = (\|\mathbf{y}_{,\xi_1}\|)^{-1} \mathbf{y}_{,\xi_1}) \quad (27)$$

with  $S_1(\xi) = \xi(\xi - 1)/2$ ,  $S_2(\xi) = 1 - \xi^2$ ,  $S_3(\xi) = \xi(\xi + 1)/2$ .

NUMERICAL EXAMPLE A circular crack of radius  $a$  located in the  $(Oy_1y_2)$  plane is loaded by two symmetrical point-forces  $\pm F\mathbf{e}_3$  applied at points  $(0, 0, \pm h)$ . This example has an analytical solution for  $G$  [12]:

$$G = \frac{1 - \nu^2}{E} \frac{P^2}{(\pi h)^3} \frac{\alpha(\kappa + \alpha^2)}{(1 + \alpha^2)^4} \quad \left( \alpha = \frac{a}{h}, \quad \kappa = \frac{2 - \nu}{1 - \nu} \right) \quad (28)$$

Three crack meshes  $\mathcal{M}(8, 2)$ ,  $\mathcal{M}(12, 3)$  and  $\mathcal{M}(16, 5)$  have been used, with  $\mathcal{M}(n, p)$  being made of  $p$  rings of  $n$  9-noded quadrilateral elements each; hence the mesh  $\mathcal{M}(n, p)$  has are  $2n$  nodes on the crack front, associated to the same number of nodal normal velocities. Likewise, the discretized  $Q(\boldsymbol{\theta}, \boldsymbol{\mu})$  is associated with a  $2n \times 2n$  symmetric matrix.



- First, Fig. 1 displays the relative error made on the values of  $G$ , computed by means of either extrapolation of  $\varphi$  or the discretized variational equation (5-13), against  $h/a \in [2, 2.5]$ , using the mesh  $\mathcal{M}(12, 3)$ .
- Next, an easy consequence of (28) is that:

$$\frac{dG}{d\alpha} > 0 \quad (\alpha > \alpha_m) \quad \text{and} \quad \frac{dG}{d\alpha} < 0 \quad (0 < \alpha < \alpha_m)$$

$$\text{with} \quad \alpha_m^2 = \frac{\sqrt{16\nu^2 - 72\nu + 105} - 2\nu + 9}{2(2 - \nu)}$$

thus the circular growth is stable for  $\alpha > \alpha_m$  and unstable otherwise. The critical value  $\alpha = \alpha_m$  corresponds to a loss of positive definiteness of the quadratic form  $Q(\boldsymbol{\theta}, \boldsymbol{\theta})$  (6). The latter has been numerically computed for  $h/a = \alpha^{-1} \in [2., 2.5]$  with a 0.005 step and for the mesh  $\mathcal{M}(12, 3)$ ; we found  $\alpha \approx 2.27$ . Then, in order to refine the search,  $Q$  has been computed for  $\alpha^{-1} \in [2.26, 2.28]$  with a 0.0005 step and for the three meshes  $\mathcal{M}(8, 2)$ ,  $\mathcal{M}(12, 3)$ ,  $\mathcal{M}(16, 5)$ . The numerical values so obtained for  $\alpha_m$  are given in table 1 As can be seen, they approximate very closely the analytical  $\alpha_m \approx 2.27589$  (with the present choice  $\nu = 0.3$ ). When  $\alpha > \alpha_m$ , the

Mesh	Brackets for $\alpha$	relative error
$\mathcal{M}(8, 2)$	$2.267 \leq \alpha_m^{-1} \leq 2.2675$	$\leq 3.910^{-3}$
$\mathcal{M}(12, 3)$	$2.268 \leq \alpha_m^{-1} \leq 2.2685$	$\leq 3.510^{-3}$
$\mathcal{M}(16, 5)$	$2.277 \leq \alpha_m^{-1} \leq 2.275$	$\leq 7.010^{-4}$

TABLE 1. Numerical values obtained for  $\alpha_m^{-1}$

radius increment  $da$  is given in terms of the load increment  $dF$  by:

$$da = 2 \frac{dF}{F} \frac{\alpha(1 + \alpha^2)(\kappa + \alpha^2)}{3\alpha^4 + (7\kappa - 5)\alpha^2 - \kappa} \quad (29)$$

The relative error between the computed values (using mesh  $\mathcal{M}(16, 5)$ ) of the extension velocity and the exact solution defined by (29) are shown in Fig. 2. The deterioration of the error as  $h/a$  approaches the degeneracy value  $\alpha_m^{-1}$ , caused by progressive ill-conditioning of  $Q$ , is clearly visible.

• Finally, table 2 displays relative errors between computed and exact values (with  $\alpha = 0.5$ ) of  $W = W(\alpha, F)$ , eqn. (1);  $G(s)$  (extrapolation of  $\varphi$ );  $G(s)$  (using the formulation (5)); the extension velocity  $\mu$ .

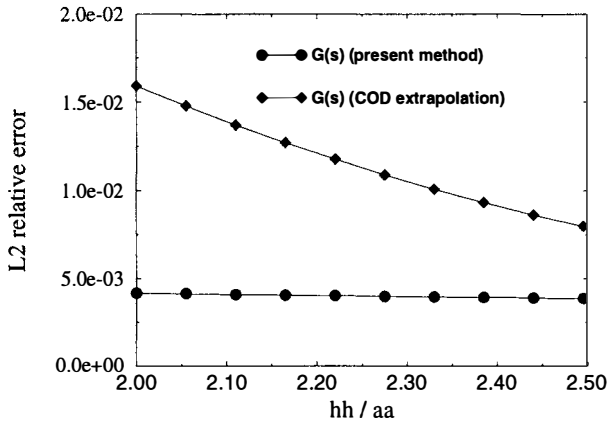


Figure 1. Relative  $L_2$  error on  $G$  by (a) the present approach (b) extrapolation of  $\phi_3$

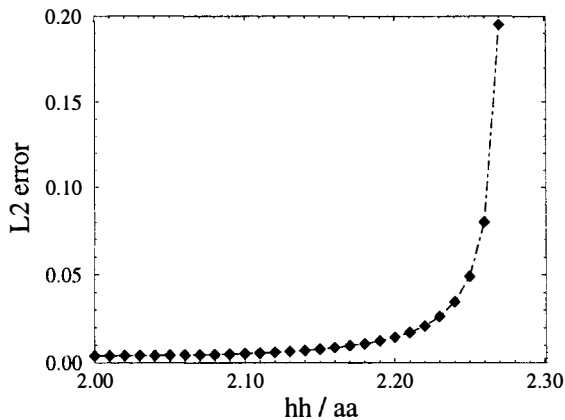


Figure 2. Relative  $L^2$  error on the extension velocity: mesh  $\mathcal{M}(16, 5)$ ,  $h/a \in [2, 2.3]$ .



Mesh	$\Delta P/P$	$\frac{\ \Delta G_{\text{present}}\ _{L^2}}{\ G\ _{L^2}}$	$\frac{\ \Delta G_{\text{extrapolation}}\ _{L^2}}{\ G\ _{L^2}}$	$\frac{\ \Delta \mu\ _{L^2}}{\ \mu\ _{L^2}}$
$\mathcal{M}(8, 2)$	2.905E-03	1.975E-02	5.463E-02	2.677E-02
$\mathcal{M}(12, 3)$	3.823E-05	7.897E-03	2.910E-02	1.965E-02
$\mathcal{M}(16, 5)$	-7.176E-04	3.789E-03	2.142E-02	4.466E-03

TABLE 2. Relative errors on  $P, G, \mu$  for  $\alpha = 0.5$

Note that the axisymmetric character of this particular example is not taken into account by the actual computer implementation, but is very well reproduced by the nodal values of  $G, \mu_\nu$  along  $\partial\Gamma$  numerically obtained.

CONCLUSION. The idea of combining boundary integral equations and analytical domain derivatives is new, to our best knowledge. The numerical example presented shows that in practice an excellent accuracy can be achieved for  $G(s), \overset{\star V}{W}$  and stability or instability of crack growth. Although the Griffith criterion alone is insufficient for mixed-mode three-dimensional crack propagation simulation, our approach can be combined with other techniques in more complex propagation criterions.

## References

1. P. BONNEMAY. *Equations intégrales pour l'élasticité plane*. PhD thesis, Université Paris VI, France, 1979.
2. M. BONNET. Regularized direct and indirect symmetric variational BIE formulations for three-dimensional elasticity. *Engng. Anal. Bound. Elem.*, 15:93–102, 1995.
3. M. BONNET, G. MAIER, C. POLIZZOTTO. On symmetric galerkin boundary element method. *Appl. Mech. Rev.*, 1997. (in preparation).
4. H. D. BUI. An integral equation method for solving the problem of a plane crack of arbitrary shape. *J. Mech. Phys. Solids*, 25:29–39, 1977.
5. PH. DESTUYNDER, M. DJAOUA, S LESCURE. Quelques remarques sur la mécanique de la rupture élastique. *J. Mécan. Théor. Appl.*, 2:113–135, 1983.
6. S. LI, M.E. MEAR, L. XIAO. Symmetric weak-form integral equation method for three-dimensional fracture analysis. *Comp. Meth. Appl. Mech. Engng.*, 1997. (to appear).
7. P. MIALON. Calcul de la dérivée d'une grandeur par rapport à un fond de fissure par la méthode  $\theta$ . Bulletin EDF/DER série c vol. 3, Electricité de France, 1987.
8. J. C. NEDELEC. Integral equations with non integrable kernels. *Integral equations and operator theory*, 5:562–572, 1982.
9. Q. S. NGUYEN, R. M. PRADELLES-DUVAL, C. STOLZ. Sur une loi régularisante en rupture et endommagement fragile. *C.R. Acad. Sci. Paris*, II-309:1515–1520, 1989.
10. R. M. PRADELLES-DUVAL. *Evolution de systèmes avec surfaces de discontinuité mobiles : application au délaminage*. PhD thesis, Ecole Polytechnique, Palaiseau, France, 1992.
11. X. Z. SUO AND A COMBESURE. Sur une formulation mathématique de la dérivée de l'énergie potentielle en théorie de la rupture fragile. *C.R. Acad. Sci. Paris, série II*, 308:1119, 1122–1989.
12. H. TADA, P. PARIS, G. IRWIN. The stress analysis of cracks handbook. Technical report, Del. Research Corporation, Hellertown, Pennsylvania, USA, 1973.