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ON THE IDENTIFICATION OF ELASTIC MODULI IN PLATES

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ABSTRACT

This paper addresses an identification problem for a thin linear elastic plate under the Love-Kirchhoff plate hypothesis. The inhomogeneous bending rigidities and the underlying elastic moduli of the material are identified from over specified displacement-force measurement data. The data represents a partial knowledge of the Dirichlet-to-Neumann data map. From a numerical point of view the problem is presented as a constrained minimization problem for the error on constitutive law over kinematically and statically admissible fields. A numerical reconstruction procedure is presented and illustrated on some examples for isotropic plates.

KEYWORDS

Plate, Love-Kirchhoff theory, Identification, Elastic moduli, Error on constitutive law

INTRODUCTION

Two important problems related to composites materials are:

- *identification of their mechanical properties*
- *identification of defects and damage*

These problems are complex as anisotropy and inhomogeneity are current characteristics of these materials and traditional direct measuring techniques cannot be always employed.

A characteristic static test for the identification of elastic moduli can be described in the following way:

Elastic moduli of a fixed plate are to be identified from displacements and forces measured simultaneously on a certain number of interior and boundary points (for a review see [6]).

This type of test has traditionally been used in order to identify a *homogeneous* plate [4, 5, 6]. Changing the distributions of applied forces gives rise to a series of deflection-force measurements, representing a partial knowledge of the Dirichlet-to-Neumann data map characterizing the boundary response of a body. This map gives generally the correspondence between displacement, the Dirichlet boundary condition, and forces the Neumann boundary conditions. A

series of mathematical results [13] showed that the knowledge of the Dirichlet-to-Neumann data map permits the identification of the *nonhomogenous* distribution of elastic moduli. Therefore one can expect to identify in a similar way for elastic plates a *nonhomogenous* distribution of elastic moduli from a series of boundary measurements.

The purpose of this paper is to place this identification problem in the classical framework of elastic impedance tomography and present a possible solution using the minimization of the error on constitutive law [2].

The first introductory part of the paper presents the mathematical inverse problem and a short discussion of the uniqueness of the identification problem in the anisotropic case under the static test.

The second part of the paper presents a numerical reconstruction procedure based on the minimization of the error on the constitutive law. This error functional can be expressed for general elastic bodies under the following form:

$$\int_{\Omega} \|\mathbf{C}^{-\frac{1}{2}}\sigma - \mathbf{C}^{\frac{1}{2}}\epsilon\|^2 dx \quad (1)$$

where \mathbf{C} , σ , ϵ , denote the tensor field of elastic moduli, of stresses and respectively of strains. In order to apply this formula to anisotropic plates a simple transformation will be made using the plate kinematics hypothesis and the representation of the elasticity tensor in the base of eigenelastic moduli represented by the eigenvalues and eigenvectors of the elasticity tensor.

The third part of the paper presents a series of examples using simulated measurements obtained by direct computations and addition of white noise. The identification corresponds to composite plates having different kinds of material symmetry.

THE IDENTIFICATION PROBLEM

Let us consider a thin elastic plate in the classical framework of the Love-Kirchhoff plate theory occupying in the reference configuration a regular domain Ω with boundary $\partial\Omega$. \mathbf{n} and \mathbf{t} will denote the normal and tangent unit vector on the $\partial\Omega$.

Let w , \mathbf{k} , \mathbf{M} stand respectively for the deflection field of the plate, the second-order tensor of generalized strain, corresponding to curvature, and the the second-order tensor of generalized stress, corresponding to bending moments. Considering a stress-free initial state and no body forces, the governing equations on Ω are:

$$\mathbf{k} = \nabla\nabla w \quad \mathbf{M} = \mathbf{D}\mathbf{k} \quad \text{divdiv}\mathbf{M} = 0 \quad (2)$$

where \mathbf{D} is the forth rank tensor of the bending rigidities, related to the Hooke elasticity tensor in the classical way. \mathbf{D} is positive definite and for which the classical symetry relations:

$$M_{\alpha\beta\gamma\phi} = M_{\beta\alpha\gamma\phi} = M_{\gamma\phi\alpha\beta} \quad \alpha, \beta, \gamma, \phi = 1, 2 \quad (3)$$

\mathbf{D} is supposed to be inhomogenous, i.e. $\mathbf{D} = \mathbf{D}(\mathbf{x})$, $\mathbf{x} \in \Omega$ unless specified otherwise.

The above equations and the prescription of two of the following quantities on the boundary:

- deflection: $w = \phi$
- normal derivative of the deflection: $\frac{\partial w}{\partial \mathbf{n}} = \psi$
- moment: $\mathbf{nMn} = \Phi$

- forces: $\frac{\partial}{\partial t} (\mathbf{nMt}) + (\text{divM}) \mathbf{n} = \Psi$

describes completely the classical *direct problem* for elastical plates. For this problem uniqueness of the solution as well as continuous dependence of the boundary values and elastic moduli are assured.

The mathematical *inverse problem for plates* (see [8, 9]), supposes that the tensor of the bending rigidities D is not known and that it has to be recovered from the knowledge of the Dirichlet-to-Neumann data map:

$$\Pi_M : \begin{pmatrix} w \\ \frac{\partial w}{\partial \mathbf{n}} \end{pmatrix} \longrightarrow \begin{pmatrix} \mathbf{nMn} \\ \frac{\partial}{\partial t} (\mathbf{nMt}) + (\text{divM}) \mathbf{n} \end{pmatrix} \quad (4)$$

This idealized problem of non-destructive testing and its electrical analogue have extensively been discussed in the mathematical literature. One of the important addressed questions is uniqueness, stated in the context of plates as follows: *If two plates having bending rigidities D and D' give the same response to measurements, are necessary D and D' equal ?*. This problem is more difficult than the global elastic one as explained in [9].

For the isotropic elastic plates a uniqueness result for the linearized problem has been given by Ikehata [7]. He has showed that the Young modulus and the Poisson ratio can be identified from the Dirichlet-to-Neumann data map if their boundary values and their derivatives are known and if they are close enough to a constant.

For anisotropic elastic plates general uniqueness can not be expected as in the case of three dimensional elasticity [1], as explained by Ikehata in [9]. The uniqueness of the linearized problem is related to two classes of homogeneous elasticity tensors which coincide with the ones defined by Lehnitskii [10] for the study of the general solution of the plate equation. The global uniqueness questions remains to our knowledge still an open problem.

From an engineering point of view, one can remark that problems are stated in a slightly different manner. One has to recover the bending rigidities D from a series of simultaneous displacement and force measurements and generally a series of additional information is given. In a certain number of cases one can accept that bending rigidities are homogeneous or deflection can be measured on a portion of the surface of the plate. A series of experiments and different methods for determining homogeneous bending rigidities have already been described in a series of papers [4, 5, 6]. However, as it will be shown in the sequel, these information are sufficient under a certain number of assumptions to obtain some informations about a inhomogeneous distribution of bending rigidities.

THE RECONSTRUCTION METHOD

The reconstruction procedure presented here is based on a variational reconstruction method of the minimization of the error on the constitutive law. This technique has already been applied in the case of electric impedance tomography [11] and the 2D elasticity identification problem [2].

The measured data will be represented by a series of simultaneous measured boundary fields $\{\phi_i, \psi_i, \Phi_i, \Psi_i\}_{i=1, N}$ and eventually the corresponding series of measured deflections and concentrated forces $\{v_i, f_i\}_{i=1, N}$ on a certain number of interior points.

Considering the constitutive equation apart, it is natural to form the space of kinematically and statically admissible fields, $\text{KA}(\phi, \psi, v)$ and respectively $\text{SA}(\Phi, \Psi, f)$. This permits to

decouple the general laws of dynamics and kinematics from the general problem and to state the inverse problem as a constrained minimization problem:

Find the distribution of bending rigidities \mathbf{D} and deflections $w_i \in \mathbf{KA}(\phi_i, \psi_i, v_i)$ ($i = 1, N$) and the bending moments $\mathbf{M}_i \in \mathbf{SA}(\Phi, \Psi, f)$ ($i = 1, N$) minimizing the error on the constitutive law (ECL):

$$I(\mathbf{D}, \{w_i\}_{i=1,N}, \{\mathbf{M}_i\}_{i=1,N}) = \sum_{i=1,N} \int_{\Omega} \|\mathbf{D}^{-1/2} \mathbf{M}_i - \mathbf{D}^{1/2} \mathbf{k}_i\|^2 dv \quad (5)$$

This form of the ECL can be changed by successive integration by parts and usage of equilibrium equations and boundary conditions to the following expression:

$$\begin{aligned} I(\mathbf{D}, \{w_i\}_{i=1,N}, \{\mathbf{M}_i\}_{i=1,N}) &= \sum_{i=1,N} \int_{\Omega} \mathbf{M}_i \mathbf{D}^{-1} \mathbf{M}_i + \mathbf{k}_i \mathbf{D} \mathbf{k}_i dv \\ &- 2 \sum_{i=1,N} \int_{\partial\Omega} \phi \Phi + \psi \Psi ds - 2 \sum_{i=1,N} v_i f_i \end{aligned}$$

As the last two terms are the known measurements, it follows that the only part of the ECL playing a role in the minimization procedure is:

$$J(\mathbf{D}, \{w_i\}_{i=1,N}, \{\mathbf{M}_i\}_{i=1,N}) = \sum_{i=1,N} \int_{\Omega} \mathbf{M}_i \mathbf{D}^{-1} \mathbf{M}_i + \mathbf{k}_i \mathbf{D} \mathbf{k}_i dv \quad (6)$$

The fact that J represents the sum of a generalized stress (moment) and strain (curvature) energy conducts naturally to the following minimization algorithm:

1. *with given \mathbf{D} minimize J over $\mathbf{KA}(\phi, \psi, v)$ and $\mathbf{SA}(\Phi, \Psi, f)$*

This step corresponds to solving a series of classical boundary value problems for plates.

2. *with given \mathbf{k}_i and \mathbf{M}_i ($i = 1, N$) minimize J over \mathbf{D}*

In special cases of elastic anisotropies it has been shown [2], using the concept of eigenelastic moduli, that the last step of the minimization process can be reduced, as in the case of isotropic electricity, to a straightforward minimization. Let us denote, using the same technique, $\mathbf{A}^{(j)}$, $j = 1, 3$, the projections of a second rank tensor \mathbf{A} on the basis of eigentensors corresponding to the tensor of bending rigidities \mathbf{D} . Thus the constitutive equations can be expressed in 3 uncoupled relations:

$$\mathbf{M}^{(j)} = d_{(j)} \mathbf{k}^{(j)} \quad j = 1, 3 \quad (7)$$

and the partial minimization of J with respect to $d^{(j)}$ has the following expression:

$$d_{(j)}^2 = \frac{\sum_{i=1,N} (\mathbf{M}_i^{(j)})^2}{\sum_{i=1,N} (\mathbf{k}_i^{(j)})^2} \quad j = 1, 3 \quad (8)$$

for given \mathbf{k}_i and \mathbf{M}_i .

NUMERICAL RESULTS

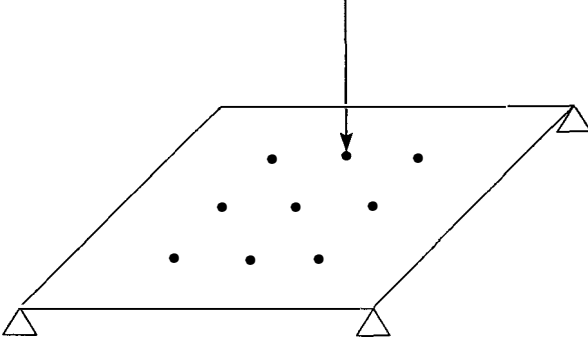


Figure 1: The tested plate and the positions of the applied forces in different experiments

The numerical results present the identification of an inclusion in case of an isotropic elastic plate. The tensor of bending rigidities has in this case the following form in the Cartesian coordinate system:

$$\mathbf{M} = \begin{pmatrix} a + b & a & 0 \\ a & a + b & 0 \\ 0 & 0 & 2b \end{pmatrix} \quad (9)$$

where $a(x)$ and $b(x)$ are related to the Young modulus E and the Poisson coefficient ν by:

$$a = \frac{h^3 E \nu}{12(1 - \nu^2)} \quad b = \frac{h^3 E(1 - \nu)}{12(1 - \nu^2)} \quad (10)$$

In this case the eigenelastic constants are defined as follows:

$$d_{(1)} = 2a + b \quad d_{(2)} = 2b \quad d_{(3)} = 2b \quad (11)$$

A straightforward calculation using the corresponding eigenelastic tensors gives the optimal eigenelastic moduli in the minimization of J :

$$d_{(1)}^2 = (2a + b)^2 = \frac{(M_{11} + M_{22})^2}{(k_{11} + k_{22})^2} \quad d_{(2)}^2 = 4b^2 = \frac{M_{12}^2}{k_{12}^2} \quad d_{(3)}^2 = 4b^2 = \frac{(M_{11} - M_{22})^2}{(k_{11} - k_{22})^2}$$

and defines therefore completely the minimization algorithm.

The plate with the loading configuration is presented schematically in fig.1. In a first step displacement-force measurements were generated by direct elastic computations, corresponding to a concentrated forces applied on 9 different locations. The measured displacement data consists of the deflection w of the point where the force has been applied and of the deflection distribution on the boundary of the plate. In some cases the normal derivative of the deflection $\frac{\partial w}{\partial \mathbf{n}}$ has also been considered as measured, this additional information has not changed drastically the obtained results. In a second step the elastic moduli have been reconstructed (fig. 2-3) using the simulated measurements and the algorithm presented in the preceding section.

All computations have been programmed using *gibiane*, a object oriented language specialized for finite element computations (FEM code Castem 2000), on a HP9000 workstation. The domain, 1 unit \times 1 unit, was divided in $2 \times 20 \times 20$ triangular elements of the DKT (Discrete Kirchhoff Triangle) type. The 9 direct computations took a couple of seconds and the

inversion took about 1 minute for 1 iteration, i.e. 2×9 direct computations plus the internal calculations of the eigenelastic moduli.

The primary focus of these reconstructions was to assess the differences in the identification of the young modulus using different eigenelastic moduli as the identification parameter.

An initial number of tests has been performed to test the robustness of the algorithm for a homogeneous distribution of the young modulus. The real value has generally been recovered after 5 iterations from starting values ranging up to 2 times the real value. This permits to expect robust behaviour of the functional in applications as the one presented by Grediac [4, 5, 6].

A series of tests have been done in order to identify different distributions of nonhomogeneous young moduli corresponding to shape as presented in fig.2-3. The inclusions have been generally localized even if the exact shape was difficult to attain. It is important to remark that the localization property was dependent on the eigenelastic moduli used for the identification, even if the same Young modulus had to be reconstructed. In a certain number of cases (see for example fig. 3) a certain interference between the boundary conditions and the form of the inclusion occurred.

Comparing these results with previous ones obtained in 2D elasticity [2] one can remark that the reconstruction in the case of plates was less precise. This could be partially explained by the different number of experiments used in the inversion (9 versus 40 or 96). However the minimisation of the error functional gave the same huge descent in the first steps before attending an almost flat valley, where a degradation of the reconstruction can be expected.

The ring-shaped inclusion was reconstructed very badly (fig. 4). The identification presented the similar problems as in 2D elasticity. It is important to remark that the concentrated forces were all applied on the ring. It is therefore possible that a more precise image could be obtained from a larger number of measurement points.

The same reconstructions, done using a slightly different error functional:

$$I(\mathbf{D}, \{w_i\}_{i=1,N}, \{\mathbf{M}_i\}_{i=1,N}) = \sum_{i=1,N} \int_{\Omega} |\mathbf{D}^{-1/2} \mathbf{M}_i - \mathbf{D}^{1/2} \mathbf{k}_i| dv \quad (12)$$

presented the same overall tendencies. This is to be expected as the two norms are equivalent from a mathematical point of view. However no a priori judgement can be done on the numerical performance of the norms. A complete report of the efficiency of this method in different cases and using a priori information on the shape of the inclusion will be presented in a further publication.

CONCLUSION

This paper presented a numerical reconstruction method for nonhomogeneous elasticity tensors in thin elastic plates. It has been shown that the minimization of the error on the constitutive law can also be applied in the case of plates and a splitting of the error functional using eigenelastic tensors has been applied. Identifying the same parameter from different parts of the error functional showed the importance of the energy distribution in different components of the solution during the reconstruction process. This shows also the importance of the experimental configuration, i.e. boundary conditions, location of applied forces, in this identification problem.

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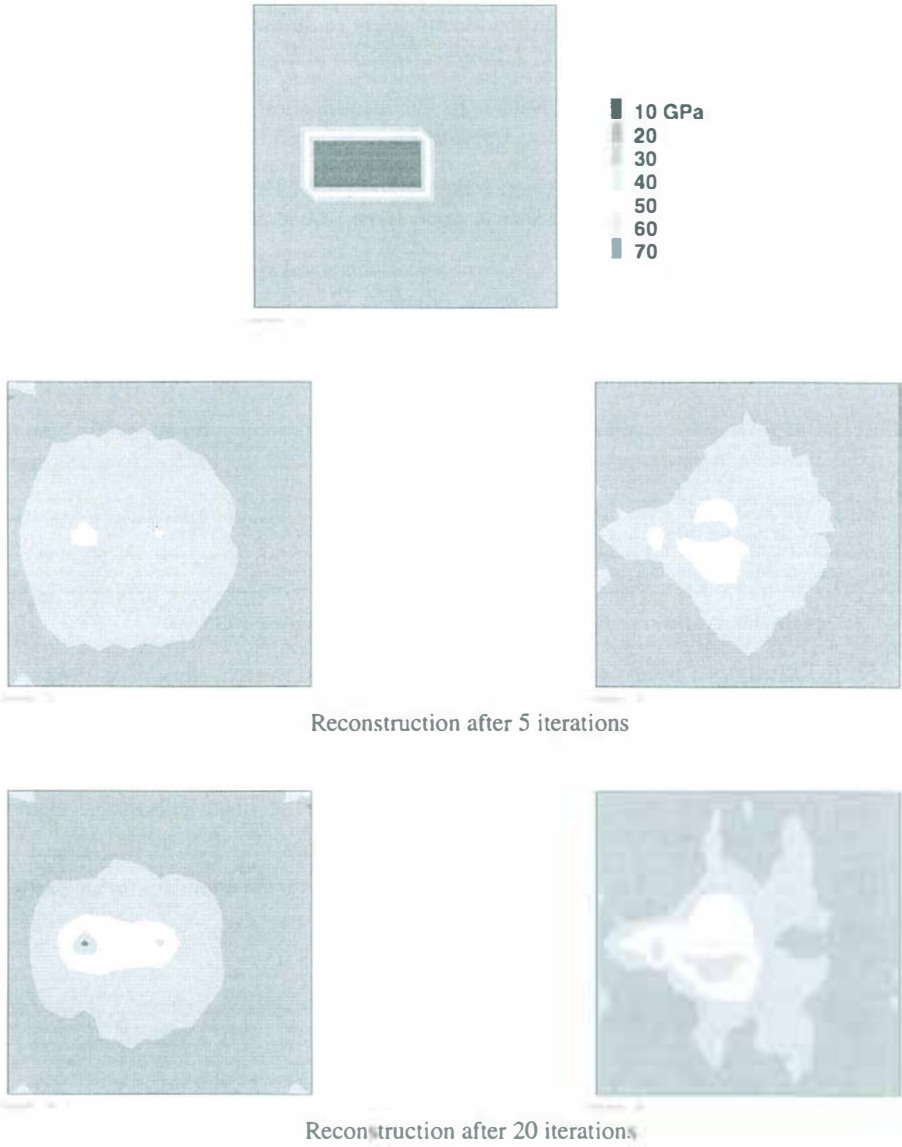


Figure 2: Real and reconstructed values of the young modulus using the $d_{(1)} = 2a + b = \frac{M_{11} + M_{22}}{k_{11} + k_{22}}$ (left column) and the $d_{(2)} = 2b = \frac{M_{12}}{k_{12}}$ (right column) eigenelastic moduli

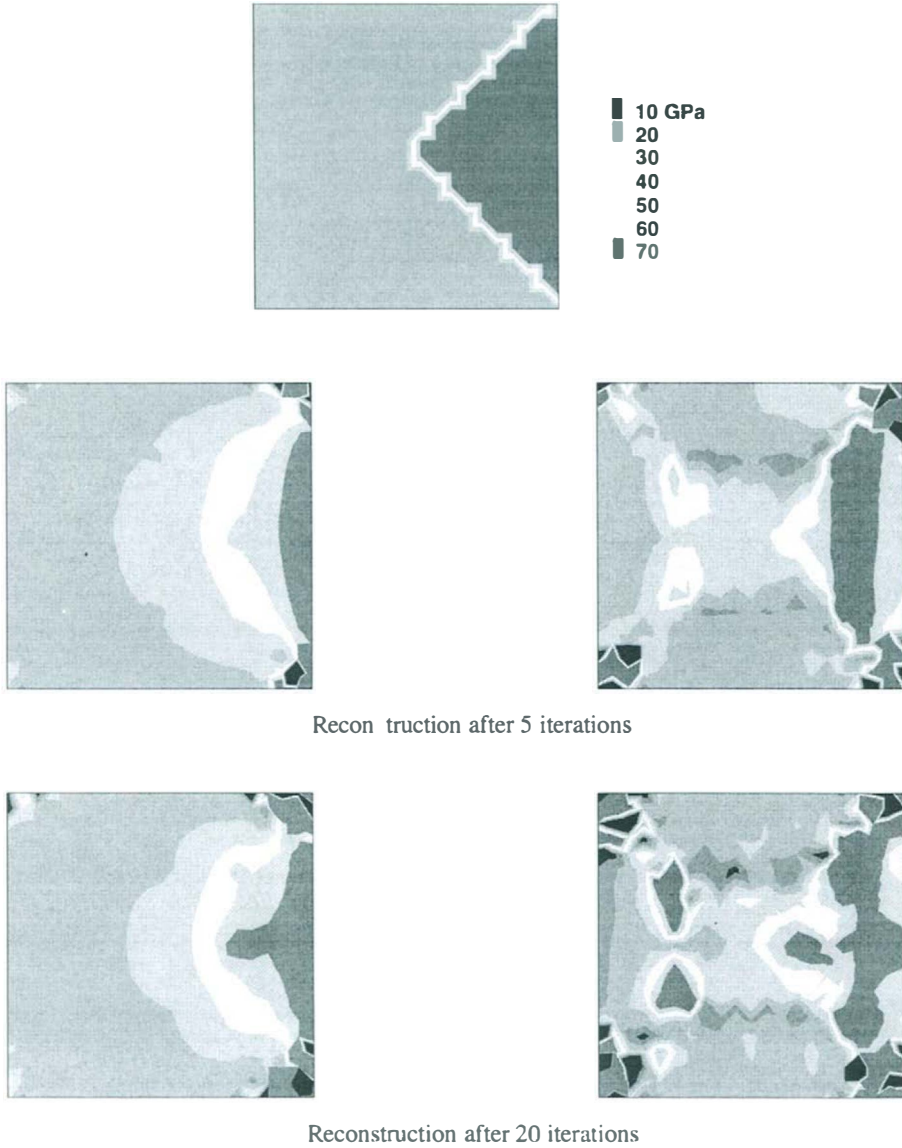


Figure 3: Real and reconstructed values of the young modulus using the $d_{(1)} = 2a + b = \frac{M_{11} + M_{22}}{k_{11} + k_{22}}$ (left column) and the $d_{(3)} = 2b = \frac{M_{11} - M_{22}}{k_{11} - k_{22}}$ (right column) eigenelastic moduli

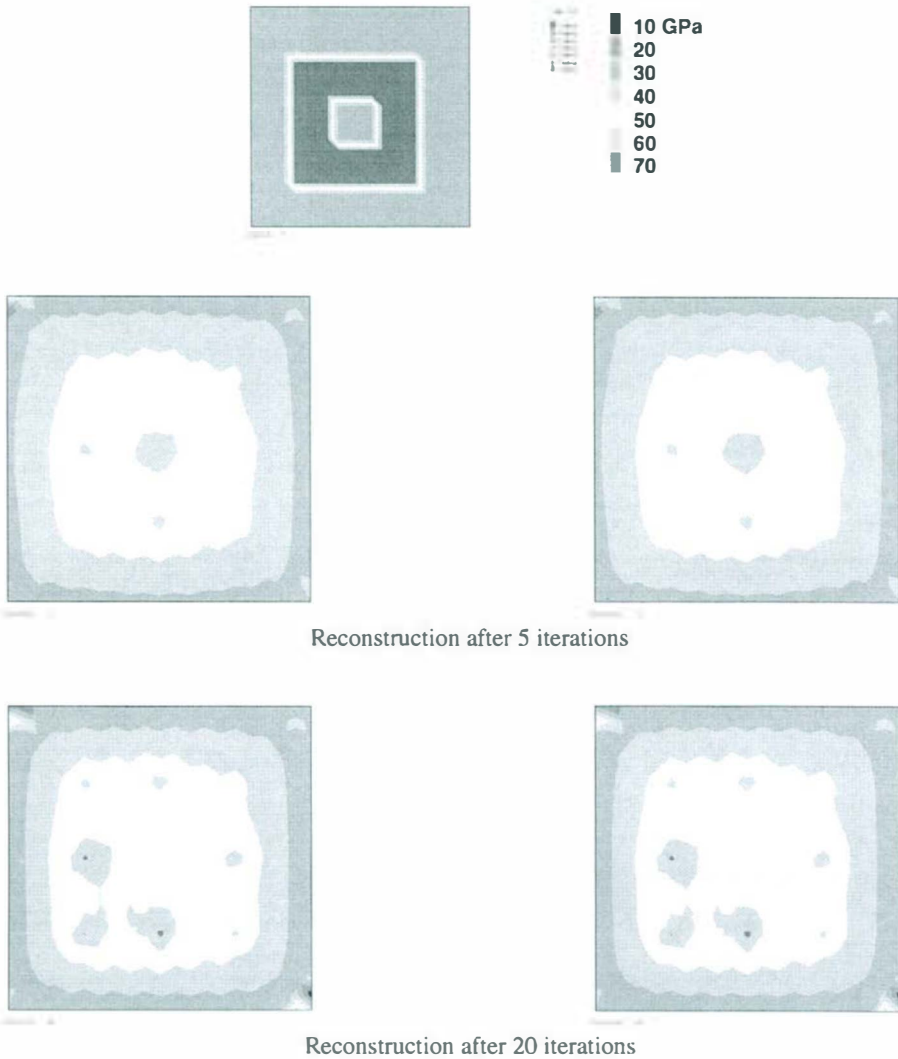


Figure 4: Real and reconstructed values of the young modulus using the $d_{(1)} = 2a + b = \frac{M_{11} + M_{22}}{k_{11} + k_{22}}$ (left column) and the $d_{(3)} = 2b = \frac{M_{11} - M_{22}}{k_{11} - k_{22}}$ (right column) eigenelastic moduli

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