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# A SINGULAR GIERER-MEINHARDT SYSTEM WITH DIFFERENT SOURCE TERMS

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## Abstract

We study the existence and nonexistence of classical solutions to a general Gierer-Meinhardt system with Dirichlet boundary condition. The main feature of this paper is that we are concerned with a model in which both the activator and the inhibitor have different sources given by general nonlinearities. Under some additional hypotheses and in case of pure powers in nonlinearities, regularity and uniqueness of the solution in one dimension is also presented.

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**Key words:** Gierer-Meinhardt; Elliptic system; Activator-inhibitor; Maximum principle.

## 1 Introduction and the main results

The systems of nonlinear equations of Gierer-Meinhardt type have received a considerable attention in the last decade. These problems arise in the study of biological pattern formation by auto and cross catalysis being related to known biochemical processes and cellular properties. The general model proposed by Gierer and Meinhardt [7, 12] may be written as

$$\begin{cases} u_t = d_1 \Delta u - \alpha u + c \rho \frac{u^p}{v^q} + \rho_0 \rho & \text{in } \Omega \times (0, T), \\ v_t = d_2 \Delta v - \beta v + c' \rho' \frac{u^r}{v^s} & \text{in } \Omega \times (0, T), \end{cases} \quad (1.1)$$

subject to Neumann boundary conditions. Here  $\Omega \subset \mathbb{R}^N$  ( $N \geq 1$ ) is a bounded domain,  $u, v$  represent the concentrations of the activator and inhibitor with the source distributions  $\rho$  and  $\rho'$  respectively. Also  $d_1, d_2$  are diffusion coefficients with  $d_1 \ll d_2$  and  $\alpha, \beta, c, c', \rho_0$  are positive constants. The exponents  $p, q, r, s \geq 0$  verify the relation  $qr > (p-1)(s+1) > 0$ . The system (1.1) is of reaction-diffusion type and involves the determination of an activator and an inhibitor concentration field. In a biological context, the Gierer-Meinhardt system (1.1) has been used to model several phenomena arising in morphogenesis and cellular differentiation.

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The model presented by Gierer and Meinhardt [7] originates in the Turing's one [20] for morphogenesis in the linear case and is based on the short range of activation and on the long range of inhibition. Also the model introduced in [7] takes into account the classification between the concentration of activators and inhibitors, on the one hand, and the densities of their sources, on the other hand. A complete description of entire dynamics of system (1.1) is given in the recent paper of Ni, Suzuki and Takagi [15], where it is shown that the dynamics of the system (1.1) exhibit various interesting behaviors such as periodic solutions, unbounded oscillating global solutions, and finite time blow-up solutions.

Many recent works have been devoted to the study of the steady-states solutions of (1.1), that is, solutions of the stationary system

$$\begin{cases} d_1 \Delta u - \alpha u + c \rho \frac{u^p}{v^q} + \rho_0 \rho = 0 & \text{in } \Omega, \\ d_2 \Delta v - \beta v + c' \rho' \frac{u^r}{v^s} = 0 & \text{in } \Omega, \end{cases} \quad (1.2)$$

subject to Neumann boundary conditions. Such systems are difficult to treat due to the lack of a variational structure or a priori estimates. In this case it is more convenient to consider the *shadow system* associated to (1.2). More exactly, dividing the second equation of (1.2) by  $d_2$  and then letting  $d_2 \rightarrow \infty$ , we reduce the system (1.2) to a single equation. The nonconstant solutions of such equation present interior or boundary peaks or spikes, i.e., they exhibit a *point concentration* phenomenon. Among the great number of works in this direction, we refer the reader to [16, 17, 18, 21, 22, 23] and the reference therein, as well as to the survey paper of Ni [14]. For the study of instability of solutions to (1.2), we also mention here the works of Miyamoto [13] and Yanagida [24].

In the case  $\Omega = \mathbb{R}^N$  ( $N = 1, 2$ ) it has been shown in [3, 4] that there exist ground state solutions of (1.3) with single or multiple bumps in the activator which, after a rescaling of  $u$ , are approaching a universal profile.

Let  $\Omega \subset \mathbb{R}^N$  ( $N \geq 1$ ) be a bounded domain with smooth boundary. In this paper we consider the stationary Gierer-Meinhardt system for a wide class of nonlinearities subject to homogeneous Dirichlet boundary conditions. More exactly, we are concerned with the following elliptic system

$$\begin{cases} \Delta u - \alpha u + \frac{f(u)}{g(v)} + \rho(x) = 0, u > 0 & \text{in } \Omega, \\ \Delta v - \beta v + \frac{h(u)}{k(v)} = 0, v > 0 & \text{in } \Omega, \\ u = 0, v = 0 & \text{on } \partial\Omega, \end{cases} \quad (\mathcal{S})$$

where  $\alpha, \beta > 0$ ,  $\rho \in C^{0,\gamma}(\Omega)$ , ( $0 < \gamma < 1$ ),  $\rho \geq 0$ ,  $\rho \not\equiv 0$  and  $f, g, h, k \in C^{0,\gamma}[0, \infty)$  are nonnegative and nondecreasing functions such that  $g(0) = k(0) = 0$ . This last assumption on  $g$  and  $k$ , together with the Dirichlet conditions on  $\partial\Omega$  makes the system singular at the boundary. Another difficulty is due to the non-cooperative (i.e., non-quasimonotone) character of our system.

We are mainly interested in the case where the activator and inhibitor have different source terms, that is, the mappings  $t \mapsto f(t)/h(t)$  and  $t \mapsto g(t)/k(t)$  are not constant on  $(0, \infty)$ . Our study is motivated by some questions addressed by Choi and McKenna [1, 2] or Kim [10, 11] concerning existence and nonexistence

or even uniqueness of the classical solutions for the model system

$$\begin{cases} \Delta u - \alpha u + \frac{u^p}{v^q} + \rho(x) = 0 & \text{in } \Omega, \\ \Delta v - \beta v + \frac{u^r}{v^s} = 0 & \text{in } \Omega, \\ u = 0, v = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.3)$$

In [1, 10] it is assumed that the activator and inhibitor have common sources and the approach rely on the Schauder's fixed point theorem through a decouplization of the system. More precisely, subtracting the two equations of (1.3) we obtain in the case  $p = r$  and  $q = s$  a linear equation in  $w = u - v$ . This is suitable to obtain a priori estimates in order to control the map whose fixed points are solutions of (1.3).

In Choi and McKenna [2] it is obtained the existence of radially symmetric solutions of (1.3) in the case  $\Omega = (0, 1)$  or  $\Omega = B_1 \subset \mathbb{R}^2$  and  $p = r > 1$ ,  $q = 1$ ,  $s = 0$ . In [2] *a priori* bounds are obtained via sharp estimates of the associated Green function.

In Section 2 we give a nonexistence result for classical solutions to  $(\mathcal{S})$ . To our best knowledge, there are no results of this type in the literature. The main idea is to speculate the asymptotic behavior of  $v$  in the second equation of  $(\mathcal{S})$ . This will be then used in the first equation of the system and by classical arguments (see, e.g., [5, Theorem 1.1]) we obtain the desired nonexistence result. A special attention is payed to the case of pure powers in nonlinearities. In this sense we obtain some relations between the exponents  $p, q, r$  and  $s$  for which the system (1.3) has no classical solutions.

In Section 3 we give an existence result for classical solutions of  $(\mathcal{S})$  under the additional hypothesis  $\beta \leq \alpha$ . In fact, this assumption is quite natural if we look at the steady-state system (1.2). We have only to divide the first equation by  $d_1$ , the second one by  $d_2$  and to take into account the fact that  $d_1 \ll d_2$ . The existence in our case is obtained without assuming any growth condition on  $\rho$  near the boundary since we are able to provide more general bounds for the regularized system associated to  $(\mathcal{S})$ . In particular, we obtain that (1.3) has solutions provided that  $r - p = s - q \geq 0$  and  $q > p - 1$ .

The uniqueness of the solution is a delicate matter. Actually, there is only one result in the literature in this direction (see [1, Theorem 1]) and concerns the one dimensional case of system (1.3) with  $\rho \equiv 0$  and  $p = q = r = s = 1$ . Using the same idea as in [1], we are able to extend the uniqueness of the solution to  $(\mathcal{S})$  in one dimension to the following range of exponents:  $0 < q \leq p \leq 1$  and  $r - p = s - q \geq 0$ . It is worth pointing out here that the uniqueness of the solution for systems like  $(\mathcal{S})$  seems to be a particular feature of the Dirichlet boundary conditions. As we can see in the above mentioned works, in the case of Neumann boundary conditions the Gierer-Meinhardt system does not have a unique solution.

## 2 A nonexistence result

Several times in this paper we shall use the following result. We refer the reader to [6, Lemma 2.1] for a complete proof.

**Lemma 2.1.** *Let  $\Psi : \Omega \times (0, \infty) \rightarrow \mathbb{R}$  be a Hölder continuous function such that the mapping  $(0, \infty) \ni t \mapsto \frac{\Psi(x, t)}{t}$  is strictly decreasing for each  $x \in \Omega$ . Assume that there exist  $v_1, v_2 \in C^2(\Omega) \cap C(\overline{\Omega})$  such that*

- (a)  $\Delta v_1 + \Psi(x, v_1) \leq 0 \leq \Delta v_2 + \Psi(x, v_2)$  in  $\Omega$ ;
- (b)  $v_1, v_2 > 0$  in  $\Omega$  and  $v_2 \leq v_1$  on  $\partial\Omega$ ;

(c)  $\Delta v_1 \in L^1(\Omega)$  or  $\Delta v_2 \in L^1(\Omega)$ .

Then  $v_2 \leq v_1$  in  $\Omega$ .

Another useful tool is the following result which is a direct consequence of the maximum principle.

**Lemma 2.2.** *Let  $k \in C(0, \infty)$  be a positive nondecreasing function and  $a_1, a_2 \in C(\Omega)$  with  $0 < a_2 \leq a_1$  in  $\Omega$ . Assume that there exist  $\beta > 0$ ,  $v_1, v_2 \in C^2(\Omega) \cap C(\overline{\Omega})$  such that  $v_1, v_2 > 0$  in  $\Omega$ ,  $v_1 \geq v_2$  on  $\partial\Omega$  and*

$$\Delta v_1 - \beta v_1 + \frac{a_1(x)}{k(v_1)} \leq 0 \leq \Delta v_2 - \beta v_2 + \frac{a_2(x)}{k(v_2)} \quad \text{in } \Omega.$$

Then  $v_2 \leq v_1$  in  $\Omega$ .

Let  $\Phi : [0, 1) \rightarrow [0, \infty)$  defined by

$$\Phi(t) = \int_0^t \frac{1}{\sqrt{2 \int_\tau^1 \frac{1}{k(\theta)} d\theta}} d\tau, \quad 0 \leq t < 1.$$

Set  $a = \lim_{t \rightarrow 1} \Phi(t)$  and consider  $\Psi : [0, a) \rightarrow [0, 1)$  the inverse of  $\Phi$ . The main result of this section is the following nonexistence property.

**Theorem 2.1.** *Assume that*

$$\int_0^a \frac{tf(mt)}{g(M\Psi(t))} dt = +\infty, \quad (2.1)$$

for all  $0 < m < 1 < M$ . Then the system (S) has no classical solutions.

*Proof.* Assume, by contradiction, that there exists a classical solution  $(u, v)$  of the system (S) and let  $\varphi_1$  be the normalized first eigenfunction of  $-\Delta$  in  $H_0^1(\Omega)$ . As it is well known,  $\varphi_1 \in C^2(\overline{\Omega})$  and we can assume that  $\varphi_1 > 0$  in  $\Omega$ . Let  $\zeta$  denote the unique solution of the problem

$$\begin{cases} \Delta \zeta - \alpha \zeta + \rho(x) = 0 & \text{in } \Omega, \\ \zeta = 0 & \text{on } \partial\Omega. \end{cases} \quad (2.2)$$

By standard elliptic arguments and the classical maximum principle we deduce that  $\zeta \in C^2(\overline{\Omega})$  and  $\zeta > 0$  in  $\Omega$ .

In view of Hopf's maximum principle and taking into account the regularity of the domain, there exist  $c_1, c_2 > 0$  such that

$$c_1 d(x) \leq \varphi_1, \zeta \leq c_2 d(x) \quad \text{in } \Omega, \quad (2.3)$$

where  $d(x) = \text{dist}(x, \partial\Omega)$ .

Since

$$\begin{cases} \Delta(u - \zeta) - \alpha(u - \zeta) \leq 0 & \text{in } \Omega, \\ u - \zeta = 0 & \text{on } \partial\Omega, \end{cases}$$

by the weak maximum principle [8, Corollary 3.2] we have  $u \geq \zeta$  in  $\Omega$ . Hence, by (2.3) it follows that

$$u(x) \geq md(x) \quad \text{in } \Omega, \quad (2.4)$$

for some  $m > 0$  small enough. Set  $C = \max_{x \in \bar{\Omega}} h(u(x)) > 0$ . Then  $v$  satisfies

$$\begin{cases} \Delta v - \beta v + \frac{C}{k(v)} \geq 0 & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \quad (2.5)$$

Let  $c > 0$  be such that

$$c\varphi_1 \leq \min\{a, d(x)\} \quad \text{in } \Omega. \quad (2.6)$$

We need the following auxiliary result.

**Lemma 2.3.** *There exists  $M > 1$  large enough such that  $\bar{v} = M\Psi(c\varphi_1)$  satisfies*

$$\Delta \bar{v} - \beta \bar{v} + \frac{C}{k(\bar{v})} \leq 0 \quad \text{in } \Omega. \quad (2.7)$$

*Proof.* Since  $\Phi(\Psi(t)) = t$  for all  $0 \leq t < a$ , we get  $\Psi(0) = 0$  and  $\Psi \in C^1(0, a)$  with

$$\Psi'(t) = \sqrt{2 \int_{\Psi(t)}^1 \frac{1}{k(\tau)} d\tau} \quad \text{for all } 0 < t < a. \quad (2.8)$$

This yields

$$\begin{cases} -\Psi''(t) = \frac{1}{k(\Psi(t))} & \text{for all } 0 < t < a, \\ \Psi'(t), \Psi(t) > 0 & \text{for all } 0 < t < a, \\ \Psi(0) = 0. \end{cases} \quad (2.9)$$

By Hopf's maximum principle, there exist  $\omega \Subset \Omega$  and  $\delta > 0$  such that

$$|\nabla \varphi_1| > \delta \quad \text{in } \Omega \setminus \omega \quad \text{and} \quad \varphi_1 > \delta \quad \text{in } \omega. \quad (2.10)$$

Fix  $M > 1$  large enough such that

$$M(c\delta)^2 > C \quad \text{and} \quad Mc\lambda_1\delta\Psi'(c\|\varphi_1\|_\infty) > \frac{C}{\min_{x \in \omega} k(\Psi(c\varphi_1))}. \quad (2.11)$$

We have

$$-\Delta \bar{v} = \frac{Mc^2}{k(\Psi(c\varphi_1))} |\nabla \varphi_1|^2 + Mc\lambda_1\varphi_1\Psi'(c\varphi_1) \quad \text{in } \Omega.$$

By (2.11) we get

$$\begin{aligned} -\Delta \bar{v} &\geq Mc\lambda_1\varphi_1\Psi'(c\varphi_1) \geq Mc\lambda_1\delta\Psi'(c\|\varphi_1\|_\infty) \geq \frac{C}{k(\bar{v})} & \text{in } \omega, \\ -\Delta \bar{v} &\geq \frac{Mc^2}{k(\Psi(c\varphi_1))} |\nabla \varphi_1|^2 \geq \frac{C}{k(\Psi(c\varphi_1))} \geq \frac{C}{k(\bar{v})} & \text{in } \Omega \setminus \omega. \end{aligned}$$

The last two inequalities imply that  $\bar{v}$  satisfies (2.7). This finishes the proof of the Lemma.  $\square$

By virtue of Lemma 2.2, relations (2.5) and (2.7) yield  $v \leq \bar{v}$  in  $\Omega$ . Using (2.4) we get

$$\frac{f(u)}{g(v)} \geq \frac{f(md(x))}{g(M\Psi(c\varphi_1))} \quad \text{in } \Omega.$$

Furthermore,  $u$  satisfies

$$\begin{cases} \Delta u - \alpha u + \frac{f(md(x))}{g(M\Psi(c\varphi_1))} \leq 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (2.12)$$

In order to avoid the singularities in (2.12) near the boundary, we consider the approximated problem

$$\begin{cases} \Delta w - \alpha w + \frac{f(md(x))}{g(M\Psi(c\varphi_1)) + \varepsilon} = 0 & \text{in } \Omega, \\ w > 0 & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \quad (2.13)$$

Clearly  $\bar{w} = u$  is a super-solution of (2.13) while  $\underline{w} = 0$  is a sub-solution. By classical results, the problem (2.13) has a unique solution  $w_\varepsilon \in C^2(\bar{\Omega})$  such that  $w_\varepsilon \leq u$  in  $\Omega$ . Moreover, the maximum principle yields  $w_\varepsilon > 0$  in  $\Omega$ .

In order to get a contradiction, we multiply by  $\varphi_1$  in (2.13) and then we integrate over  $\Omega$ . We obtain

$$(\alpha + \lambda_1) \int_{\Omega} w_\varepsilon \varphi_1 dx = \int_{\Omega} \varphi_1 \frac{f(md(x))}{g(M\Psi(c\varphi_1)) + \varepsilon} dx.$$

Since  $w_\varepsilon \leq u$  in  $\Omega$  we have

$$(\alpha + \lambda_1) \int_{\Omega} u \varphi_1 dx \geq \int_{\omega} \varphi_1 \frac{f(md(x))}{g(M\Psi(c\varphi_1)) + \varepsilon} dx \quad \text{for all } \omega \Subset \Omega.$$

Let  $\tilde{C} = (\alpha + \lambda_1) \int_{\Omega} u \varphi_1 dx$ . Passing to the limit in the above inequality we deduce

$$\int_{\omega} \varphi_1 \frac{f(md(x))}{g(M\Psi(c\varphi_1))} dx \leq \tilde{C} < +\infty \quad \text{for all } \omega \Subset \Omega.$$

Hence,

$$\int_{\Omega} \varphi_1 \frac{f(md(x))}{g(M\Psi(c\varphi_1))} dx \leq \tilde{C} < +\infty.$$

Let now  $\Omega_0 = \{x \in \Omega : d(x) < a\}$ . The above inequality combined with (2.6) produces

$$\int_{\Omega_0} d(x) \frac{f(md(x))}{g(M\Psi(d(x)))} dx < +\infty,$$

but this clearly contradicts (2.1). Hence the system  $(\mathcal{S})$  has no positive classical solutions. This completes the proof of Theorem.  $\square$

If  $k(t) = t^s$ ,  $s > 0$ , condition (2.1) can be written more explicitly by describing the asymptotic behavior of  $\Psi$ . We have.

**Corollary 2.1.** *Assume that  $k(t) = t^s$ ,  $s > 0$ , and one of the following conditions hold*

- (i)  $s > 1$  and  $\int_0^a \frac{tf(mt)}{g(Mt^{2/(1+s)})} dt = +\infty$ , for all  $0 < m < 1 < M$ ;
- (ii)  $s = 1$  and  $\int_0^{\min\{a, 1/2\}} \frac{tf(mt)}{g(Mt\sqrt{-\ln t})} dt = +\infty$ , for all  $0 < m < 1 < M$ ;
- (iii)  $0 < s < 1$  and  $\int_0^a \frac{tf(mt)}{g(Mt)} dt = +\infty$ , for all  $0 < m < 1 < M$ .

Then, the system  $(\mathcal{S})$  has no positive classical solutions.

*Proof.* The main idea is to describe the asymptotic behavior of  $\Psi$  near the origin. Notice that in our case the mapping  $\Psi : [0, a) \rightarrow [0, 1)$  satisfies

$$\begin{cases} -\Psi''(t) = \Psi^{-s}(t) & \text{for all } 0 < t < a, \\ \Psi'(t), \Psi(t) > 0 & \text{for all } 0 < t < a, \\ \Psi(0) = 0. \end{cases} \quad (2.14)$$

(i) If  $s > 1$  then the mapping

$$(0, \infty) \ni t \mapsto \left[ \frac{(1+s)^2}{2(1-s)} \right]^{1/(1+s)} \cdot t^{2/(s+1)},$$

satisfies (2.14). Hence, there exist two positive constants  $c_1, c_2 > 0$  such that

$$c_1 t^{2/(s+1)} \leq \Psi(t) \leq c_2 t^{2/(s+1)} \quad \text{for all } 0 < t < a.$$

Now, (i) follows directly from the above inequality.

(ii) Using the fact that  $\Psi$  is concave, we deduce that  $\Psi(t) > t\Psi'(t)$ , for all  $0 < t < a$ . From (2.14) it follows that

$$-\Psi''(t) < \frac{1}{t\Psi'(t)} \quad \text{for all } 0 < t < a.$$

We multiply by  $\Psi'$  in the last inequality and then we integrate over  $[t, b]$ ,  $0 < b < a$ . We get

$$(\Psi')^2(t) - (\Psi')^2(b) \leq 2(\ln b - \ln t) \quad \text{for all } 0 < t \leq b < a.$$

Hence, there exist  $c_1 > 0$  and  $\delta_1 \in (0, b)$  such that  $\Psi'(t) \leq c_1 \sqrt{-\ln t}$  for all  $0 < t \leq \delta_1$ . Integrating over  $[0, t]$  we obtain

$$\Psi(t) \leq c_1 \int_0^t \sqrt{-\ln \tau} d\tau = c_1 t \sqrt{-\ln t} + \frac{c_1}{2} \int_0^t \frac{1}{\sqrt{-\ln \tau}} d\tau \quad \text{for all } 0 < t \leq \delta_1. \quad (2.15)$$

Since the last integral in (2.15) is finite, there exist  $c_2 > 0$  and  $\delta_2 \in (0, \delta_1)$  such that

$$\Psi(t) \leq c_2 t \sqrt{-\ln t} \quad \text{for all } 0 < t \leq \delta_2. \quad (2.16)$$

From (2.14) and (2.16) we deduce  $-\Psi''(t) = \frac{1}{\Psi(t)} \geq \frac{1}{c_2} \cdot \frac{1}{t\sqrt{-\ln t}}$  for all  $0 < t \leq \delta_2$ .

An integration over  $[t, \delta_2]$  in the last inequality yields

$$\Psi'(t) \geq \frac{2}{c_2} \left( \sqrt{-\ln t} - \sqrt{-\ln \delta_2} \right) \quad \text{for all } 0 < t \leq \delta_2.$$

Therefore, there exist  $c_3 > 0$  and  $\delta_3 \in (0, \delta_2)$  such that  $\Psi'(t) \geq c_3 \sqrt{-\ln t}$  for all  $0 < t \leq \delta_3$ . Proceeding in the same manner as above, there exist  $c_4 > 0$  and  $\delta_4 \in (0, \delta_3)$  such that

$$\Psi(t) \geq c_4 t \sqrt{-\ln t} \quad \text{for all } 0 < t \leq \delta_4. \quad (2.17)$$

From (2.16) and (2.17) we get

$$c_3 t \sqrt{-\ln t} \leq \Psi(t) \leq c_4 t \sqrt{-\ln t} \quad \text{for all } 0 < t \leq \delta_4.$$



Now, (ii) follows from the above estimates.

(iii) By (2.8) we have

$$\Psi'(t) = \sqrt{2 \int_{\Psi(t)}^1 \tau^{-s} d\tau} = \sqrt{\frac{2}{1-s}(1 - \Psi^{1-s}(t))}, \quad \text{for all } 0 < t < a.$$

Hence  $0 < \Psi'(0) = \sqrt{2/(1-s)} < +\infty$  which implies  $\Psi \in C^1[0, a)$  and  $c_1 t \leq \Psi(t) \leq c_2 t$  in  $(0, a)$  for some  $c_1, c_2 > 0$ . This proves (iii).  $\square$

In the case of pure powers in the nonlinearities, we have the following nonexistence result for (1.3).

**Corollary 2.2.** *Let  $p, q, r, s > 0$  be such that one of the following conditions hold*

- (i)  $s > 1$  and  $2q \geq (s+1)(p+2)$ ;
- (ii)  $s = 1$  and  $q > p+2$ ;
- (iii)  $0 < s < 1$  and  $q \geq p+2$ .

*Then, the system (1.3) has no positive classical solutions.*

*Proof.* The proofs of (i) and (iii) are simple exercises of calculus. For (ii), by Corollary 2.2 we have that (1.3) has no classical solutions provided  $s = 1$  and

$$\int_0^{1/2} t^{1+p-q} (-\ln t)^{-q/2} dt = +\infty. \quad (2.18)$$

On the other hand, for  $a, b \in \mathbb{R}$  we have  $\int_0^{1/2} t^a (-\ln t)^b dt < +\infty$  if and only if  $a > -1$  or  $a = -1$  and  $b < -1$ . Now condition (2.18) reads  $q > p+2$ . This concludes the proof.  $\square$

### 3 Existence results

For all  $t_1, t_2 > 0$  we define

$$A(t_1, t_2) = \frac{f(t_1)}{h(t_1)} - \frac{g(t_2)}{k(t_2)}.$$

In this section we assume that  $A$  fulfills

$$(A_1) \quad A(t_1, t_2) \leq 0 \text{ for all } t_1 \geq t_2 > 0.$$

We also assume that

$$(A_2) \quad k \in C^1(0, \infty) \text{ is nonnegative and nondecreasing function such that } \lim_{t \rightarrow +\infty} \frac{K(t)}{h(t+c)} = +\infty, \text{ for all } c > 0, \text{ where } K(t) = \int_0^t k(\tau) d\tau.$$

Here are some examples of nonlinearities that fulfill (A1) and (A2).

- (i)  $f(t) = t^p, g(t) = t^q, h(t) = t^r, k(t) = t^s, t \geq 0, p, q, r, s > 0, r - p = s - q \geq 0$  and  $p - q < 1$ ;
- (ii)  $f(t) = \ln(1 + t^p), g(t) = e^{t^q} - 1, h(t) = t^p$  and  $k(t) = t^q, t \geq 0, p, q > 0, p - q < 1$ ;
- (iii)  $f(t) = \log(1 + at), g(t) = \log(1 + t), h(t) = at$  and  $k(t) = t, t \geq 0, a \geq 1$ ;

We give in what follows a general method to construct nonlinearities  $f, g, h, k$  that verify hypotheses (A1) and (A2). Let  $f, g, h, k : [0, \infty) \rightarrow [0, \infty)$  be nondecreasing functions such that  $k$  and  $h$  verify (A2) and one of the following assumptions hold:

- (a)  $fk = gh$  and the mapping  $(0, \infty) \ni t \mapsto f(t)/h(t)$  is nonincreasing;
- (b) there exists  $m > 0$  such that  $f(t)/h(t) \leq m \leq g(t)/k(t)$ , for all  $t > 0$ .

Then the mapping  $A$  verifies (A1).

For instance, the mappings given in example (i) satisfy the condition (a) while the mappings given in example (ii) verify the condition (b).

The first result of this section concerns the existence of classical solutions for the general system (S).

**Theorem 3.1.** *Assume that the hypotheses (A1) – (A2) are fulfilled. Then the system (S) has classical solutions.*

The existence of a solution to (S) is obtained by considering the regularized system

$$\begin{cases} \Delta u - \alpha u + \frac{f(u + \varepsilon)}{g(v + \varepsilon)} + \rho(x) = 0 & \text{in } \Omega, \\ \Delta v - \beta v + \frac{h(u + \varepsilon)}{k(v + \varepsilon)} = 0 & \text{in } \Omega, \\ u = 0, v = 0 & \text{on } \partial\Omega. \end{cases} \quad (\mathcal{S})_\varepsilon$$

**Lemma 3.1.** *Let  $u_\varepsilon, v_\varepsilon \in C^2(\Omega) \cap C(\overline{\Omega})$  be a positive solution of  $(\mathcal{S})_\varepsilon$ . Then, there exists  $M > 0$  which does not depend on  $\varepsilon$  such that*

$$\max\{\|u_\varepsilon\|_\infty, \|v_\varepsilon\|_\infty\} \leq M. \quad (3.1)$$

*Proof.* Let  $w_\varepsilon = u_\varepsilon - v_\varepsilon$  and  $\omega = \{x \in \Omega : w_\varepsilon > 0\}$ . In order to prove the Lemma, it suffices to provide an uniform upper bound for  $v_\varepsilon$  and  $w_\varepsilon$ . From  $(\mathcal{S})_\varepsilon$  we get

$$\begin{aligned} \Delta w_\varepsilon - \alpha w_\varepsilon + \rho(x) &= (\alpha - \beta)v_\varepsilon - \frac{f(u_\varepsilon + \varepsilon)}{g(v_\varepsilon + \varepsilon)} + \frac{h(u_\varepsilon + \varepsilon)}{gk(v_\varepsilon + \varepsilon)} \\ &= (\alpha - \beta)v_\varepsilon - \frac{h(u_\varepsilon + \varepsilon)}{g(v_\varepsilon + \varepsilon)} A(u_\varepsilon + \varepsilon, v_\varepsilon + \varepsilon) \quad \text{in } \Omega. \end{aligned}$$

Let us notice that  $A(u_\varepsilon + \varepsilon, v_\varepsilon + \varepsilon) \geq 0$  in  $\omega$  and  $w_\varepsilon = 0$  on  $\partial\omega$ . This yields

$$\Delta w_\varepsilon - \alpha w_\varepsilon \geq \rho(x) \quad \text{in } \omega.$$

Let  $\zeta \in C^2(\overline{\Omega})$  be the unique solution of (2.2). Then

$$\begin{cases} \Delta(w_\varepsilon - \zeta) - \alpha(w_\varepsilon - \zeta) \geq 0 & \text{in } \omega, \\ w_\varepsilon - \zeta \leq 0 & \text{on } \partial\omega. \end{cases}$$

Furthermore, by the weak maximum principle [8, Corollary 3.2] we have  $w_\varepsilon \leq \zeta$  in  $\omega$ . Since  $w_\varepsilon \leq 0$  in  $\Omega \setminus \omega$ , it follows that

$$w_\varepsilon \leq \zeta \quad \text{in } \Omega. \quad (3.2)$$

We multiply by  $k(v_\varepsilon)$  in the second equation of  $(S)_\varepsilon$  and we deduce that

$$k(v_\varepsilon)\Delta v_\varepsilon - \beta v_\varepsilon k(v_\varepsilon) + \frac{k(v_\varepsilon)}{k(v_\varepsilon + \varepsilon)}h(u_\varepsilon + \varepsilon) = 0 \quad \text{in } \Omega. \quad (3.3)$$

But

$$k(v_\varepsilon)\Delta v_\varepsilon = \Delta K(v_\varepsilon) - k'(v_\varepsilon)|\nabla v_\varepsilon|^2 \quad \text{in } \Omega. \quad (3.4)$$

Since  $k$  is nondecreasing, we have

$$K(v_\varepsilon) = \int_0^{v_\varepsilon} k(t)dt \leq v_\varepsilon k(v_\varepsilon) \quad \text{in } \Omega. \quad (3.5)$$

Using now (3.4) and (3.5) in (3.3) we deduce

$$\Delta K(v_\varepsilon) - k'(v_\varepsilon)|\nabla v_\varepsilon|^2 - \beta K(v_\varepsilon) + \frac{k(v_\varepsilon)}{k(v_\varepsilon + \varepsilon)}h(u_\varepsilon + \varepsilon) \geq 0 \quad \text{in } \Omega.$$

Hence

$$\Delta K(v_\varepsilon) - \beta K(v_\varepsilon) + h(u_\varepsilon + \varepsilon) \geq 0 \quad \text{in } \Omega. \quad (3.6)$$

By [8, Theorem 3.7], there exists a positive constant  $C > 1$  depending only on  $\Omega$  such that

$$\sup_{\overline{\Omega}} K(v_\varepsilon) \leq C \sup_{\overline{\Omega}} h(u_\varepsilon + \varepsilon) \leq C \sup_{\overline{\Omega}} h(v_\varepsilon + \|\zeta\|_\infty + 1).$$

Using the assumption (A2) we deduce that  $(v_\varepsilon)_\varepsilon$  is uniformly bounded, i.e.,  $\|v_\varepsilon\|_\infty \leq m$  for some  $m > 0$  independent on  $\varepsilon$ . This yields  $u_\varepsilon = v_\varepsilon + w_\varepsilon \leq m + \|\zeta\|_\infty$  in  $\Omega$  and the proof of Lemma 3.1 is now complete.  $\square$

**Lemma 3.2.** *For all  $0 < \varepsilon < 1$  there exists a solution  $(u_\varepsilon, v_\varepsilon) \in C^2(\overline{\Omega}) \times C^2(\overline{\Omega})$  of the system  $(S)_\varepsilon$ .*

*Proof.* We use topological degree arguments. Consider the set

$$\mathcal{U} := \left\{ (u, v) \in C^2(\overline{\Omega}) \times C^2(\overline{\Omega}) : \begin{array}{l} \|u\|_\infty, \|v\|_\infty \leq M + 1 \\ u, v \geq 0 \text{ in } \Omega, u|_{\partial\Omega} = v|_{\partial\Omega} = 0 \end{array} \right\},$$

where  $M > 0$  is the constant in (3.1). Define

$$\Phi_t : \mathcal{U} \rightarrow \mathcal{U}, \quad \Phi_t(u, v) = (\Phi_t^1(u, v), \Phi_t^2(u, v)),$$

by

$$\begin{aligned} \Phi_t^1(u, v) &= u - t(-\Delta + \alpha)^{-1} \left( \frac{f(u + \varepsilon)}{g(v + \varepsilon)} + \rho \right), \\ \Phi_t^2(u, v) &= v - t(-\Delta + \beta)^{-1} \left( \frac{h(u + \varepsilon)}{k(v + \varepsilon)} \right). \end{aligned}$$

Using Lemma 3.1 we have  $\Phi_t(u, v) \neq (0, 0)$  on  $\partial\mathcal{U}$ , for all  $0 \leq t \leq 1$ . Therefore, by the invariance of the topological degree at homotopy we have

$$\deg(\Phi_1, \mathcal{U}, (0, 0)) = \deg(\Phi_0, \mathcal{U}, (0, 0)) = 1.$$

Hence, there exists  $(u, v) \in \mathcal{U}$  such that  $\Phi_1(u, v) = (0, 0)$ . This means that the system  $(S)_\varepsilon$  has at least one classical solution.  $\square$

**Proof of Theorem 3.1.** Let  $(u_\varepsilon, v_\varepsilon) \in C^2(\overline{\Omega}) \times C^2(\overline{\Omega})$  be a solution of  $(\mathcal{S})_\varepsilon$ . Then

$$\begin{cases} \Delta(u_\varepsilon - \zeta) - \alpha(u_\varepsilon - \zeta) \leq 0 & \text{in } \Omega, \\ u_\varepsilon - \zeta = 0 & \text{on } \partial\Omega, \end{cases}$$

where  $\zeta$  is the unique solution of (2.2). Hence  $\zeta \leq u_\varepsilon$  in  $\Omega$ . By (3.2) it follows that

$$w_\varepsilon \leq \zeta \leq u_\varepsilon \quad \text{in } \Omega. \quad (3.7)$$

Let  $\xi \in C^2(\overline{\Omega})$  be the unique positive solution of the boundary value problem

$$\begin{cases} \Delta\xi - \beta\xi + \frac{h(\zeta)}{k(\xi+1)} = 0 & \text{in } \Omega, \\ \xi = 0 & \text{on } \partial\Omega. \end{cases} \quad (3.8)$$

In view of Lemma 2.2 we have  $\xi \leq v_\varepsilon$  in  $\Omega$ , so that, by Lemma 3.1, the following estimates hold

$$\begin{cases} \zeta(x) \leq u_\varepsilon(x) \leq M & \text{in } \Omega, \\ \xi(x) \leq v_\varepsilon(x) \leq M & \text{in } \Omega. \end{cases} \quad (3.9)$$

Now, standard Hölder and Schauder estimates can be employed in order to deduce that  $\{(u_\varepsilon, v_\varepsilon)\}_{0 < \varepsilon < 1}$  converges (up to a subsequence) in  $C_{\text{loc}}^2(\Omega) \times C_{\text{loc}}^2(\Omega)$  to  $(u, v) \in C^2(\Omega) \times C^2(\Omega)$ . It remains only to obtain an upper bound near  $\partial\Omega$  for  $(u_\varepsilon, v_\varepsilon)$  which leads us to the continuity up to the boundary of the solution  $(u, v)$ . This will be done by combining standard arguments with the estimate (3.7). First, by (3.6) we have

$$\Delta K(v_\varepsilon) + h(M+1) \geq 0 \quad \text{in } \Omega. \quad (3.10)$$

Fix  $x_0 \in \partial\Omega$ . Since  $\partial\Omega$  is smooth, there exist  $y \in \mathbb{R}^N \setminus \Omega$  and  $R > 0$  such that  $\overline{\Omega} \cap \overline{B}(y, R) = \partial\Omega \cap \overline{B}(y, R) = \{x_0\}$ .

Let  $\delta(x) = |x - y| - R$  and  $\Omega_0 = \{x \in \Omega : 4(N-1)\delta(x) < R\}$ .

Consider  $\psi \in C^2(0, \infty)$  such that  $\psi' > 0$  and  $\psi'' < 0$  on  $(0, \infty)$  and set  $\phi(x) = \psi(\delta(x))$ ,  $x \in \Omega_0$ . Then

$$\begin{aligned} \Delta\phi(x) &= \psi'(\delta(x))\Delta\delta(x) + \psi''(\delta(x))|\nabla\delta(x)|^2 \\ &= \frac{N-1}{|x-y|}\psi'(\delta(x)) + \psi''(\delta(x)) \\ &\leq \frac{N-1}{R}\psi'(\delta(x)) + \psi''(\delta(x)) \quad \text{in } \Omega_0. \end{aligned}$$

Let us choose now  $\psi(t) = C\sqrt{t}$ ,  $t > 0$ , where  $C > 0$ . Therefore

$$\Delta\phi(x) \leq \frac{C}{4}\delta^{-3/2}(x) \left[ \frac{2(N-1)\delta(x)}{R} - 1 \right] \leq -\frac{C}{8}\delta^{-3/2}(x) < 0 \quad \text{in } \Omega.$$

We choose  $C > 0$  large enough such that

$$\Delta\phi \leq -h(M+1) \quad \text{in } \Omega_0 \quad (3.11)$$

and

$$\phi|_{\partial\Omega_0 \setminus \partial\Omega} > K(M) \geq \sup_{\overline{\Omega}_0} K(u_\varepsilon). \quad (3.12)$$

Furthermore, by (3.10), (3.11) and (3.12) we obtain

$$\begin{cases} \Delta(\phi - K(v_\varepsilon)) \leq 0 & \text{in } \Omega_0, \\ \phi - K(v_\varepsilon) \geq 0 & \text{on } \partial\Omega_0. \end{cases}$$

This implies  $\phi(x) \geq K(v_\varepsilon)$  in  $\Omega_0$ , that is,

$$0 \leq v_\varepsilon \leq K^{-1}(\phi(x)) \quad \text{in } \Omega_0.$$

Passing to the limit with  $\varepsilon \rightarrow 0$  in the last inequality we have  $0 \leq v \leq K^{-1}(\phi(x))$  in  $\Omega_0$ . Hence

$$0 \leq \lim_{x \rightarrow x_0} v(x) \leq \lim_{x \rightarrow x_0} K(\phi(x)) = 0.$$

Since  $x_0 \in \partial\Omega$  was arbitrary choosen, it follows that  $v \in C(\overline{\Omega})$ . Using the fact that  $u_\varepsilon = w_\varepsilon + v_\varepsilon \leq \zeta + v_\varepsilon$  in  $\Omega$ , in the same manner we conclude  $u \in C(\overline{\Omega})$ . This finishes the proof of Theorem 3.1.  $\square$

The next result concerns the following system

$$\begin{cases} \Delta u - \alpha u + \frac{u^p}{v^q} + \rho(x) = 0 & \text{in } \Omega, \\ \Delta v - \beta v + \frac{u^{p+\sigma}}{v^{q+\sigma}} = 0 & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \quad (3.13)$$

where  $\sigma \geq 0$  is a non-negative real number.

**Theorem 3.2.** *Assume that  $p, q \geq 0$  satisfy  $p - q < 1$ .*

- (i) *Then the system (3.13) has solutions for all  $\sigma \geq 0$ ;*
- (ii) *For any solution  $(u, v)$  of (3.13), there exist  $c_1, c_2 > 0$  such that*

$$c_1 d(x) \leq u, v \leq c_2 d(x) \quad \text{in } \Omega. \quad (3.14)$$

*Moreover, the following properties hold.*

- (ii1) *If  $-1 < p - q < 0$  then  $u, v \in C^2(\Omega) \cap C^{1,1+p-q}(\overline{\Omega})$ ;*
- (ii2) *If  $0 \leq p - q < 1$  then  $u, v \in C^2(\overline{\Omega})$ .*

*Proof.* Existence follows directly from Theorem 3.1 since conditions (A1) and (A2) are fulfilled.

(ii) Recall that from (2.3) we have  $u \geq \zeta \geq \bar{c}\varphi_1$  in  $\Omega$ . From the second equation in (3.13) we deduce

$$\Delta v - \beta v + \bar{c}^{p+\sigma} \frac{\varphi_1^{p+\sigma}}{v^{q+\sigma}} \leq 0 \quad \text{in } \Omega.$$

Since  $p - q < 1$ , we also get that  $\underline{v} = \underline{c}\varphi_1$  satisfies

$$\Delta \underline{v} - \beta \underline{v} + \bar{c}^{p+\sigma} \frac{\varphi_1^{p+\sigma}}{\underline{v}^{q+\sigma}} \leq 0 \quad \text{in } \Omega,$$

provided  $\underline{c} > 0$  is sufficiently small. Therefore, by virtue of Lemma 2.2, we obtain  $v \geq \underline{c}\varphi_1$  in  $\Omega$ .

Let us prove now the second inequality in (3.14). To this aim, set  $w = u - v$ . With the same idea as in Lemma 3.1 one gets  $\Delta w - \alpha w + \rho(x) \geq 0$  in the set  $\{x \in \Omega : w(x) > 0\}$ . Hence

$$w \leq \zeta \leq c\varphi_1 \quad \text{in } \Omega. \quad (3.15)$$

Let  $w^+ = \max\{w, 0\}$ . Then  $v$  satisfies

$$\begin{cases} \Delta v - \beta v + \frac{(w^+ + v)^{p+\sigma}}{v^{q+\sigma}} \geq 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Consider now the problem

$$\begin{cases} \Delta z - \beta z + 2^{p+\sigma} z^{p-q} = 0 & \text{in } \Omega, \\ z > 0 & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega. \end{cases} \quad (3.16)$$

The existence of a classical solution to (3.16) follows from [19, Lemma 2.4]. Moreover, if  $0 \leq p - q < 1$  then  $z \in C^2(\overline{\Omega})$  and with the same arguments as in [9, Theorem 1.1] we have  $z \in C^2(\Omega) \cap C^{1,1+p-q}(\overline{\Omega})$  in the case  $-1 < p - q < 0$ . Furthermore  $z \leq m\varphi_1$  in  $\Omega$  for some  $m > 0$ . On the other hand,  $\tilde{c}\varphi_1$  is a subsolution of (3.16) provided  $\tilde{c} > 0$  is small enough. Therefore, by Lemma 2.1 we get  $z \geq \tilde{c}\varphi_1$  in  $\Omega$ . This last inequality together with (3.15) allows us to choose  $M > 1$  large enough such that  $Mz \geq w^+$  in  $\Omega$ . Hence

$$\begin{aligned} \Delta(Mz) - \beta(Mz) + \frac{(w^+ + Mz)^{p+\sigma}}{(Mz)^{q+\sigma}} &\leq \Delta(Mz) - \beta(Mz) + 2^{p+\sigma}(Mz)^{p-q} \\ &= M \left( \Delta z - \beta z + 2^{p+\sigma} z^{p-q} \right) = 0 \quad \text{in } \Omega. \end{aligned}$$

This means that  $\bar{v} := Mz$  verifies

$$\Delta \bar{v} - \beta \bar{v} + \frac{(w^+ + \bar{v})^{p+\sigma}}{\bar{v}^{q+\sigma}} \leq 0 \quad \text{in } \Omega \quad \text{and} \quad \bar{v} = 0 \quad \text{on } \partial\Omega.$$

Remark now that  $\Psi(x, t) = -\beta t + \frac{(w^+(x) + t)^{p+\sigma}}{t^{q+\sigma}}$ ,  $(x, t) \in \overline{\Omega} \times (0, \infty)$  satisfies the hypotheses in Lemma 2.1 since  $p - q < 1$ . Furthermore, we have

$$\Delta \bar{v} + \Psi(x, \bar{v}) \leq 0 \leq \Delta v + \Psi(x, v) \quad \text{in } \Omega,$$

$$\bar{v}, v > 0 \quad \text{in } \Omega, \quad \bar{v} = v = 0 \quad \text{on } \partial\Omega \quad \text{and} \quad \Delta \bar{v} \in L^1(\Omega).$$

Hence, by Lemma 2.1 we obtain

$$v \leq \bar{v} \leq \tilde{c}\varphi_1 \quad \text{in } \Omega. \quad (3.17)$$

Combining (3.15) and (3.17) we deduce  $u = w + v \leq C\varphi_1$  in  $\Omega$ , for some  $C > 0$ . This completes the proof of (ii1). As a consequence, there exists  $M > 1$  such that

$$0 \leq \frac{u^p}{v^q}, \frac{u^{p+\sigma}}{v^{q+\sigma}} \leq M\varphi_1^{p-q} \quad \text{in } \Omega.$$

If  $0 \leq p - q < 1$  then by classical regularity arguments we have  $u, v \in C^2(\overline{\Omega})$ . If  $-1 < p - q < 0$ , then the same method as in [9, Theorem 1.1] can be employed in order to obtain  $u, v \in C^2(\Omega) \cap C^{1,1+p-q}(\overline{\Omega})$ .

This finishes the proof of Theorem 3.2.  $\square$

## 4 Uniqueness of the solution in one dimension

In this section we are concerned with the uniqueness of the solution associated to the one dimensional system

$$\begin{cases} u'' - \alpha u + \frac{u^p}{v^q} + \rho(x) = 0 & \text{in } (0, 1), \\ v'' - \beta v + \frac{u^{p+\sigma}}{v^{q+\sigma}} = 0 & \text{in } (0, 1), \\ u(0) = u(1) = 0, v(0) = v(1) = 0. \end{cases} \quad (4.1)$$

Our approach is inspired by the methods developed in [1], where a  $C^2$ -regularity of the solution up to the boundary is needed. So, we restrict our attention to the case  $0 < q \leq p \leq 1$ . Thus, by virtue of Theorem 3.2, any solution of (4.1) belongs to  $C^2[0, 1] \times C^2[0, 1]$ . By Hopf's maximum principle we also have that  $u'(0) > 0$ ,  $v'(0) > 0$ ,  $u'(1) < 0$  and  $v'(1) < 0$  for any solution  $(u, v)$  of system (4.1).

The main result of this section is the following

**Theorem 4.1.** *Assume that  $0 < q \leq p \leq 1$ ,  $\sigma \geq 0$ . Then the system (4.1) has a unique solution  $(u, v) \in C^2(\overline{\Omega}) \times C^2(\overline{\Omega})$ .*

*Proof.* Existence follows from Theorem 3.2. We prove here only the uniqueness. Suppose that there exist  $(u_1, v_1), (u_2, v_2) \in C^2[0, 1] \times C^2[0, 1]$  two distinct solutions of (4.1).

First we claim that we can not have  $u_2 \geq u_1$  or  $v_2 \geq v_1$  in  $[0, 1]$ . Indeed, let us assume that  $u_2 \geq u_1$  in  $[0, 1]$ . Then

$$v_2'' - \beta v_2 + \frac{u_2^{p+\sigma}}{v_2^{q+\sigma}} = 0 = v_1'' - \beta v_1 + \frac{u_1^{p+\sigma}}{v_1^{q+\sigma}} \quad \text{in } (0, 1),$$

and by Lemma 2.2 we get  $v_2 \geq v_1$  in  $[0, 1]$ . On the other hand

$$u_1'' - \alpha u_1 + \frac{u_1^p}{v_2^q} + \rho(x) \leq 0 = u_2'' - \alpha u_2 + \frac{u_2^p}{v_2^q} + \rho(x) \quad \text{in } (0, 1). \quad (4.2)$$

Note that the mapping  $\Psi(x, t) = -\alpha t + \frac{t^p}{v_2(x)^q} + \rho(x)$ ,  $(x, t) \in (0, 1) \times (0, \infty)$  satisfies the hypotheses in Lemma 2.1 since  $p \leq 1$ . Hence  $u_2 \leq u_1$  in  $[0, 1]$ , that is  $u_1 \equiv u_2$ . This also implies  $v_1 \equiv v_2$ , contradiction. Replacing  $u_1$  by  $u_2$  and  $v_1$  by  $v_2$ , we also get that the situation  $u_1 \geq u_2$  or  $v_1 \geq v_2$  in  $[0, 1]$  is not possible.

Set  $U = u_2 - u_1$  and  $V = v_2 - v_1$ . From the above arguments, both  $U$  and  $V$  change sign in  $(0, 1)$ . The key result in the approach is the following.

**Proposition 4.1.**  *$U$  and  $V$  vanish only at finitely many points in the interval  $[0, 1]$ .*

*Proof.* We write the system (4.1) as

$$\begin{cases} \mathbf{W}''(x) + A(x)\mathbf{W}(x) = \mathbf{0} & \text{in } (0, 1), \\ \mathbf{W}(0) = \mathbf{W}(1) = \mathbf{0}, \end{cases}$$

where  $\mathbf{W} = (U, V)$  and  $A(x) = (A_{ij}(x))_{1 \leq i, j \leq 2}$  is a  $2 \times 2$  matrix defined as

$$A_{11}(x) = -\alpha + \begin{cases} \frac{1}{v_2^q(x)} \cdot \frac{u_2^p(x) - u_1^p(x)}{u_2(x) - u_1(x)}, & u_1(x) \neq u_2(x) \\ p \frac{u_1^{p-1}(x)}{v_1^q(x)}, & u_1(x) = u_2(x) \end{cases}$$

$$\begin{aligned}
A_{12}(x) &= \begin{cases} -\frac{u_1^p(x)}{v_1^q(x)v_2^q(x)} \cdot \frac{v_2^q(x) - v_1^q(x)}{v_2(x) - v_1(x)}, & v_1(x) \neq v_2(x) \\ -q \frac{u_1^p(x)}{v_1^{q+1}(x)}, & v_1(x) = v_2(x) \end{cases} \\
A_{21}(x) &= \begin{cases} \frac{1}{v_2^{q+\sigma}(x)} \cdot \frac{u_2^{p+\sigma}(x) - u_1^{p+\sigma}(x)}{u_2(x) - u_1(x)}, & u_1(x) \neq u_2(x) \\ (p + \sigma) \frac{u_1^{p+\sigma-1}(x)}{v_1^{q+\sigma}(x)}, & u_1(x) = u_2(x) \end{cases} \\
A_{22}(x) &= -\beta - \begin{cases} \frac{u_1^{p+\sigma}(x)}{v_1^{q+\sigma}(x)v_2^{q+\sigma}(x)} \cdot \frac{v_2^{q+\sigma}(x) - v_1^{q+\sigma}(x)}{v_2(x) - v_1(x)}, & v_1(x) \neq v_2(x) \\ (q + \sigma) \frac{u_1^{p+\sigma}(x)}{v_1^{q+\sigma+1}(x)}, & v_1(x) = v_2(x) \end{cases}
\end{aligned}$$

Therefore,  $A \in C(0, 1)$  and  $A_{12}(x) \neq 0$ ,  $A_{21}(x) \neq 0$  for all  $x \in (0, 1)$ . Moreover,  $xA(x)$ ,  $(1-x)A(x)$  are bounded in  $L^\infty(0, 1)$ . Indeed, let us notice first that, by (3.14) in Theorem 3.2, there exist  $c_1, c_2 > 0$  such that

$$c_1 \leq \frac{u_i}{\min\{x, 1-x\}}, \frac{v_i}{\min\{x, 1-x\}} \leq c_2, \quad (i = 1, 2) \quad \text{in } (0, 1).$$

Then, by the mean value theorem, we have

$$\begin{aligned}
x|A_{12}(x)| &\leq qx \frac{u_1^p(x)}{v_1^q(x)v_2^q(x)} \max\{v_1^{q-1}(x), v_2^{q-1}(x)\} \\
&\leq qx^{p-q} \left(\frac{u_1(x)}{x}\right)^p \max\left\{\left(\frac{x}{v_1(x)}\right)^{q+1}, \left(\frac{x}{v_2(x)}\right)^{q+1}\right\} \\
&\leq cx^{p-q} \quad \text{for all } 0 < x \leq 1/2.
\end{aligned}$$

We obtain similar estimates for  $xA_{11}$ ,  $xA_{21}$  and  $xA_{22}$ . This allows us to employ Lemma 7 and Lemma 8 in [1]. Note that condition  $xA(x) \in L^\infty(0, 1)$  suffices in order to obtain the same conclusion as in [1, Lemma 8]. In particular, we get that  $U$  and  $V$  vanish only at finitely many points in any compact interval  $[a, b] \subset (0, 1)$ .

It remains to show that  $U$  and  $V$  can not have infinitely many zeroes in the neighborhood of  $x = 0$  and  $x = 1$ . We shall consider only the case  $x = 0$ , the situation where  $u$  or  $v$  vanishes for infinitely many times near  $x = 1$  being similar.

Without losing the generality, we may assume that  $U$  has infinitely many zeroes in a neighborhood of  $x = 0$ . Since  $U \in C^2[0, 1]$  by Rolle's Theorem we get that both  $U'$  and  $U''$  have infinitely many zeros near  $x = 0$ . As a consequence, we obtain  $U'(0) = 0$ , that is,  $u_1'(0) = u_2'(0)$ .

If  $V'(0) = 0$ , then  $\mathbf{W}(0) = \mathbf{W}'(0) = \mathbf{0}$  and by [1, Lemma 8] we deduce  $\mathbf{W} \equiv \mathbf{0}$  in  $[0, 1/2]$  which is a contradiction. Hence  $V'(0) \neq 0$ . Subtracting the first equation in (4.1) corresponding to  $u_1$  and  $u_2$  we have

$$\begin{aligned}
U''(x) &= \alpha U(x) + \frac{u_1^p(x)}{v_1^q(x)} - \frac{u_2^p(x)}{v_2^q(x)} \\
&= x^{p-q} \left\{ \alpha \frac{U(x)}{x^{p-q}} + \left(\frac{u_1(x)}{x}\right)^p \left(\frac{x}{v_1(x)}\right)^q - \left(\frac{u_2(x)}{x}\right)^p \left(\frac{x}{v_2(x)}\right)^q \right\}.
\end{aligned}$$

Since  $0 \leq p - q < 1$ ,  $u_1'(0) = u_2'(0)$  and  $v_1'(0) \neq v_2'(0)$  we get

$$\lim_{x \rightarrow 0^+} \left\{ \alpha \frac{U(x)}{x^{p-q}} + \left(\frac{u_1(x)}{x}\right)^p \left(\frac{x}{v_1(x)}\right)^q - \left(\frac{u_2(x)}{x}\right)^p \left(\frac{x}{v_2(x)}\right)^q \right\} = u_1^p(0) \left( \frac{1}{v_1^{q/q}(0)} - \frac{1}{v_1^{q/q}(0)} \right) \neq 0.$$

Therefore,  $U''$  has constant sign in a small neighborhood of  $x = 0$  which contradicts the above arguments. The proof of Proposition 4.1 is now complete.  $\square$



**Proof of Theorem 4.1 completed.**

Set

$$\begin{aligned} \mathcal{I}^+ &= \{x \in [0, 1] : U(x) \geq 0\}, & \mathcal{I}^- &= \{x \in [0, 1] : U(x) \leq 0\}, \\ \mathcal{J}^+ &= \{x \in [0, 1] : V(x) \geq 0\}, & \mathcal{J}^- &= \{x \in [0, 1] : V(x) \leq 0\}. \end{aligned}$$

According to Proposition 4.1, the above sets consist of finitely many disjoint closed intervals. Therefore,  $\mathcal{I}^+ = \cup_{i=1}^m I_i^+$ . For simplicity, let  $I^+$  denote any interval  $I_i^+$  and we use similar notations for  $I^-$ ,  $J^+$  and  $J^-$ . We have

**Lemma 4.1.** *For any intervals  $I^+$ ,  $I^-$ ,  $J^+$  and  $J^-$  defined above, the following situations can not occur:*

- (i)  $I^+ \subset J^+$ ; (ii)  $I^- \subset J^-$ ; (iii)  $J^+ \subset I^-$ ; (iv)  $J^- \subset I^+$ .

*Proof.* (i) Assume that  $I^+ \subset J^+$ . Since  $v_2 \geq v_1$  in  $I^+$  we deduce that the inequality (4.2) holds in  $I^+$ . Using the fact that  $u_2 = u_1$  on  $\partial I^+$ , by virtue of Lemma 2.1 we get  $u_2 \leq u_1$  in  $I^+$ . Hence,  $u_2 \equiv u_1$  in  $I^+$ , which contradicts Proposition 4.1. Similarly we can prove the statement (ii).

(iii) Assume that  $J^+ \subset I^-$ . Then  $u_1^{p+\sigma}/v_1^{q+\sigma} \geq u_2^{p+\sigma}/v_2^{q+\sigma}$  in  $J^+$ . Notice that  $V = v_2 - v_1$  verifies

$$\begin{cases} V'' - \beta V = \frac{u_1^{p+\sigma}}{v_1^{q+\sigma}} - \frac{u_2^{p+\sigma}}{v_2^{q+\sigma}} \geq 0 & \text{in } J^+, \\ V = 0 & \text{on } \partial J^+. \end{cases}$$

By the maximum principle, it follows that  $V \leq 0$  in  $J^+$ , i.e.,  $v_2 \leq v_1$  in  $J^+$ . This yields  $v_2 \equiv v_1$  in  $J^+$  which again contradicts Proposition 4.1. The proof of (iv) follows in the same manner.  $\square$

From now on, the proof of Theorem 4.1 is the same as in [1, Theorem 6].  $\square$

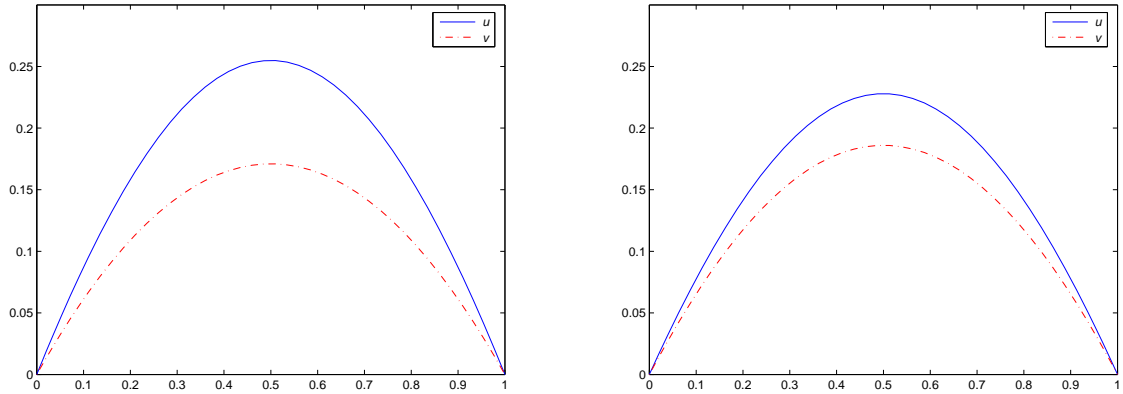


Figure 1: The solution  $(u, v)$  of the system  $(S_\varepsilon)$  with  $\alpha = 1$ ,  $\beta = 0.5$ ,  $p = q = 1$ ,  $\varepsilon = 10^{-2}$  and  $\rho(x) = \sin(\pi x)$ . We have chosen  $\sigma = 0$  (on the left) and  $\sigma = 2$  (on the right).

**Remark.** As a consequence of Theorem 3.1, the solution  $(u, v)$  of the system (3.13) can be approximated by the solutions of  $(S)_\varepsilon$ . Furthermore, the shooting method combined with the Broyden method in order to avoid the derivatives, are suitable to numerically approximate the solution of (3.13). We have considered  $\alpha = 1$ ,

$\beta = 0.5$ ,  $p = q = 1$ ,  $\varepsilon = 10^{-2}$  and  $\rho(x) = \varphi_1(x) = \sin(\pi x)$ . In the above figure we have plotted the solution  $(u, v)$  of  $(S_\varepsilon)$  for  $\sigma = 0$  (on the left) and  $\sigma = 2$  (on the right) respectively.

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