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Quasicontinuous Atom Laser in the Presence of Gravity

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We analyze the extraction from a trapped Bose-Einstein condensate to a nontrapping state of a coherent atomic beam and its subsequent fall, at $T = 0$ K. Our treatment fully takes gravity into account but neglects the mean-field potential exerted on the free falling beam by the trapped atoms. We derive analytical expressions for the output rate and the output mode of the quasicontinuous “atom laser.” Comparison with experimental data of Bloch *et al.* [Phys. Rev. Lett. **82**, 3008 (1999)] is satisfactory.

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Bose-Einstein condensates (BEC) of dilute alkali vapors [1] constitute a potential source of matter waves for atom interferometry, since they are first-order coherent [2]. Various schemes for “atom lasers” have been used to extract a coherent matter wave out of a trapped BEC. Pulsed devices have been demonstrated by using an intense spin-flip radio frequency (rf) pulse [3], gravity-induced tunneling from an optically trapped BEC [4] or Raman transitions [5]. Quasicontinuous output has been obtained by using a weak rf field that continuously couples atoms into a free falling state [6]. This “quasicontinuous” atom laser promises spectacular improvements in application of atom optics, for example, in the performances of atom-interferometer-based inertial sensors [7].

Gravity plays an important role in outcoupling: in the case of rf outcouplers, it lies at the very heart of the extraction process, as will be shown in this Letter. However, most of the theoretical studies do not take gravity into account (for an up-to-date review, see Ref. [8]). Although gravity has been included in numerical treatments [9] relevant for the pulsed atom laser of [3], to our knowledge, only the one-dimensional (1D) simulations of Refs. [8,10] treat the quasicontinuous case in the presence of gravity, and the results of [10] compare only qualitatively to the experimental data of [6].

In this Letter, we present a three-dimensional (3D) analytical treatment taking gravity fully into account. To compare with the experimental results of [6], we specifically focus on rf outcouplers. The extraction of the atoms from the trapped BEC and their subsequent propagation under gravity are analytically treated in the case of quasicontinuous outcoupling with appropriate approximations, discussed in the text. We derive an expression for the atom laser wave function and a generalized rate equation for the trapped atoms that agrees quantitatively with the experimental results of [6].

We consider a ^{87}Rb BEC in the $F = 1$ hyperfine level at $T = 0$ K. The $m = -1$ state is confined in a harmonic magnetic potential $V_{\text{trap}} = \frac{1}{2}M(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2)$. A rf magnetic field $\mathbf{B}_{\text{rf}} = B_{\text{rf}} \cos(\omega_{\text{rf}} t) \mathbf{e}_x$ can induce transitions to $m = 0$ (nontrapping state) and $m = +1$ (expelling state). The condensate three-component spinor

wave function $\Psi' = [\psi'_m]_{m=-1,0,+1}$ obeys a set of coupled nonlinear Schrödinger equations [11]. We consider the “weak coupling limit” [12] relevant to all experiments done so far with a quasicontinuous atom laser [6]. In this regime to be defined more precisely later, the coupling strength is low enough that the populations N_m of the three Zeeman sublevels obey the following inequality: $N_{+1} \ll N_0 \ll N_{-1}$. In the rest of this paper, we therefore restrict ourselves to $m = -1$ and $m = 0$, and set the total atomic density $n(\mathbf{r}) \approx |\psi_{-1}(\mathbf{r}, t)|^2$.

At this stage, the components $\psi_m = \psi'_m e^{-im\omega_{\text{rf}} t}$ obey, under the rotating wave approximation, the following two coupled equations:

$$i\hbar \frac{\partial \psi_{-1}}{\partial t} = [\hbar \delta_{\text{rf}} + \mathcal{H}_{-1}] \psi_{-1} + \frac{\hbar \Omega_{\text{rf}}}{2} \psi_0, \quad (1)$$

$$i\hbar \frac{\partial \psi_0}{\partial t} = \mathcal{H}_0 \psi_0 + \frac{\hbar \Omega_{\text{rf}}}{2} \psi_{-1}, \quad (2)$$

with $\mathcal{H}_{-1} = \mathbf{p}^2/2M + V_{\text{trap}} + U|\psi_{-1}|^2$ and $\mathcal{H}_0 = \mathbf{p}^2/2M - Mgz + U|\psi_{-1}|^2$. The strength of interactions is fixed by $U = 4\pi\hbar^2 aN/M$, $a \approx 5$ nm being the diffusion length, and N the initial number of trapped atoms. The rf outcoupler is described by the detuning from the bottom of the trap $\hbar \delta_{\text{rf}} = V_{\text{off}} - \hbar\omega_{\text{rf}}$ and the Rabi frequency $\hbar \Omega_{\text{rf}} = \mu_{\text{B}} B_{\text{rf}}/2\sqrt{2}$ (taking the Landé factor $g_F = -1/2$). The origin of the z axis is at the center of the condensate, displaced by gravity from the trap center by $z_{\text{sag}} = g/\omega_{\perp}^2$. We have taken the zero of energy at $z = 0$ in $m = 0$, so that the level splitting at the bottom of the trap is $V_{\text{off}} = \mu_{\text{B}} B_0/2 + Mg^2/2\omega_{\perp}^2$ (B_0 is the bias field).

The evolution of the $m = -1$ sublevel at $T = 0$ K is described [13] by a decomposition of the wave function into a condensed part and an orthogonal one that describes elementary excitations (quasiparticles). As \mathcal{H}_{-1} depends on time through $|\psi_{-1}|$, both the coefficients and the eigenmodes depend on time. However, we assume an adiabatic evolution of the trapped BEC [14], so that the uncondensed component remains negligible and

$$\psi_{-1}(\mathbf{r}, t) \approx \alpha(t) \phi_{-1}(\mathbf{r}, t) e^{-i \int_0^t [\mu(t')/\hbar + \delta_{\text{rf}}] dt'}, \quad (3)$$

where $\alpha(t) = [N_{-1}(t)/N]^{1/2}$. The time-dependent ground state ϕ_{-1} , of energy μ is, in the Thomas-Fermi (TF) approximation [13], $\phi_{-1}(\mathbf{r}, t) = (\mu/U)^{1/2}[1 - \tilde{r}_\perp^2 - \tilde{z}^2]^{1/2}$, where $\tilde{r}_\perp^2 = (x/x_0)^2 + (y/y_0)^2$ and $\tilde{z} = z/z_0$ are such that $\tilde{r}_\perp^2 + \tilde{z}^2 \leq 1$. The BEC dimensions are, respectively, $x_0 = (2\mu/M\omega_x^2)^{1/2}$ and $y_0 = z_0 = (2\mu/M\omega_\perp^2)^{1/2}$. The condensate energy is given by $\mu = (\hbar\omega/2)(15aN_{-1}/\sigma)^{2/5}$ [we have set $\omega = (\omega_x\omega_\perp^2)^{1/3}$ and $\sigma = (\hbar/M\omega)^{1/2}$]. The time dependence of ϕ_{-1} is contained in μ , x_0 , y_0 , and z_0 , which decrease with $N_{-1}(t)$. In the following, we will take typical values corresponding to the situation of [6] (reported in [10]): $N = 7 \times 10^5$ atoms initially, $\omega_x = 2\pi \times 13$ Hz and $\omega_\perp = 2\pi \times 140$ Hz, which gives $x_0 \approx 55.7 \mu\text{m}$, $z_0 \approx 5.3 \mu\text{m}$, and $\mu/h \approx 2.2$ kHz.

For the $m = 0$ state, the Hamiltonian of Eq. (2) with $\Omega_{\text{rf}} = 0$ becomes in the TF approximation [15]: $\mathcal{H}_0 = \mathbf{p}^2/2M - Mgz + \mu \max[0, 1 - \tilde{r}_\perp^2 - \tilde{z}^2]$. There is a clear hierarchy in the energy scales associated with each term in \mathcal{H}_0 . For the atom numbers involved in current experiments, μ/Mgz_0 is small, and the mean-field term is only a perturbation as compared with gravity. The leading term is therefore the gravitational potential that subsequently converts into kinetic energy along z . By contrast, the *transverse* kinetic energy (along x and y) remains small. Therefore, we neglect the spatial variations of the mean field potential and consider in the following the approximated Hamiltonian $H_0 \approx \mathbf{p}^2/2M - Mgz$.

The eigenstates of H_0 are factorized products of one-dimensional wave functions along the three axes. In the horizontal plane (x, y) , the eigenstates are plane waves with wave vectors k_x, k_y that we quantize with periodic boundary conditions in a 2D box of size L . Consequently, the wave function is $\phi_0^\perp(x, y) = L^{-1}e^{i(k_x x + k_y y)}$ and the density of states $\rho_{xy} = L^2/4\pi^2$. Along the vertical direction z , the normalizable solution of the 1D Schrödinger equation in a gravitational field is [16] $\phi_0^{(z_{E_z})} = A \cdot \text{Ai}(-\zeta_{E_z})$, where Ai is the Airy function of the first kind and $\zeta_{E_z} = (z - z_{E_z})/l$. The classical turning point $z_{E_z} = -E_z/Mg$ associated with the vertical energy E_z labels the vertical solution; $l = (\hbar^2/2M^2g)^{1/3}$ is a length scale, such that $l \ll x_0, y_0, z_0$ (for ^{87}Rb , $l \approx 0.28 \mu\text{m}$). In order to normalize Ai and to work out the density of states, we restrict the wave function to the domain $]-\infty, H]$, where the boundary at $z = H$ can be arbitrarily far from the origin. In $]-\infty, z_{E_z}]$, Ai falls off exponentially over a distance l , while it can be identified with its asymptotic form $\text{Ai}(-s) \approx \pi^{1/2}s^{-1/4} \cos(2s^{3/2}/3 - \pi/4)$ for $s \geq 0$. To leading order in H , by averaging the fast oscillating \cos^2 term to 1/2, we obtain the normalization factor $A = (\pi H^{-1/2}/l)^{1/2}$. The longitudinal density of states follows from $\rho_z(E_z) = h^{-1}d\mathcal{V}/dE_z$, where $\mathcal{V} = \int dz dp_z \theta(E_z - p_z^2/2M + Mgz)$ (with θ the step function) is the volume in phase space associated with

energies lower than E_z . We restrict the z space to the domain defined above, and the p_z space to $p_z \geq 0$. In this way, we do not count the component of Ai that propagates opposite to gravity [17]. We obtain $\rho_z(z_{E_z}) = (1/2\pi l)H^{1/2}$. Finally, the output modes are given by $\phi_0^{(n)}(\mathbf{r}) = \phi_0^\perp(x, y)\phi_0^{(z_{E_z})}(z)$, where n stands for the quantum numbers (k_x, k_y, z_{E_z}) , and the density of modes is $\rho_{3\text{D}}^{(n)} = (1/8\pi^3)L^2H^{1/2}/l$.

Thus, the problem is reduced to the coupling of an initially populated bound state $m = -1$, of energy $E_{-1} = \hbar\delta_{\text{rf}} + \mu$ to a quasicontinuum of final states $m = 0$, with a total energy $E_0^{(n)} = \hbar^2(k_x^2 + k_y^2)/2M - Mgz_{E_z}$. A crucial feature in this problem is the resonant bell shape of the coupling matrix element $W_{\text{fi}} = (\hbar\Omega_{\text{rf}}/2)\langle\phi_0^{(n)}|\phi_{-1}\rangle$. We work out the overlap integral $I^{(n)} = \langle\phi_0^{(n)}|\phi_{-1}\rangle$ in reduced cylindrical coordinates $(\tilde{r}_\perp, \theta, \tilde{z})$ and set $\tilde{k}^2 = (k_x x_0)^2 + (k_y y_0)^2$. We integrate over θ and \tilde{r}_\perp , and transform the sum over z with the Parseval relation [18] to obtain $I^{(n)} = 2\pi A x_0 y_0 / L(\mu/U)^{(1/2)} \int_{-\infty}^{\infty} \tilde{g}(\tilde{k}, \nu) e^{i(z_0/l)(\nu^3/3 - \nu z_{E_z}/l)} d\nu$. In this integral, \tilde{g} is the Fourier transform with respect to z of $g(\tilde{k}, \tilde{z}) = (p \cos p - \sin p)/\tilde{k}^3$, with $p(\tilde{k}, \tilde{z}) = \tilde{k}(1 - \tilde{z}^2)^{1/2}$, and $(2\pi)^{-(1/2)} e^{i(\nu^3/3 - \nu z_{E_z}/l)}$ the Fourier transform of Ai. Since $z_0/l \gg 1$, the integrand averages to zero out of a small neighborhood of the origin, where the linear term in ν is dominant. We obtain in this way

$$I^{(n)} = 2\pi \frac{Al}{L} \sqrt{\frac{\mu}{U}} x_0 y_0 g(\tilde{k}, z_{E_z}). \quad (4)$$

This overlap integral is nonvanishing only if the accessible final energies $E_0^{(n)} \approx E_{-1}$ are approximately restricted to an interval $[-\Delta/2, \Delta/2]$, where $\Delta = 2Mgz_0 \sim 22.7$ kHz. This gives a resonance condition for the frequency ω_{rf} [6,14]:

$$|\hbar\delta_{\text{rf}}| \lesssim Mgz_0. \quad (5)$$

Two different behaviors can be expected in such a situation [14,19]. In the strong coupling regime ($\hbar\Gamma \gg \Delta$, where Γ is the decay rate worked out with the Fermi golden rule), Rabi oscillations occur between the BEC bound state and the narrow-band continuum. This describes the pulsed atom laser experimentally realized by Mewes *et al.* [3]. Conversely, in the weak coupling limit ($\hbar\Gamma \ll \Delta$), oscillations persist only for $t \leq t_c = \hbar/\Delta$, while for $t \geq t_c$, the evolution of the BEC bound state is a monotonic decay. We have numerically verified this behavior on a 1D simulation analogous to [10] (see Fig. 1a).

Equation (2) can be formally integrated [14] with the help of the propagator G_0 of H_0 : $\Psi_0(\mathbf{r}, t) = \frac{\Omega_{\text{rf}}}{2i} \int_0^t dt' \int d^3\mathbf{r}' G_0(\mathbf{r}, t; \mathbf{r}', t') \Psi_{-1}(\mathbf{r}', t')$. Together with Eqs. (1) and (3), we can derive an exact integro-differential equation on α^2 , the fraction of atoms remaining in the BEC. In the weak coupling regime, the condition $\hbar\Gamma \ll \Delta$ expresses the fact that the memory time of the continuum t_c is much shorter than the damping time of the condensate level. This allows one to make

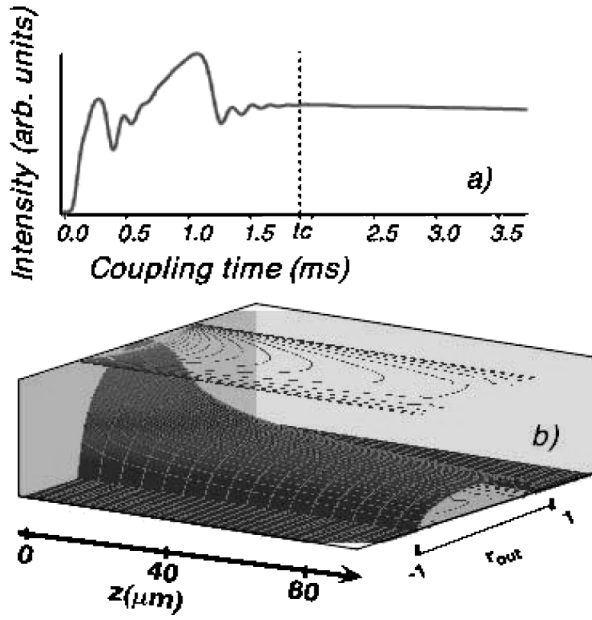


FIG. 1. (a) A 1D numerical integration of the time evolution of the laser intensity starting with ~ 2000 atoms for $\Omega_{\text{rf}} = 300$ Hz shows that, for $t \gg t_c$, a quasisteady state is achieved. (b) The spatial intensity profile of the atom laser according to Eq. (9) is shown for $t \gg t_c$.

a Born-Markov approximation [14,20], which yields the rate equation:

$$\frac{dN_{-1}}{dt} = -\Gamma(N_{-1})N_{-1}. \quad (6)$$

The output rate Γ is given by the Fermi golden rule. By neglecting the transverse kinetic energy as compared to E_{-1} , we obtain

$$\frac{\Gamma}{\Omega_{\text{rf}}^2} \approx \frac{15\pi}{32} \frac{\hbar}{\Delta} \max[0, 1 - \eta^2]^2, \quad (7)$$

with $\eta = -2E_{-1}/\Delta$. The rate equation (6) is nonlinear, since Γ depends on N_{-1} through $\Delta = 2Mgz_0$. From Eq. (7) and the condition for weak coupling $h\Gamma \ll \Delta$, we deduce a critical Rabi frequency $\Omega_{\text{rf}}^C \sim 0.8\Delta/\hbar$ below which a quasicontinuous output is obtained. Moreover, we have assumed from the beginning that the BEC decay was adiabatic. This requires $|\partial h_{-1}/\partial t| \sim \Gamma\mu \ll \epsilon_{(i)}^2/\hbar$, where $\epsilon_{(i)}$ is the energy of the i th quasiparticle level in the trap. Taking $\epsilon_{(i)} \gtrsim \hbar\omega_{\perp}$, we deduce from Eq. (7) a condition on the Rabi frequency: $\Omega_{\text{rf}} \ll 1.6(g/z_0)^{1/2}$. This condition turns out to be much more stringent than the condition for weak coupling $\Omega_{\text{rf}} \ll \Omega_{\text{rf}}^C$.

To compare our model to the data of Bloch *et al.* [6], we have numerically integrated Eq. (6) with the output rate (7) and their experimental parameters. We show in Fig. 2a (respectively, Fig. 2b) the number of atoms remaining in the $|F = 1; m_F = -1\rangle$ (respectively, $|F = 2; m_F = 2\rangle$) condensate after a fixed time as a function of the detuning. In the latter case, the condensate is coupled to the continuum via the intermediate state $|F = 2; m_F = 1\rangle$. If we neglect the mutual interaction between these two bound states, the

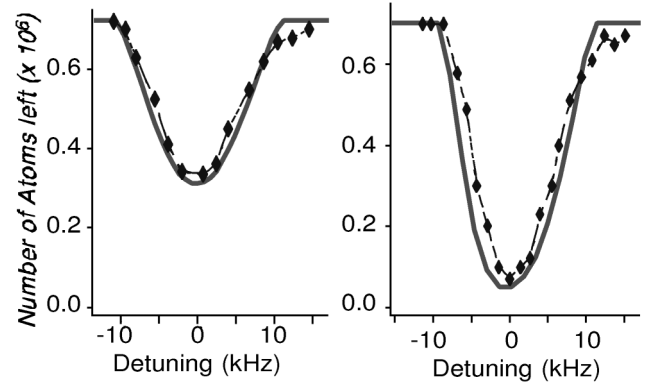


FIG. 2. The number of trapped atoms after 20 ms of rf out-coupling, starting with $\approx 7.2 \times 10^5$ condensed atoms, $\Omega_{\text{rf}} = 312$ Hz for the $|1; -1\rangle$ sublevel (left), and $\approx 7.0 \times 10^5$ atoms, $\Omega_{\text{rf}} = 700$ Hz for the $|2; 2\rangle$ sublevel (right) is plotted versus the coupler frequency. Diamonds are the experimental points from [6]; solid line is the prediction based upon our model using the experimental parameters. The experimental curves have been shifted in frequency to match theory, since V_{off} is not experimentally known precisely enough (within a kHz uncertainty).

upper level acquires a decay rate $\Gamma_{2,2} \sim \Gamma_{2,1}/2$ for near resonant coupling and for $\Omega_{\text{rf}} \gg \Gamma_{2,1}$ [19]. Here $\Gamma_{2,1}$ results from Eq. (7) applied to the relay state.

The quantitative agreement between theory and experiments underlines the importance of gravity and dimensionality in this problem. On one hand, the width and depth of the resonance curve depend directly on the strength of gravity g . On the other hand, in a 1D calculation, the output rate is found to be roughly 3 times higher [10]. The discrepancies on the wings might be explained by the presence of uncondensed atoms near the condensate boundaries [14].

Finally, we want to point out that, since $h\Gamma \ll \Delta$, the spatial region where outcoupling takes place, of vertical extension $\delta z \sim h\Gamma/mg$, is very thin compared to the BEC size. Outcoupling can thus be understood semiclassically, in analogy with a Franck-Condon principle [21]: the coupling happens at the turning point of the classical trajectory of the free falling atoms. Neglecting the transverse kinetic energy in the calculation of Γ amounts to approximating the Franck-Condon surfaces, whose curvature should be small over the size of the BEC, by planes.

We have analyzed so far the extraction of the atoms from the trapped BEC. We address now the question of the propagation of the outcoupled atoms under gravity. For times $t \gg t_c$, the explicit expression of the propagator G_0 of H_0 is [22]

$$G_0(\mathbf{r}, \mathbf{r}', \tau = t - t') = \left(\frac{M}{2\pi\hbar\tau} \right)^{3/2} e^{iS_{\text{class}}(\mathbf{r}, \mathbf{r}', \tau = t - t')} \theta(\tau). \quad (8)$$

Here, the classical action is given by $S_{\text{class}}(\mathbf{r}, \mathbf{r}', \tau = t - t') = (M/2\hbar\tau)[(\mathbf{r} - \mathbf{r}')^2 + g\tau^2(z + z') - g^2\tau^4/12]$ and θ is the step function. We compute the output wave function Ψ_0 with stationary phase approximations. This amounts to propagating the wave function along the

classical trajectory: in the limit where $S_{\text{class}} \gg \hbar$ (this is true as soon as the atom has fallen from a height l), this is the Feynman path that essentially contributes. We obtain the following expression for the outcoupled atomic wave function i.e., the atom laser mode [23] (see Fig. 1b):

$$\begin{aligned} \psi_0(\mathbf{r}, t) \approx & \sqrt{\pi} \frac{\hbar \Omega_{\text{rf}}}{2Mgl} \\ & \times \Psi_{-1}(r_{\perp}, z_{\text{res}}, t - t_{\text{fall}}) \frac{e^{i2/3|\zeta_{\text{res}}|^{3/2} - i(\delta_{\text{rf}} t_{\text{fall}})}}{\sqrt{|\zeta_{\text{res}}|^{1/2}}} \\ & \times M(t, t_{\text{fall}}). \end{aligned} \quad (9)$$

In this expression, $z_{\text{res}} = \eta z_0$ is the point of extraction, $t_{\text{fall}} = (2/g)^{1/2}(z - z_{\text{res}})^{1/2}$ is the time of fall from this point, $\zeta_{\text{res}} = (z - z_{\text{res}})/l$, and $M(t, t_{\text{fall}})$ is unity if $0 \leq t_{\text{fall}} \leq t$ and zero elsewhere: it describes the finite extent of the atom laser due to the finite coupling time. We can deduce from Eq. (9) the size of the laser beam in the x direction $x_{\text{out}} = x_0[1 - (2E_{-1}/\Delta)^2]^{1/2}$, and a similar formula for y_{out} . Because the semiclassical approximation neglects the quantum velocity spread due to the spatial confinement of the trapped BEC, this wave function has planar wave fronts. This property will not persist beyond a distance $z_R \approx y_{\text{out}}^4/4l^3 \sim$ a few mm, analog to the Rayleigh length in photonic laser beams, where the transverse diffraction $(\hbar/M y_{\text{out}})(2z_R/g)^{1/2}$ becomes comparable to the size y_{out} . Beyond z_R , diffraction has to be taken into account.

In conclusion, we have obtained analytical expressions for the output rate and the output mode of a quasicontinuous atom laser based on rf outcoupling from a trapped BEC. Our treatment, which fully takes the gravity and the 3D geometry into account, leads to a quantitative agreement with the experimental results of Ref. [6]. This points out the crucial role played by gravity in the atom laser behavior. Generalization to other coupling schemes is straightforward and should allow one to investigate the role of gravity in these cases. In more elaborated treatments, interactions between the trapped and the outcoupled atoms should be taken into account to describe the transverse dynamics. Finite temperature effects [14] might be relevant as well. Nevertheless, the good agreement with experimental data shows that the present zero-temperature theory is a valuable starting point to describe experiments with the atom laser.

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