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Abstract

We study the behavior of perturbations of small nonlinear Dirac standing waves. We assume that the linear Dirac operator of reference $H = D_m + V$ has only two double eigenvalues, this degeneracy is due to a theorem of Kramers. In this case, we can build a small 8-dimensional manifold of stationary solutions tangent to the first eigenspace of H.

Then we assume that a resonance condition holds for the first eigenvalue. We build a center manifold of real codimension 16 around each stationary solution. Inside this center manifold any $H^{s'}$ perturbation of stationary solutions, with $s' > 2$, stabilizes towards a standing wave. We also build center-stable and center-unstable manifolds each one of real codimension 8. Inside each manifold, we obtain stabilization towards the center manifold in one direction of time, while in the other, we have instability. Eventually, outside all these manifolds, we have instability in the two directions of time.

Introduction

We study the asymptotic stability of stationary solutions of a time-dependent nonlinear Dirac equation.

A localized stationary solution of a given time-dependent equation represents a bound state of a particle. Like Rañada [Ran], we call it a *particle-like solution* (PLS). Many works have been devoted to the proof of the existence of such solutions for a wide variety of equations. Although their stability is a crucial problem (in particular in numerical computation or experiment), a smaller attention has been deserved to this issue.

In this paper, we deal with the problem of stability of small PLS of the following nonlinear Dirac equation:

$$\mathrm{i}\partial_t\psi = (D_m + V)\psi + \nabla F(\psi) \tag{NLDE}$$

where ∇F is the gradient of $F : \mathbb{C}^4 \mapsto \mathbb{R}$ for the standard scalar product of $\mathbb{R}^8$. Here, D_m is the usual Dirac operator, see Thaller [Tha92], acting on $L^2(\mathbb{R}^3, \mathbb{C}^4)$

$$D_m = \alpha \cdot (-\mathrm{i}\nabla) + m\beta = -\mathrm{i}\sum_{k=1}^{3} \alpha_k \partial_k + m\beta$$

where $m \in \mathbb{R}_+^*$, $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, β are $\mathbb{C}^4$ hermitian matrices satisfying the following properties:

$$\begin{cases} \alpha_i\alpha_k + \alpha_k\alpha_i = 2\delta_{ik}\mathbf{1}_{\mathbb{C}^4}, & i,k \in \{1,2,3\}, \\ \alpha_i\beta + \beta\alpha_i = \mathbf{0}_{\mathbb{C}^4}, & i \in \{1,2,3\}, \\ \beta^2 = \mathbf{1}_{\mathbb{C}^4}. \end{cases}$$

Here, we choose

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} I_{\mathbb{C}^2} & 0 \\ 0 & -I_{\mathbb{C}^2} \end{pmatrix}$$

$$\text{where } \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In (NLDE), V is the external potential field and $F : \mathbb{C}^4 \mapsto \mathbb{R}$ is a nonlinearity with the following gauge invariance:

$$\forall(\theta, z) \in \mathbb{R} \times \mathbb{C}^4, \quad F(e^{i\theta} z) = F(z). \tag{0.1}$$

Some additional assumptions on F and V will be made in the sequel. Stationary solutions (PLS) of (NLDE) take the form $\psi(t, x) = e^{-\mathrm{i}Et}\phi(x)$ where ϕ satisfies

$$E\phi = (D_m + V)\phi + \nabla F(\phi). \tag{PLSE}$$

We show that there exists a manifold of small solutions to (PLSE) tangent to the first eigenspace of $D_m + V$ (see Proposition 1.1 below).

In [Bou06], we prove that there are stable directions for the PLS manifold under a non resonance assumption on the spectrum of $H := D_m + V$. This gives a stable manifold, containing the PLS manifold. But we were not able to say anything about solutions starting outside the stable manifold.

In the Schrödinger case, orbital stability results (see e.g [CL82], [Wei85, Wei86] or [SS85, GSS87]) give that any solution stays near the PLS manifold. Unfortunately, orbital stability criteria applied to Schrödinger equations use the fact that Schrödinger operators are bounded from below. Hence the question of orbital stability for Dirac standing waves cannot be solved by a straightforward application of the methods used in the Schrödinger case.

Concerning the asymptotic stability, in the Schrödinger equation, the question has been solved in several cases. For small stationary solutions in the simple eigenvalue case it has been studied by Soffer and Weinstein [SW90, SW92], Pillet and Wayne [PW97] or Gustafson, Nakanishi and Tsai [GNT04]. For the two eigenvalue case under a resonance condition for an excited state, the problem has been studied by Tsai and Yau [TY02a, TY02c, TY02d, TY02b, Tsa03] or Soffer and Weinstein [SW04, SW05]. Another problem has been studied by Cuccagna [Cuc01, Cuc03, Cuc05], he considered the case of big PLS, when the linearized operator has only one eigenvalue and obtained the asymptotic stability of the manifold of ground states. Schlag [Sch04] proved that any ground state of the cubic nonlinear Schrödinger equation in dimension 3 is orbitally unstable but posses a stable manifold of codimension 9.

We also would like to mention the works of Buslaev and Perel'mann [BP95, BP92b, BP92c, BP92a], Buslaev and Sulem [BS03, BS02], Weder [Wed00] or Krieger and Schlag [KS05] in the one dimensional Schrödinger case. Krieger and Schlag [KS05] proved a result similar to [Sch04] in the one dimensional case.

The result we present here, states the existence of a stable manifold and describes the behavior of solutions starting outside of it. In fact, we prove the instability of the stable manifold. We also prove stabilization towards stationary solutions inside the stable manifold for $H^{s'}$ perturbation with $s' > 2$. We have been able to obtain it since we impose a resonance condition for the first eigenvalue, while in [Bou06], we assumed there is no resonance phenomena.

This paper is organized as follow.

In Section 1, we present our main results and the assumptions we need. Subsection 1.1, is devoted to the statement of the time decay estimates of the propagator associated with $H = D_m + V$ on the continuous subspace. One is a kind of smoothness result, in the sense of Kato (see e.g. [Kat66]), the other is a Strichartz type result. We prove these estimates with the propagation and dispersive estimates proved in [Bou06]. In subsection 1.2, we state the existence of small stationary states forming a manifold tangent to an eigenspace of H. The study of the dynamics around such states leads us to our main result, see Subsection 1.3, where we split a neighborhood of a stationary state in different parts, each one giving rise to stabilization or instability.

To prove our main theorem, we consider our nonlinear system as a small perturbation of a linear equation. More precisely in Subsection 2.2, we show that the spectral properties of the linearized operator around a stationary state, presented in Section 2, permits to obtain, like in the linear case, some properties of the dynamics around a stationary state. Eventually, in Section 3, we obtain, with our time decay estimates, a stabilization result for $H^{s'}$ perturbation with $s' > 2$. Contrarily to [Bou06], we do not restrict ourself to localized perturbations. Our results are an adaptation to the Dirac case of some results of Tsai and Yau [TY02d] and Gustafson, Nakanishi and Tsai [GNT04].

1 Assumptions and statements

1.1 Time decay estimates

We generalize to small nonlinear perturbations, stability results for linear systems. These results, like in [Bou06], follow from linear decay estimates. Here we use smoothness type and Strichartz type estimates deduced from propagation and dispersive estimates of [Bou06]. Hence, we work within the

Assumption 1.1. *The potential $V : \mathbb{R}^3 \mapsto S_4(\mathbb{C})$ (self-adjoint 4×4 matrices) is a $\mathcal{C}^\infty$ function such that there exists $\rho > 5$ with*

$$\forall \alpha \in \mathbb{N}^3,\ \exists C > 0,\ \forall x \in \mathbb{R}^3,\ |\partial^\alpha V|(x) \leq \frac{C}{\langle x \rangle^{\rho+|\alpha|}}.$$

We notice that by the Kato-Rellich theorem, the operator

$$H := D_m + V$$

is essentially self-adjoint on $\mathcal{C}_0^\infty(\mathbb{R}^3, \mathbb{C}^4)$ and self-adjoint on $H^1(\mathbb{R}^3, \mathbb{C}^4)$.

We also mention that Weyls's theorem gives us that the essential spectrum of H is $(-\infty, -m] \cup [+m, +\infty)$ and the work of Berthier and Ggeorgescu [BG87, Theorem 6, Theorem A], gives us that there is no embedded eigenvalue. Hence the thresholds $\pm m$ are the only points of the continuous spectrum which can be associated with wave of zero velocity. These waves perturb the spectral density and diminish the decay rate in the propagation and the dispersive estimates. Hence, we also work within the

Assumption 1.2. *The operator H presents no resonance at thresholds and no eigenvalue at thresholds.*

A resonance is a stationary solution in $H^{1/2}_{-\sigma} \setminus H^{1/2}$ for any $\sigma \in (1/2, \rho - 2)$, where H^t_σ is given by

Definition 1.1 (Weighted Sobolev space). *The weighted Sobolev space is defined by*

$$H^t_\sigma(\mathbb{R}^3, \mathbb{C}^4) = \left\{ f \in \mathcal{S}'(\mathbb{R}^3),\ \|\langle Q \rangle^\sigma \langle P \rangle^t f\|_2 < \infty \right\}$$

for $\sigma, t \in \mathbb{R}$. We endow it with the norm

$$\|f\|_{H^t_\sigma} = \|\langle Q \rangle^\sigma \langle P \rangle^t f\|_2.$$

If $t = 0$, we write L^2_σ instead of H^0_σ.

We have used the usual notations $\langle u \rangle = \sqrt{1 + u^2}$, $P = -i\nabla$, and Q is the operator of multiplication by x in $\mathbb{R}^3$.

Now let

$$\mathbf{P}_c(H) = \mathbb{1}_{(-\infty, -m| \cup [+m, +\infty)}(H)$$

be the projector associated with the continuous spectrum of H and $\mathcal{H}_c$ its range. Using [Bou06, Theorem 1.1], we obtain a Limiting Absorption Principle which gives the

Theorem 1.1 (H-smoothness of $\langle Q \rangle^{-1}$). *If Assumptions 1.1 and 1.2 hold. Then for any $\sigma \geq 1$ and $s \in \mathbb{R}$, one has:*

$$\left\| \langle Q \rangle^{-\sigma} e^{-\mathrm{i}tH} \mathbf{P}_c(H) \psi \right\|_{L^2_t(\mathbb{R}, H^s(\mathbb{R}^3, \mathbb{C}^4))} \leq C \|\psi\|_{H^s(\mathbb{R}^3, \mathbb{C}^4)}, \tag{i}$$

$$\left\| \int_{\mathbb{R}} e^{\mathrm{i}tH} \mathbf{P}_c(H) \langle Q \rangle^{-\sigma} F(t)\, dt \right\|_{H^s(\mathbb{R}^3, \mathbb{C}^4)} \leq C \|F\|_{L^2_t(\mathbb{R}, H^s)}, \tag{ii}$$

$$\left\| \int_{s<t} \langle Q \rangle^{-\sigma} e^{-\mathrm{i}(t-s)H} \mathbf{P}_c(H) \langle Q \rangle^{-\sigma} F(s)\, ds \right\|_{L^2_t(\mathbb{R}, H^s(\mathbb{R}^3, \mathbb{C}^4))} \leq C \|F\|_{L^2_t(\mathbb{R}, H^s(\mathbb{R}^3, \mathbb{C}^4))}, \tag{iii}$$

Proof. We first prove (i). For $s = 0$, it is (see *e.g.* [ABdMG96, Proposition 7.11] or [RS78, Theorem XIII.25]) a consequence of the limiting absorption principle:

$$\sup_{|\Re z|\geq m, \Im z\in(0,1)} \left\{ \left\| \langle Q\rangle^{-\sigma} (H-z)^{-1} P_c(H) \langle Q\rangle^{-\sigma} \right\|_2^2 \right\} < \infty \tag{1.1}$$

this follows from [Bou06, Theorem 1.1] for $\sigma > 5/2$.Then we use Born expansion

$$R_V^+(\lambda) = R_0^+(\lambda) - R_0^+(\lambda) V R_0^+(\lambda) + R_0^+(\lambda) V R_V^+(\lambda) V R_0^+(\lambda)$$

and [IM99, Theorem 2.1(i)] to conclude the proof for $s = 0$ and $\sigma \geq 1$. For $s \in 2\mathbb{Z}$ and $\sigma \geq 1$ it follows from the previous cases using boundedness of $< H >^s < D_m >^{-s}$ and $< H >^{-s} < D_m >^s$ (which follow from the boundedness of V and its derivatives) and the boundedness of $\langle Q\rangle^{\mp\sigma}[\langle Q\rangle^{\pm\sigma}, < H >^s]\langle H\rangle^{-s}$ (which follow from multicommutator estimates see [HS00, Appendix B]). The rest follows by interpolation.

Estimates (i) and (ii) are equivalent by duality.

To prove estimate (iii) when $s = 0$, we notice, using the Fourier transform in time, that it is equivalent to

$$\left| \iint_{\mathbb{R}^2} \left\langle \widehat{G}(\lambda), \langle Q\rangle^{-\sigma} R_V^+(\lambda) \mathbf{P}_c(H) \langle Q\rangle^{-\sigma} \widehat{F}(\lambda) \right\rangle \, ds dt \right| \leq C \left\| \widehat{G} \right\|_{L^2_\lambda(\mathbb{R}, L^2(\mathbb{R}^3,\mathbb{C}^4))} \left\| \widehat{F} \right\|_{L^2_\lambda(\mathbb{R}, L^2(\mathbb{R}^3,\mathbb{C}^4))}$$

for any F, G in $L^2_t(\mathbb{R}, L^2(\mathbb{R}^3, \mathbb{C}^4)$. This in turn follows from the Limiting Absorption just proved.

To prove the estimates for $s \neq 0$, we work like we do for the previous estimates. □

To state the next result, we need the

Definition 1.2 (Besov space). *For $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$, the Besov space $B^s_{p,q}(\mathbb{R}^3, \mathbb{C}^4)$ is the space of all $f \in \mathcal{S}'(\mathbb{R}^3, \mathbb{C}^4)$ (dual of the Schwartz space) such that*

$$\|f\|_{B^s_{p,q}} = \left(\sum_{j\in\mathbb{N}} 2^{jsq} \|\varphi_j * f\|_p^q \right)^{\frac{1}{q}} < +\infty$$

with $\widehat{\varphi} \in \mathcal{D}(\mathbb{R}^n \setminus \{0\})$ such that $\sum_{j\in\mathbb{Z}} \widehat{\varphi}(2^{-j}\xi) = 1$ for all $\xi \in \mathbb{R}^3 \setminus \{0\}$, $\widehat{\varphi}_j(\xi) = \widehat{\varphi}(2^{-j}\xi)$ for all $j \in \mathbb{N}^*$ and for all $\xi \in \mathbb{R}^3$, and $\widehat{\varphi_0} = 1 - \sum_{j\in\mathbb{N}^*} \widehat{\varphi}_j$. It is endowed with the natural norm $f \in B^s_{p,q}(\mathbb{R}^3, \mathbb{C}^4) \mapsto \|f\|_{B^s_{p,q}}$.

Using the Dispersive estimates of [Bou06, Theorem 1.2] and [KT98, Theorem 10.1], we obtain the

Theorem 1.2 (Strichartz-type estimates). *If Assumptions 1.1 and 1.2 hold. Then for any $2 \leq p, q \leq \infty$, $\theta \in [0, 1]$, with $(1 - \frac{2}{q})(1 \pm \frac{\theta}{2}) = \frac{2}{p}$ and $(p, \theta) \neq (2, 0)$, and for any reals s, s' with $s' - s \geq \alpha(q)$ where $\alpha(q) = (1 + \frac{\theta}{2})(1 - \frac{2}{q})$, there exists a positive constant C such that*

$$\|e^{-\mathrm{i}tH} P_c(H)\psi\|_{L^p_t(\mathbb{R}, B^s_{q,2}(\mathbb{R}^3,\mathbb{C}^4))} \leq C\|\psi\|_{H^{s'}(\mathbb{R}^3,\mathbb{C}^4)}, \tag{i}$$

$$\| \int e^{\mathrm{i}tH} P_c(H) F(t)\, dt\|_{H^s} \leq C\|F\|_{L^{p'}_t(\mathbb{R}, B^{s'}_{q',2}(\mathbb{R}^3,\mathbb{C}^4))}, \tag{ii}$$

$$\| \int_{s<t} e^{-\mathrm{i}(t-s)H} P_c(H) F(s)\, ds\|_{L^p_t(\mathbb{R}, B^{-s}_{q,2}(\mathbb{R}^3,\mathbb{C}^4))} \leq C\|F\|_{L^{\tilde{p}'}_t(\mathbb{R}, B^{\tilde{s}}_{\tilde{q}',2}(\mathbb{R}^3,\mathbb{C}^4))}, \tag{iii}$$

for any $r \in [1, \infty]$, $(\tilde{q}, \tilde{p})$ chosen like (q, p) and $s + \tilde{s} \geq \alpha(q) + \alpha(\tilde{q})$.

Proof. This is a consequence of [KT98, Theorem 10.1] applied to $U(t) = e^{-\mathrm{i}tH} P_c(H)$, using [Bou06, Theorem 1.2] and

$$B^{(1+\frac{\theta}{2})(1-\frac{2}{q})}_{q,2} \hookrightarrow (L^2, B^{1+\theta/2}_{1,2})_{2/((1\pm\theta/2)p),2}$$

continuously for $p \geq 2$ ($p \neq 2$ if $\theta = 0$) and $1/q = 1 - 1/((1 \pm \theta/2)p)$, see proof of [BL76, Theorem 6.4.5] as well as the properties of the real interpolation (see [BL76] or [Tri78]).

In the case $\theta \neq 0$, the proof is actually simpler. We prove it using the usual TT^* method and the hölder inequality instead of the Hardy-Littlewood-Sobolev inequality.

□

1.2 The manifold of PLS

We study the following nonlinear Dirac equation

$$\begin{cases} \mathrm{i}\partial_t\psi = H\psi + \nabla F(\psi) \\ \psi(0,\cdot) = \psi_0. \end{cases} \tag{1.2}$$

with $\psi \in \mathcal{C}^1(I, H^1(\mathbb{R}^3, \mathbb{C}^4))$ for some open interval I which contains 0 and $H = D_m + V$. The nonlinearity $F : \mathbb{C}^4 \mapsto \mathbb{R}$ is a differentiable map for the real structure of $\mathbb{C}^4$ and hence the ∇ symbol has to be understood for the real structure of $\mathbb{C}^4$. For the usual hermitian product of $\mathbb{C}^4$, one has

$$DF(v)h = \Re\langle \nabla F(v), h\rangle.$$

If F has a gauge invariance (see Equation (0.1) or Assumption 1.4 below), this equation can have stationary solutions *i.e.* solution of the form $e^{-\mathrm{i}Et}\phi_0$ where ϕ_0 satisfies the nonlinear stationary equation:

$$E\phi_0 = H\phi_0 + \nabla F(\phi_0).$$

Then we notice the Dirac operator D_m have an interesting invariance property due to its matrix structure. This invariance can be shared by some perturbed Dirac operators and gives a consequence of a theorem of Kramers, see [Par90]. Let us precise it, we introduce K an antilinear operator defined by:

$$K\begin{pmatrix} \psi \\ \chi \end{pmatrix} = \begin{pmatrix} \sigma_2\overline{\psi} \\ \sigma_2\overline{\chi} \end{pmatrix} \text{ with } \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}. \tag{1.3}$$

The operator D_m commutes with K. So if V also commutes with K, we obtain that the eigenspaces of H are always of even dimension. Here we work with the

Assumption 1.3. *The potential V commutes to K. The operator $H := D_m + V$ has only two double eigenvalues $\lambda_0 < \lambda_1$, with $\{\phi_0, K\phi_0\}$ and $\{\phi_1, K\phi\}$ as associated orthonormalized basis.*

We also need the

Assumption 1.4. *The function $F : \mathbb{C}^4 \mapsto \mathbb{R}$ is in $\mathcal{C}^\infty(\mathbb{R}^8, \mathbb{R})$ and satisfies $F(z) = O(|z|^4)$ as $z \to 0$. Moreover, it has the following invariance properties:*

$$\forall z \in \mathbb{C}^4,\ \forall \theta \in \mathbb{R},\ F(Kz) = F(z),\ F(e^{i\theta}z) = F(z).$$

We obtain the

Proposition 1.1 (PLS manifold). *If Assumptions 1.1–1.4 hold. Then for any $\sigma \in \mathbb{R}^+$, there exists Ω a neighborhood of $0 \in \mathbb{C}^2$, a $\mathcal{C}^\infty$ map*

$$h : \Omega \mapsto \{\phi_0, K\phi_0\}^\perp \cap H^2(\mathbb{R}^3, \mathbb{C}^4) \cap L^2_\sigma(\mathbb{R}^3, \mathbb{C}^4)$$

and a $\mathcal{C}^\infty$ map $E : \Omega \mapsto \mathbb{R}$ such that $S((u_1,u_2)) = u_1\phi_0 + u_2K\phi_0 + h((u_1,u_2))$ satisfy for all $U \in \Omega$,

$$HS(U) + \nabla F(S(U)) = E(U)S(U), \tag{1.4}$$

with the following properties

$$\begin{cases} h((u_1,u_2)) = \left(\frac{u_1}{|(u_1,u_2)|}Id_{\mathbb{C}^4} + \frac{u_2}{|(u_1,u_2)|}K\right) h\left((|(u_1,u_2)|,0)\right), \quad \forall U = (u_1,u_2) \in \Omega, \\ h(U) = O(|U|^2), \\ E(U) = E(|U|), \\ E(U) = \lambda_0 + O(|U|^2). \end{cases}$$

Proof. This result is adapted from [PW97, Proposition 2.2] after the reduction due to the invariance of the problem with respect to K. □

Moreover, we have

Lemma 1.1 (exponential decay). *For any $\beta \in \mathbb{N}^4$, $s \in \mathbb{R}^+$ and $p, q \in [1, \infty]$. There is $\gamma > 0$, $\varepsilon > 0$ and $C > 0$ such that for all $U \in B_{\mathbb{C}^2}(0,\varepsilon)$ one has*

$$\|e^{\gamma\langle Q\rangle}\partial^\beta_U S(U)\|_{B^s_{p,q}} \le C\|S(U)\|_2,$$

where $\partial^\beta_{(u_1,u_2)} = \frac{\partial^{|\beta|}}{\partial^{\beta_1}\Re u_1\partial^{\beta_2}\Im u_1\partial^{\beta_3}\Re u_2\partial^{\beta_4}\Im u_2}$.

Proof. This is like in [Bou06, Lemma 4.1], where we used ideas of [His00]. □

1.3 The unstable manifold, the stabilization and the nonlinear scattering

Each stationary solution previously introduced has, like in [Bou06], a stable manifold. Under the following assumptions, we can prove that the stable manifold is unstable, that is to say that a small perturbation of a stationary solution starting outside of this manifold leaves from any neighborhood of this stationary solution. We work with the

Assumption 1.5. *The resonant condition*

$$|\lambda_1 - \lambda_0| > \min\{|\lambda_0 + m|,\, |\lambda_0 - m|\}$$

holds. Moreover, we have the Fermi Golden Rule

$$\Gamma(\phi) = \lim_{\substack{\varepsilon \to 0,\\ \varepsilon > 0}} \left\langle d^2F(\phi)\phi_1, \Im\left((H-\lambda_0) + (\lambda_1 - \lambda_0) - \mathrm{i}\varepsilon\right)^{-1} P_c(H) d^2F(\phi)\phi_1 \right\rangle > 0 \tag{1.5}$$

for any non zero eigenvector ϕ associated with λ_0.

In this assumption, the notation d^2F denotes the differential of ∇F.

Let us introduce the linearized operator $JH(U)$ around a stationary state $S(U)$:

$$H(U) = H + d^2F(S(U)).$$

We notice that the operator $H(U)$ is only $\mathbb{R}$-linear but not $\mathbb{C}$-linear. Hence we work with the space $L^2(\mathbb{R}^3, \mathbb{R}^4 \times \mathbb{R}^4)$ instead of $L^2(\mathbb{R}^3, \mathbb{C}^4)$ by writing

$$\begin{pmatrix} \Re\phi \\ \Im\phi \end{pmatrix}$$

instead of ϕ. The multiplication by $-\mathrm{i}$ gives the operator

$$J = \begin{pmatrix} 0 & -I_{\mathbb{R}^4} \\ I_{\mathbb{R}^4} & 0 \end{pmatrix}.$$

Now we mention some spectral properties of $JH(U)$ in $L^2(\mathbb{R}^3, \mathbb{R}^4 \times \mathbb{R}^4)$ which are needed to state and to understand our main theorem. These properties will be proven in subsection 2 below.

The operator $JH(U)$ has a four dimensional geometric kernel and four double eigenvalues $E_1(U)$, $\overline{E_1}(U)$, $-E_1(U)$ and $-\overline{E_1}(U)$ with $\Re E_1(U) > 0$.

The eigenspaces associated with $E_1(U)$ and $\overline{E_1}(U)$ are conjugated via the complex conjugation, we will work on the real part of their sum: $X_u(U)$, we introduce a basis $(\xi_i(U))_{i=1,\dots,4}$ of $X_u(U)$.

The same holds for $-E_1(U)$ and $-\overline{E_1}(U)$ and working on the real part of the sum of the associated eigenspaces : $X_s(U)$, we introduce a basis $(\xi_i(U))_{i=5,\dots,8}$ of $X_s(U)$.

The rest of the spectrum is the essential or continuous spectrum. We write $\mathcal{H}_c(U)$ for the space associated with the continuous spectrum. The space $\mathcal{H}_c(U)$ is the orthogonal of the previous eigenspaces with respect to the product $(f,g) \mapsto \Re\langle f, Jg\rangle_{L^2(\mathbb{R}^3,\mathbb{C}^4)}$. We introduce $\mathcal{H}_c^r(U)$ the intersection of $\mathcal{H}_c = \mathcal{H}_c(0)$ with the orthogonal with respect to J of

$$\left\{ \Im\left(H - E(U) + \lambda_1 - \lambda_0 - \frac{\mathrm{i}}{2}\Gamma(S(U)) \right)^{-1} d^2F(S(U))|U| \left(\frac{u_1}{|U|} I_{\mathbb{C}^4} + \frac{u_2}{|U|} K \right) \phi_1, \forall U = (u_1, u_2) \in \mathbb{C}^2 \right\}.$$

The space $\mathcal{H}_c(U)$ is isomorphic to $\mathcal{H}_c^r(U)$. The isomorphism is the orthogonal projector onto $\mathcal{H}_c^r(U)$ with respect to the product $(f,g) \mapsto \Re\langle f, Jg\rangle_{L^2(\mathbb{R}^3,\mathbb{C}^4)}$ and this projector is bounded from $H^s_\sigma(\mathbb{R}^3, \mathbb{C}^4)$ into itself for any reals s and σ.

We can state our

Theorem 1.3. *If Assumptions 1.1–1.5 hold. Then, for $s' > \beta + 2 > 2$ and $\sigma > 3/2$, there exist $\varepsilon > 0$ and a continuous map $r : B_{\mathbb{C}^2}(0,\varepsilon) \mapsto \mathbb{R}$ with $r(U) = O(\Gamma(U))$, such that for any initial condition of the form*

$$\psi_0 = S(U_0) + z_0 + A \cdot \xi(U_0)$$

with $U_0 \in B_{\mathbb{C}^2}(0,\varepsilon)$, $z_0 \in \mathcal{H}_c(U_0) \cap B_{H^{s'}}(0, r(U))$ and $A \in \mathbb{C}^4$, the following holds.

(i) For the set

$$S = \{(U, z)\,;\ U \in B_{\mathbb{C}^2}(0,\varepsilon),\ z \in \mathcal{H}_c(U) \cap B_{H^{s'}}(0, r(U))\}$$

endowed with the metric of $\mathbb{C}^2 \times H^{s'}$*, there exist* $C > 0$*,* $\mathcal{V}$ *a neighborhood of* $(0,0)$ *in* S*, and a smooth map* $\Psi : \mathcal{V} \mapsto \mathbb{C}^4$ *with graph* $\mathcal{CM}$*,* $\Psi(\cdot, 0) = 0$ *and* $\|\Psi(0,0)\| = \|D\Psi(0,0)\| = 0$ *such that if* $(U_0, z_0) \in \mathcal{V}$ *if* $A = \Psi(U_0, z_0)$*, we have:*

(a) there exists a unique solution $\psi \in \mathcal{C}\left(\mathbb{R}, H^{s'}\right) \cap \mathcal{C}^1\left(\mathbb{R}, H^{s'-1}\right)$ *of* (1.2) *with initial condition* ψ_0 *and this solution is unique in* $L^\infty((-T,T), H^{s'}(\mathbb{R}^3, \mathbb{C}^4))$ *for any* $T > 0$*;*

(b) there exist $\mathcal{V}^\pm$ *open neighborhoods of* $(0,0)$ *in* S *and bijective maps*

$$(U_0; z_0) \in \mathcal{V} \mapsto (U_{\pm\infty}; z_{\pm\infty}) \in \mathcal{V}^\pm,$$

such that

$$|U_{\pm\infty} - U_0| \leq C\|z_0\|^2_{H^{s'}},\ \|z_{\pm\infty} - z_0\|_{H^{s'}} \leq C\,|U_0|\,\|z_0\|_{H^{s'}},$$

and for all $t \in \mathbb{R}$

$$\psi(t) = e^{-\mathrm{i}\int_0^t E(U(v))\,dv} S(U(t)) + e^{JtE(U_{\pm\infty})} e^{JtH(U_{\pm\infty})} z_{\pm\infty} + \varepsilon_\pm(t)$$

with $\dot{U} \in L^q(\mathbb{R})$ *for all* $q \in [1,\infty]$*,* $\lim\limits_{t\to\pm\infty} U(t) = U_{\pm\infty}$*,*

$$\max\left\{ \|\varepsilon_\pm\|_{L^\infty(\mathbb{R}^\pm, H^{s'})},\ \|\varepsilon_\pm\|_{L^2(\mathbb{R}^\pm, H^s_{-\sigma})}, \|\varepsilon_\pm\|_{L^2(\mathbb{R}^\pm, B^\beta_{\infty,2})} \right\} \leq C\,|U_0|\,\|z_0\|_{H^{s'}},$$

and

$$\lim_{t\to\pm\infty} \|\varepsilon_\pm(t)\|_{H^{s'}} = 0.$$

(ii) For the sets

$$\widetilde{S_+} = \left\{ (U, z, p)\,;\ U \in B_{\mathbb{C}^2}(0,\varepsilon),\ z \in \mathcal{H}_c(U) \cap B_{H^{s'}}(0, r(U)),\right.$$
$$\left. p \in B_{\mathbb{R}^8}(0, r(U)),\ \text{with } p_i = 0 \text{ for } i = 1, \ldots, 4 \right\}$$

and

$$\widetilde{S_-} = \left\{ (U, z, p)\,;\ U \in B_{\mathbb{C}^2}(0,\varepsilon),\ z \in \mathcal{H}_c(U) \cap B_{H^{s'}}(0, r(U)),\right.$$
$$\left. p \in B_{\mathbb{R}^8}(0, r(U)),\ \text{with } p_i = 0 \text{ for } i = 5, \ldots, 8 \right\}$$

endowed with the metric of $\mathbb{C}^2 \times H^{s'} \times \mathbb{R}^8$*, there exist neighborhoods* $\mathcal{W}_\pm$ *of* $(0,0,0)$ *in* $\tilde{S}_\pm$ *and smooth maps* $\Phi_\pm : \mathcal{W}_\pm \mapsto \mathbb{C}^4$ *with* $\|\Phi_\pm(0,0,0)\| = \|D\Phi_\pm(0,0,0)\| = 0$*, and* $(\Phi_+(\cdot,\cdot,p))_i = p_i$ *if* $i = 5, \ldots, 8$*,* $(\Phi_-(\cdot,\cdot,p))_i = p_i$ *if* $i = 1, \ldots, 4$*, such that*

(a) if $A \in \Phi_\pm(\mathcal{W}_\pm)$ *and* $A \notin \Psi(\mathcal{V})$ *then there exist* $t_\pm(\psi_0) > 0$ *and a unique solution* ψ *of* (1.2) *with initial condition* ψ_0 *such that for any small neighborhood* $\mathcal{O}$ *of* $S(U_0)$ *containing* ψ_0*,* $\phi \in \mathcal{C}([-t_+; +\infty), H^{s'}) \cap \mathcal{C}^1((-t_+; +\infty), H^{s'-1})$ *(resp.* $(-\infty; t_-]$*) and there exist* $C > 0$*,* $\psi(t) \in \mathcal{CM}$ *and* $\rho(t) \in X_s(U_0)$ *(resp.* $\rho(t) \in X_u(U_0)$*)) for all* $t > -t_-$ *(resp for all* $t < t_+$*) such that* $\phi(t) = \psi(t) + \rho(t)$ *with*

$$\|\rho(t)\|_{H^{s'}} \leq C\,\|\rho(0)\|_{H^{s'}}\, e^{\mp\gamma t} \text{ as } t \to \pm\infty \quad \text{and} \quad \psi(\mp t_\pm) \notin \mathcal{O}$$

where γ *is in a ball around* $1/2\Gamma(S(U_0))$*, the radius of which is* $O(|U_0|^6)$*), moreover this solution is unique in* $L^\infty((-T', T), H^{s'}(\mathbb{R}^3, \mathbb{C}^4))$ *for any* $T > 0$ *(resp* $T \in (0, t_-)$*) and any* $T' \in (0, t_+)$ *(resp* $T' < 0$*) and there exist* $(U_{\pm\infty}; z_{\pm\infty}) \in S$ *such that*

$$|U_{\pm\infty} - U_0| \leq C\left(\|z_0\|_{H^{s'}} + \|\rho(0)\|_{H^{s'}}\right)^2,\ \|z_{\pm\infty} - z_0\|_{H^{s'}} \leq C\,|U_0|\left(\|z_0\|_{H^{s'}} + \|\rho(0)\|_{H^{s'}}\right),$$

and for all $t > -t_-$

$$\psi(t) = e^{-\mathrm{i}\int_0^t E(U_+(v))\,dv} S(U_+(t)) + e^{JtE(U_{+\infty})} e^{JtH(U_{+\infty})} z_{+\infty} + \varepsilon_+(t)$$

(resp for all $t < t+$

$$\psi(t) = e^{-\mathrm{i}\int_0^t E(U_-(v))\,dv} S(U_-(t)) + e^{JtE(U_{-\infty})} e^{JtH(U_{-\infty})} z_{-\infty} + \varepsilon_-(t))$$

with $\dot{U}_\pm \in L^q((-t_-, +\infty))$ *(resp.* $L^q((-\infty, t_+))$*) for all* $q \in [1, \infty]$, $\lim_{t\to\pm\infty} U_\pm(t) = U_{\pm\infty}$,

$$\max\left\{ \|\varepsilon_+\|_{L^\infty((-t_-,+\infty),H^{s'})},\ \|\varepsilon_+\|_{L^2((-t_-,+\infty),H^s_{-\sigma})}, \|\varepsilon_+\|_{L^2((-t_-,+\infty),B^\beta_{\infty,2})} \right\} \leq C\,|U_0|\,(\|z_0\|_{H^{s'}} + \|\rho(0)\|_{H^{s'}}),$$

(resp.

$$\max\left\{ \|\varepsilon_-\|_{L^\infty((-\infty,t_+),H^{s'})},\ \|\varepsilon_-\|_{L^2((-\infty,t_+),H^s_{-\sigma})}, \|\varepsilon_-\|_{L^2((-\infty,t_+),B^\beta_{\infty,2})} \right\} \leq C\,|U_0|\,(\|z_0\|_{H^{s'}} + \|\rho(0)\|_{H^{s'}}))$$

and

$$\lim_{t\to\pm\infty} \|\varepsilon_\pm(t)\|_{H^{s'}} = 0;$$

(b) if $A \notin \Phi_+(\mathcal{W}^+) \cup \Phi_-(\mathcal{W}_-)$ *then there exist* $t_+(\psi_0) > 0$, $t_-(\psi_0) < 0$ *and a unique solution* ψ *of* (1.2) *with initial condition* ψ_0 *such that for any small neighborhood* $\mathcal{O}$ *of* $S(U_0)$ *containing* ψ_0, $\phi \in \mathcal{C}([t_-; t_+], H^{s'}) \cap \mathcal{C}^1((t_-; t_+), H^{s'})$ *with* $\psi(t_+) \notin \mathcal{O}$ *and* $\psi(t_-) \notin \mathcal{O}$, *moreover this solution is unique in* $L^\infty((T', T), H^{s'}(\mathbb{R}^3, \mathbb{C}^4))$ *for any* $T \in (0, t_+)$ *and any* $T' \in (t_-, 0)$.

The first part of this theorem shows, as in [Bou06], that perturbations in the direction of the continuous subspace, except eight (real) directions, relax towards a stationary solution. We have excluded eight directions in the continuous subspace, which, due to resonance phenomena, induce orbital instability. The second part of the theorem tells us what happens for perturbations in the directions of an excited state and in the eight directions of the continuous spectrum for which we haven't proved stabilization. We thus study sixteen directions: eight of them give a manifold on which there holds exponential stabilization in positive time and orbital instability in negative time, while the eight others give a manifold on which there holds exponential stabilization in negative time and orbital instability in positive time. Outside these manifolds, we have orbital instability in both negative and positive time.

Remark 1.1. *One notice that in* (ib) *and* (iia), *we obtain an asymptotic profile of the form*

$$e^{-\mathrm{i}(tE(U_{\pm\infty}))}\left(S(U_{\pm\infty}) + e^{JtH(U_{\pm\infty})} z_{\pm\infty}\right)$$

with a convergence in $H^{s'}$.

In fact, we also have in the previous expansions

$$\max\left\{ \left\|e^{JtH(U_{\pm\infty})} z_{\pm\infty}\right\|_{L^\infty(\mathbb{R}^\pm,H^{s'})},\ \left\|e^{JtH(U_{\pm\infty})} z_{\pm\infty}\right\|_{L^2(\mathbb{R}^\pm,H^s_{-\sigma})}, \left\|e^{JtH(U_{\pm\infty})} z_{\pm\infty}\right\|_{L^2(\mathbb{R}^\pm,B^\beta_{\infty,2})} \right\} \leq C\,(\|z_0\|_{H^{s'}} + \|\rho(0)\|_{H^{s'}})).$$

This follows from Theorem 1.1 and Theorem 1.2 applied to the Duhamel formula :

$$e^{JtH(U_{\pm\infty})} z_{\pm\infty} = e^{-\mathrm{i}t(H - E(U_{\pm\infty}))} z_{\pm\infty} + \int_0^t e^{-\mathrm{i}(t-s)(H-E(U_{\pm\infty}))} Jd^2\nabla F(S(U_{\pm\infty})) e^{JsH(U_{\pm\infty})} z_{\pm\infty}\,ds.$$

We can obtain an expansion of the form $\psi(t) = e^{-\mathrm{i}(tE(U_{\pm\infty}))} S(v_\pm) + e^{-\mathrm{i}tD_m} z_\pm + \varepsilon_\pm(t)$ *with the same conclusion but we will only have* $\|z_{\pm\infty} - z_0\|_{H^{s'}} \leq C\|z_0\|_{H^{s'}}$ *and*

$$\max\left\{ \|\varepsilon_\pm\|_{L^\infty(\mathbb{R}^\pm,H^{s'})},\ \|\varepsilon_\pm\|_{L^2(\mathbb{R}^\pm,H^s_{-\sigma})}, \|\varepsilon_\pm\|_{L^2(\mathbb{R}^\pm,B^\beta_{\infty,2})} \right\} \leq C\|z_0\|_{H^{s'}},$$

or of the form $\psi(t) = e^{-\mathrm{i}(tE(U_{\pm\infty}))}S(U_\pm(t)) + e^{-\mathrm{i}tH}z_\pm + \varepsilon_\pm(t)$ *or* $\psi(t) = e^{-\mathrm{i}(tE(U_{\pm\infty}))}S(v_\pm) + e^{-\mathrm{i}tD_m}z_\pm + \varepsilon_\pm(t)$ *with* $\|z_{\pm\infty} - z_0\|_{H^{s'}} \leq C\,|U_0|\,\|z_0\|_{H^{s'}}$ *and*

$$\max\left\{ \|\varepsilon_\pm\|_{L^\infty(\mathbb{R}^\pm, H^{s'})}\,,\ \|\varepsilon_\pm\|_{L^2(\mathbb{R}^\pm, H^s_{-\sigma})}\,, \|\varepsilon_\pm\|_{L^2(\mathbb{R}^\pm, B^\beta_{\infty,2})} \right\} \leq C\,|U_0|\,\|z_0\|_{H^{s'}}.$$

We notice that when $U_0 = 0$ *then* $z_0 = 0$ *and* $p = 0$ *so the theorem do not say anything for this case. In fact, the charge conservation gives the orbital stability of* 0*. But one cannot obtain asymptotic stability since we can build a manifold of stationary states tangent to the eigenspace associated with* λ_1 *similarly to Proposition 1.1.*

2 Linearized operator and exponentially stable and unstable manifolds

We study the dynamics associated with (1.2) around a stationary state. We will use spectral properties of the linearized operator around a stationary state.

2.1 The spectrum of the linearized operator

Here we study the spectrum of the linearized operator associated with Equation (1.2) around a stationary state $S(U)$. Let us introduce

$$H(U) = H + d^2F(S(U)) - E(U)$$

where d^2F is the differential of ∇F. The operator $H(U)$ is $\mathbb{R}-$linear but not $\mathbb{C}-$linear. Replacing $L^2(\mathbb{R}^3, \mathbb{C}^4)$ by $L^2(\mathbb{R}^3, \mathbb{R}^4 \times \mathbb{R}^4)$ with the inner product obtained by taking the real part of the inner product of $L^2(\mathbb{R}^3, \mathbb{C}^4)$, we obtain a symmetric operator. We then complexify this real Hilbert space and obtain $L^2(\mathbb{R}^3, \mathbb{C}^4 \times \mathbb{C}^4)$ with its canonical hermitian product. This process transforms the operator $-\mathrm{i}$ into

$$J = \begin{pmatrix} 0 & Id_{\mathbb{C}^4} \\ -Id_{\mathbb{C}^4} & 0 \end{pmatrix}.$$

For $\phi \in L^2(\mathbb{R}^3, \mathbb{R}^4 \times \mathbb{R}^4) \subset L^2(\mathbb{R}^3, \mathbb{C}^4 \times \mathbb{C}^4)$, we still write ϕ instead of

$$\begin{pmatrix} \Re\phi \\ \Im\phi \end{pmatrix}.$$

The extension of $H(U)$ over $L^2(\mathbb{R}^3, \mathbb{C}^4 \times \mathbb{C}^4)$ is also written $H(U)$ and is now a real operator. The extension of K is also written K.

The linearized operator associated with Equation (1.2) around the stationary state $S(U)$ is given by $JH(U)$. We shall now study its spectrum.

Differentiating (1.4), we have that for $U = (u_1, u_2)$

$$\mathcal{H}_0(u_1, u_2) = \mathrm{Span}\left\{ \frac{\partial}{\partial \Re u_1} S(u_1, u_2),\ \frac{\partial}{\partial \Im u_1} S(u_1, u_2), \frac{\partial}{\partial \Re u_2} S(u_1, u_2),\ \frac{\partial}{\partial \Im u_2} S(u_1, u_2) \right\}$$

is invariant under the action of $JH(U)$. Using the gauge invariance, we notice that $S(U) \in \mathcal{H}_0(U)$ and differentiating (1.4), we obtain for any $\beta \in \mathbb{N}^4$ with $|\beta| = 1$:

$$JH(U)JS(U) = 0 \quad \text{and} \quad JH(U)\partial_U^\beta S(U) = (\partial_U^\beta E)(U)JS(U).$$

The space $\mathcal{H}_0(U)$ is contained in the geometric null space of $JH(U)$, in fact it is exactly the geometric null space as proved in the sequel.

Now we state our results on the spectrum of $JH(U)$. The first deals with the excited states part. We have the

Proposition 2.1. *If Assumptions 1.1–1.5 hold. Then there exists a map* $E_1 :\ B_{\mathbb{C}^2}(0, \varepsilon) \mapsto\ \mathbb{R}$ *with*

$$\begin{cases} \Im E_1(U) = (\lambda_1 - E(U)) + O(|U|^4) \\ \Re E_1(U) = 1/2\Gamma(S(U)) + O(|U|^6). \end{cases}$$

such that $E_1(U)$, $\overline{E_1}(U)$, $-E_1(U)$ *and* $-\overline{E_1}(U)$ *are double eigenvalue of* $JH(U)$. *For any* $s \in \mathbb{R}$, *there exist smooth maps* $k^{\pm}:\ B_{\mathbb{C}^2}(0,\varepsilon) \mapsto \left\{ \begin{pmatrix} \phi_1 \\ -i\phi_1 \end{pmatrix} \right\}^{\perp} \cap H^s$ *such that*

$$k_{\pm}(U) - ((H - E(U)) + \mathrm{i}E_1(U))^{-1} P_c(H) d^2F(S(U))|U| \left(\frac{u_1}{|U|} Id_{\mathbb{C}^4} + \frac{u_2}{|U|} K \right) \begin{pmatrix} \phi_1 \\ -i\phi_1 \end{pmatrix} \tag{2.1}$$

is $o(|U|^2)$ *in* $B(L^2_{-\sigma})$ *for any* $\sigma \in \mathbb{R}$. *For*

$$\Phi_{\pm}(U) = |U| \left(\frac{u_1}{|U|} Id_{\mathbb{C}^4} + \frac{u_2}{|U|} K \right) \begin{pmatrix} \phi_1 \\ -i\phi_1 \end{pmatrix} + k_{\pm}(U),$$

we have

- $\{\Phi_+(U), K\Phi_+(U)\}$ *is a basis of the eigenspace associated with* $E_1(U)$,
- $\left\{\overline{\Phi_+(U)}, K\overline{\Phi_+(U)}\right\}$ *is a basis of the eigenspace associated with* $\overline{E_1(U)}$,
- $\{\Phi_-(U), K\Phi_-(U)\}$ *is a basis of the eigenspace associated with* $-E_1(U)$,
- $\left\{\overline{\Phi_-(U)}, K\overline{\Phi_-(U)}\right\}$ *is a basis of the eigenspace associated with* $-\overline{E_1(U)}$.

Moreover for any $\beta \in \mathbb{N}^4$, $s \in \mathbb{R}^+$ *and* $p, q \in [1, \infty]$. *There is* $\gamma > 0$, $\varepsilon > 0$ *and* $C > 0$ *such that for all* $U \in B_{\mathbb{C}^2}(0,\varepsilon)$, *for*

$$\xi(U) \in \left\{ \Phi_+(U), K\Phi_+(U), \overline{\Phi_+(U)}, K\overline{\Phi_+(U)}, \Phi_-(U), K\Phi_-(U), \overline{\Phi_-(U)}, K\overline{\Phi_-(U)} \right\}$$

one has

$$\|e^{\gamma\langle Q\rangle} \partial_U^{\beta} \xi(U)\|_{B^s_{p,q}} \leq C \|S(U)\|_2, \tag{2.2}$$

where $\partial^{\beta}_{u_1,u_2} = \frac{\partial^{|\beta|}}{\partial^{\beta_1} \Re u_1 \partial^{\beta_2} \Im u_1 \partial^{\beta_1} \Re u_2 \partial^{\beta_2} \Im u_2}$.

Proof. Using Weyl's sequences, we prove that the essential spectrum of $JH(U)$, for small U, is the essential spectrum of JH. So z with non zero real part is in the spectrum of $JH(U)$ if and only if it is an isolated eigenvalue. The equation to solve for excited states is:

$$(JH(U) - z)\,\phi = 0. \tag{2.3}$$

Since the proof is similar for all cases, we restrict the study to solutions the form $\phi = S_1 + \eta$ where S_1 is the normalized eigenvector:

$$S_1 = \begin{pmatrix} \phi_1 \\ -i\phi_1 \end{pmatrix}$$

and $\eta \in \{S_1\}^{\perp}$, the orthogonal relation is taken in fact with respect to J, but since $JS_1 = \mathrm{i}S_1$, we can take it in the standard way. We obtain the equation

$$\eta = (JH - z)^{-1} P_1^{\perp} W(U) \{S_1 + \eta\} \tag{2.4}$$

with $P_1^{\perp}$ the orthogonal projector, with respect to J, into $\{S_1\}^{\perp}$ and $W(U) = JH(U) - J(H - E(U))$. We remark that $\{S_1\}^{\perp}$ is invariant under the action of $J(H - E(U))$. To solve this equation in η for a fixed u and z, we notice that if

$$\Re z > 0 \text{ and } |\Im z| \geq m,$$

the series

$$k(U,z) = (JH - z)^{-1} P_1^{\perp} \sum_{k \geq 0} \left(-W(U) (JH - z)^{-1} P_1^{\perp} \right)^k W(U) S_1,$$

is convergent for sufficiently small $|U|$ using the Limiting Absorption Principle (1.1). Hence, we have a solution of (2.4).

Then we solve the equation in z, we obtain from Equation (2.3) the equation

$$\langle (JH(U) - z)\,\phi, S_1\rangle = 0.$$

With $\phi = S_1 + k(U,z)$, we infer

$$\begin{aligned} z &= \langle JH(U)S_1, S_1\rangle + \langle JH(U)k(U,z), S_1\rangle \\ &= \mathrm{i}\,(\lambda_1 - \lambda_0) + \langle W(U)S_1, S_1\rangle \\ &\quad + \sum_{k\geq 0} \left\langle JH(U)\,(JH - z)^{-1}\,P_1^{\perp}\left(-W(U)\,(J(H - E(U)) - z)^{-1}\,P_1^{\perp}\right)^k W(U)S_1, S_1\right\rangle \\ &= \mathrm{i}\,(\lambda_1 - \lambda_0) \\ &\quad + \langle W(U)S_1, S_1\rangle + \sum_{k\geq 0}\left\langle P_1^{\perp}\left(-W(U)\,(J(H - E(U)) - z)^{-1}\,P_1^{\perp}\right)^k W(U)S_1, S_1\right\rangle \\ &\qquad + \sum_{k\geq 0}\Bigg\langle (W(U) - z)\,(J(H - E(U)) - z)^{-1}\,P_1^{\perp}\times \\ &\qquad\quad \times\left(-W(U)\,(J(H - E(U)) - z)^{-1}\,P_1^{\perp}\right)^k W(U)S_1, S_1\Bigg\rangle. \end{aligned}$$

Since $P_1^{\perp}S_1 = 0$, we introduce the function

$$\begin{aligned} f(z) &= \mathrm{i}\,(\lambda_1 - \lambda_0) + \langle W(U)S_1, S_1\rangle \\ &\quad + \sum_{k\geq 0}\left\langle W(U)\,(J(H - E(U)) - z)^{-1}\,P_1^{\perp}\left(-W(U)\,(J(H - E(U)) - z)^{-1}\,P_1^{\perp}\right)^k W(U)S_1, S_1\right\rangle. \end{aligned}$$

Since $JS_1 = -\mathrm{i}S_1$ we obtain that $\Re\,\langle W(U)S_1, S_1\rangle = 0$, so with the Limiting Absorption Principle (1.1), we have

$$\begin{aligned} \Re f(z) &= \Re\left\langle W(U)\,(J(H - E(U)) - z)^{-1}\,P_1^{\perp}W(U)S_1, S_1\right\rangle + O(|U|^6) \\ &= \Im\left\langle d^2F(S(U))\,((H - E(U)) + zJ)^{-1}\,P_1^{\perp}d^2F(S(U))S_1, S_1\right\rangle + O(|U|^6) \end{aligned}$$

using (1.5) and

$$((H - E(U)) + zJ)^{-1} = \frac{1}{2}\left(((H - E(U)) - \mathrm{i}z)^{-1}\,(I_{C^2} + \mathrm{i}J) + ((H - E(U)) + \mathrm{i}z)^{-1}\,(I_{C^2} - \mathrm{i}J)\right),$$

we obtain

$$\begin{aligned} &\Im\left\langle d^2F(S(U))\,((H - E(U)) + zJ)^{-1}\,P_1^{\perp}d^2F(S(U))S_1, S_1\right\rangle \\ &\qquad = \frac{1}{2}\Im\left\langle d^2F(S(U))\,((H - E(U)) - \mathrm{i}z)^{-1}\,d^2P_1^{\perp}F(S(U))S_1, S_1\right\rangle \\ &\qquad + \frac{1}{2}\Im\left\langle d^2F(S(U))\,((H - E(U)) + \mathrm{i}z)^{-1}\,P_1^{\perp}d^2F(S(U))S_1, S_1\right\rangle \\ &\qquad\qquad - \Im\left\langle d^2F(S(U))\left((H - E(U))^2 + z^2\right)^{-1} zJP_1^{\perp}d^2F(S(U))S_1, S_1\right\rangle, \end{aligned}$$

and so

$$\begin{aligned} &\Im\left\langle d^2F(S(U))\,((H - E(U)) + (\mathrm{i}(\lambda_1 - \lambda_0) + 0)\,J)^{-1}\,P_1^{\perp}d^2F(S(U))S_1, S_1\right\rangle \\ &\qquad = \frac{1}{2}\Im\left\langle d^2F(S(U))\,((H - E(U)) + (\lambda_1 - \lambda_0) - \mathrm{i}0)^{-1}\,P_{\mathrm{c}}(H)d^2F(S(U))S_1, S_1\right\rangle \end{aligned}$$

Using Assumption 1.5, the local limiting absorption principle (1.1) and regularity results of [GM01][Theorem 1.7], we obtain

$$\Re f(z) = 1/2\Gamma(S(U)) + O(|U|^6)$$

for z in a ball of radius of order $|U|^2$ around $\mathrm{i}\,(\lambda_1-\lambda_0)$ and for small U. We also prove by the same way

$$\Im f(z) = (\lambda_1-\lambda_0) + O(|U|^4)$$

for z in a ball of radius of order $|U|^2$ around $\mathrm{i}\,(\lambda_1-\lambda_0)$ and for small U.

So we have proved that for sufficiently small U, f leaves a ball around $\mathrm{i}\,(\lambda_1-\lambda_0)+1/2\Gamma(S(U))$ invariant and is a contraction. Therefore, we have a fixed point $E_1(U)$ of each U. Then we choose $k_+(U)=k(U,E_1(U))$. Using the complex conjugation, we obtain the eigenvalue $\overline{E_1(U)}$ and its associated eigenspace.

To obtain $-E_1(U)$ and $-\overline{E_1(U)}$, we notice that $E_1(U)$ and $\overline{E_1(U)}$ are eigenvalues of $(JH(U))^*$. Hence using the symmetry : $J(JH(U))=-(JH(U))^*J$, we show that any eigenvector ϕ of $(JH(U))^*$ associated with λ, $J\phi$ is an eigenvector of $JH(U)$ associated with $-\lambda$ and this gives k_-.

The exponential decay works like in Lemma 1.1 □

We introduce $\mathcal{H}_c^r(U)$ is the intersection of $\mathcal{H}_c$ with the orthogonal with respect to J of

$$\left\{\Im\left(H-E(U)+\lambda_1-\lambda_0+\frac{\mathrm{i}}{2}\Gamma(S(U))\right)d^2F(S(U))|V|\left(\frac{v_1}{|V|}I_{\mathbb{C}^4}+\frac{v_2}{|V|}K\right)\begin{pmatrix}\phi_1\\ -i\phi_1\end{pmatrix},\forall V=(v_1,v_2)\in\mathbb{C}^2\right\},$$

and $P_c^r(U)$ the orthogonal projector with respect to J onto $\mathcal{H}_c^r(U)$. The following result deals with the essential spectrum of our linearized operator.

Proposition 2.2. *If Assumptions 1.1–1.5 hold. For any sufficiently small non zero $U\in\mathbb{C}^2$, let*

$$\mathcal{H}_1(U)=\mathrm{span}\left\{\Phi_+(U),K\Phi_+(U),\overline{\Phi_+(U)},K\overline{\Phi_+(U)},\Phi_-(U),K\Phi_-(U),\overline{\Phi_-(U)},K\overline{\Phi_-(U)}\right\}.$$

The orthogonal space of $\mathcal{H}_0(U)\oplus\mathcal{H}_c(U)$ with respect to the product associated to J

$$\mathcal{H}_c(U)=\{\mathcal{H}_0(U)\oplus\mathcal{H}_1(U)\}^{\perp}$$

is invariant under the action of $JH(U)$.

We have for $U'\in B_{\mathbb{C}^2}(U,\varepsilon)$, with sufficiently small $\varepsilon>0$, that

$$\mathbf{P}_c^r(U)|_{\mathcal{H}_c(U')}:\mathcal{H}_c(U')\mapsto\mathcal{H}_c^r(U)$$

is an isomorphism and is a bounded operator from $H^s_\sigma(\mathbb{R}^3,\mathbb{C}^8)$ or $B^s_{p,q}(\mathbb{R}^3,\mathbb{C}^8)$ to itself for any reals s and σ and $p,q\in[1,\infty]$, the inverse $R^r(U',U)$ is continuous with respect to U and U' for these norms.

We also have for $\mathbf{P}_c(U)$, the orthogonal projector onto $\mathcal{H}_c(U)$ with respect to J, and for $U'\in B_{\mathbb{C}^2}(U,\varepsilon)$, with sufficiently small $\varepsilon>0$, that

$$\mathbf{P}_c(U)|_{\mathcal{H}_c(U')}:\mathcal{H}_c(U')\mapsto\mathcal{H}_c(U)$$

is an isomorphism and is a bounded operator from $H^s_\sigma(\mathbb{R}^3,\mathbb{C}^8)$ or $B^s_{p,q}(\mathbb{R}^3,\mathbb{C}^8)$ to itself for any reals s and σ and $p,q\in[1,\infty]$, the inverse $R(U',U)$ is continuous with respect to U and U' for these norms.

Moreover, we have

$$\int_{\mathbb{R}}\|\langle Q\rangle^{-\sigma}e^{sJH(U)}\mathbf{P}_c(U)\psi\|^2\,ds\le C\|\psi\|_2^2,\ \forall\psi\in L^2. \tag{2.5}$$

and $\mathcal{H}_c(U)$ contains no eigenvector. We also have

$$\|e^{tJH(U)}\mathbf{P}_c(U)\psi\|\le C\|\psi\|_2,\ \forall t\in\mathbb{R},\ \forall\psi\in L^2. \tag{2.6}$$

Proof. We prove that there is no other eigenvector, by proving that smoothness estimate (2.5) takes place over

$$\mathcal{H}_c(U)=\{\mathcal{H}_0(U)\oplus\mathcal{H}_1(U)\}^{\perp}.$$

First we prove that

$$\mathbf{P}_c((U))|_{\mathcal{H}_c(U')}:\mathcal{H}_c(U')\mapsto\mathcal{H}_c(U)$$

is an isomorphism. To prove it, we exhibit an inverse $R(U',U)$ which is of the form

$$R(U',U) = Id + \sum_i |\alpha_i(U',U)\rangle \langle J\xi_i(U)|$$

where $\xi_i(U)$ is a basis of the eigenspaces of $JH(U)$ and $\alpha_i(U',U)$ solve the equations

$$J\xi_j(U') + \sum_i \langle J\xi_i(U), J\xi_j(U')\rangle \, \alpha_i(U',U) = 0.$$

Such α exist because this constraint reduce to invert a Gramm matrix when $U = U'$ and otherwise a small perturbation of such matrices for U and U' close one to each other.The boundedness of R follows from the exponential decay of eigenvectors and their derivatives, the continuity of R follows from the continuity of the eigenvector with respect to the parameters U and U' see Proposition 2.1, the paragraph above and Proposition 1.1.

The statement for $\mathbf{P}_c(U)^r|_{\mathcal{H}_c(U')}$ follows by the same way.

Therefore since $R(U,U)\mathbf{P}_c^r(U)\mathbf{P}_c(U) = \mathbf{P}_c(U)$.

$$\begin{aligned}
&\|\langle Q\rangle^{-\sigma} e^{tJH(U)}\mathbf{P}_c(U)\psi\| \\
&= C\|\langle Q\rangle^{-\sigma} R(U,U)\mathbf{P}_c^r(U)e^{tJH(U)}\mathbf{P}_c(U)\psi\| \\
&\leq \|\langle Q\rangle^{-\sigma} R(U,U)\mathbf{P}_c^r(U)e^{-\mathrm{i}t(H-E(U))}\mathbf{P}_c(U)\psi\| \\
&\quad + \int_0^t \|\langle Q\rangle^{-\sigma} R(U,U)\mathbf{P}_c^r(U)e^{-\mathrm{i}(t-s)(H-E(U))}D\nabla F(S(U))e^{sJH(U)}\mathbf{P}_c(U)\psi\|\, ds \\
&\leq \|\langle Q\rangle^{-\sigma} R(U,U)\langle Q\rangle^{\sigma}\|\|\langle Q\rangle^{-\sigma}\mathbf{P}_c^r(U)e^{-\mathrm{i}t(H-E(U))}\mathbf{P}_c(U)\psi\| \\
&\quad + C|z|^2\|\langle Q\rangle^{-\sigma} R(U,U)\langle Q\rangle^{\sigma}\| \times \\
&\quad \times \int_0^t \|\langle Q\rangle^{-\sigma}\mathbf{P}_c^r(U)e^{-\mathrm{i}(t-s)(H-E(U))}\langle Q\rangle^{-\sigma}\|\|\langle Q\rangle^{-\sigma}e^{sJH(U)}\mathbf{P}_c(U)\psi\|\, ds. \qquad (2.7)
\end{aligned}$$

Using estimate (i), we obtain the estimate (2.5) for U sufficiently small:

$$\int_{\mathbb{R}} \|\langle Q\rangle^{-\sigma} e^{-sJH(U)}\mathbf{P}_c(U)\psi\|^2\, ds \leq C\|\psi\|^2.$$

Hence there is no eigenvector in the range of $P_c(U)$.

Using the inequalities (2.7) and the conservation law for H, we prove the estimate (2.6). □

Since $\mathrm{codim}\mathcal{H}_c(U) = \dim\{\mathcal{H}_0(U)\oplus\mathcal{H}_1(U)\}$ and $\mathcal{H}_c(U)\cap\{\mathcal{H}_0(U)\oplus\mathcal{H}_1(U)\} = \emptyset$, we obtain $\mathcal{H}_0(U)\oplus\mathcal{H}_1(U)\oplus\mathcal{H}_c(U) = L^2(\mathbb{R}^3,\mathbb{C}^8)$ and the

Proposition 2.3. *Suppose that Assumptions 1.1–1.3 and 1.5 hold. Then the space $\mathcal{H}_0(U)$ is the geometric null space of $JH(U)$.*

2.2 Stable, unstable and center manifold

We can now obtain results similar to those of Bates and Jones [BJ89].

We have that $JH(U)$ as an operator in $L^2(\mathbb{R}^3,\mathbb{R}^8)$ is a closed densely defined operator that generates a continuous semigroup on $L^2(\mathbb{R}^3,\mathbb{R}^8)$. The spectrum of $JH(U)$ in $L^2(\mathbb{R}^3,\mathbb{R}^8)$ is the same as $JH(U)$ in $L^2(\mathbb{R}^3,\mathbb{C}^8)$ and so it splits in three parts:

$$\begin{aligned}
\sigma_s(U) &= \{\lambda\in\sigma(JH(u)),\ \Re\lambda<0\} = \left\{-E_1(U), -\overline{E_1(U)}\right\} \\
\sigma_c(U) &= \{\lambda\in\sigma(JH(u)),\ \Re\lambda=0\} = \mathrm{i}\,\{\mathbb{R}\setminus(-E(U),E(U))\} \\
\sigma_u(U) &= \{\lambda\in\sigma(JH(u)),\ \Re\lambda>0\} = \left\{E_1(U), \overline{E_1(U)}\right\}
\end{aligned}$$

each one is associated with a spectral real subspace

$$\begin{aligned}
X_s(U) &= \mathrm{span}_{\mathbb{R}}\{\Re\Phi_-(U), \Im\Phi_-(U), K\Re\Phi_-(U), K\Im\Phi_-(U)\} \\
X_u(U) &= \mathrm{span}_{\mathbb{R}}\{\Re\Phi_+(U), \Im\Phi_+(U), K\Re\Phi_+(U), K\Im\Phi_+(U)\} \\
X_c(U) &= \Re\mathcal{H}_0(U)\oplus\Re\mathcal{H}_c(U)
\end{aligned}$$

where we used the notation $\Re\Psi = (1/2)\left(\Psi + \overline{\Psi}\right)$ and $\Im\Psi = -(\mathrm{i}/2)\left(\Psi - \overline{\Psi}\right)$ and $\Re X = \{\Re\Psi, \Psi \in X\}$. The spaces $X_s(U)$ and $X_u(U)$ are finite dimensional. Let us write $\pi^c(U)$, $\pi^s(U)$ and $\pi^u(U)$ for the projector associated with the decomposition $X_c(U) \oplus X_s(U) \oplus X_u(U)$. Since the eigenvectors belongs also to L^2_σ for any $\sigma \in \mathbb{R}$, the projector $P_c(U)$ and $\pi^c(U)$ can be defined in L^2_σ for any real σ. We can hence defined an extend the spaces $\mathcal{H}_c(U)$ and $X_c(U)$ to L^2_σ for any $\sigma \in \mathbb{R}$. We have the

Lemma 2.1. *If Assumptions 1.1–1.5 hold. Then there exist $C_1, C_2 > 0$ such that for all $t \in \mathbb{R}$, we have*

$$C_1 e^{-\gamma(U)t} \leq \left\| e^{tJH(U)}\pi^s(U) \right\|_{\mathcal{B}(L^2_\sigma)} \leq C_2 e^{-\gamma(U)t}, \tag{2.8}$$

$$C_1 e^{\gamma(U)t} \leq \left\| e^{tJH(U)}\pi^u(U) \right\|_{\mathcal{B}(L^2_\sigma)} \leq C_2 e^{\gamma(U)t}, \tag{2.9}$$

$$\left\| e^{tJH(U)}\pi^c(U) \right\|_{\mathcal{B}(L^2_\sigma)} \leq C_2 \langle t \rangle^r, \tag{2.10}$$

for some power r, for any $\sigma \in \mathbb{R}$ and where $\gamma(U) = \Re E_1(U)$.

Proof. The statements for the spaces $X^s(U)$ and $X^u(U)$ follows from (2.2).

The statement about $X^c(U)$ is a few more complicate. We are not looking for an optimal r.

First, let us prove the result for $e^{-\mathrm{i}t(D_m+V)}$ in L^2_σ with $\sigma \in 2\mathbb{N}$, the general case follows by duality and interpolation. In fact, the result in this case follows from iterating several times the proof of [Tha92, Theorem 1.3], which is based on the charge conservation.

Then for $e^{tJH(U)}\pi^c(U)$ follows by the same way using also the charge conservation of $e^{tJH(U)}P_c(U)$ (see (2.6)) and the fact that $e^{tJH(U)}S(U) = S(U)$, $e^{tJH(U)}\partial_U^\beta S(U) = \partial_U^\beta S(U) + t\partial_U^\beta E(U)S(U)$ and Lemma 1.1. □

By now we do not restrict to the norm of $L^2(\mathbb{R}^3, \mathbb{R}^8)$, we extend our study to $L^2_\sigma(\mathbb{R}^3, \mathbb{R}^8)$ for any $\sigma \in \mathbb{R}$, but we still write $\mathcal{H}_c(U)$ and $X_c(U)$ for the extensions of these spaces to $L^2_\sigma(\mathbb{R}^3, \mathbb{R}^8)$ for any $\sigma \in \mathbb{R}$.

We now study the behavior of the solutions in L^2_σ of (1.2) centered around $S(U)$:

$$\partial_t \phi = JH(U)\phi + JN(U, \phi) \tag{2.11}$$

where $H(U) = H + d^2F(S(U)) - E(U)$ and $N(U, \phi) = \nabla F(S(U) + \phi) - \nabla F(S(U)) - d^2F(S(U))\phi$ and d^2F is the differential of ∇F.

In this subsection, we study a modified equation which coincides with (2.11) as long as the solutions stays small:

$$\partial_t \phi = JH(U)\phi + JN_\varepsilon(U, \phi) \tag{2.12}$$

where $N_\varepsilon(U, \eta) = \rho(\varepsilon^{-1}\eta)N(U, \eta)$ and ρ is a smooth function with compact support around 0.

We state the

Proposition 2.4 (Center-Stable Manifold). *If Assumptions 1.1–1.5 hold. Then for any sufficiently small non zero U, there exists around $S(U)$ a unique invariant smooth center-stable manifold $W^{cs}(U)$ for (2.12) build as a graph and tangent to $S(U) + X_c(U) \oplus X_s(U)$ at $S(U)$.*

Any solution ϕ of (2.12) in the neighborhood of $S(U)$ tends as $t \to -\infty$ to $W^{cs}(U)$ with

$$dist_{H^s_\sigma}(\phi(t), W^{cs}(U)) = O(e^{\gamma t}) \text{ as } t \to -\infty$$

for any $\gamma \in (0, \gamma(U))$, any $s, \sigma \in \mathbb{R}$ and for any sufficiently small neighborhood V of $S(U)$ any solution in V not in $W^{cs}(U)$ leaves V in finite positive time.

Remark 2.1. *Even if $\phi \notin H^s_\sigma$, there exist $\psi \in W^{cs}(U)$ such that $\phi - \psi \in H^s_\sigma$ as shown in the following proof.*

If we only consider small solutions, we obtain a locally invariant manifold for the equation (2.11), that is to say that for any initial condition in the manifold there exist an associated solution of (2.11) which stays in this manifold in a small interval of time around 0. We notice that in the following proofs the size of this invariant manifold, which is given by ε, is a function of U and this function is $O(\gamma(U))$. By now, we call this function r.

Proof. Our proof is an adaptation of the one of Bressan [Bre] and we refer to it for more details. We make the proof only for the case $\sigma = 0$, the proof in the general case is similar.

First we prove that there is a global solution of the equation (2.12) which do not grow much as $t \to +\infty$. We look for solution as a fixed point:

$$y(t) = \mathcal{G}_\varepsilon(y_0, y)(t)$$

for any $y_0 \in X_s(U) \oplus X_c(U)$ where for small positive ε

$$\begin{aligned}\mathcal{G}_\varepsilon(y_0,\eta)(t) = e^{tJH(U)}y_0 &+ \int_0^t e^{(t-s)JH(U)}\pi^s(U)JN_\varepsilon(U,\eta(s))\,ds \\ &+ \int_0^t e^{(t-s)JH(U)}\pi^c(U)JN_\varepsilon(U,\eta(s))\,ds - \int_t^{+\infty} e^{(t-s)JH(U)}\pi^u(U)JN_\varepsilon(U,\eta(s))\,ds\end{aligned}$$

when $t \geq 0$ and

$$\begin{aligned}\mathcal{G}_\varepsilon(y_0,\eta)(t) = e^{tJH(U)}y_0 &+ \int_t^0 e^{(t-s)JH(U)}\pi^s(U)JN_\varepsilon(U,\eta(s))\,ds \\ &+ \int_t^0 e^{(t-s)JH(U)}\pi^c(U)JN_\varepsilon(U,\eta(s))\,ds - \int_t^{+\infty} e^{(t-s)JH(U)}\pi^u(U)JN_\varepsilon(U,\eta(s))\,ds\end{aligned}$$

when $t \leq 0$, with $\pi^*(U)$ the projector into $X^*(U)$ with respect to the decomposition $\oplus_{*\in\{c,s,u\}}X^*(U)$.

Let us introduce for $\gamma(U) = \Re E_1(U)$ and any Γ smaller than $\gamma(U)$, the space

$$Y_\Gamma = \left\{\eta : \mathbb{R} \mapsto L^2(\mathbb{R}^3,\mathbb{C}^4),\ \exists C > 0,\ \|\eta(t)\|_2 \leq e^{\Gamma|t|},\ \forall t \in \mathbb{R}\right\}.$$

For sufficiently small $\varepsilon > 0$, the map $\mathcal{G}_\varepsilon(y_0,\cdot)$ leaves Y_Γ invariant and is continuous for the norm

$$y \mapsto \sup_{t\in\mathbb{R}}\left\{\|\eta(t)\|_2 e^{-\Gamma|t|}\right\}.$$

Moreover, it is a strict contraction for sufficiently small U and $\varepsilon > 0$. In fact, ε is a $o(\gamma(U))$. This proves the existence of the fixed point y.

Then we fix $h_U^{cs}(y_0) = y(0) - y_0$. The invariance of the graph of h_U^{cs} by the flow is immediate.

Now we prove the smoothness property. In fact $N_\varepsilon(U,\eta)$ is l times differentiable in η from $Y_{\Gamma'}$ to Y_Γ if $(l+1)\Gamma' \leq \Gamma$ and $\mathcal{G}_\varepsilon$ is l times differentiable from $X_c(U) \oplus X_s(U) \times Y_{\Gamma''}$ to Y_Γ if $2l\Gamma'' \leq \Gamma$, see [Bre]. We introduce the family y_n satisfying

$$\eta_0 = 0 \text{ and } \eta_{n+1} = \mathcal{G}_\varepsilon(y_0,\eta_n).$$

This sequence converge to y in Y_Γ. Moreover, as functions of y_0, the convergence is uniform in Y_Γ on bounded sets of $X_s(U) \oplus X_c(U)$.

We want to prove that the sequence of their derivatives of order k with respect to η also converges in Y_Γ on bounded sets for any $\Gamma < \gamma(U)$. We prove it by induction in k. So suppose that $(\partial^j\eta_n)_{n\in\mathbb{N}}$ is converging in Y_Γ for all $j < k$ and any $\Gamma < \gamma(U)$. Then we have that (see [Bre])

$$\begin{aligned}\partial\eta_n &= \partial\mathcal{G}_\varepsilon(y_0,\eta_{n-1}) = L + M\left(\partial_\eta N_\varepsilon(U,\eta_{n-1})\partial\eta_{n_1}\right) \\ \partial^k\eta_n &= \partial^k\mathcal{G}_\varepsilon(y_0,\eta_{n-1}) = M\left(\partial_\eta N_\varepsilon(U,\eta_{n-1})\partial^k\eta_{n-1} + \Psi_k(\eta_{n-1},\dots,\partial^{k-1}\eta_{n-1})\right),\ \forall k \geq 2\end{aligned}$$

with $L = e^{tJH(U)}$ and

$$(M\eta)(t) = -\int_0^t e^{(t-s)JH(U)}\pi^{cs}(U)J\eta(s)\,ds + \int_t^{+\infty} e^{(t-s)JH(U)}\pi^u(U)J\eta(s)\,ds$$

and Ψ_k a smooth function of k parameter. Hence since $M \circ \partial_\eta N_\varepsilon(U, y_{n-1})$ is a strict contraction in Y_Γ for sufficiently small ε and U (once more ε is a $o(\gamma(U))$), this proves the convergence of the sequence of k-th derivatives in Y_Γ on bounded sets for any $\Gamma < \gamma(U)$. Hence the sequences of derivatives of $(\eta_n)_{n\in\mathbb{N}}$ in Y_Γ on bounded sets for any $\Gamma < \gamma(U)$. This gives the differentiability at any order of $y(0) = h(y_0)$. This also gives, since $N(U,\eta) = O(|\eta|^2)$ around zero, that $h(y_0) = O(|y_0|^2)$ around zero.

Now we want prove that $W^{cs}(U)$ is attractive in negative time. In fact $W^{cs}(U)$ is the graph of a smooth function $h : X^{cs} \mapsto X^u(U)$. Let η be such that $S(U) + \eta$ is a solution of (1.2), we have

$$\partial_t \eta = JH(U)\eta + JN(U, \eta).$$

$$\eta = y + r = y + h(y) + z$$

with $y = \pi^{cs}(U)\eta$ and we have the following equation for $z \in X^u(U)$

$$\partial_t z = JH(U)z + M(U, y, z)$$

where

$$M(U, y, z) = \pi^u(U) \left\{ JN(U, \eta) - JN(U, y + h(y)) \right\} - Dh(y)\pi^{cs}(U) \left\{ JN(U, \eta) - JN(U, y + h(y)) \right\}.$$

Using Duhamel's formula, we obtain

$$z(t) = e^{tJH(U)} z(0) + \int_0^t e^{(t-s)JH(U)} M(U, y(s), z(s))\, ds$$

We obtain since $z \in X^u(U)$

$$\|z(t)\| \leq e^{\gamma(U)t} \|z(0)\| + C \int_0^t e^{(t-s)\gamma(U)} \|M(U, y(s), z(s))\|\, ds$$

and so for $\gamma \in (0, \gamma(U))$

$$e^{-\gamma t} \|z(t)\| \leq \|z(0)\| + C|U| \sup_{s \in [0,t]} \left\{ e^{-\gamma s} \|z(s)\| \right\} + o(\sup_{s \in [0,t]} \left\{ e^{-\gamma s} \|z(s)\| \right\})$$

where C do not depend of U and z. Hence if $z(0)$ is small, we have that there exists $c > 0$ such that $\|z(t)\| \leq ce^{\gamma t}$ for all $t \leq 0$. We notice that since $X^u(U) \in H^s_\sigma$ and finite dimensional (see Lemma 2.1) in fact the'time decay in L^2 gives also a time decay in H^s_σ for any $s, \sigma \in \mathbb{R}$.

Now choose V a sufficiently small neighborhood of 0 and ϕ a solution of (2.12) initially in V but not in $W^{cs}(U)$. Suppose it stays in V in positive time. We obtain that $\phi \in Y_\Gamma$. We have

$$\begin{aligned}
\phi(t) = e^{tJH(U)} \left(\pi^s(U) + \pi^c(U)\right) \phi(0) \\
+ \int_0^t e^{-sJH(U)} \pi^s(U) JN_\varepsilon(U, \phi(s))\, ds + \int_0^t e^{-sJH(U)} \pi^c(U) JN_\varepsilon(U, \phi(s))\, ds \\
+ e^{tJH(U)} \pi^u(U) \left(y(0) + \int_0^\infty e^{-sJH(U)}(U) JN_\varepsilon(U, \phi(s))\, ds \right) \\
- \int_t^\infty e^{-sJH(U)} \pi^u(U) JN_\varepsilon(U, \phi(s))\, ds.
\end{aligned}$$

with

$$\phi(0) + \int_0^\infty e^{-sJH(U)}(U) JN_\varepsilon(U, \phi(s))\, ds \neq 0.$$

Hence we obtain with (2.9), that $\phi(t)$ exponentially tends to infinity in norm. This is a contradiction so ϕ leaves V in finite time. □

Then reversing the time direction that is to say replacing H by $-H$ and F by $-F$, we obtain with this theorem a locally invariant center unstable manifold with the corresponding properties:

Proposition 2.5 (Center-Unstable Manifold). *If Assumptions 1.1–1.5 hold. Then for any sufficiently small non zero U, there exists around $S(U)$ a unique smooth invariant center unstable manifold $W^{cu}(U)$ for (2.12), build as a graph and tangent to $S(U) + X_c(U) \oplus X_u(U)$ at $S(U)$.*

Any solution ϕ of (2.12) in the neighborhood of $S(U)$ tends as $t \to +\infty$ to $W^{cu}(U)$ with

$$dist_{H^s_\sigma}(\phi(t), W^{cu}(U)) = O(e^{-\gamma t}) \text{ as } t \to +\infty$$

and for $\gamma \in (0, \gamma(U))$, any $s, \sigma \in \mathbb{R}$ and for any V sufficiently small neighborhood of $S(U)$ any solution in V not in $W^{cu}(U)$ leaves V in finite negative time.

We can build by the same way a center manifold which is the intersection of the two previous:

Proposition 2.6 (Center Manifold). *If Assumptions 1.1–1.5 hold. Then for any sufficiently small non zero U, there exists around $S(U)$ a unique smooth invariant center manifold $W^c(U)$ for (2.12), build as a graph and tangent to $S(U) + X_c(U)$ at $S(U)$.*

Moreover, we have $W^c(U) = W^{cs}(U) \cap W^{cu}(U)$ and $W^c(U)$ contains the part of the PLS manifold which is in a neighborhood of $S(U)$.

Proof. We build the center manifold with the same method as in the previous cases. We can also build a center-unstable manifold inside center-stable manifold. More precisely, let $h^s_U : X_c(U) \oplus X_s(U) \mapsto X_u(U)$ be the map defining center-stable manifold and $h^u_U : X_c(U) \oplus X_u(U) \mapsto X_s(U)$ be the map defining center-unstable manifold. A solution $y = S(U) + y_c + y_s + y_u$ with $y_* \in X_*(U)$ for $* \in \{c, s, u\}$ is in the center-stable manifold if $y_u = h^s_U(y_c, y_s)$. Hence to obtain a center-unstable manifold inside center-stable manifold one has to solve, for each y_c, the equation

$$y_s = h^u_U(y_c, h^s_U(y_c, y_s))$$

this could be solve inside a small ball for small y_c and small U by means of the fixed point theorem, since $h^*_U(y_c, z)$ is a $O(|y_c|^2 + |z|^2)$ around zero for $* \in \{s, u\}$.

By the same way, we can also build a center-stable manifold inside the center-unstable manifold.

Using the uniqueness of the center manifold, we obtain that this two manifolds are equal to the center manifold and $W^c(U) = W^{cs}(U) \cap W^{cu}(U)$.

Then any stationary states in a small neighborhood of $S(U)$ converges to $W^{cs}(U)$ and $W^{cu}(U)$ using the stabilization results of the previous lemmas. Hence, we have that it belongs to $W^{cs}(U) \cap W^{cu}(U) = W^c(U)$. □

In the following section, we precise the dynamic inside the center manifold.

3 The dynamic inside the center manifold

In this section, we prove that the dynamic inside the center manifold around $S(V_0)$, for small non zero V_0, relaxes towards the PLS manifold. To this end, we use Theorem 1.1 and Theorem 1.2 about the time decay of the propagator associated with H.

3.1 Decomposition of the system

Like in [Bou06], we decompose a solution $\phi \in W^c(V_0)$ of the equation (1.2) with respect to the spectrum of $JH(U)$, with U specified in the sequel, and we study the equations for these different parts of the decomposition.

For any solution of (1.2) over an interval of time I containing 0, we write for $t \in I$

$$\phi(t) = e^{-\mathrm{i}\int_0^t E(U(s))\,ds}\left(S(U(t)) + \eta(t)\right).$$

where $\eta \in \mathcal{H}_0^\perp(U)$ and

$$\mathcal{H}_0^\perp(u_1, u_2) = \Bigg\{ \eta \in L^2(\mathbb{R}^3, \mathbb{C}^8), \left\langle J\eta, \frac{\partial}{\partial \Re u_1} S(u_1, u_2) \right\rangle = \left\langle J\eta, \frac{\partial}{\partial \Im u_1} S(u_1, u_2) \right\rangle$$
$$= \left\langle J\eta, \frac{\partial}{\partial \Re u_2} S(u_1, u_2) \right\rangle = \left\langle J\eta, \frac{\partial}{\partial \Im u_2} S(u_1, u_2) \right\rangle = 0 \Bigg\},$$

we notice that the orthogonality is taken with respect to J. In fact, we have

$$\mathcal{H}_0^\perp(U) = \mathcal{H}_1(U) \oplus \mathcal{H}_c(U)$$

which is invariant under the action of $JH(U)$. We recall that $\mathcal{H}_1(U)$ is defined in Proposition 2.1 and $\mathcal{H}_c(U)$ in Proposition 2.2. We impose the orthogonality condition

$$\eta(t) \in \mathcal{H}_0^\perp(U(t)). \tag{3.1}$$

So we want to solve the equation

$$\begin{aligned} \mathrm{i}\partial_t\eta &= \{H - E(U)\}\,\eta + \{\nabla F(S(U)+\eta) - \nabla F(S(U))\} - \mathrm{i}dS(U)\dot{U} \\ &= \left\{H + d^2F(S(U)) - E(U)\right\}\eta + N(U,\eta) - \mathrm{i}dS(U)\dot{U} \end{aligned} \tag{3.2}$$

for $\eta \in \mathcal{H}_0^{\perp}(U(t))$. Here d^2F is the differential of ∇F and dS the differential of S in $\mathbb{R}^2$. To close the system, we need the equation for U. This follows from the condition

$$\langle \eta(t), JdS(U(t))\rangle = 0.$$

After a time derivation (like in[Bou06]), we obtain the desired equation:

$$\dot{U}(t) = -A(U(t),\eta(t))\langle N(U(t),\eta(t)), dS(U(t))\rangle.$$

where

$$A(U,\eta) = [\langle JdS(U), dS(U)\rangle - \langle J\eta, d^2S(U)\rangle]^{-1}$$

the matrix $[\langle JdS(U(t)), dS(U(t))\rangle - \langle J\eta(t), d^2S(U(t))\rangle]$ is invertible for small $|U(t)|$ and $\|\eta(t)\|_2$ since we have

$$[\langle JdS(U(t)), dS(U(t))\rangle - \langle J\eta(t), d^2S(U(t))\rangle] = \begin{pmatrix} J & 0_2 \\ 0_2 & J \end{pmatrix} + O(|U(t)| + \|\eta(t)\|_2),$$

Lemma 3.1. *For any $s, s', \sigma \in \mathbb{R}$ and $p, q \in [1,\infty]$, let*

$$\mathcal{S}(V_0,\varepsilon) = \left\{(U,z);\ U \in B_{\mathbb{C}^2}(V_0,\varepsilon), z \in \mathcal{H}_c(U) \cap B_{H^{s'}_\sigma}(0,\varepsilon')\right\}$$

There exist $\varepsilon, \varepsilon' > 0$ and a unique map $g : S(V_0,\varepsilon) \mapsto B^s_{p,q}(\mathbb{R}^3,\mathbb{C}^4)$ which is smooth and such that for all $(U,z) \in S(V_0,\varepsilon)$, $g(U,z) \in \mathcal{H}_1(U)$, $z + g(U,z) \in \mathcal{H}_0(U)^{\perp}$, and $S(U) + z + g(U,z) \in W^c(V_0)$. Moreover, we have $g(U,0) = 0$.

Proof. if h^c is the function for which $W^c(V_0)$ is the graph. Any $\phi \in L^2(\mathbb{R}^3,\mathbb{R}^8)$ can be written in the form $S(V_0) + \tilde{U}\cdot DS(V_0) + \xi + \rho$ with $\rho \in \mathcal{H}_1(V_0)$ and $\xi \in \mathcal{H}_c(V_0)$. It can be also written in the form $S(U) + z + r$ with $r \in \mathcal{H}_1(U)$ and $z \in \mathcal{H}_c(U)$. These two decompositions in fact defines two bijective smooth maps in sufficiently small sets. We write Ψ for the first and Φ for the second. Then $f = \Psi \circ \Phi^{-1}$ has 3 components following the decomposition $\mathcal{H}_0(V_0) \oplus \mathcal{H}_1(V_0) \oplus \mathcal{H}_c(V_0)$, we write them (f_1, f_2, f_3). Then g is the solution of the implicit equation in r

$$F(U,z,r) = f_2(U,z,r) - h^c(f_1(U,z,r), f_3(U,z,r)) = 0$$

which can be solved by the implicit function theorem in L^2_σ for $\sigma \in \mathbb{R}$ since $\partial_r F(V_0,0,0)$ is invertible from $\mathcal{H}_1(V_0)$ to itself because $\partial_r f_2(V_0,0,0)$ is invertible from $\mathcal{H}_1(V_0)$ to itself and $\partial_r h_c(0,0) = 0$.

The smoothness of g follows from the fact that $g(U,z) \in \mathcal{H}_1(U)$ and the exponential decay for excited states and their derivatives given by (2.2).

The last assertion follows by the fact that around $S(V_0)$, the PLS manifold is contained in center manifold associated with $S(V_0)$. □

Hence decomposing η with respect to the spectrum of $JH(U)$, we write

$$\eta(t) = g(U(t), z(t)) + z(t)$$

with $z \in \mathcal{H}_c(U) \cap L^2(\mathbb{R}^3,\mathbb{R}^8)$. We obtain the system

$$\begin{cases} \dot{U} = -A(U,\eta)\langle N(U,\eta), dS(U)\rangle \\ \partial_t z = JH(U)z + \mathbf{P}_c(U)JN(U,\eta) \\ \qquad +\mathbf{P}_c(U(v))dS(U(v))A(U(v),\eta(v))\langle N(U(v),\eta(v)), dS(U(v))\rangle + (dP_c(U))A(U,\eta)\langle N(U,\eta), dS(U)\rangle\eta \end{cases}$$

with

$$\eta(t) = z(t) + g\left(U(t), z(t)\right).$$

We notice that this equation is defined only for z small with real values and U small. We now study this system.

3.2 The stabilization towards the PLS manifold

We now show that any solution of (1.2) which belongs to the center manifold $W^c(V_0)$ stabilizes as $t \to +\infty$ towards the manifold of the stationary states inside $W^c(V_0)$. The proof for $t \to -\infty$ is similar. To this end, we will use Theorem 1.1 and Theorem 1.2 to prove that z tends to zero in some sense.

Let us define for any ε, $\delta > 0$

$$\mathcal{U}(\varepsilon,\delta) = \left\{ U \in \mathcal{C}^\infty(\mathbb{R}, B_{\mathbb{C}^2}(V_0,\varepsilon)),\ \|\dot U\|_{L^1(\mathbb{R})\cap L^\infty(\mathbb{R})} \le \delta^2 \right\}$$

and for any $U \in \mathcal{U}(\varepsilon)$, let s', β be such that $s' > \beta + 2 > 2$ and $\sigma > 3/2$,

$$\mathcal{Z}(U,\delta) = \Bigg\{ z \in \mathcal{C}(\mathbb{R}, L^2(\mathbb{R}^3,\mathbb{R}^8)),\ z(t) \in \mathcal{H}_c(U(t)), \\ \max\left[\|z\|_{L^\infty(\mathbb{R},H^{s'})}, \|z\|_{L^2(\mathbb{R},H^{s'}_{-\sigma})}, \|z\|_{L^2(\mathbb{R},B^\beta_{\infty,2})} \right] \le \delta \Bigg\},$$

and ε, δ are small enough to ensure that for $U \in \mathcal{U}(\varepsilon,\delta)$ and $z \in \mathcal{Z}(U,\delta)$

$$S(U) + z + g(U,z) \in W^c(V_0) \cap B_{H^{s'}}(S(V_0), r(V_0)).$$

where g is defined by Lemma 3.1 and r in Remark 2.1.

3.2.1 some useful lemma

We will need some technical lemmas, which we collect here.

Lemma 3.2. *If Assumptions 1.1–1.5 hold. Let $\sigma, \sigma' \in \mathbb{R}$, $s > 1$ and $p, \tilde p_1, p_1, p_2, q \in [1,\infty]$ such that*

$$\frac{1}{p} + \frac{s}{3} \ge \frac{1}{p_1} + \frac{1}{p_2} \ge \frac{1}{p}.$$

and

$$\frac{1}{p} + \frac{s}{3} \ge \frac{1}{\tilde p_1}.$$

Then there exist $\varepsilon > 0$ and $C > 0$ such that for all $U \in B_{\mathbb{C}}(0,\varepsilon)$ and $\eta \in B^s_{p_2,q}(\mathbb{R}^3,\mathbb{R}^8) \cap L^\infty(\mathbb{R}^3,\mathbb{R}^8)$ with $\langle Q\rangle^\sigma \eta \in B^s_{p_1,q}(\mathbb{R}^3,\mathbb{R}^8)$ and $\langle Q\rangle^{\sigma'} \eta \in B^s_{\tilde p_1,q}(\mathbb{R}^3,\mathbb{R}^8)$, we have

$$\|\langle Q\rangle^\sigma N(U,\eta)\|_{B^s_{p,q}} \le C\left(s,F,|U| + \|\eta\|_{L^\infty}\right)|U|\,\|\eta\|_{L^\infty} \left\|\langle Q\rangle^{\sigma'}\eta\right\|_{B^s_{\tilde p_1,q}} \\ + C\left(s,F,|U| + \|\eta\|_{L^\infty\cap B^s_{p2,q}}\right)\|\eta\|^2_{L^\infty}\,\|\langle Q\rangle^\sigma \eta\|_{B^s_{p_1,q}}. \quad (3.3)$$

Proof. We recall the definition

$$N(U,\eta) = \nabla F(S(U)+\eta) - \nabla F(S(U)) - d^2F(S(U))\eta.$$

We have

$$N(U,\eta) = \int_0^1\int_0^1 d^3F(S(U)+\theta'\theta\eta)\cdot\eta\cdot\theta\eta\,d\theta' d\theta,$$

or

$$N(U,\eta) = \int_0^1\int_0^1 d^3F(S(U))\cdot\eta\cdot\theta\eta\,d\theta' d\theta + \int_0^1\int_0^1 d^4F(S(U)+\theta''\theta'\theta\eta)\cdot\theta'\theta\eta\cdot\eta\cdot\theta\eta\,d\theta'' d\theta' d\theta,$$

Then we use for $s \in \mathbb{R}^*_+$, p, p_1, p_2, $\in [1,\infty]$ such that $\frac{1}{p} + \frac{s}{3} \ge \frac{1}{p_1} + \frac{1}{p_2} \ge \frac{1}{p}$,

$$\|uv\|_{B^s_{p,q}} \le C\|u\|_{B^s_{p_1,q}}\|v\|_{B^s_{p_2,q}},$$

and for $s > 1$, we use [EV97, Proposition 2.1]

$$\|d^k F(\psi)\|_{B^s_{p_2,q}} \leq C\left(s, F, \|\psi\|_{L^\infty}\right) \|\psi\|_{B^s_{p_2,q}},$$

for $k = 3$ or $k = 4$ and $d^4F(z) = O(|z|)$, otherwise we decompose $d^4F(z) = A + O(|z|)$ where A is the constant operator.

Eventually using Lemma 1.1 and

$$\left\| \langle Q\rangle^\sigma |\eta|^l \right\|_{B^s_{p_1,q}} \leq C \|\eta\|^{l-1}_{L^\infty} \|\langle Q\rangle^\sigma \eta\|_{B^s_{p_1,q}},$$

for $l \in \mathbb{N}$, we conclude the proof. □

Lemma 3.3. *If Assumptions 1.1–1.5 hold. Let be $\sigma \in \mathbb{R}$, $s > 1$, $p, p_1, p_2, q \in [1, \infty]$ and $\sigma_1, \sigma_2 \in \mathbb{R}$ such that*

$$\frac{1}{p} + \frac{s}{3} \geq \frac{1}{p_1} + \frac{1}{p_2} \geq \frac{1}{p}.$$

Then there exist $\varepsilon > 0$ and $C > 0$ such that for all $U \in B_{\mathbb{C}}(0, \varepsilon)$ and $\eta \in B^s_{p,q}(\mathbb{R}^3, \mathbb{R}^8) \cap L^\infty(\mathbb{R}^3, \mathbb{R}^8)$ with $\langle Q\rangle^{\sigma_1}\eta \in B^s_{p_1,q}(\mathbb{R}^3, \mathbb{R}^8)$ and $\langle Q\rangle^{\sigma_2}\eta \in B^s_{p_2,q}(\mathbb{R}^3, \mathbb{R}^8)$, we have

$$\begin{aligned}\|< Q >^\sigma \left(\nabla F(S(U) + \eta) - \nabla F(S(U) - \nabla F(\eta))\right)\|_{B^s_{p,q}} \\ \leq C(s, F, |U| + \|\eta\|_{L^\infty}) \left(|U| + \|< Q >^{\sigma_1} \eta\|_{B^s_{p_1,q}}\right) |U| \, \|< Q >^{\sigma_2} \eta\|_{B^s_{p_2,q}}.\end{aligned}$$

Lemma 3.4. *If Assumptions 1.1–1.5 hold. Let be $\sigma \in \mathbb{R}$, $s > 1$ and $p, q \in [1, \infty]$ such that $sp \geq 3$. Then there exist $\varepsilon > 0$ and $C > 0$ such that for all U, $U' \in B_{\mathbb{C}^2}(0, \varepsilon)$ and η, $\eta' \in B^s_{p,q}(\mathbb{R}^3, \mathbb{R}^8)$, we have*

$$\begin{aligned}\|\langle Q\rangle^\sigma \{N(U, \eta) - N(U', \eta')\}\|_{B^s_{p,q}} \leq C\left(s, F, |U| + |U'| + \|\eta\|_{B^s_{p,q}} + \|\eta'\|_{B^s_{p,q}}\right) \times \\ \times \Bigg\{ \left(\|\langle Q\rangle^{\sigma_1}\eta\|_{B^s_{p,q}} + \|\langle Q\rangle^{\sigma_1}\eta'\|_{B^s_{p,q}}\right)^2 \left(|U - U'| + \|\langle Q\rangle^{\sigma_2}(\eta - \eta')\|_{B^s_{p,q}}\right) \\ + \left(|U| + |U'| + \left\|\langle Q\rangle^{\sigma'_1}\eta\right\|_{B^s_{p,q}} + \left\|\langle Q\rangle^{\sigma'_1}\eta'\right\|_{B^s_{p,q}}\right) \times \\ \times \left(\|\langle Q\rangle^{\sigma'_2}\eta\|_{B^s_{p,q}} + \|\langle Q\rangle^{\sigma'_2}\eta'\|_{B^s_{p,q}}\right) \|\langle Q\rangle^{\sigma'_3}(\eta - \eta')\|_{B^s_{p,q}} \Bigg\},\end{aligned}$$

with $2\sigma_1 + \sigma_2 = \sigma'_1 + \sigma'_2 + \sigma'_3 = \sigma$ if $< Q >^w \eta$, $< Q >^w \eta' \in B^s_{p,q}(\mathbb{R}^3, \mathbb{R}^8)$ for $w \in \{\sigma_1, \sigma_2, \sigma'_1, \sigma'_2, \sigma'_3\}$.

Proof. Using the identity

$$N(u, \eta) = \int_0^1 \int_0^1 d^3F(S(u) + \theta'\theta\eta) \cdot \eta \cdot \theta\eta \, d\theta' d\theta.$$

we can restrict the study to $d^3F(\phi) - d^3F(\phi')$. If $F = O(|z|^5)$, we have

$$\|\langle Q\rangle^\sigma \left(d^3F(\phi) - d^3F(\phi')\right)\|_{B^s_{p,q}} \leq \int_0^1 \|d^4F(\phi + t(\phi - \phi'))\|_{B^s_{p,q}} \|\langle Q\rangle^\sigma(\phi - \phi')\|_{B^s_{p,q}} \, dt.$$

Then since $s > 1$ and $sp \geq 3$, we use

$$\|d^4F(\psi)\|_{B^s_{p,q}} \leq C(s, F, \|\psi\|_{B^s_{p,q}}).$$

Using Lemma 1.1, we conclude the proof when $F = O(|z|^5)$.

Otherwise, if F is an homogeneous polynomial of order 4, the proof is easily adaptable since d^4F is a constant tensor.

The case $F = O(|z|^4)$ follows by summing the two previous one since as a function of $u \in \mathbb{R}^8$, $F(u) = Au^{\otimes 4} + O(|u|^5)$. □

Lemma 3.5. *If Assumptions 1.1–1.5 hold. Let be $\sigma \in \mathbb{R}$, $s \in \mathbb{R}$ and $p, q \in [1, \infty]$. Then there exist $\varepsilon > 0$, $M > 0$ and $C > 0$ such that for all $U, U' \in B_{\mathbb{C}^2}(0, \varepsilon)$ and $\eta, \eta' \in B_{L^2(\mathbb{R}^3, \mathbb{R}^8)}(0, M)$ with $\langle Q \rangle^\sigma \{\eta - \eta'\} \in B^s_{p,q}(\mathbb{R}^3, \mathbb{R}^8)$, one has*

$$|A(U, \eta) - A(U', \eta')| \leq C \left\{ |U - U'| + \|\langle Q \rangle^\sigma \{\eta - \eta'\}\|_{B^s_{p,q}} \right\} \tag{3.4}$$

Proof. We recall that

$$A(U, \eta) = [\langle JdS(U), dS(U) \rangle - \langle J\eta, d^2 S(U) \rangle]^{-1}.$$

We have

$$\begin{aligned} A(U, \eta) - A(u', \eta') = -[\langle JdS(U), dS(U) \rangle - \langle J\eta, d^2 S(U) \rangle]^{-1} \times \\ \times \left\{ \langle JdS(U), dS(U) \rangle - \langle J\eta, d^2 S(U) \rangle - \langle JdS(U'), dS(U') \rangle + \langle J\eta', d^2 S(U') \rangle \right\} \times \\ \times [\langle JdS(U'), dS(U') \rangle - \langle J\eta', d^2 S(U') \rangle]^{-1}. \end{aligned}$$

The lemma then follows from Lemma 1.1. □

3.2.2 Global wellposedness for z and stabilization

Let be $U \in \mathcal{U}(\varepsilon, \delta)$ and $z_0 \in \mathcal{H}_c(U(0)) \cap H^{s'}$. Let us write $U_\infty = \lim_{t \to +\infty} U(t)$, we define $\mathcal{T}_{U, z_0}(z)$ by

$$\begin{aligned} \mathcal{T}_{U,z_0}(z)(t) = e^{-\mathrm{i}tH + \mathrm{i}\int_0^t E(U(r))\, dr} z_0 + \int_0^t e^{-\mathrm{i}(t-v)H + \mathrm{i}\int_v^t E(U(r))\, dr} \mathbf{P}_c(U(v)) J \nabla F(\eta(v))\, dv \\ + \int_0^t e^{-\mathrm{i}(t-v)H + \mathrm{i}\int_v^t E(U(r))\, dr} \mathbf{P}_c(U(v)) J \{\nabla F(S(U(v)) + \eta(v)) - \nabla F(S(U(v)) - \nabla F(\eta(v))\}\, dv \\ + \int_0^t e^{-\mathrm{i}(t-v)H + \mathrm{i}\int_v^t E(U(r))\, dr} \mathbf{P}_c(U(v)) dS(U(v)) A(U(v), \eta(v)) \langle N(U(v), \eta(v)), dS(U(v)) \rangle\, dv \\ - \int_0^t e^{-\mathrm{i}(t-v)H + \mathrm{i}\int_v^t E(U(r))\, dr} (d\mathbf{P}_c(U(v)) \dot{U}(v) \eta(v)\, dv. \end{aligned}$$

with

$$\eta(t) = z(t) + g\,(U(t), z(t))$$

First, we have a local wellposedness result for z with the

Lemma 3.6. *If Assumptions 1.1–1.5 hold. Then there exist $\delta > 0$ and $\varepsilon > 0$ such that for any $U \in \mathcal{U}(\delta, \varepsilon)$ and $z_0 \in B_{H^{s'}(0,\delta)} \cap \mathcal{H}_c(U(0))$ there are $T^\pm(z_0, U) > 0$ and a unique solution*

$$z \in \mathcal{C}((-T^-(z_0, U); +T^+(z_0, U)), H^{s'}(0, \delta))$$

of the equation

$$\begin{cases} \partial_t z = JH(U)z + \mathbf{P}_c(U) JN(u, \eta) - (d\mathbf{P}_c(U))\dot{U}\eta, \\ z(0) = z_0, \end{cases} \tag{3.5}$$

where $\eta(t) = z(t) + g\,(U(t), z(t))$.

Moreover, z is unique in $L^\infty((-T', T), H^{s'})$ for any $T \in (0, T^+(z_0, U))$ and $T' \in (0, T^-(z_0, U))$ and we have $T^+(z_0, U) = +\infty$ or $\lim_{t \to T^+(z_0,U)} \|z(t)\|_{H^{s'}} \geq \delta$ and $T^-(z_0, U) = +\infty$ or $\lim_{t \to -T^-(z_0,U)} \|z(t)\|_{H^{s'}} \geq \delta$.

Proof. It is a consequence of the fix point theorem applied to $\mathcal{T}_{U,z_0}$.

Using Lemmas 3.2, 3.4 and 3.5 with the Estimate (2.8)–(2.10) and the properties of g given by Lemma 3.1 we obtain that $\mathcal{T}_{U,z_0}$ leaves a small ball in $H^{s'}$ invariant and is a contraction inside this ball.

Hence there exists a unique solution defined on the interval $[-T, T]$. Classical arguments permit to extend the solution over a maximal interval $(-T^-(z_0, U), T^+(z_0, U))$ such that if $T^+(z_0, U) < \infty$ then necessarily the solution should leave a small ball in $H^{s'}$ at time $T^+(z_0, U)$. □

We have now a global wellposedness result as stated in the

Lemma 3.7. *If Assumptions 1.1–1.5 hold. There exist $\delta_0 > 0$, $\varepsilon_0 > 0$ and $C > 0$ such that for any $\delta \in (0, \delta_0)$, $\varepsilon \in (0, \varepsilon_0)$, $U \in \mathcal{U}(\varepsilon, \delta)$ and $z_0 \in B_{H^{s'}}(0, \delta) \cap \mathcal{H}_c(U(0))$ we obtain for the Cauchy problem (3.5), $T^+(U, z_0) = +\infty$, $T^+(U, z_0) = -\infty$, $z \in \mathcal{Z}(U, \delta)$ and*

$$\max\left[\|z\|_{L^\infty(\mathbb{R},H^{s'})}, \|z\|_{L^2(\mathbb{R},H^{s'}_{-\sigma})}, \|z\|_{L^2(\mathbb{R},B^\beta_{\infty,2})}\right] < C\,\|z_0\|_{H^{s'}}\,.$$

Proof. The smoothness of z follows from a bootstrapping argument on the fixed point theorem and the fact that $(1 - P_c(U))z \equiv 0$ follows from a derivation and $(1 - P_c(U(0)))z(0) = 0$.

Let us introduce for any $0 < T < T^+(U, z_0)$, the function

$$m(T) = \sup_{t\in(-T,T)} \left\{ \|z\|_{L^\infty((-T,T),H^{s'})}\,,\ \|z\|_{L^2((-T,T),H^{s'}_{-\sigma})}\,,\ \|z\|_{L^2((-T,T),B^\beta_{\infty,2})} \right\}$$

First, we study the estimation of $L^2((-T,T), H^{s'}_{-\sigma})$. We use only the estimates of the Theorem B.1.

$$\begin{aligned}
&\|z\|_{L^2((-T,T),H^{s'}_{-\sigma})}\\
&\leq\ C_0\,\|z_0\|_{H^{s'}_{-\sigma}} + C\left\|\mathbf{P}_c\int_0^t e^{-\mathrm{i}(t-v)H+\mathrm{i}\int_v^t E(U(r))\,dr}\mathbf{P}_c(U(v))J\nabla F(\eta(v))\,dv\right\|_{L^2((-T,T),H^{s'}_{-\sigma})}\\
&\quad +C\,\|\nabla F(S(U)+\eta) - \nabla F(S(U) - \nabla F(\eta)\|_{L^2((-T,T),H^{s'}_{\sigma})}\\
&\quad +C\,\|dS(U)A(U,\eta)\langle N(U,\eta), dS(U)\rangle\|_{L^2((-T,T),H^{s'}_{\sigma})}\\
&\quad +C\left\|(d\mathbf{P}_c(U)\dot{U}\eta\right\|_{L^2((-T,T),H^{s'}_{\sigma})}.
\end{aligned}$$

We now study the estimation of the third term of the right hand side

$$\begin{aligned}
&\left\|\int_0^t e^{-\mathrm{i}(t-v)H+\mathrm{i}\int_v^t E(U(r))\,dr}\mathbf{P}_c\mathbf{P}_c(U(v))J\nabla F(\eta(v))\,dv\right\|_{L^2_t((-T,T),H^{s'}_{-\sigma})}\\
&\leq\ \int_{-T}^T \left\|e^{-\mathrm{i}(t-v)H+\mathrm{i}\int_v^t E(U(r))\,dr}\mathbf{P}_c\mathbf{P}_c(U(v))J\nabla F(\eta(v))\right\|_{L^2_t((-T,T),H^{s'}_{-\sigma})}\,dv\\
&\leq\ C(U)\,\|\nabla F(\eta)\|_{L^1((-T,T),H^{s'})}\\
&\leq\ C(U)\,\|\eta\|^2_{L^2((-T,T),L^\infty)}\,\|\eta\|_{L^\infty((-T,T),H^{s'})}\,,
\end{aligned}$$

where we used Theorem B.1 Estimate (ii).

Hence for the $L^2H^{s'}_{-\sigma}$ estimate, we obtain

$$\begin{aligned}
\|z\|_{L^2((-T,T),H^{s'}_{-\sigma})}\ \leq\ & C_0\,\|z_0\|_{H^{s'}_{-\sigma}} + C\,\|\eta\|^2_{L^2((-T,T),L^\infty)}\,\|\eta\|_{L^\infty((-T,T),H^{s'})}\\
&+C\left(\|U\|_\infty + \|\eta(v)\|_{L^\infty((-T,T),H^{s'}_{-\sigma})}\right)\|U\|_\infty\,\|\eta\|_{L^2((-T,T),H^{s'}_{-\sigma})}\\
&+C\,\|\eta\|^2_{L^2((-T,T),L^\infty)} + C\left\|\dot{U}\right\|_{L^2}\|\eta\|_{L^\infty((-T,T),H^{s'})}\,,
\end{aligned}$$

using Lemma 3.1, we obtain

$$\|z\|_{L^2((-T,T),H^{s'}_{-\sigma})} \leq C_0\,\|z_0\|_{H^{s'}_{-\sigma}} + Cm(T)^3 + Cm(T)^2 + C\sqrt{m(T)}m(T)^2$$

Then, we estimate the $H^{s'}$ norm

$$\begin{aligned}
\|z(t)\|_{H^{s'}} \leq\ & \|z_0\|_{H^{s'}} + \int_{-T}^T \|\nabla F(\eta(v))\|_{H^{s'}}\,dv\\
&+\Big\|\int_0^t e^{-\mathrm{i}(t-v)H+\mathrm{i}\int_v^t E(U(r))\,dr}\mathbf{P}_c(U(v))\times\\
&\quad\times J\{\nabla F(S(U(v))+\eta(v)) - \nabla F(S(U(v)) - \nabla F(\eta(v))\}\,dv\Big\|_{H^{s'}}\\
&+\int_{-T}^T \|dS(U(v))A(U(v),\eta(v))\langle N(U(v),\eta(v)), dS(U(v))\rangle\|_{H^{s'}}\,dv\\
&+\int_{-T}^T \left\|(d\mathbf{P}_c(U(v))\dot{U}(v)\eta(v)\right\|_{H^{s'}}\,dv.
\end{aligned}$$

to estimate the third term of the right hand side, we use the H-smoothness estimates, more precisely Theorem B.1 Estimate (ii) and then we use Lemma B.14:

$$\begin{aligned}\left\|\int_0^t e^{-\mathrm{i}(t-v)H+\mathrm{i}\int_v^t E(U(r))\,dr}\mathbf{P}_c(U(v))J\{\nabla F(S(U(v))+\eta(v))-\nabla F(S(U(v)))-\nabla F(\eta(v))\}\,dv\right\|_{H^{s'}}\\ \leq C\left\|\{\nabla F(S(U)+\eta)-\nabla F(S(U)-\nabla F(\eta)\right\|_{L^2((-T,T),H^{s'}_\sigma)}\\ \leq C\left(\|U\|_\infty+\|\eta(v)\|_{L^\infty((-T,T),H^{s'})}\right)\|U\|_\infty\|\eta\|_{L^2((-T,T),H^{s'}_{-\sigma})}\end{aligned}$$

Hence for the $L^\infty H^{s'}$ estimate, we obtain

$$\begin{aligned}\|z(t)\|_{H^{s'}}\leq\|z_0\|_{H^{s'}}+C\|\eta\|_{L^\infty((-T,T),H^{s'})}\|\eta\|^2_{L^2((-T,T),L^\infty)}\\ C\left(\|U\|_{L^\infty((-T,T))}+\|\eta(v)\|_{L^\infty((-T,T),H^{s'})}\right)\|U\|_{L^\infty((-T,T))}\|\eta\|_{L^2((-T,T),H^{s'}_{-\sigma})}\\ +C\|\eta\|^2_{L^2((-T,T),L^\infty)}+\left\|\dot U\right\|_{L^1((-T,T))}\|\eta\|_{L^\infty((-T,T),H^{s'})},\end{aligned}$$

using Lemma 3.1, we obtain

$$\|z(t)\|_{H^{s'}}\leq\|z_0\|_{H^{s'}}+C\varepsilon m(T)+Cm(T)^3+Cm(T)^2.$$

For the $L^2B^\beta_{\infty,2}$ estimate, by Proposition 2.2 and Theorem 1.2, we have for any $\varepsilon>0$, any $p_\varepsilon>3/\varepsilon$ and $\theta_\varepsilon=\frac{4}{\partial_\varepsilon-2}$

$$\begin{aligned}\|z\|_{L^2((-T,T),B^\beta_{\infty,2})} &\leq \|z\|_{L^2((-T,T),B^{\beta+\varepsilon}_{p_\varepsilon,2})}\\ &\leq C_0\|z_0\|_{H^{\beta+1+\varepsilon+\theta_\varepsilon/2}}+C\left\|d^2F(S(U))\cdot\eta\right\|_{L^2((-T,T),B^{\beta+2+\varepsilon+\theta_\varepsilon}_{p'_\varepsilon,2})}\\ &\quad+C\|N(U,\eta)\|_{L^1((-T,T),H^{\beta+1+\varepsilon+\theta_\varepsilon/2})}\\ &\quad+C\|dS(U)A(U,\eta)\langle N(U,\eta),dS(U)\rangle\|_{L^1((-T,T),H^{\beta+1+\varepsilon+\theta_\varepsilon/2})}\\ &\quad+C\left\|(d\mathbf{P}_c(U))\dot U\eta\right\|_{L^1((-T,T),H^{\beta+1+\varepsilon+\theta_\varepsilon/2})}dv.\end{aligned}$$

With Lemma 3.2 and 3.3, we infer

$$\begin{aligned}\|z\|_{L^2(\mathbb{R},B^\beta_{\infty,2})}\leq C_0\|z_0\|_{H^{\beta+1+\varepsilon+\theta_\varepsilon/2}}+C|U|_\infty\|\eta\|_{L^2((-T,T),H^{\beta+2+\varepsilon+\theta_\varepsilon}_{-\sigma})}\\ +C|U|_\infty\|z\|_{L^2((-T,T),L^\infty)}\|z\|_{L^2((-T,T),H^{\beta+1+\varepsilon+\theta_\varepsilon/2})}\\ +C(|U|_\infty+\|\eta\|_{L^\infty((-T,T),H^{\beta+1+\varepsilon+\theta_\varepsilon/2})})\|\eta\|^2_{L^2((-T,T),L^\infty)}\|z\|_{L^\infty((-T,T),H^{\beta+1+\varepsilon+\theta_\varepsilon/2})}\\ +C(|U|_\infty+\|\eta\|_{L^\infty((-T,T),H^{\beta+1+\varepsilon+\theta_\varepsilon/2})})\|\eta\|_{L^2((-T,T),H^{\beta+1+\varepsilon+\theta_\varepsilon/2}_{-\sigma})}\|\eta\|_{L^\infty((-T,T),H^{\beta+1+\varepsilon+\theta_\varepsilon/2})}\\ +C|\dot U|_\infty\|\eta\|_{L^2((-T,T),H^{\beta+1+\varepsilon+\theta_\varepsilon/2}_{-\sigma})}\end{aligned}$$

we infer since $s'\geq\beta+2+\varepsilon+\theta_\varepsilon$ and using Lemma 3.1,

$$\|z\|_{L^2((-T,T),B^\beta_{\infty,2})} \leq C_0\|z_0\|_{H^{\beta+1+\varepsilon+\theta_\varepsilon/2}}+\varepsilon m(T)+Cm(T)^3+Cm(T)^2.$$

Hence we obtain

$$m(T) \leq C_0\|z_0\|_{H^{\beta+1+\varepsilon+\theta_\varepsilon/2}}+C\varepsilon m(T)+Cm(T)^3+Cm(T)^2+C\sqrt{m(T)}m(T)^2,$$

where C_0 do not depend of m and C is a nondecreasing function of $\|z\|_{L^\infty((-T,T),H^{\beta+1+\varepsilon+\theta_\varepsilon/2})}$ and hence it can be bounded by a nondecreasing function of m.

If $\|z_0\|_{H^{s'}}$ is small then $m(0)$ is small and $m(T)$ stay small. Therefore we have that $z\in\mathcal{Z}(U,\delta)$ if $\|z_0\|_{H^{s'}}$is small enough for any δ and ε are small enough and

$$\max\left[\|z\|_{L^\infty(\mathbb{R},H^{s'})},\|z\|_{L^2(\mathbb{R},H^{s'}_{-\sigma})},\|z\|_{L^2(\mathbb{R},B^\beta_{\infty,2})}\right]\leq f(\|z_0\|_{H^{s'}})$$

where f is such that there exists $C>0$ with

$$f(\|z_0\|_{H^{s'}})\leq C\|z_0\|_{H^{s'}}.$$

□

The solution z just found is a function of z_0 and U, writing it $z[z_0, U]$, we have the following important property given by the

Lemma 3.8. *If Assumptions 1.1–1.5 hold. Then there exists $\delta_0 > 0$, $\varepsilon_0 > 0$, $C > 0$, $\kappa \in (0,1)$ such that for any $\delta \in (0, \delta_0)$, $\varepsilon \in (0, \varepsilon_0)$, U, $U' \in \mathcal{U}(\varepsilon, \delta)$, $z_0 \in \mathcal{H}_c(U(0))$, $z_0' \in \mathcal{H}_c(U'(0))$, $z \in \mathcal{Z}(U, \delta)$ and $z' \in \mathcal{Z}(U', \delta)$, one has*

$$\left\| e^{-\mathrm{i}\int_0^t E(U(r))\,dr} z[z_0', U'] - e^{-\mathrm{i}\int_0^t E(U'(r))\,dr} z[z_0, U] \right\|_{L^\infty(\mathbb{R}, H^{s'}) \cap L^2(\mathbb{R}, L^\infty) \cap L^2(\mathbb{R}, H^{s'}_{-\sigma})}$$
$$\leq C \left\| z_0 - z_0' \right\|_{H^{s'}} + \kappa \left\{ \left\| U - U' \right\|_{L^\infty} + \left\| \dot{U} - \dot{U}' \right\|_{L^\infty} \right\}.$$

Proof. We use the technics of the previous lemma on the identity

$$e^{-\mathrm{i}\int_0^t E(U(r))\,dr} z - e^{-\mathrm{i}\int_0^t E(U'(r))\,dr} z' = e^{-\mathrm{i}\int_0^t E(U(r))\,dr} \mathcal{T}_{U, z_0}(z) - e^{-\mathrm{i}\int_0^t E(U'(r))\,dr} \mathcal{T}_{U', z_0'}(z').$$

□

Since $U \in \mathcal{U}(\varepsilon, \delta)$, we notice that $\lim_{\pm\infty} U = U_{\pm\infty}$ exist. Then we notice that we can also obtain an asymptotic profile for $e^{\mathrm{i}tH + \mathrm{i}\int_0^t E(U(v))\,dv} z(t)$. But we prefer to obtain a scattering result with respect to $e^{JtH(U_\infty))\,dv}$. We have the

Lemma 3.9. *If Assumptions 1.1–1.5 hold. Then for any $U \in \mathcal{U}(\delta, \varepsilon)$ and solution z of (3.5), with $z_0 \in H^{s'}$ small, the following limit*

$$z_\infty = \lim_{t \to \pm\infty} e^{-JtH(U_{\pm\infty})} e^{J\int_0^t (E(U(v)) - E(U_{\pm\infty}))\,dv} z(t)$$

exists in $H^{s'}$. Moreover, we have $z_\infty \in \mathcal{H}_c(0)$ and there exists $c > 0$ such that

$$\| e^{-J\int_0^t (E(U(v)) - E(U_{\pm\infty}))\,dv} e^{JtH(U_\infty)\,dr} z_{\pm\infty} - z(t) \|_{L_t^\infty(\mathbb{R}^\pm, H^{s'}) \cap L_t^2(\mathbb{R}^\pm, H^{s'}_{-\sigma}) \cap L_t^2(\mathbb{R}^\pm, B^\beta_{\infty,2})} \leq C\,|U_0|\,\|z_0\|_{H^{s'}}$$

and

$$\lim_{t \to \pm\infty} \| e^{JtH(U_\infty)\,dr} z_{\pm\infty} - z(t) \|_{H^{s'}} = 0.$$

Proof. Let

$$t \mapsto \xi_\pm(t) = e^{J\int_0^t (E(U(v)) - E(U_{\pm\infty}))\,dv} z(t)$$

and

$$t \mapsto V_\pm(t) = e^{-\mathrm{i}\int_0^t (E(U(v)) - E(U_{\pm\infty}))\,dv} U(t).$$

Using exactly the same method as the one of Lemma 3.7, applied to

$$\begin{aligned}
e^{-JtH(U_{\pm\infty})}\xi_\pm(t) = z_0 &+ \int_0^t e^{-JsH(U_{\pm\infty})} \mathbf{P}_c(V_\pm(v)) J \left(d^2F(S(V_\pm(v))) - d^2F(S(U_{\pm\infty})) \right) \xi_\pm(v)\,dv \\
&+ \int_0^t e^{-JsH(U_{\pm\infty})} \mathbf{P}_c(V_\pm(v)) J N(V_\pm(v), \tilde{\eta}_\pm(v))\,dv \\
&+ \int_0^t e^{-JsH(U_{\pm\infty})} \mathbf{P}_c(V_\pm(v)) dS(V(v)) A(V_\pm(v), \tilde{\eta}_\pm(v)) \langle N(V_\pm(v), \tilde{\eta}_\pm(v)), dS(V_\pm(v)) \rangle\,dv \\
&- \int_0^t e^{-JsH(U_{\pm\infty})} (d\mathbf{P}_c(V_\pm(v))) A(V_\pm(v), \tilde{\eta}_\pm(v)) \langle N(V_\pm(v), \tilde{\eta}_\pm(v)), dS(V_\pm(v)) \rangle \tilde{\eta}_\pm(v)\,dv,
\end{aligned}$$

with $\tilde{\eta}_\pm(t) = e^{J\int_0^t (E(U(v)) - E(U_{\pm\infty}))\,dv} (z(t) + g(U(t), z(t)))$. We prove that the limits

$$\lim_{t \to \pm\infty} e^{-JtH(U_{\pm\infty})} \xi_\pm(t) = z_{\pm\infty}$$

exist by the same way we also obtain the estimates and the convergence of

$$\begin{aligned}
e^{JtH(U_{\pm\infty})} z_{\pm\infty} - \xi_\pm(t) = e^{JtH(U_{\pm\infty})} (z_{\pm\infty} - z_0) &- \left(\xi_\pm(t) - e^{-\mathrm{i}t(H - E(U_{\pm\infty}))} z_0 \right) \\
&+ \left(e^{JtH(U_{\pm\infty})} - e^{-\mathrm{i}t(H - E(U_{\pm\infty}))} \right) z_0.
\end{aligned}$$

Then for multiplying by $e^{-J\int_0^t (E(U(v))-E(U_{\pm\infty}))\,dv}$, we obtain the estimates and the convergence of

$$e^{-J\int_0^t (E(U(v))-E(U_{\pm\infty}))\,dv} e^{-JtH(U_{\pm\infty})} z_{\pm\infty} - z(t).$$

Since $(1 - P_c(U(t)))\, z(t) = 0$, we have $(1 - P_c(U_{\pm}\infty))\, z_{\pm\infty} = 0$ and hence $z_{\pm\infty}$ belongs to $\mathcal{H}_c(U_{\pm\infty})$. □

3.2.3 Global wellposedness for U and stabilization

Here we want to solve the equation for U. We notice that z and α have been built in the previous section and are functions of U and $z_0 \in \mathcal{H}_c(U(0))$. Let us introduce for any $U_0 \in B_{\mathbb{C}}(0,\varepsilon)$ the function on $\mathcal{U}(\varepsilon,\delta)$:

$$f_{U_0}(U)(t) = U_0 - \int_0^t A(U(v),\eta(v))\langle N(U(v),\eta(v)), dS(U(v))\rangle\, dv,$$

where $\eta = z(t) + g\,[U(t), z(t)]$. We have the

Lemma 3.10. *If Assumptions 1.1–1.5 hold. There exists $\delta_0 > 0$ and $\varepsilon_0 > 0$ such that for any $\delta \in (0,\delta_0)$, for any $\varepsilon \in (0,\varepsilon_0)$, the function f_{U_0} maps $\mathcal{U}(\varepsilon,\delta)$ into itself if U_0 and $z_0 \in H^{s'} \cap \mathcal{H}_c(U_0)$ are small enough.*

Proof. By means of Lemma 3.2, we obtain

$$\|\partial_t f_{U_0}(U)\|_{L^1(\mathbb{R})\cap L^\infty(\mathbb{R})} \le C\,\|N(U(v),\eta(v))\|_{L^1(\mathbb{R},H^{s'}_{-\sigma})\cap L^\infty(\mathbb{R},H^{s'})} \le \delta^2.$$

and

$$\|f_{U_0}(U)\|_{L^\infty(\mathbb{R})} \le |U_0| + C\,\|N(U(v),\eta(v))\|_{L^1(\mathbb{R},H^{s'})} \le |U_0| + \delta^2,$$

hence for sufficiently small U_0 and δ, we obtain the lemma. □

The function f_{U_0} has also a local Lipshitz property as stated by the

Lemma 3.11. *If Assumptions 1.1–1.5 hold. For any $T > 0$, there exists $\delta_0 > 0$, $\varepsilon_0 > 0$ and $\kappa \in (0,1)$ such that for any $\delta \in (0,\delta_0)$, for any $\varepsilon \in (0,\varepsilon_0)$, for any u, $u' \in \mathcal{U}(\varepsilon,\delta)$, for any $z_0 \in \mathcal{H}_c(U(0)) \cap H^{s'}$, for any $z_0' \in \mathcal{H}_c(U'(0)) \cap H^{s'}$ small enough, for U_0, U_0' small enough, such that*

$$\begin{aligned}\left|f_{U_0}(U) - f_{U_0'}(U')\right|_{L^\infty((-T;T))} &+ \left|\partial_t f_{U_0}(U) - \partial_t f_{U_0'}(U')\right|_{L^1((-T;T))} \\ &\le |U_0 - U_0'| + \kappa\left(\|U - U'\|_{L^\infty((-T;T))} + \left\|\dot U - \dot U'\right\|_{L^1((-T;T))} + \|z_0 - z_0'\|_{H^{s'}}\right).\end{aligned}$$

Proof. This a straightforward consequence of Lemma 3.4, 3.5 and 3.8. □

We now obtain the

Lemma 3.12. *If Assumptions 1.1–1.5 hold. There exists $\delta_0 > 0$ such that for any $\delta \in (0,\delta_0)$ such that for any $U_0 \in \mathbb{C}$ small and $z_0 \in \mathcal{H}_c(U_0) \cap H^{s'}_\sigma$ small, the equation*

$$\begin{cases} \dot U &= -A(U,\eta)\langle N(U,\eta), dS(U)\rangle, \\ U(0) &= U_0, \end{cases}$$

where $\eta(t) = z(t) + g\,[U(t), z(t)]$, has a unique solution in $\mathcal{U}(\delta,\varepsilon)$. Moreover, there exists $C > 0$ such that

$$|U_{\pm\infty} - U_0| \le C\,\|z_0\|^2_{H^{s'}}$$

Proof. This is also a the fixed point result for f_{U_0}. Let us fix $T > 0$ and consider, for any $V \in \mathcal{U}(\delta,\varepsilon)$ with sufficiently small $\delta > 0$ and $\varepsilon > 0$, the sequence:

$$\begin{cases} V_{n+1} = f_{U_0}(V_n),\ \forall n \in \mathbb{N} \\ V_0 = V; \end{cases}$$

for any $n \in \mathbb{N}$, $V_n \in \mathcal{U}(\delta,\varepsilon)$. With Lemma 3.11, the fixed point theorem give us the convergence for the norms of $L^\infty((-T,T))$ and $\dot W^{1,1}((-T,T))$ of (V_n).

Then we notice that for any $T' \in \mathbb{R}$, we have

$$V_{n+1}(t) = f_{f_{U_0}(V_n)(T')}(V_n)(t - T').$$

Since for $T' \in (-T; T)$, $(f_{U_0}(V_n)(T'))$ is a Cauchy sequence, the Lemma 3.11 give us the convergence of (V_n) for the norms of $L^\infty((T' - T; T' + T))$ and $\dot{W}^{1,1}((T' - T; T' + T))$.

Iterating this process, we prove the Lemma since the other statements are classical. We just notice that the last statement follow from the fact that there exists $C > 0$ such that

$$\left|\dot{U}\right| \leq |A(U, \eta)\langle N(U, \eta), dS(U)\rangle| \leq C \|z_0\|^2_{H^{s'}}$$

□

4 The dynamic outside the center manifold

We can make the same study in the center stable manifold and the center unstable manifold but only in one direction of time. Let us explain it for the center stable manifold in positive time since it is similar for the center unstable manifold. Actually it is equivalent if we revert the time direction.

We just give a sketch of the proof since it is similar to the previous study. Using the idea of the proof of exponential stabilization for Proposition 2.4, we write any solution ϕ in the form $\psi + \rho$ with ψ in the center manifold and $\rho \in X_s(V_0)$.

Indeed $W^c(V_0)$ is the graph of a smooth function $h^c : X_c(V_0) \mapsto X_s(V_0) \oplus X_u(V_0)$. Let ν be such that $S(V_0) + \nu$ satisfy (1.2), we have

$$\partial_t \nu = JH(V_0)\nu + JN(V_0, \nu).$$

$$\nu = y + h^c(y) + \rho = \psi - S(V_0) + \rho$$

with $y = \pi^c(V_0)\nu$. We notice that using the fact the intersection of the center stable and center unstable manifolds is the center stable manifold (see Proposition 2.6), we obtain $\rho \in X_s(V_0)$ and we have the following equation for ρ

$$\partial_t \rho = JH(V_0)\rho + M(V_0, y, \rho)$$

where

$$M(V_0, y, \rho) = \pi^s(V_0)\left\{JN(V_0, \nu) - JN(V_0, y + h^c(y))\right\} - \pi^s(V_0)Dh^c(y)\pi^c(V_0)\left\{JN(V_0, \nu) - JN(V_0, y + h^c(y))\right\}.$$

Then we obtain for ψ the equation

$$\partial_t \psi = JH\psi + J\nabla F(\psi) + R(\psi, \rho)$$

$$R(\psi, \rho) = J\nabla F(\psi + \rho) - J\nabla F(\psi) - Jd^2F(S(V_0))\rho - M(V_0, \pi^c(V_0)(\psi - S(V_0)), \rho)$$

with notice that $|R(\psi, \rho)| \leq C(\|\phi\|_{H^s}, \|\rho\|_{L^\infty})|\rho|$.

Then working like in 3, we obtain an equation for U and z , namely

$$\begin{cases} \dot{U} = -A(U, \eta)\langle N(U, \eta) - JR(U, \eta, \rho), dS(U)\rangle \\ \partial_t z = JH(U)z + \mathbf{P}_c(U)JN(U, \eta) + \mathbf{P}_c(U(v))dS(U(v))A(U(v), \eta(v))\langle N(U(v), \eta(v)) - JR(U, \eta, \rho), dS(U(v))\rangle \\ \quad + (dP_c(U))A(U, \eta)\langle N(U, \eta) - JR(U, \eta, \rho), dS(U)\rangle\eta + \mathbf{P}_c(U)R(U, \eta, \rho) \end{cases}$$

with

$$\eta(t) = z(t) + g\left(U(t), z(t)\right).$$

where g is defined by Lemma 3.1 and

$$R(U, \eta, \rho) = R(S(U) + \eta, \rho)$$

These equations are similar to those we have studied but with an extra term coming from R which is exponentially decaying in positive time.

Indeed for ρ, we study its equation in

$$\mathcal{R}_{T_0,\gamma} = \left\{\rho \in \mathcal{C}^\infty((T_0, +\infty), X_s(V_0)),\ \exists C > 0\, |\rho(t)|_{H^{s'}} < \frac{r(V_0)}{2}e^{-\gamma t}, \forall t > T_0\right\}$$

for any $T_0 < 0$ and $\gamma \in (0, \gamma(V_0))$.

Let us define for any $\varepsilon > 0$

$$\mathcal{U}_{T_0}(\varepsilon, \delta) = \left\{ U \in \mathcal{C}^\infty((T_0, +\infty), B_{\mathbb{C}^2}(V_0, \varepsilon)),\ \|\dot{U}\|_{L^1((T_0,+\infty))\cap L^\infty((T_0,+\infty))} \leq \delta^2 \right\}$$

and for any $U \in \mathcal{U}_{T_0}(\varepsilon)$, let s', β be such that $s' > \beta + 2 > 2$ and $\sigma > 3/2$,

$$\mathcal{Z}_{T_0}(U, \delta) = \Bigg\{ z \in \mathcal{C}^\infty((T_0, +\infty), L^2(\mathbb{R}^3, \mathbb{R}^8)),\ z(t) \in \mathcal{H}_c(U(t)),$$
$$\max\left[\|z\|_{L^\infty((T_0,+\infty),H^{s'})}, \|z\|_{L^2((T_0,+\infty),H^{s'}_{-\sigma})}, \|z\|_{L^2((T_0,+\infty),B^\beta_{\infty,2})} \right] < \delta \Bigg\},$$

and ε, δ are small enough to ensure that for $U \in \mathcal{U}(\varepsilon, \delta)$ and $z \in \mathcal{Z}(U, \delta)$

$$S(U) + z + g(U, z) \in W^c(V_0) \cap B_{H^{s'}}(S(V_0), \frac{r(V_0)}{2}).$$

We solve the equation for z first and then the one for ρ and eventually the one for U using the idea of Section 3. This gives us the desired exponential decay for r as well as similar results for U and z.

We notice that instead of Lemma 3.8, we obtain the

Lemma 4.1. *If Assumptions 1.1–1.5 hold. Then there exists $\delta_0 > 0$, $\varepsilon_0 > 0$, $C > 0$, $\kappa \in (0, 1)$ such that for any $\delta \in (0, \delta_0)$, $\varepsilon \in (0, \varepsilon_0)$, $U, U' \in \mathcal{U}_{T_0}(\varepsilon, \delta)$, $\rho, \rho' \in \mathcal{R}_{T_0,\gamma}$ $z_0 \in \mathcal{H}_c(U(0))$, $z'_0 \in \mathcal{H}_c(U'(0))$, $z \in \mathcal{Z}(U, \delta)$ and $z' \in \mathcal{Z}(U', \delta)$, one has*

$$\left\| e^{-\mathrm{i}\int_0^t E(U(r))\,dr} z[z'_0, U', \rho'] - e^{-\mathrm{i}\int_0^t E(U'(r))\,dr} z[z_0, U, \rho] \right\|_{L^\infty((T_0,+\infty),H^{s'})\cap L^2((T_0,+\infty),L^\infty)\cap L^2((T_0,+\infty),H^{s'}_{-\sigma})}$$
$$\leq C \|z_0 - z'_0\|_{H^{s'}} + \kappa \left\{ \|U - U'\|_{L^\infty((T_0,+\infty),\mathbb{C}^2} + \left\|\dot{U} - \dot{U}'\right\|_{L^\infty((T_0,+\infty),\mathbb{C}^2)} + \left\|e^{\gamma t}(\rho - \rho')(t)\right\|_{L^\infty_t((T_0,+\infty),X^s(V_0))} \right\}.$$

Then for ρ as a function of U, z_0 and ρ_0 (the initial condition for ρ), we obtain the

Lemma 4.2. *If Assumptions 1.1–1.5 hold. Then there exists $\delta_0 > 0$, $\varepsilon_0 > 0$, $C > 0$, $\kappa \in (0, 1)$ such that for any $\delta \in (0, \delta_0)$, $\varepsilon \in (0, \varepsilon_0)$, $U, U' \in \mathcal{U}_{T_0}(\varepsilon, \delta)$, $r_0, r'_0 \in X_s(V_0)$ $z_0 \in \mathcal{H}_c(U(0))$, $z'_0 \in \mathcal{H}_c(U'(0))$, $z \in \mathcal{Z}(U, \delta)$ and $z' \in \mathcal{Z}(U', \delta)$, one has*

$$\left\| e^{\gamma t} \left(\rho[z'_0, U', \rho'_0] - \rho[z_0, U, \rho_0]\right) \right\|_{L^\infty_t((T_0,+\infty),X^s(V_0))}$$
$$\leq C \|z_0 - z'_0\|_{H^{s'}} + \kappa \left\{ \|U - U'\|_{L^\infty((T_0,+\infty),\mathbb{C}^2} + \left\|\dot{U} - \dot{U}'\right\|_{L^\infty((T_0,+\infty),\mathbb{C}^2)} + \|\rho_0 - \rho'_0\|_{L^2} \right\}.$$

We also notice that we need during the proof ρ to be small. Hence this impose some smallness to r, which is slightly modified. Such assumptions, gives that there exists $C > 0$ with

$$|\rho(t)|_{H^{s'}} \leq C\rho(0)e^{-\gamma t}, \forall t > T_0.$$

5 End of the proof of Theorem 1.3

Then we notice that the small *locally invariant* center manifold build in Section 2.2 for Equation (2.11) is now a small *invariant* (globally in time) center manifold. Indeed , we have just proved the stabilization towards the PLS manifold, this ensures that a solution in the center manifold will stay inside this manifold in the two direction of time.

Now let us consider $\mathcal{C}M \setminus \{0\}$ the union of all this small globally invariant center manifolds using the uniqueness of center manifold with help of Lemma 3.1, we prove that this is a manifold. Now we generalize Lemma 3.1 by the

Lemma 5.1. *For any $s, s' \in \mathbb{R}$ and $p, q \in [1, \infty]$, there exist $\varepsilon > 0$, a continuous map $r : B^2_{\mathbb{C}}(0, \varepsilon) \mapsto \mathbb{R}^+$ with $r(U) = O(\Gamma(U))$ and a bijective map $\Psi : S \mapsto \mathcal{C}M$ where*

$$\mathcal{S} = \{(U, z);\ U \in B_{\mathbb{C}^2}(0, \varepsilon), z \in \mathcal{H}_c(U) \cap B_{H^{s'}}(0, r(U))\}.$$

Moreover Ψ is smooth and there exists $C > 0$ such that for all $(U, z) \in S$, $\Psi(U, z) \in \mathcal{H}_1(U)$, $z + \Psi(U, z) \in \mathcal{H}_0(U)^{\perp}$, $S(U) + z + \Psi(U, z) \in \mathcal{C}M$ with $|\Psi(U, z)|_{B^s_{p,q}} \leq C \|z\|^2_{H^{s'}}$.

Proof. The proof works like for Lemma 3.1. The last statement follows from the fact that each central manifold at $S(U)$ is tangent to $X^c(U)$.

The statements for r follow from Remark 2.1.

□

The scattering result follows from a one to one correspondence of the initial profile with the asymptotic profile as stated in the

Proposition 5.1. *If Assumptions 1.1–1.5 hold. There exist $\varepsilon > 0$ and a continuous map $r : B^2_{\mathbb{C}}(0, \varepsilon) \mapsto \mathbb{R}^+$ with $r(U) = O(\Gamma(U))$ and $\mathcal{V}_0$, $\mathcal{V}_{\pm}$ neighborhoods of $(0, 0)$ in*

$$S = \left\{(U, z)\,;\ U \in \mathbb{C}^2,\ z \in \mathcal{H}_c(U) \cap B_{H^{s'}}(0, r(U))\right\}$$

which is endowed with the norm of $\mathbb{C}^2 \times H^{s'}$ such that the maps

$$\mathcal{P}_{\pm} : \begin{pmatrix} U_0 \\ z_0 \end{pmatrix} \in \mathcal{V}_0 \mapsto \begin{pmatrix} U_{\pm\infty} \\ z_{\pm\infty} \end{pmatrix} \in \mathcal{V}_{\pm}$$

are smooth bijections.

Proof. We choose

$$\mathcal{V}_0 = \{(U, z)\,;\ U \in B_{\mathbb{C}^2}(0, \varepsilon),\ z \in \mathcal{H}_c(U) \cap B_{H^{s'}}(0, r(U))\}$$

for some positive ε and we work on the manifold $\mathcal{V}_0 \setminus \{(0, 0)\}$ which is locally isomorphic to an open set of $\mathbb{C}^2 \times \mathcal{H}_c(U) \cap H^{s'}$.

Since

$$\|(U_\infty, z_\infty) - (U_0, z_0)\| = O\left(|U_0|^2 + \|z_0\|^2_{H^{s'}}\right),$$

we only need to prove the statement locally. Hence we prove that in a neighborhood of $(U_0, 0)$ the maps $\mathcal{P}^{U_0}_{\pm}(U, z) \mapsto (Id_{\mathbb{C}^2}, P_c(U_0))\mathcal{P}_{\pm}(U, R(U, U_0)z)$ are bijective (P_c and R are defined in Proposition 2.2).

We write

$$\mathcal{P}^{U_0}_{\pm}(U, z) = (U, z) + \mathcal{R}^{U_0}_{\pm}(U, z)$$

with $\mathcal{R}^{U_0}_{\pm}$ a lipshitz function. With the Proposition 3.8 and 3.11, we obtain that the Lipshitz norms of $\mathcal{R}^{U_0}_{\pm}$ are $O(\varepsilon)$. The proposition follows for sufficiently small positive ε.

□

For the same reasons the small locally invariant center-stable manifold build in Section 2.2 is invariant in positive time. We can also consider the union of these manifolds, and we can obtain a map Φ_+ similar to the map Φ just built. The instability in negative time is in fact a consequence of Proposition 2.5.

The corresponding conclusion holds for the center stable manifold.

The statements on the instability outside these manifolds follow from Propositions 2.4 and 2.5.

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References

[ABdMG96] W. O. Amrein, A. Boutet de Monvel, and V. Georgescu. *C_0-groups, commutator methods and spectral theory of N-body Hamiltonians*, volume 135 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1996.

[BG87] Anne Berthier and Vladimir Georgescu. On the point spectrum of Dirac operators. *J. Funct. Anal.*, 71(2):309–338, 1987.

[BJ89] P. W. Bates and C. K. R. T. Jones. Invariant manifolds for semilinear partial differential equations. In *Dynamics reported, Vol. 2*, volume 2 of *Dynam. Report. Ser. Dynam. Systems Appl.*, pages 1–38. Wiley, Chichester, 1989.

[BL76] J. Bergh and J. Löfström. *Interpolation spaces. An introduction*. Springer-Verlag, Berlin, 1976. Grundlehren der Mathematischen Wissenschaften, No. 223.

[Bou06] N. Boussaid. Stable directions for small nonlinear dirac standing waves. *Comm. Math. Phys.*, 268(3):757–817, 2006.

[BP92a] V. S. Buslaev and G. S. Perel′man. Nonlinear scattering: states that are close to a soliton. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 200(Kraev. Zadachi Mat. Fiz. Smezh. Voprosy Teor. Funktsii. 24):38–50, 70, 187, 1992.

[BP92b] V. S. Buslaev and G. S. Perel′man. On nonlinear scattering of states which are close to a soliton. *Astérisque*, (210):6, 49–63, 1992. Méthodes semi-classiques, Vol. 2 (Nantes, 1991).

[BP92c] V. S. Buslaev and G. S. Perel′man. Scattering for the nonlinear Schrödinger equation: states that are close to a soliton. *Algebra i Analiz*, 4(6):63–102, 1992.

[BP95] V. S. Buslaev and G. S. Perel′man. On the stability of solitary waves for nonlinear Schrödinger equations. In *Nonlinear evolution equations*, volume 164 of *Amer. Math. Soc. Transl. Ser. 2*, pages 75–98. Amer. Math. Soc., Providence, RI, 1995.

[Bre] A. Bressan. A tutorial on the center manifold theorem. C.I.M.E. course in Cetraro,2003, P.Marcati ed. to appear in Springer Lecture Notes in Math.

[BS02] V. S. Buslaev and C. Sulem. Asymptotic stability of solitary waves for nonlinear Schrödinger equations. In *The legacy of the inverse scattering transform in applied mathematics (South Hadley, MA, 2001)*, volume 301 of *Contemp. Math.*, pages 163–181. Amer. Math. Soc., Providence, RI, 2002.

[BS03] V. S. Buslaev and C. Sulem. On asymptotic stability of solitary waves for nonlinear Schrödinger equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 20(3):419–475, 2003.

[CL82] T. Cazenave and P.-L. Lions. Orbital stability of standing waves for some nonlinear Schrödinger equations. *Comm. Math. Phys.*, 85(4):549–561, 1982.

[Cuc01] S. Cuccagna. Stabilization of solutions to nonlinear Schrödinger equations. *Comm. Pure Appl. Math.*, 54(9):1110–1145, 2001.

[Cuc03] S. Cuccagna. On asymptotic stability of ground states of NLS. *Rev. Math. Phys.*, 15(8):877–903, 2003.

[Cuc05] S. Cuccagna. Erratum: "Stabilization of solutions to nonlinear Schrödinger equations" [Comm. Pure Appl. Math. **54** (2001), no. 9, 1110–1145; mr1835384]. *Comm. Pure Appl. Math.*, 58(1):147, 2005.

[EV97] M. Escobedo and L. Vega. A semilinear Dirac equation in $H^s(\mathbf{R}^3)$ for $s > 1$. *SIAM J. Math. Anal.*, 28(2):338–362, 1997.

[GM01] V. Georgescu and M. Măntoiu. On the spectral theory of singular Dirac type Hamiltonians. *J. Operator Theory*, 46(2):289–321, 2001.

[GNT04] S. Gustafson, K. Nakanishi, and T.-P. Tsai. Asymptotic stability and completeness in the energy space for nonlinear Schrödinger equations with small solitary waves. *Int. Math. Res. Not.*, (66):3559–3584, 2004.

[GSS87] M. Grillakis, J. Shatah, and W. Strauss. Stability theory of solitary waves in the presence of symmetry. I. *J. Funct. Anal.*, 74(1):160–197, 1987.

[His00] P. D. Hislop. Exponential decay of two-body eigenfunctions: a review. In *Proceedings of the Symposium on Mathematical Physics and Quantum Field Theory (Berkeley, CA, 1999)*, volume 4 of *Electron. J. Differ. Equ. Conf.*, pages 265–288 (electronic), San Marcos, TX, 2000. Southwest Texas State Univ.

[HS00] W. Hunziker and I. M. Sigal. Time-dependent scattering theory of N-body quantum systems. *Rev. Math. Phys.*, 12(8):1033–1084, 2000.

[IM99] A. Iftimovici and M. Măntoiu. Limiting absorption principle at critical values for the Dirac operator. *Lett. Math. Phys.*, 49(3):235–243, 1999.

[Kat66] T. Kato. Wave operators and similarity for some non-selfadjoint operators. *Math. Ann.*, 162:258–279, 1965/1966.

[KS05] J. Krieger and W. Schlag. Stable manifolds for all supercritical monic nls in one dimension. preprint, 2005.

[KT98] M. Keel and T. Tao. Endpoint Strichartz estimates. *Amer. J. Math.*, 120(5):955–980, 1998.

[Par90] B. Parisse. Résonances paires pour l'opérateur de Dirac. *C. R. Acad. Sci. Paris Sér. I Math.*, 310(5):265–268, 1990.

[PW97] C.-A. Pillet and C. E. Wayne. Invariant manifolds for a class of dispersive, Hamiltonian, partial differential equations. *J. Differential Equations*, 141(2):310–326, 1997.

[Ran] A. F Ranada. Classical nonlinear dirac field models of extended particles. In *Quantum theory, groups, fields and particles,*, volume 198, pages 271–291. A. O. Barut, Amsterdam, Reidel.

[RS78] M. Reed and B. Simon. *Methods of modern mathematical physics. IV. Analysis of operators.* Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1978.

[Sch04] W. Schlag. Stable manifolds for an orbitally unstable nls. preprint, 2004.

[SS85] J. Shatah and W. Strauss. Instability of nonlinear bound states. *Comm. Math. Phys.*, 100(2):173–190, 1985.

[SW90] A. Soffer and M. I. Weinstein. Multichannel nonlinear scattering for nonintegrable equations. *Comm. Math. Phys.*, 133(1):119–146, 1990.

[SW92] A. Soffer and M. I. Weinstein. Multichannel nonlinear scattering for nonintegrable equations. II. The case of anisotropic potentials and data. *J. Differential Equations*, 98(2):376–390, 1992.

[SW04] A. Soffer and M. I. Weinstein. Selection of the ground state for nonlinear Schrödinger equations. *Rev. Math. Phys.*, 16(8):977–1071, 2004.

[SW05] A. Soffer and M. I. Weinstein. Theory of Nonlinear Dispersive Waves and Selection of the Ground State. *Physical Review Letters*, 95(21):213905–+, November 2005.

[Tha92] B. Thaller. *The Dirac equation.* Texts and Monographs in Physics. Springer-Verlag, Berlin, 1992.

[Tri78] H. Triebel. *Interpolation theory, function spaces, differential operators*, volume 18 of *North-Holland Mathematical Library.* North-Holland Publishing Co., Amsterdam, 1978.

[Tsa03] T.-P. Tsai. Asymptotic dynamics of nonlinear Schrödinger equations with many bound states. *J. Differential Equations*, 192(1):225–282, 2003.

[TY02a] T.-P. Tsai and H.-T. Yau. Asymptotic dynamics of nonlinear Schrödinger equations: resonance-dominated and dispersion-dominated solutions. *Comm. Pure Appl. Math.*, 55(2):153–216, 2002.

[TY02b] T.-P. Tsai and H.-T. Yau. Classification of asymptotic profiles for nonlinear Schrödinger equations with small initial data. *Adv. Theor. Math. Phys.*, 6(1):107–139, 2002.

[TY02c] T.-P. Tsai and H.-T. Yau. Relaxation of excited states in nonlinear Schrödinger equations. *Int. Math. Res. Not.*, (31):1629–1673, 2002.

[TY02d] T.-P. Tsai and H.-T. Yau. Stable directions for excited states of nonlinear Schrödinger equations. *Comm. Partial Differential Equations*, 27(11-12):2363–2402, 2002.

[Wed00] R. Weder. Center manifold for nonintegrable nonlinear Schrödinger equations on the line. *Comm. Math. Phys.*, 215(2):343–356, 2000.

[Wei85] M. I. Weinstein. Modulational stability of ground states of nonlinear Schrödinger equations. *SIAM J. Math. Anal.*, 16(3):472–491, 1985.

[Wei86] M. I. Weinstein. Lyapunov stability of ground states of nonlinear dispersive evolution equations. *Comm. Pure Appl. Math.*, 39(1):51–67, 1986.