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AN EXAMPLE OF STICK-SLIP WAVES

Franck Moiro

PSA, Centre Technique La Garenne-Colombe

Paris, France

Quoc-Son Nguyen

Laboratoire de Mécanique des Solides, CNRS-umr7649

Ecole Polytechnique, Paris, France

An analytical solution representing a family of stick-slip waves is obtained in a simple example modelling the dynamic behaviour of an elastic cylindrical tube in contact with Coulomb's friction with a rigid and rotating cylinder. This family of waves, representing the non-trivial periodic responses of a continuous system of one space variable, is not classical in the literature.

1. Introduction

It is well known that in frictional contact of solids, a contact point may have a slip or stick or separation regime according to Coulomb's law of dry friction. For dynamical problems, the study of the propagation of these zones on the contact surface may be interesting in different applications : ultrasonic motors (cf. Zharii, 1996), seismic predictions and simulations (cf. Cochard and Madariaga, 1995), mechanical vibrations and instabilities induced by friction (cf. Martins and Simoes, 1995, Vola et al., 1999, Oancea and Laursen, 1997, Oestreich et al., 1996). In the particular case of a solid in contact with a moving obstacle, some recent results of the literature on the flutter instability of the steady sliding solution (cf. for example Adams, 1995 or Martins and Simoes, 1995 or Vola et al., 1999) have suggested that we investigate dynamic periodic responses. A possible transition of this response to a cyclic response in the spirit of Poincaré-Andronov-Hopf bifurcation, cf. (Nguyen, 2000), is the long term goal. As an example, the study of the phenomenon of brake squeal has been discussed in this context, Moiro, 1998. A simple modelling of a drum brake leads to the study of an academic example of

encased cylinders, and has enabled us to obtain an interesting example of stick-slip waves. Such an example is new in the literature.

2. An encased-cylinder problem and the reduced equations

The mechanical response in plane strain of an elastic cylinder, of internal radius R and external radius R^* , in frictional contact on its inner surface with a rotating rigid cylinder of radius $R + d$ and of angular rotation Ω as shown in Figure 1, is considered when the displacement is assumed to be homogeneous on its outer surface. Coulomb's law of dry friction is assumed with a constant friction coefficient f . An interesting approximation reducing this two-dimensional problem to an one-dimensional problem of contact with friction is discussed here. Our objective is to demonstrate the existence of periodic dynamic responses in the form of stick-slip waves. Although stick-slip solutions have often been discussed in statics, we emphasize that such a non-trivial example of stick-slip vibrations is not classically known in the literature for continuous systems. The construction of periodic dynamic responses has principally been discussed numerically, (cf. for example Oancea and Laursen, 1997, Oestreich et al., 1996, Vola et al., 1999), the work of Zharii, 1996 on surface waves generated by piezo-electrical sources on a half-space boundary in contact with an obstacle remains an exception, and his results deal only with slip waves. Non-dimensional variables are introduced $u = \frac{\bar{u}}{R}$, $\sigma = \frac{\bar{\sigma}}{E}$, $r = \frac{\bar{r}}{R}$, $\gamma = \frac{\rho R^2 \Omega^2}{E}$, $\xi = \frac{R^*}{R}$, $\delta = \frac{d}{R}$, $t = \Omega \bar{t}$, $\dot{u} = \frac{du}{dt}$. Governing equations are the kinetic relations, the fundamental law, the linear elastic constitutive equations, the boundary conditions, the unilateral contact conditions and Coulomb's laws :

$$\left\{ \begin{array}{l} \epsilon = (\nabla u)_s, \\ \text{Div } \sigma = \gamma \ddot{u}, \\ \sigma = \frac{\nu}{(1+\nu)(1-2\nu)} \text{Tr}(\epsilon) I + \frac{1}{1+\nu} \epsilon, \\ u(\xi, \theta, t) = v(\xi, \theta, t) = 0, \\ \sigma_{rr}(1, \theta, t) = -p(\theta, t), \quad \sigma_{r\theta}(1, \theta, t) = -q(\theta, t), \\ u \geq \delta, \quad p \geq 0, \quad p(u - \delta) = 0, \\ |q| \leq fp, \quad q(1 - \dot{v}) - fp|1 - \dot{v}| = 0. \end{array} \right. \quad (1)$$

The steady sliding solution is

$$\left\{ \begin{array}{l} u_e = \delta \frac{1}{\xi^2 - 1} \left(\frac{\xi^2}{r} - r \right), \quad v_e = \delta f \frac{1}{\xi^2 - 1} \left(\frac{\xi^2}{r} - r \right) \left(1 + \frac{1}{\xi^2(1-2\nu)} \right), \\ p_e = \delta \frac{1}{\xi^2 - 1} \frac{1}{1+\nu} \left(\xi^2 + \frac{1}{1-2\nu} \right) > 0, \quad q_e = fp_e. \end{array} \right. \quad (2)$$

An interesting modelling of the problem is obtained when the displacement is approximated in the form

$$u = U(\theta, t)X(r), \quad v = V(\theta, t)X(r), \quad X(r) = \frac{1}{\xi^2 - 1} \left(\frac{\xi^2}{r} - r \right). \quad (3)$$

In this approximation, the following local equations are obtained from virtual work equation

$$\begin{cases} \ddot{U} - bU'' - DV' + gU = P, \\ \ddot{V} - aV'' + DU' + hV = Q, \\ P \geq 0, \quad U - \delta \geq 0, \quad P(U - \delta) = 0, \\ |Q| \leq fP, \quad Q(1 - \dot{V}) - fP|1 - \dot{V}| = 0. \end{cases} \quad (4)$$

where $'$ denotes the derivative with respect to θ and $a = \frac{\tilde{a}A}{\gamma B}$, $b = \frac{\tilde{b}A}{\gamma B}$, $g = \frac{2\tilde{a} + 2(\xi^2 - 1)\tilde{b}}{\gamma B}$, $h = \frac{2\xi^2\tilde{b}}{\gamma B}$, $\tilde{a} = \frac{1 - \nu}{(1 + \nu)(1 - 2\nu)}$, $\tilde{b} = \frac{1}{2(1 + \nu)}$, $A = -\frac{2\xi^2 \ln \xi}{\xi^2 - 1} + \frac{1 + \xi^2}{2} > 0$, $B = \frac{\xi^4 \ln \xi}{\xi^2 - 1} + \frac{1 - 3\xi^2}{4} > 0$, $D = \frac{aC_1 - bC_2}{A}$, $C_1 = \frac{2\xi^2 \ln \xi}{\xi^2 - 1} - 1 > 0$, $C_2 = -\frac{2\xi^2 \ln \xi}{\xi^2 - 1} - 1 + 2\xi^2 > 0$. The coupling coefficient D between the normal and tangential displacements can be positive or negative according to the values of ν and ξ . The steady sliding solution is given by $U_e = \delta$, $V_e = \delta f g / h$, $P = P_e$ and $Q_e = f P_e$.

The steady sliding response is unstable, cf. Martins and Simoes, 1995 or Adams, 1995. The proof of this result is straightforward under the assumption of sliding motions. In this case a small perturbation motion is described by $U = U_e$, $V = V_e + V_*$, $P = P_e + P_*$ and $Q = Q_e + Q_*$. It follows that

$$\ddot{V}_* - aV_*'' + fDV_*' + hV_* = 0. \quad (5)$$

A general solution is sought in the form $V_* = e^{st} e^{ik\theta}$. It follows that $-s^2 = ak^2 + h + ikfD$. If $f = 0$, then $s = \pm i\omega_k$ with $\omega_k^2 = ak^2 + h$. Thus two harmonic waves propagating in opposite senses of the form $\cos(k\theta \pm \omega_k t + \varphi)$ are obtained as in classical elasticity. If $f > 0$ and $D > 0$, then $s = \pm(s_{rk} + is_{ik})$, $s_{rk} > 0$, $s_{ik} < 0$, thus a general solution of the form $V_* = e^{\pm s_{rk} t} \cos(k\theta \pm s_{ik} t + \varphi)$ is obtained and represents two waves propagating in opposite senses : an exploding wave in the sense of the implied rotation, and a damping wave propagating in the opposite sense. If $f > 0$ and $D < 0$, the exploding wave propagates in the opposite sense since the previous expression of s is still valid with $s_{rk} > 0$ and $s_{ik} > 0$.

3. Existence of stick-slip waves

It is expected that in some particular situations, there is a dynamic bifurcation of Poincaré-Andronov-Hopf's type. This means that since the

steady sliding response is unstable and there is a flutter instability, the perturbed motion may eventually become a periodic response. This transition has been obtained numerically in many examples, cf. for example Oestreich et al., 1996 or Vola et al., 1999, but a mathematical proof is still lacking since the classical proof in Hopf's theorem cannot be applied to the dynamic equations with unilateral contact and dry friction. The existence of possible periodic dynamic solutions is our centre of interest. A periodic solution is sought in the form of a wave propagating at constant velocity :

$$U = U(\phi), \quad V = V(\phi), \quad \phi = \theta - ct \quad (6)$$

where c is the non-dimensional wave velocity, U and V are periodic functions of period $T = \frac{2\pi}{k}$. The physical velocity of the wave is thus $\bar{c} = |c|R\Omega$ and the associated dynamic response is periodic of frequency $|c|k\Omega$. The propagation occurs in the sense of the rotation when $c > 0$. According to the regime of contact, a slip wave, a stick-slip wave, a slip-separation wave or a stick-slip-separation wave can be discussed. Only the two first cases will be discussed here. The governing equations of such a wave follow from (1) :

$$\begin{cases} (c^2 - b)U'' - DV' + gU = P, \\ (c^2 - a)V'' + DU' + hV = Q, \\ P \geq 0, \quad U \geq \delta, \quad P(U - \delta) = 0, \\ |Q| \leq fP, \quad Q(1 - \dot{V}) - fP|1 - \dot{V}| = 0. \end{cases} \quad (7)$$

For a slip wave, there is no separation and $Q = fP$ everywhere when there is positive slip ($1 - \dot{V} \geq 0$). Thus the governing equations of a positive slip wave are :

$$U = \delta, \quad -DV' + g\delta = P, \quad (c^2 - a)V'' + hV = fP \quad (8)$$

It follows that $V = V_e + \Delta$ with

$$(c^2 - a)\Delta'' + fD\Delta' + h\Delta = 0 \quad (9)$$

Thus $\Delta = e^{r\phi}$ with $r^2 - 2\alpha r + \beta = 0$, $\alpha = \frac{-fD}{2(c^2 - a)}$, $\beta = \frac{h}{c^2 - a}$. It follows that $r = \alpha \pm \sqrt{\alpha^2 - \beta}$. This expression shows that no periodic solution can be obtained and thus no periodic response under the form of a positive slip wave can be found.

A solution of (7) in the form of a stick-slip wave is now sought. It is assumed that there is a positive slip regime in the interval $(0, \Psi T)$ and a stick regime in the interval $(\Psi T, T)$:

- in the stick region $(\Psi T, T)$, governing equations are

$$U = \delta, \quad V' = -\frac{1}{c}, \quad P = \frac{D}{c} + g\delta > 0, \quad Q = hV, \quad (10)$$

- in the slip region $(0, \Psi T)$, governing equations are

$$U = \delta, \quad P = g\delta - DV' \geq 0, \quad Q = fP, \quad 1 + cV' \geq 0, \quad (11)$$

$$(c^2 - a)V'' + fDV' + hV - fg\delta = 0. \quad (12)$$

The last equation shows that $V = V_e + \Delta$ in the interval $(0, \Psi T)$ with (9). The case $\alpha^2 < \beta$ is considered thus $c^2 > a + \frac{f^2 D^2}{4h}$. Let $\omega = \sqrt{\beta - \alpha^2}$. Since $r = \alpha \pm i\omega$, the general expression of Δ is $\Delta = e^{\alpha\phi}(M \cos \omega\phi + N \sin \omega\phi)$. There are 4 unknowns (c, Ψ, M, N) . The continuity of the displacement gives $V(0) = V(T)$. The stick region leads to two supplementary equations $V'(0) = V'(\Psi T) = -1/c$. Thus, the following system of equations must be satisfied :

$$\begin{cases} M = e^{\alpha\Psi T}(M \cos \omega\Psi T + N \sin \omega\Psi T) - \frac{T(1-\Psi)}{c}, \\ \frac{-1}{c} = \alpha M + \omega N, \\ \frac{-1}{c} = \alpha e^{\alpha\Psi T}(M \cos \omega\Psi T + N \sin \omega\Psi T) + \\ \omega e^{\alpha\Psi T}(-M \sin \omega\Psi T + N \cos \omega\Psi T) \end{cases} \quad (13)$$

$$\begin{cases} c^2 > a + \frac{f^2 D^2}{4h}, \\ |fg\delta + h\Delta| < f(\frac{D}{c} + g\delta) \quad \forall \phi \in (\Psi T, T), \\ g\delta - D\Delta' \geq 0, \quad 1 + c\Delta' \geq 0 \quad \forall \phi \in (0, \Psi T). \end{cases} \quad (14)$$

A family of stick-slip waves is considered for which the continuity of Q at $\phi = T$ is ensured :

$$Q(T_-) = Q(0_+). \quad (15)$$

This condition implies that $hV(T_-) = fg\delta + \frac{fD}{c}$ and leads to the following system of governing equations and inequalities after the elimination of M and N :

$$\begin{cases} T(1 - \Psi) + \frac{1}{\omega} e^{\alpha\Psi T} \sin \omega\Psi T = 0, \\ 1 + \frac{\alpha}{\omega} e^{\alpha\Psi T} \sin \omega\Psi T - e^{\alpha\Psi T} \cos \omega\Psi T = 0, \\ \alpha = \frac{-fD}{2(c^2 - a)}, \quad \beta = \frac{h}{c^2 - a}, \quad \omega = \sqrt{\beta - \alpha^2}, \quad c^2 > a + \frac{f^2 D^2}{4h}, \\ |fg\delta + h\Delta_\Psi| \leq \frac{fD}{c} + fg\delta, \quad 1 + c\Delta' \geq 0. \end{cases} \quad (16)$$

The equations can be solved numerically by Mathematica for different values of k and f . It is found that c must have the sign of D ; these waves propagate in the sense of the previous exploding perturbed motions. The value of δ must be strong enough to avoid possible separations.

The results (phase diagram, displacement and reactions) are given respectively in Figure 2 and Figure 3 for the particular case of a common metal with $f = 1$, $k = 8$ and $D < 0$. In this case, the physical velocity is $\bar{c} = 1255m/s$ and the wave propagates in the opposite sense to the implied rotation.

4. Transition of the steady sliding to a stick-slip wave after perturbations

Since the steady sliding solution is unstable, a small or finite perturbation of this solution will lead to a different response. Some numerical calculations of the system (4) have been performed in order to study the possibility of transition to a periodic response. It has been found that the rate of convergence and the limit response depend strongly on the initial conditions. Some stick-slip solutions are stable and present a specific domain of attraction. Figures 4,5 and 6 present the numerical results obtained after a direct integration of the system following an explicit scheme of time integration with a common regularization of the friction force-slip velocity relation.

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Figure 1 – An example of encased cylinders with a stick-slip wave in the case $k = 2$.

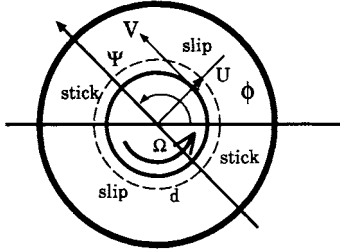


Figure 2 – The phase diagram of the periodic solution associated with a stick-slip wave is given for $k = 8$, $f = 1$ and $\delta = 0.005$.

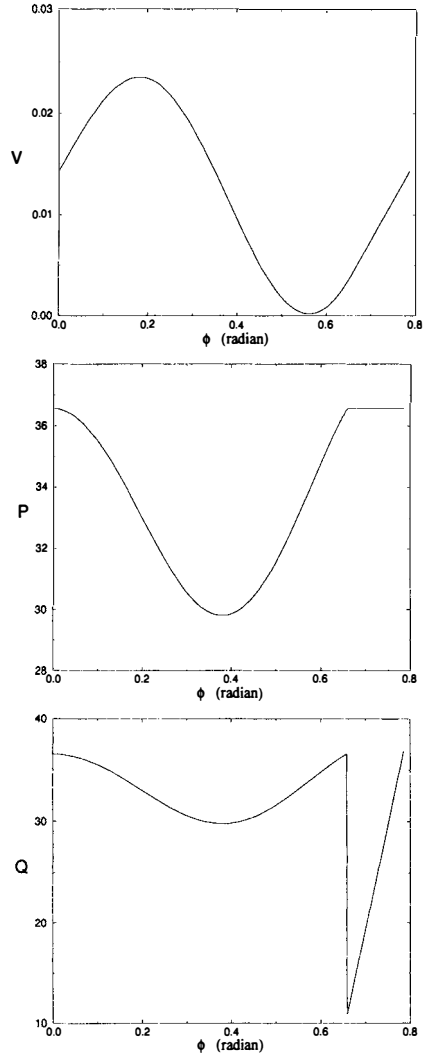
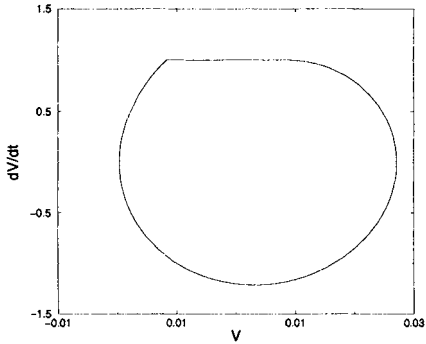


Figure 3 – An example of stick-slip wave with $k = 8$, $f = 1$ and $\delta = 0.005$. In this case, $\Psi = 0.839$, $c = 12.55$. The variations of V , P and Q are given as a function of $0 \leq \phi \leq 2\pi/k$.

Figure 4 – A numerical illustration of the transition to a periodic limit response. A mode-3 stick-slip wave is obtained here from a particular initial condition of displacement and velocity

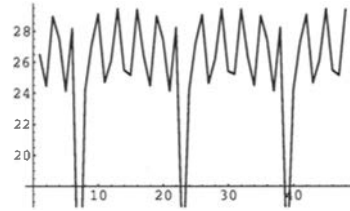
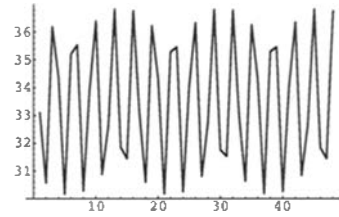
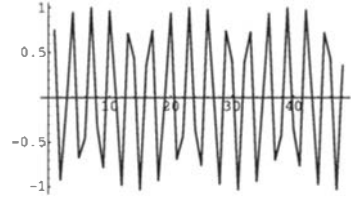
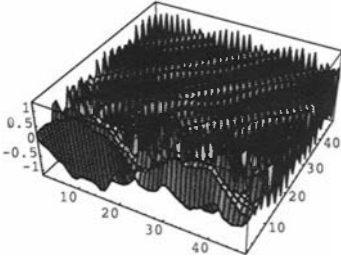


Figure 5 – The phase diagram of the limit response (mode-3 stick-slip wave)

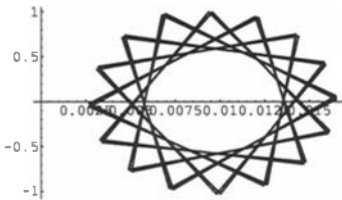


Figure 6 – Limit response : relative velocity, normal pressure and tangent force obtained by numerical calculations