A general algorithm for the direct numerical evaluation of element integrals in the 2D symmetric Galerkin BEM

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A GENERAL ALGORITHM FOR THE DIRECT NUMERICAL EVALUATION OF ELEMENT INTEGRALS IN THE 2D GALERKIN BEM

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Keywords: Symmetric Galerkin BEM, singular integration

Abstract This paper deals with a direct approach for the evaluation of singular element integrals arising from the discretization of hypersingular boundary integral equations (BIEs) in symmetric Galerkin form, for two-dimensional problems.

1. INTRODUCTION

The direct evaluation of element integrals arising from the discretization of strongly singular or hypersingular boundary integral equations (BIEs) in collocation form, summarized in [4], is well-established and widely used.

In this paper, the direct approach is developed for hypersingular BIEs in weighted-residual form, i.e. for Galerkin boundary element methods (GBEMs). In line with previous works, the limiting form of the integral identity as a small parameter ε vanishes must be derived, and this is again done *after* discretization. References [1, 6] implement, in a different way, the same general principles.

The present investigation differs from previous ones in that GBEMs involve *double* element integrals. Hence, two cases of potentially singular integrations arise: coincident (double integral on twice the same element) and adjacent (double integral on a pair of touching elements). The algorithm presented in this communication deals with both the coincident and adjacent cases, for 2D problems. Indeed, it appears that

they must be considered together: each one generates a potentially infinite term, which cancel each other. Double integrals are treated as a whole (i.e. not as inner singular integrals followed by outer nonsingular ones), through the introduction of suitable coordinate transformations in the *two-dimensional* space of intrinsic coordinates.

The usual direct approach to collocation BEM does not resort to prior regularization [4]. Similarly, the method presented here is not based on any prior analytical manipulation, like the double integration by parts on the hypersingular kernel which is used in other contributions, see e.g. [2].

The proposed algorithm is in particular applicable to symmetric GBEMs, which are under intense investigation (see the survey paper [3]), and is devised so as to define in that case a perfectly symmetric integration procedure, even when the numerical quadrature is not exact. However, its definition is valid whether the GBEM formulation considered is symmetric or not, the key features being the *hypersingularity* of the kernels involved and the *double* integration.

2. INTEGRAL STATEMENTS

Let Ω be a 2D bounded domain with boundary $\Gamma = \partial\Omega$ (possibly with a finite number of corners). With $\mathbf{n}(\mathbf{y})$ and $\mathbf{t}(\mathbf{y})$ we indicate, respectively, the normal and tangent unit vectors at $\mathbf{y} \in \Gamma$ (Figure 1).

The starting point is the 3rd Green identity for φ and $q = \partial\varphi/\partial n$ on the punctured domain Ω_ε, with boundary $\Gamma - e_\varepsilon + s_\varepsilon$ (Figure 1). In this case we deal with the hypersingular BIE

$$\int_{\Gamma - e_\varepsilon + s_\varepsilon} \left[V(\mathbf{x}, \mathbf{y})\varphi(\mathbf{y}) - W(\mathbf{x}, \mathbf{y})q(\mathbf{y}) \right] ds_y = 0. \tag{1}$$

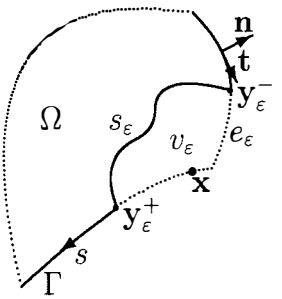

Figure 1 Exclusion of the singular point $\mathbf{x}$ by a vanishing neighbourhood v_ε.

We assume for simplicity that the free-plane fundamental solution for the Laplace equation is being used, i.e. $G(\mathbf{x},\mathbf{y}) = -1/(2\pi)\ln r$ and hence

$$\begin{aligned} W(\mathbf{x},\mathbf{y}) &= -G_{,i}(\mathbf{x},\mathbf{y})n_i(\mathbf{x}) = \frac{1}{2\pi}\frac{r_i n_i(\mathbf{x})}{r^2}, \\ V(\mathbf{x},\mathbf{y}) &= W_{,j}n_j(\mathbf{y}) = -G_{,ij}(\mathbf{x},\mathbf{y})n_i(\mathbf{x})n_j(\mathbf{y}) = O(r^{-2}), \end{aligned} \tag{2}$$

where $\mathbf{r} = \mathbf{y} - \mathbf{x}$, $r = |\mathbf{r}|$ and r_i is the i-th Cartesian component of $\mathbf{r}$.

Let $\psi(\mathbf{x})$ be any sufficiently regular function defined on Γ. *Before* taking the limit for $\varepsilon \to 0$, we can use $\psi(\mathbf{x})$ to weigh the integral identity (1):

$$\int_\Gamma \psi(\mathbf{x}) \left\{ \int_{\Gamma - e_\varepsilon + s_\varepsilon} \left[V(\mathbf{x},\mathbf{y})\varphi(\mathbf{y}) - W(\mathbf{x},\mathbf{y})q(\mathbf{y}) \right] ds_y \right\} ds_x = 0. \tag{3}$$

As usual the density function φ is assumed to be $C^{(1,\alpha)}$ at $\mathbf{x}$. For the selected fundamental solution the integrals on s_ε can be evaluated analytically. This step is not influenced by the presence of the weight function ψ and has to be performed for the collocation BEM as well.

In principle any shape may be used for v_ε, since the overall result will be independent of this shape. Therefore, it is just more convenient to select a circular shape or, more precisely, $|y_\varepsilon^+ - \mathbf{x}| = |y_\varepsilon^- - \mathbf{x}| = \varepsilon$, thereby avoiding a $\ln\varepsilon$ contribution from $\left[G_{,i}(\mathbf{x},\mathbf{y})\right]_{y_\varepsilon^-}^{y_\varepsilon^+}$. The direct SGBEM formulation is thus sought as the limiting form as $\varepsilon \to 0$ of the identity

$$\begin{aligned} 0 = \int_\Gamma \psi(\mathbf{x}) \Bigg\{ & \frac{1}{2}q(\mathbf{x}) + \varphi(\mathbf{x})t_i(\mathbf{x})\left[G_{,i}(\mathbf{x},\mathbf{y})\right]_{y_\varepsilon^-(\mathbf{x})}^{y_\varepsilon^+(\mathbf{x})} \\ & + \int_{\Gamma - e_\varepsilon} \Big[V(\mathbf{x},\mathbf{y})\varphi(\mathbf{y}) - W(\mathbf{x},\mathbf{y})q(\mathbf{y}) \Big] ds_y \Bigg\} ds_x + o(1), \end{aligned} \tag{4}$$

where $o(1)$ accounts for all vanishing contributions. It should be noted that within curly braces we have precisely the hypersingular boundary integral equation as obtained for the collocation BEM by Guiggiani et al [5].

3. PRELIMINARY DEFINITIONS

Let the boundary Γ be modeled by (curvilinear) boundary elements and let $\mathbf{y} \in E$ and $\mathbf{x} \in \tilde{E}$, E and $\tilde{E}$ being two such elements. In the Galerkin BEM (regardless of its symmetry) we have to deal with double integrals on $\tilde{E} \times E$.

If the two elements are *disjoint*, that is, do not share a common end-point, the double integration is performed using ordinary means since the integrand is continuous.

On the other hand, singularities in the integrand function arise either when the two elements share one common endpoint (i.e., they are *adjacent*), or when they are *coincident* (i.e., $E = \tilde{E}$). In these cases the appropriate setting is

$$\int_{\tilde{E}} \int_{E-e_\varepsilon(\mathbf{x})} \{\cdots\} ds_y ds_x,$$

where, although not strictly necessary, we can use the following restricted definition for $e_\varepsilon(\mathbf{x})$

$$e_\varepsilon(\mathbf{x}) = \{\mathbf{y} \in E : |\mathbf{y} - \mathbf{x}| \leq \varepsilon, \text{ with } \mathbf{x} \in \tilde{E}\}, \tag{5}$$

Each boundary element is analytically defined by means of suitable parametric equations. Typically the Cartesian coordinates of a point $\mathbf{y}$ of E are given by $\mathbf{y} = \mathbf{y}(\xi) = \sum_{p=1}^{N_e} N_p(\xi)\mathbf{z}^p$, where $-1 \leq \xi \leq 1$ is the parameter (or intrinsic coordinate), $N_p(\xi)$ are cardinal shape functions and $\mathbf{z}^p$ are the geometric nodes. We will denote by $\mathbf{a}(\xi)$ and $\mathbf{t}(\xi)$, respectively, the natural and normalized tangent vector to an element:

$$\mathbf{a}(\xi) = \frac{d\mathbf{y}}{d\xi} = \sum_{p=1}^{N_e} N_p'(\xi)\mathbf{z}^p, \qquad \text{so that} \qquad \mathbf{t}(\xi) = \frac{\mathbf{a}(\xi)}{|\mathbf{a}(\xi)|} = \frac{\mathbf{a}(\xi)}{a(\xi)}, \tag{6}$$

and by $\mathbf{m}(\xi)$ and $\mathbf{n}(\xi)$ the natural and normalized normal vectors:

$$\mathbf{m}(\xi) = \mathbf{e}_3 \wedge \mathbf{a}(\xi), \qquad \mathbf{n}(\xi) = \frac{\mathbf{m}(\xi)}{|\mathbf{m}(\xi)|} = \frac{\mathbf{m}(\xi)}{a(\xi)}. \tag{7}$$

Moreover we have $\mathbf{n}(\mathbf{y})ds_y = \mathbf{n}(\xi)J(\xi)d\xi = \mathbf{m}(\xi)d\xi$. Also define

$$\mathbf{b}(\xi) = \frac{d\mathbf{a}}{d\xi} = \frac{d^2\mathbf{y}}{d\xi^2} = \sum_{p=1}^{N_e} N_p''(\xi)\mathbf{z}^p. \tag{8}$$

4. COINCIDENT ELEMENTS

Let $\mathbf{y}$ and $\mathbf{x}$ belong to the same boundary element $E = \tilde{E}$. We will consider the hypersingular integral

$$I_C = \int_E \int_{E-e_\varepsilon(\mathbf{x})} V_{ij}(\mathbf{y} - \mathbf{x}) n_i(\mathbf{x}) n_j(\mathbf{y}) \psi(\mathbf{x}) \varphi(\mathbf{y}) ds_y ds_x, \tag{9}$$

where $V_{ij}(\mathbf{y} - \mathbf{x}) = -G_{,ij}(\mathbf{y} - \mathbf{x})$ and

$$e_\varepsilon(\mathbf{x}) = \{\mathbf{y} \in E : |\mathbf{y} - \mathbf{x}| \leq \varepsilon, \text{ with } \mathbf{x} \in E\}. \tag{10}$$

The distance vector $\mathbf{r}$ now becomes

$$\mathbf{r} = \mathbf{y}(\xi) - \mathbf{x}(\eta) = \sum_{p=1}^{N_e} \big[N_p(\xi) - N_p(\eta)\big]\mathbf{z}^p, \tag{11}$$

and $(r = 0) \Leftrightarrow (\mathbf{y} = \mathbf{x}) \Leftrightarrow (\xi = \eta)$. Therefore, in the parameter space we have to integrate over the square $(\eta, \xi) \in [-1, 1] \times [-1, 1]$ minus a (narrow, non uniform) strip across the $\xi = \eta$ diagonal. The strip is the image of $E \times e_\varepsilon(\mathbf{x})$.

The double integration (9) will be treated by subdividing the square $(\eta, \xi) \in [-1, 1] \times [-1, 1]$ into two triangular regions, labeled (1) and (2). In each region a new pair of coordinates (u, v) is introduced according to the following scheme

$$\begin{aligned} &\text{region (1):} \quad \begin{cases} \eta = u(1 - v) - v \\ \xi = u(1 - v) + v \end{cases} \qquad \text{with } u \in [-1, 1],\ v \in [0, 1], \\ &\text{region (2):} \quad \begin{cases} \eta = u(1 + v) - v \\ \xi = u(1 + v) + v \end{cases} \qquad \text{with } u \in [-1, 1],\ v \in [-1, 0]. \end{aligned} \tag{12}$$

It should be noted that in both cases if $v = 0$ we have $\xi = \eta = u$. Another feature of these coordinate transformations is that u and v vary between fixed values, that is the range of variation of each coordinate does not depend on the other one. Also useful are the relations $d\xi d\eta = 2(1 \mp v)dudv$.

In terms of the new coordinates, the distance vector $\mathbf{r}$ is

$$\mathbf{r} = \mathbf{r}(u, v) = \sum_{p=1}^{N_e} \big[N_p(\xi(u, v)) - N_p(\eta(u, v))\big]\mathbf{z}^p. \tag{13}$$

We are also interested in the Taylor expansion of the function $\mathbf{r}(u, v)$ as a function of v and near $v = 0$

$$\begin{aligned} \mathbf{r}(u, v) &= 2\mathbf{a}(u)\, v \mp \frac{1}{2} 4u\mathbf{b}(u)\, v^2 + o(v^2) \\ &= 2v\big[\mathbf{a}(u) \mp uv\mathbf{b}(u) + o(v)\big] = 2v\hat{\mathbf{r}}(u, v), \end{aligned} \tag{14}$$

where, since $u = \xi = \eta$ when $v = 0$, $\mathbf{a}$ and $\mathbf{b}$ are precisely the functions defined in equations (6) and (8).

Similarly, the distance $r = |\mathbf{r}|$ has the following Taylor expansion

$$r = 2va(u)\big[\pm 1 - uvc(u) + o(v)\big] = \pm 2v\hat{r}(u, v), \tag{15}$$

where $a(u) = |\mathbf{a}(u)|$ and

$$c(u) = \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}}(u) = \frac{a'}{a}(u). \tag{16}$$

The integrand function in equation (9) as a function of u and v becomes

$$V_{ij}(\mathbf{r})n_i(\mathbf{x})n_j(\mathbf{y})\psi(\mathbf{x})\varphi(\mathbf{y})ds_y ds_x = \frac{1}{v^2}F(u,v)dudv \tag{17}$$

where $F(u,v)$ is, in each triangular region, a regular function. We will use $F^{(1)}(u,v)$ and $F^{(2)}(u,v)$ to refer to the restrictions of $F(u,v)$ on subregion (1) and (2), respectively.

The direct algorithm (Guiggiani et al. [5]) relies on a two-term Taylor expansion of $F(u,v)$ around $v=0$

$$F(u,v) = F(u,0) + \frac{\partial F}{\partial v}(u,0)\, v + O(v^2). \tag{18}$$

In the present case (Laplace equation) the two terms are found to be:

$$\begin{aligned} F(u,0) &= \frac{1}{4\pi}\psi(u)\varphi(u), \\ \frac{\partial F}{\partial v}(u,0) &= \frac{1}{4\pi}\Big\{\big[\psi\varphi' - \psi'\varphi\big](u) \mp \big[u\psi\varphi\big]'(u)\Big\}. \end{aligned} \tag{19}$$

It is worth noting that $F(1,0)=0$ if $\psi(1)\varphi(1)=0$, and $F(-1,0)=0$ if $\psi(-1)\varphi(-1)=0$.

The boundary of the exclusion neighbourhood $e_\varepsilon(\mathbf{x})$ is defined by condition (10), which together with expansion (15), leads to

$$\varepsilon = \pm 2v\hat{r}(u,v) = 2va(u)\big[\pm 1 - uvc(u) + o(v)\big], \tag{20}$$

which, upon reversion, gives

$$\begin{aligned} v_\varepsilon &= \pm\alpha(\varepsilon,u) = \pm\big[\varepsilon\beta(u) + \varepsilon^2\gamma(u) + o(\varepsilon^2)\big] \\ &= \pm\frac{\varepsilon}{2a(u)}\left[1 + \frac{uc(u)}{2a(u)}\varepsilon + o(\varepsilon)\right], \end{aligned} \tag{21}$$

where

$$\beta(u) = \frac{1}{2a(u)} \qquad \text{and} \qquad \gamma(u) = \frac{uc(u)}{(2a(u))^2}. \tag{22}$$

According to equations (17) and (21), the hypersingular double integral (9) becomes in terms of the coordinates u and v

$$\begin{aligned} I_C &= \int_E \int_{E-e_\varepsilon(\mathbf{x})} V_{ij}(\mathbf{y}-\mathbf{x})n_i(\mathbf{x})n_j(\mathbf{y})\psi(\mathbf{x})\varphi(\mathbf{y})ds_y ds_x \\ &= \int_{-1}^{1}\left\{\int_{-1}^{-\alpha(\varepsilon,u)} \frac{F^{(2)}(u,v)}{v^2}dv + \int_{\alpha(\varepsilon,u)}^{1} \frac{F^{(1)}(u,v)}{v^2}dv\right\}du. \end{aligned} \tag{23}$$

Following the direct algorithm, the first two terms of expansion (18) are added and subtracted in (23), where $F^{(1)}(u,0) = F^{(2)}(u,0)$. As typical in the direct method, all the potentially singular integrals are now trivial functions of v and can be integrated analytically.

Summing up, the following form of the double integral (23) is obtained

$$
\begin{aligned}
I_C = & \int_{-1}^{1}\int_{-1}^{0} \frac{1}{v^2}\Big\{F^{(2)}(u,v) - \Big[F^{(2)}(u,0) + \frac{\partial F^{(2)}}{\partial v}(u,0)\, v\Big]\Big\} dv du \\
& + \int_{-1}^{1}\int_{0}^{1} \frac{1}{v^2}\Big\{F^{(1)}(u,v) - \Big[F^{(1)}(u,0) + \frac{\partial F^{(1)}}{\partial v}(u,0)\, v\Big]\Big\} dv du \\
& + 2\int_{-1}^{1} F(u,0)\Big[\frac{1}{\varepsilon\beta(u)} - \frac{\gamma(u)}{\beta(u)^2} - 1\Big] du \\
& + \int_{-1}^{1}\Big[\frac{\partial F^{(2)}}{\partial v}(u,0) - \frac{\partial F^{(1)}}{\partial v}(u,0)\Big]\big[\ln|\varepsilon\beta(u)|\big] du + O(\varepsilon). \qquad (24)
\end{aligned}
$$

The distinctive feature of this expression is that all divergent terms are of ε-type. Of course they will eventually disappear.

5. ADJACENT ELEMENTS

Let E and $\tilde{E}$ be two *adjacent* boundary elements, with $\mathbf{y} \in E$ and $\mathbf{x} \in \tilde{E}$. By adjacent elements we mean two elements having a common endpoint. The singularity occurs when $\mathbf{y} = \mathbf{x}$ which may happen either when $(\eta, \xi) = (-1, 1)$ or when $(\eta, \xi) = (1, -1)$, depending on which element comes first on the boundary Γ.

As above, we will consider the integral

$$
I_A = \int_{\tilde{E}} \int_{E - e_\varepsilon(\mathbf{x})} V_{ij}(\mathbf{y}-\mathbf{x}) \tilde{n}_i(\mathbf{x}) n_j(\mathbf{y}) \tilde{\psi}(\mathbf{x}) \varphi(\mathbf{y}) ds_y ds_x, \qquad (25)
$$

where $V_{ij}(\mathbf{y}-\mathbf{x}) = -G_{,ij}(\mathbf{y}-\mathbf{x})$. The definition of $e_\varepsilon(\mathbf{x})$ was given in (5).

The distance vector $\mathbf{r}$ is such that $(r = 0) \Leftrightarrow (\mathbf{y} = \mathbf{x}) \Leftrightarrow (\xi\eta = -1)$. To make the distinction clearer, quantities defined on $\tilde{E}$ are also marked by a tilde. Therefore, in the parameter space we have to integrate over the square $(\eta, \xi) \in [-1, 1] \times [-1, 1]$ minus a small region around either vertex $(\eta, \xi) = (1, -1)$ or vertex $(-1, 1)$. This region is the image of $\tilde{E} \times e_\varepsilon(\mathbf{x})$.

The double integration (25) will be treated by subdividing again the square $(\eta, \xi) \in [-1, 1] \times [-1, 1]$ into two triangular regions, labeled (1) and (2). Moreover, the mappings $(\eta, \xi) \Leftrightarrow (u, v)$ defined in (12) could be used in this case as well.

Singularity at $(\eta,\xi)=(1,-1)$. In this case the integral over region (1) is not singular and we will consider in detail only the integral $I_A^{(2)}$ over region (2).

If the mappings defined in (12) are employed, the singularity would occur at $v=-1$. Therefore, it is merely a matter of convenience to shift the v coordinate in (12) so that the singular point is defined by $v=0$ and thus obtaining $d\xi d\eta = 2v du dv$.

As before, we are interested in the Taylor expansion of the function $\mathbf{r}(u,v) = \mathbf{y}(\xi(u,v)) - \mathbf{x}(\eta(u,v))$ as a function of v near $v=0$

$$\begin{aligned}\mathbf{r}(u,v) &= [(u+1)\mathbf{a}(-1) - (u-1)\tilde{\mathbf{a}}(+1)]\,v + o(v)\\ &= 2v\mathbf{s}(u) + o(v) = 2v\bar{\mathbf{r}}(u,v),\end{aligned} \tag{26}$$

where $\mathbf{a}(-1)=\mathbf{a}$ and $\tilde{\mathbf{a}}(1)=\tilde{\mathbf{a}}$ are the natural tangent vectors (6) to the elements E and $\tilde{E}$ at their common endpoint. Equation (26) defines the vector $\mathbf{s}(u)$. The distance $r=|\mathbf{r}|$ has the expansion

$$r = 2vs(u) + o(v) = 2v\bar{r}(u,v), \tag{27}$$

with $s(u)=|\mathbf{s}(u)|$ and $\bar{r}(u,v)=|\bar{\mathbf{r}}(u,v)|$.

The integrand function in (25) as a function of u and v becomes

$$V_{ij}(\mathbf{r})\tilde{n}_i(\mathbf{x})n_j(\mathbf{y})\tilde{\psi}(\mathbf{x})\varphi(\mathbf{y})ds_y ds_x = \frac{1}{v}Q(u,v)\,du dv, \tag{28}$$

where $Q(u,v)$ is a regular function.

Here a first-order Taylor expansion of $Q(u,v)$ around $v=0$ is sufficient: $Q(u,v)=Q(u,0)+o(1)$, where the term $Q(u,0)$ is easily found to be

$$Q(u,0) = \frac{1}{2}V_{ij}(\bar{\mathbf{r}}(u,0))\tilde{m}_i(1)m_j(-1)\tilde{\psi}(1)\varphi(-1). \tag{29}$$

It is worth noting that $Q(u,0)\equiv 0$ whenever $\tilde{\psi}(1)\varphi(-1)=0$. This observation has a strong relevance in the selection of suitable boundary elements in the Galerkin BEM with hypersingular kernels.

The boundary of the exclusion neighbourhood $e_\varepsilon(\mathbf{x})$ is defined by condition (5), which together with expansion (27), leads to

$$v_\varepsilon = \varepsilon\bar{\beta}(u) + o(\varepsilon) = \frac{\varepsilon}{2s(u)} + o(\varepsilon), \tag{30}$$

where $\bar{\beta}(u) = 1/(2s(u))$.

According to equations (28) and (30), the singular integral (25) over adjacent elements when E follows $\tilde{E}$ becomes

$$I_A = \int_{\tilde{E}} \int_{E-e_\varepsilon(\mathbf{x})} V_{ij}(\mathbf{y}-\mathbf{x})\tilde{n}_i(\mathbf{x})n_j(\mathbf{y})\tilde{\psi}(\mathbf{x})\varphi(\mathbf{y})ds_y ds_x = I_A^{(1)} + I_A^{(2)}$$

$$= I_A^{(1)} + \int_{-1}^{1} \int_{\varepsilon/(2s(u))}^{1} \frac{Q(u,v)}{v} du dv + O(\varepsilon). \quad (31)$$

Following the direct method, the first term of expansion of Q is added and subtracted in (31). Moreover, expansion (30) is considered and all singular integrals are performed analytically. The final result is

$$I_A^{(2)} = \int_0^1 \left\{ \int_{-1}^{1} \frac{Q(u,v)-Q(u,0)}{v} dv + Q(u,0)\ln|2s(u)| \right\} du$$

$$- \ln|\varepsilon| \int_{-1}^{1} Q(u,0)du + O(\varepsilon). \quad (32)$$

As in equation (24), the divergent term is of ε-type.

Singularity at $(\eta,\xi)=(-1,1)$. This time, the singularity in the integral (25) (with e_ε still defined in (5)) occurs when $(\eta,\xi)=(-1,1)$ (i.e., $\tilde{E}$ following E on γ). The integral over region (2) is then nonsingular, and essentially the same treatment as above is applied to the integral over region (1).

6. EVALUATION OF THE SINGULAR FREE TERM

Let us go back to the starting identity (4) for the direct SGBEM. It can be shown that the following result holds

$$\int_{\tilde{E}} \psi(\mathbf{x})\varphi(\mathbf{x})t_i(\mathbf{x})G_{,i}(\mathbf{x},\mathbf{y}_\varepsilon^+(\mathbf{x}))ds_x - \int_{\tilde{E}} \psi(\mathbf{x})\varphi(\mathbf{x})t_i(\mathbf{x})G_{,i}(\mathbf{x},\mathbf{y}_\varepsilon^-(\mathbf{x}))ds_x$$

$$= -\frac{1}{\pi\varepsilon}\int_{-1}^{1} \psi(u)\varphi(u)a(u)du + \frac{1}{2\pi}\left[\psi(1)\varphi(1)+\psi(-1)\varphi(-1)\right]$$

$$- \frac{\psi(1)\varphi(1)}{4\pi}\left[1+\frac{\theta^+\cos\theta^+}{\sin\theta^+}\right] - \frac{\psi(-1)\varphi(-1)}{4\pi}\left[1+\frac{\theta^-\cos\theta^-}{\sin\theta^-}\right] + o(1) \quad (33)$$

7. A NUMERICAL TEST

A MATLAB program has been written to test the present direct integration method. To demonstrate its accuracy, we present a comparison

between numerical and exact (analytical) values of double singular integrals of the form:

$$\int_{E+\tilde{E}} \int_{E+\tilde{E}-e_\varepsilon} W(\mathbf{x},\mathbf{y})\psi(\mathbf{x})\varphi(\mathbf{y})ds_y ds_x \tag{34}$$

where $E, \tilde{E}$ are straight boundary elements meeting at their common endpoint with an angle θ (figure 2) and ψ and φ are taken as the same piecewise linear 'hat function':

$$(\psi,\varphi)(\xi) = \frac{1+\xi}{2} \quad (\text{on } E) \qquad \text{and} \quad (\psi,\varphi)(\xi) = \frac{1-\xi}{2} \quad (\text{on } \tilde{E})$$

Integral (34) is scale-independent (this stems from the fact that the kernel $W(\mathbf{x},\mathbf{y})$ is homogeneous of degree -2 with respect to the position vector $\mathbf{y}-\mathbf{x}$), and hence depends only on θ and the element length ratio $\eta = |\tilde{E}|/|E|$). The exact value (in the limiting case as $\varepsilon \to 0$) of (34) is found to be:

$$-[2+(\eta+\eta^{-1})\cos\theta]\ln(2\cos\theta+\eta+\eta^{-1}) + (\eta-\eta^{-1})\cos\theta\ln\eta$$
$$-2\sin\theta\left[\eta\tan^{-1}\frac{\sin\theta}{(\cos\theta+eta)} + \eta^{-1}\tan^{-1}\frac{\sin\theta}{(\cos\theta+\eta^{-1})}\right]$$

The relative error, for various corner angles $\theta_k = k\pi/12$, $0 \le k \le 6$ (i.e. $0 \le \theta_k\pi/2$), are obtained as follows:

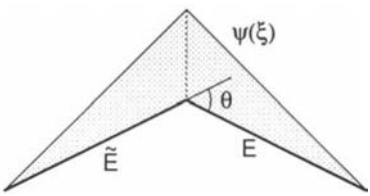

Figure 2 Adjacent straight boundary elements $E, \tilde{E}$ and the 'hat function'.

	$\theta\,(^0)$	$n_G = 4$	$n_G = 6$	$n_G = 8$	$n_G = 10$
$\eta = 1$	0	$3.56\,10^{-06}$	$4.98\,10^{-09}$	$5.96\,10^{-12}$	$2.86\,10^{-14}$
	15	$3.14\,10^{-06}$	$3.38\,10^{-09}$	$2.64\,10^{-12}$	$2.30\,10^{-14}$
	30	$2.52\,10^{-06}$	$6.49\,10^{-10}$	$-6.48\,10^{-13}$	$2.74\,10^{-14}$
	45	$-5.83\,10^{-06}$	$-1.30\,10^{-08}$	$-2.28\,10^{-11}$	$-1.30\,10^{-14}$
	60	$-1.65\,10^{-04}$	$-1.20\,10^{-06}$	$-8.03\,10^{-09}$	$-5.09\,10^{-11}$
	75	$-1.50\,10^{-03}$	$-2.88\,10^{-05}$	$-5.01\,10^{-07}$	$-8.22\,10^{-09}$
	90	$-9.28\,10^{-03}$	$-3.96\,10^{-04}$	$-1.53\,10^{-05}$	$-5.58\,10^{-07}$
$\eta = \sqrt{8}$	0	$-1.82\,10^{-04}$	$-1.12\,10^{-06}$	$-5.96\,10^{-09}$	$-2.78\,10^{-11}$
	15	$4.55\,10^{-05}$	$1.59\,10^{-06}$	$9.97\,10^{-09}$	$7.78\,10^{-12}$
	30	$4.55\,10^{-04}$	$-1.46\,10^{-06}$	$-3.60\,10^{-08}$	$2.18\,10^{-10}$
	45	$1.65\,10^{-04}$	$-1.33\,10^{-05}$	$2.52\,10^{-07}$	$-2.67\,10^{-09}$
	60	$-2.43\,10^{-03}$	$5.56\,10^{-05}$	$-1.07\,10^{-06}$	$1.74\,10^{-08}$
	75	$-9.67\,10^{-03}$	$4.31\,10^{-04}$	$-1.63\,10^{-05}$	$5.40\,10^{-07}$
	90	$-2.37\,10^{-02}$	$8.60\,10^{-04}$	$2.90\,10^{-05}$	$-7.60\,10^{-06}$

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