



**HAL**  
open science

# Averaging Theorems for Ordinary Differential Equations and Retarded Functional Differential Equations

Mustapha Lakrib, Tewfik Sari

► **To cite this version:**

Mustapha Lakrib, Tewfik Sari. Averaging Theorems for Ordinary Differential Equations and Retarded Functional Differential Equations. 2006. hal-00115539

**HAL Id: hal-00115539**

**<https://hal.science/hal-00115539>**

Preprint submitted on 21 Nov 2006

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# AVERAGING THEOREMS FOR ORDINARY DIFFERENTIAL EQUATIONS AND RETARDED FUNCTIONAL DIFFERENTIAL EQUATIONS

MUSTAPHA LAKRIB AND TEWFIK SARI

ABSTRACT. We prove averaging theorems for ordinary differential equations and retarded functional differential equations. Our assumptions are weaker than those required in the results of the existing literature. Usually, we require that the nonautonomous differential equation and the autonomous averaged equation are locally Lipschitz and that the solutions of both equations exist on some interval. We extend this result to the case of vector fields which are continuous in the spatial variable uniformly with respect to time and without any assumption on the interval of existence of the solutions of the nonautonomous differential equation. Our results are formulated in classical mathematics. Their proofs use nonstandard analysis.

## 1. INTRODUCTION

Averaging is an important method for analysis of nonlinear oscillation equations containing a small parameter. This method is well-known for ordinary differential equations (in short ODEs) and fundamental averaging results (see, for instance, [2, 5, 8, 23, 24] and references therein) assert that the solutions of a nonautonomous equation in *normal form*

$$x'(\tau) = \varepsilon f(\tau, x(\tau)), \quad (1.1)$$

where  $\varepsilon$  is a small positive parameter, are approximated by the solutions of the autonomous averaged equation

$$y'(\tau) = \varepsilon F(y(\tau)). \quad (1.2)$$

The approximation holds on time intervals of order  $1/\varepsilon$  when  $\varepsilon$  is sufficiently small. In (1.2), the function  $F$  is the average of the function  $f$  in (1.1) defined by

$$F(x) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(t, x) dt. \quad (1.3)$$

The method of averaging was extended by Hale [7] (see also Section 2.1 of [15]) to the case of retarded functional differential equations (in short RFDEs) containing a small parameter when the equations are considered in normal form

$$x'(\tau) = \varepsilon f(\tau, x_\tau), \quad (1.4)$$

where, for  $\theta \in [-r, 0]$ ,  $x_\tau(\theta) = x(\tau + \theta)$ . Equations of the form (1.4) cover a wide class of differential equations including those with pointwise delay for which a method of averaging was developed in [6, 20, 27]. Note that the averaged equation corresponding to (1.4) is the ODE

$$y'(\tau) = \varepsilon F(\tilde{y}^\tau), \quad (1.5)$$

---

2000 *Mathematics Subject Classification.* 34C29, 34C15, 34K25, 34E10, 34E18.

where, for  $\tau$  fixed and  $\theta \in [-r, 0]$ ,  $\tilde{y}^\tau(\theta) = y(\tau)$  and the average function  $F$  is defined by (1.3). Recently, Lehman and Weibel [16] proposed to retain the delay in the averaged equation and proved that equation (1.4) is approximated by the averaged RFDE

$$y'(\tau) = \varepsilon F(y_\tau). \quad (1.6)$$

They observed, using numerical simulations, that equation (1.4) is better approximated by the averaged RFDE (1.6) than by the averaged ODE (1.5). However, this equation depends nontrivially on the small parameter  $\varepsilon$  (see Remark 2.7).

The change from the slow time scale  $\tau$  to the fast time scale  $t = \tau/\varepsilon$  transforms equations (1.1) and (1.2), respectively, into

$$\dot{x}(t) = f(t/\varepsilon, x(t)) \quad (1.7)$$

and

$$\dot{y}(t) = F(y(t)). \quad (1.8)$$

Thus a method of averaging can be developed for (1.7), that is, if  $\varepsilon$  is sufficiently small, the difference between the solution  $x$  of (1.7) and the solution  $y$  of (1.8), with the same initial condition, is small on finite time intervals.

The analog of equation (1.7) for RFDEs is

$$\dot{x}(t) = f(t/\varepsilon, x_t). \quad (1.9)$$

The averaged equation corresponding to (1.9) is the RFDE

$$\dot{y}(t) = F(y_t),$$

where the average function  $F$  is defined by (1.3).

Notice that the RFDEs (1.4) and (1.9) are not equivalent under the change of time  $t = \tau/\varepsilon$ , as it was the case for the ODEs (1.1) and (1.7). Indeed, by rescaling  $\tau$  as  $t = \tau/\varepsilon$  equation (1.4) becomes

$$\dot{x}(t) = f(t/\varepsilon, x_{t,\varepsilon}), \quad (1.10)$$

where, for  $\theta \in [-r, 0]$ ,  $x_{t,\varepsilon}(\theta) = x(t + \varepsilon\theta)$ . Equation (1.10) is different from (1.9), so that the results obtained for (1.10) cannot be applied to (1.9). This last equation deserves a special attention. It was considered by Hale and Verduyn Lunel in [9] where a method of averaging is developed for infinite dimensional evolutionary equations which include RFDEs such (1.9) as a particular case (see also Section 12.8 of Hale and Verduyn Lunel's book [10] and Section 2.3 of [15]).

Following our previous works [11, 12, 13, 14, 24, 25], we consider in this paper all equations (1.7), (1.9) and (1.10). Our aim is to give theorems of averaging under weaker conditions than those of the literature. We want to emphasize that our main contribution is the weakening of the regularity conditions on the equation under which the averaging method is justified in the existing literature. Indeed, usually classical averaging theorems require that the vector field  $f$  in (1.7), (1.9) and (1.10) is at least locally Lipschitz with respect to the second variable uniformly with respect to the first one (see Remarks 2.3, 2.6 and 2.10 below). In our results this condition is weakened and it is only assumed that  $f$  is continuous in the second variable uniformly with respect to the first one. Also, it is often assumed that the solutions  $x$  and  $y$  exist on the same finite interval of time. In this paper we assume only that the solution  $y$  of the averaged equation exists on some finite interval and we give conditions on the vector field  $f$  so that, for  $\varepsilon$  sufficiently small, the solution  $x$  of (1.7), (1.9) or (1.10) will be defined at least on the same

interval. The *uniform quasi-boundedness* of the vector field  $f$  is thus introduced for this purpose. Recall that the property of quasi-boundedness is strongly related to results on continuation of solutions of RFDEs. It should be noticed that the existing literature [7, 9, 15] proposed also important results on the infinite time interval  $[0, \infty)$ , provided that more hypothesis are made on the nonautonomous system and its averaged system. For example, to a hyperbolic equilibrium point of the averaged system there corresponds a periodic solution of the original equation if  $\varepsilon$  is small. Of course, for such results, stronger assumptions on the regularity of the vector field  $f$  are required.

In this work our averaging results are formulated in classical mathematics. We prove them within *Internal Set Theory* (in short IST) [21] which is an axiomatic approach to *Nonstandard Analysis* (in short NSA) [22]. The idea to use NSA in perturbation theory of differential equations goes back to the 1970s with the Reebian school [18, 19]. It has become today a well-established tool in asymptotic theory, as attested by the the five-digits classification 34E18 of the 2000 Mathematical Subject Classification (see also [1, 4, 14, 17, 26]).

The structure of the paper is as follows. In Section 2 we introduce the notations and present our main results : Theorems 2.1, 2.4 and 2.8. We discuss also both periodic and almost periodic special cases. In Section 3 we start with a short tutorial to NSA and then state our main (nonstandard) tool, the so-called *stroboscopic method*. In Section 4. we give the proofs of Theorems 2.1, 2.4 and 2.8.

We recall that the stroboscopic method was proposed for the first time in the study of some ODEs with a small parameter which occur in the theory of nonlinear oscillations [3, 18, 25]. Here, we first present a slightly modified version of this method and then extend it in the context of RFDEs.

Let us notice that none of our proofs needs to be translated into classical mathematics, because IST is a conservative extension of ordinary mathematics, that is, any classical statement which is a theorem of IST is also a theorem of ordinary mathematics.

## 2. NOTATIONS AND MAIN RESULTS

In this section we will present our main results on averaging for fast oscillating ODEs (1.7), RFDEs in normal form (1.10) and fast oscillating RFDEs (1.9). First we introduce some necessary notations. We assume that  $r \geq 0$  is a fixed real number and denote by  $\mathcal{C} = \mathcal{C}([-r, 0], \mathbb{R}^d)$  the Banach space of continuous functions from  $[-r, 0]$  into  $\mathbb{R}^d$  with the norm  $\|\phi\| = \sup\{|\phi(\theta)| : \theta \in [-r, 0]\}$ , where  $|\cdot|$  is a norm of  $\mathbb{R}^d$ . Let  $L \geq 0$ . If  $x : [-r, L] \rightarrow \mathbb{R}^d$  is a continuous function then, for each  $t \in [0, L]$ , we define  $x_t \in \mathcal{C}$  by setting  $x_t(\theta) = x(t+\theta)$  for all  $\theta \in [-r, 0]$ . Note that when  $r = 0$  the Banach space  $\mathcal{C}$  can be identified with  $\mathbb{R}^d$  and  $x_t$  with  $x(t)$  for each  $t \in [0, L]$ .

**2.1. Averaging for ODEs.** Let  $f : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ ,  $(t, x) \mapsto f(t, x)$ , be a continuous function. Let  $x_0 \in \mathbb{R}^d$  be an initial condition. We consider the initial value problem

$$\dot{x}(t) = f(t/\varepsilon, x(t)), \quad x(0) = x_0, \quad (2.1)$$

where  $\varepsilon > 0$  is a small parameter. We state the precise assumptions on this problem.

(C1) The function  $f$  is continuous in the second variable uniformly with respect to the first one.

(C2) For all  $x \in \mathbb{R}^d$ , there exists a limit  $F(x) := \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(t, x) dt$ .

From conditions (C1) and (C2) we deduce that the average of the function  $f$ , that is, the function  $F : \mathbb{R}^d \rightarrow \mathbb{R}^d$  in (C2), is continuous (see Lemma 4.1). So, the following (averaged) initial value problem is well defined.

$$\dot{y}(t) = F(y(t)), \quad y(0) = x_0. \quad (2.2)$$

We need also the condition :

(C3) The initial value problem (2.2) has a unique solution.

The main theorem of this section is on averaging for fast oscillating ODEs. It establishes nearness of solutions of (2.1) and (2.2) on finite time intervals, and reads as follows.

**Theorem 2.1.** *Let  $f : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  be a continuous function and let  $x_0 \in \mathbb{R}^d$ . Suppose that conditions (C1)-(C3) are satisfied. Let  $y$  be the solution of (2.2) and let  $L \in J$ , where  $J$  is the positive interval of definition of  $y$ . Then, for every  $\delta > 0$ , there exists  $\varepsilon_0 = \varepsilon_0(L, \delta) > 0$  such that, for all  $\varepsilon \in (0, \varepsilon_0]$ , every solution  $x$  of (2.1) is defined at least on  $[0, L]$  and satisfies  $|x(t) - y(t)| < \delta$  for all  $t \in [0, L]$ .*

Let us discuss now the result above when the function  $f$  is periodic or more generally almost periodic in the first variable. We will see that some of the conditions in Theorem 2.1 can be removed. Indeed, in the case where  $f$  is periodic in  $t$ , from continuity plus periodicity properties one can easily deduce condition (C1). Periodicity also implies condition (C2) in an obvious way. The average of  $f$  is then given, for every  $x \in \mathbb{R}^d$ , by

$$F(x) = \frac{1}{T} \int_0^T f(t, x) dt, \quad (2.3)$$

where  $T$  is the period. In the case where  $f$  is almost periodic in  $t$  it is well-known that for all  $x \in \mathbb{R}^d$ , the limit

$$F(x) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_s^{s+T} f(t, x) dt \quad (2.4)$$

exists uniformly with respect to  $s \in \mathbb{R}$ . So, condition (C2) is satisfied when  $s = 0$ . We point out also that in a number of cases encountered in applications the function  $f$  is a finite sum of periodic functions in  $t$ . As in the periodic case above, condition (C1) is satisfied. Hence we have the following result.

**Corollary 2.2** (Periodic and Almost periodic cases). *The conclusion of Theorem 2.1 holds when  $f : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  is a continuous function which is periodic (or a sum of periodic functions) in the first variable and satisfies condition (C3). It holds also when  $f$  is continuous, almost periodic in the first variable and satisfies conditions (C1) and (C3).*

*Remark 2.3.* In the results of the classical literature, for instance [15, Theorem 1, p. 202], it is assumed that  $f$  has bounded partial derivatives with respect to the second variable.

**2.2. Averaging for RFDEs in normal form.** This section concerns the use of the method of averaging to approximate initial value problems of the form

$$\dot{x}(t) = f(t/\varepsilon, x_{t,\varepsilon}), \quad x(t) = \phi(t/\varepsilon), t \in [-\varepsilon r, 0]. \quad (2.5)$$

Here  $f : \mathbb{R}_+ \times \mathcal{C} \rightarrow \mathbb{R}^d$ ,  $(t, x) \mapsto f(t, x)$ , is a continuous function,  $\phi \in \mathcal{C}$  is an initial condition and  $\varepsilon > 0$  is a small parameter. For each  $t \geq 0$ ,  $x_{t,\varepsilon}$  denotes the element of  $\mathcal{C}$  given by  $x_{t,\varepsilon}(\theta) = x(t + \varepsilon\theta)$  for all  $\theta \in [-r, 0]$ .

We recall that the change of time scale  $t = \varepsilon\tau$  transforms (2.5) into the following initial value problem, associated to a RFDE in normal form :

$$x'(\tau) = \varepsilon f(\tau, x_\tau), \quad x_0 = \phi.$$

We make the following hypotheses:

- (H1) The function  $f$  is continuous in the second variable uniformly with respect to the first one.
- (H2) The function  $f$  is quasi-bounded in the second variable uniformly with respect to the first one, that is, for every bounded subset  $B$  of  $\mathcal{C}$ ,  $f$  is bounded on  $\mathbb{R}_+ \times B$ .

- (H3) For all  $x \in \mathcal{C}$ , the limit  $F(x) := \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(t, x) dt$  exists.

We define the averaged initial value problem associated to (2.5) by

$$\dot{y}(t) = G(y(t)), \quad y(0) = \phi(0). \quad (2.6)$$

The function  $G : \mathbb{R}^d \rightarrow \mathbb{R}^d$  is defined by  $G(x) = F(\tilde{x})$  where, for each  $x \in \mathbb{R}^d$ ,  $\tilde{x} \in \mathcal{C}$  is given by  $\tilde{x}(\theta) = x$ ,  $\theta \in [-r, 0]$ . We add the last hypothesis:

- (H4) The initial value problem (2.6) has a unique solution.

As we will see later, condition (H2) is used essentially to prove continuability of solutions of (2.5) at least on every finite interval of time on which the solution of (2.6) is defined. For more details and a complete discussion about quasi-boundedness and its crucial role in the continuability of solutions of RFDEs, we refer the reader to Sections 2.3 and 3.1 of [10].

In assumption (H4) we anticipate the existence of solutions of (2.6). This will be justified a posteriori by Lemma 4.1 where we show that the function  $F : \mathcal{C} \rightarrow \mathbb{R}^d$  in (H3), which is the average of the function  $f$ , is continuous. This implies the continuity of  $G : \mathbb{R}^d \rightarrow \mathbb{R}^d$  in (2.6) and then guaranties the existence of solutions.

The result below is our main theorem on averaging for RFDEs in normal form. It states closeness of solutions of (2.5) and (2.6) on finite time intervals.

**Theorem 2.4.** *Let  $f : \mathbb{R}_+ \times \mathcal{C} \rightarrow \mathbb{R}^d$  be a continuous function and  $\phi \in \mathcal{C}$ . Let conditions (H1)-(H4) hold. Let  $y$  be the solution of (2.6) and let  $L \in J$ , where  $J$  is the positive interval of definition of  $y$ . Then, for every  $\delta > 0$ , there exists  $\varepsilon_0 = \varepsilon_0(L, \delta) > 0$  such that, for all  $\varepsilon \in (0, \varepsilon_0]$ , every solution  $x$  of (2.5) is defined at least on  $[-\varepsilon r, L]$  and satisfies  $|x(t) - y(t)| < \delta$  for all  $t \in [0, L]$ .*

As in Section 2.1, we discuss now both periodic and almost periodic special cases. In each one, some of the conditions in Theorem 2.4 can be either removed or weakened. Let us consider the following (weak) condition which will be used hereafter instead of condition (H2):

- (H5) The function  $f$  is quasi-bounded, that is,  $f$  is bounded on bounded subsets of  $\mathbb{R}_+ \times \mathcal{C}$ .

When  $f$  is periodic it is easy to see that condition (H1) derives from the continuity and the periodicity properties of  $f$ . On the other hand, by periodicity and condition (H5), condition (H2) is also satisfied. The average  $F$  in condition (H3) exists and is now given by formula (2.3) where  $T$  is the period. When  $f$  is almost periodic, condition (H5) imply condition (H2) and the average  $F$  is given by formula (2.4). Quite often the function  $f$  is a finite sum of periodic functions so that condition (H1) is satisfied. Hence we have the following result.

**Corollary 2.5** (Periodic and Almost periodic cases). *The conclusion of Theorem 2.4 holds when  $f : \mathbb{R}_+ \times \mathcal{C} \rightarrow \mathbb{R}^d$  is a continuous function which is periodic (or a sum of periodic functions) in the first variable and satisfies condition (H4) and (H5). It holds also when  $f$  is continuous, almost periodic in the first variable and satisfies conditions (H1), (H4) and (H5).*

Consider now the special case of equations with pointwise delay of the form

$$\dot{x}(t) = f(t/\varepsilon, x(t), x(t - \varepsilon r))$$

which is obtained, by letting  $\tau = t/\varepsilon$ , from equation

$$x'(\tau) = \varepsilon f(\tau, x(\tau), x(\tau - r)).$$

In this case, for both periodic and almost periodic functions, condition (H5) follows from the continuity property and then may be removed in Corollary 2.5.

*Remark 2.6.* In the results of the literature, for instance [15, Theorem 3, p. 206],  $f$  is assumed to be locally Lipschitz with respect to the second variable. Note that local Lipschitz condition with respect to the second variable implies condition (H1). It also assures the local existence for the solution of (2.5). But, in opposition to the case of ODEs, it is well known (see Sections 2.3 and 3.1 of [10]) that without condition (H5) one cannot extend the solution  $x$  to finite time intervals where the solution  $y$  is defined in spite of the closeness of  $x$  and  $y$ . So, in the existing literature it is assumed that the solutions  $x$  and  $y$  are both defined at least on the same interval  $[0, L]$ .

*Remark 2.7.* In the introduction, we noticed that Lehman and Weibel [16] proposed to retain the delay in the averaged equation (1.6). At time scale  $t = \varepsilon\tau$ , their observation is that equation (2.5) is better approximated by the averaged RFDE

$$\dot{y}(t) = F(y_{t,\varepsilon}) \tag{2.7}$$

than by the averaged ODE (2.6). It should be noticed that the averaged RFDE (2.7) depends on the small parameter  $\varepsilon$ , which is not the case of the averaged equation (2.6).

**2.3. Averaging for fast oscillating RFDEs.** The aim here is to approximate the solutions of the initial value problem

$$\dot{x}(t) = f(t/\varepsilon, x_t), \quad x_0 = \phi, \tag{2.8}$$

where  $f : \mathbb{R}_+ \times \mathcal{C} \rightarrow \mathbb{R}^d$ ,  $(t, x) \mapsto f(t, x)$ , is a continuous function,  $\phi \in \mathcal{C}$  is an initial condition and  $\varepsilon > 0$  is a small parameter. This will be obtained under conditions (H1), (H2) and (H3) in Section 2.2 plus condition (H6) below. We define the averaged initial value problem associated to (2.8) by

$$\dot{y}(t) = F(y_t), \quad y_0 = \phi. \tag{2.9}$$

As in the previous section, conditions (H1) and (H3) imply the continuity of the function  $F : \mathcal{C} \rightarrow \mathbb{R}^d$  in (H3). So, the problem (2.9) is well defined. We need the following condition

(H6) The initial value problem (2.9) has a unique solution.

Under the above assumptions, we may state our main result on averaging for fast oscillating RFDEs. It shows that the solution of (2.9) is an approximation of solutions of (2.8) on finite time intervals.

**Theorem 2.8.** *Let  $f : \mathbb{R}_+ \times \mathcal{C} \rightarrow \mathbb{R}^d$  be a continuous function and let  $\phi \in \mathcal{C}$ . Suppose that conditions (H1)-(H3) and (H6) are satisfied. Let  $y$  be the solution of (2.9) and let  $L \in J$  be positive, where  $J$  is the interval of definition of  $y$ . Then, for every  $\delta > 0$ , there exists  $\varepsilon_0 = \varepsilon_0(L, \delta) > 0$  such that, for all  $\varepsilon \in (0, \varepsilon_0]$ , every solution  $x$  of (2.8) is defined at least on  $[-r, L]$  and satisfies  $|x(t) - y(t)| < \delta$  for all  $t \in [0, L]$ .*

In the same manner as in Section 2.2 we have the following result corresponding to the periodic and almost periodic special cases.

**Corollary 2.9** (Periodic and Almost periodic cases). *The conclusion of Theorem 2.8 holds when  $f : \mathbb{R}_+ \times \mathcal{C} \rightarrow \mathbb{R}^d$  is a continuous function which is periodic (or a sum of periodic functions) in the first variable and satisfies condition (H5) and (H6). It holds also when  $f$  is continuous, almost periodic in the first variable and satisfies conditions (H1), (H5) and (H6).*

For fast oscillating equations with pointwise delay of the form

$$\dot{x}(t) = f(t/\varepsilon, x(t), x(t-r)),$$

in the periodic case as well as in the almost periodic one, condition (H5) derives from the continuity property and then can be removed in Corollary 2.9.

*Remark 2.10.* In the results of the classical literature, for instance, [15, Theorem 4, p. 210], it is assumed that  $f$  is locally Lipschitz with respect to the second variable and the existence of the solutions  $x$  and  $y$  on the same interval  $[0, L]$  is required.

### 3. THE STROBOSCOPIC METHOD

**3.1. Internal Set Theory.** In this section we give a short tutorial of NSA. Additional informations can be found in [1, 4, 21, 22]. *Internal Set Theory* (IST) is a theory extending ordinary mathematics, say ZFC (Zermelo-Fraenkel set theory with the axiom of choice), that axiomatizes (Robinson's) *nonstandard analysis* (NSA). We adjoin a new undefined unary predicate *standard* (st) to ZFC. In addition to the usual axioms of ZFC, we introduce three others for handling the new predicate in a relatively *consistent* way. Hence *all theorems* of ZFC *remain valid* in IST. What is new in IST is an addition, not a change.

A real number  $x$  is said to be *infinitesimal* if  $|x| < a$  for all standard positive real numbers  $a$  and *limited* if  $|x| \leq a$  for some standard positive real number  $a$ . A limited real number which is not infinitesimal is said to be *appreciable*. A real number which is not limited is said to be *unlimited*. The notations  $x \simeq 0$  and  $x \simeq +\infty$  are used to denote, respectively,  $x$  is infinitesimal and  $x$  is unlimited positive.

Let  $E$  be a standard normed space. A vector  $x$  in  $E$  is *infinitesimal* (resp. *limited*, *unlimited*) if its norm  $\|x\|$  is infinitesimal (resp. limited, unlimited). Two elements



$x$  and  $y$  in  $E$  are said to be *infinitely close*, in symbols,  $x \simeq y$ , if  $\|x - y\| \simeq 0$ . An element  $x$  is said to be *nearstandard* if  $x \simeq x_0$  for some standard  $x_0 \in E$ . The element  $x_0$  is called the *standard part* or *shadow* of  $x$ . It is unique and is usually denoted by  ${}^o x$ . Note that for  $d$  standard, each limited vector in  $\mathbb{R}^d$  is nearstandard.

Let  $I \subset \mathbb{R}$  be some interval and  $f : I \rightarrow \mathbb{R}^d$  be a function, with  $d$  standard. We say that  $f$  is *S-continuous at  $x$*  in  $I$  if, for all  $y$  in  $I$ ,  $x \simeq y$  implies  $f(x) \simeq f(y)$ , and *S-continuous on  $I$*  if, for all  $x$  and  $y$  nearstandard in  $I$ ,  $x \simeq y$  implies  $f(x) \simeq f(y)$ . When  $f$  (and then  $I$ ) and  $x$  are standard, the first definition is the same as saying that  $f$  is continuous at  $x$ , and the uniform continuity of  $f$  on  $I$  is equivalent to, for all  $x$  and  $y$  in  $I$ ,  $x \simeq y$  implies  $f(x) \simeq f(y)$ .

We need the following basic result on S-continuous functions.

**Theorem 3.1** (Continuous shadow). *Let  $I \subset \mathbb{R}$  be some interval and  $f : I \rightarrow \mathbb{R}^d$  be a function, with  $d$  a standard positive integer. Let  ${}^s I$  be the standard subset of  $\mathbb{R}$  whose standard elements are those of  $I$ . There exists a standard and continuous function  $f_0 : {}^s I \rightarrow \mathbb{R}^d$  such that, for all  $x$  nearstandard in  $I$ ,  $f(x) \simeq f_0(x)$  holds, if and only if,  $f$  is S-continuous on  $I$  and  $f(x)$  is nearstandard for all  $x$  nearstandard in  $I$ .*

The function  $f_0$  in Theorem 3.1 is unique. It is called the *standard part* or *shadow* of the function  $f$  and denoted by  ${}^o f$ .

**3.2. The Stroboscopic Method for ODEs.** Let  $x_0 \in \mathbb{R}^d$  be standard and let  $F : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  be a standard and continuous function. Let  $I$  be some subset of  $\mathbb{R}$  and let  $x : I \rightarrow \mathbb{R}^d$  be a function such that  $0 \in I$  and  $x(0) \simeq x_0$ .

**Definition 3.2** (*F*-Stroboscopic property). The function  $x$  is said to satisfy the *F*-Stroboscopic property if there exists  $\mu > 0$  such that, for all limited  $t \geq 0$  in  $I$  satisfying  $[0, t] \subset I$  and  $x(s)$  is limited for all  $s \in [0, t]$ , there exists  $t' \in I$  such that  $\mu < t' - t \simeq 0$ ,  $[t, t'] \subset I$ ,  $x(s) \simeq x(t)$  for all  $s \in [t, t']$  and  $\frac{x(t') - x(t)}{t' - t} \simeq F(t, x(t))$ .

Now, if a function satisfies the *F*-stroboscopic property, the result below asserts that it can be approximated by a solution of the ODE

$$\dot{y}(t) = F(t, y(t)), \quad y(0) = x_0. \quad (3.1)$$

**Theorem 3.3** (Stroboscopic Lemma for ODEs). *Suppose that*

- (a) *The function  $x$  satisfies the  $F$ -stroboscopic property.*
- (b) *The initial value problem (3.1) has a unique solution  $y$ . Let  $J = [0, \omega)$ ,  $0 < \omega \leq \infty$ , be its maximal positive interval of definition.*

*Then, for every standard  $L \in J$ ,  $[0, L] \subset I$  and the approximation  $x(t) \simeq y(t)$  holds for all  $t \in [0, L]$ .*

The proof of Stroboscopic Lemma for ODEs needs some results which are given in the section below.

### 3.2.1. Preliminaries.

**Lemma 3.4.** *Let  $L > 0$  be limited such that  $[0, L] \subset I$ . Suppose that*

- (i) *The function  $x$  is limited on  $[0, L]$ .*
- (ii) *There exist some positive integer  $N$  and some infinitesimal partition  $\{t_n : n = 0, \dots, N+1\}$  of  $[0, L]$  such that  $t_0 = 0$ ,  $t_N \leq L < t_{N+1}$  and, for  $n = 0, \dots, N$ ,*

$t_{n+1} \simeq t_n$ ,  $x(t) \simeq x(t_n)$  for all  $t \in [t_n, t_{n+1}]$ , and  $\frac{x(t_{n+1}) - x(t_n)}{t_{n+1} - t_n} \simeq F(t_n, x(t_n))$ .  
Then the function  $x$  is S-continuous at each point in  $[0, L]$ .

*Proof.* Let  $t \in [0, L]$ . We will show that  $x$  is S-continuous at  $t$ . Let  $t' \in [0, L]$  and  $p, q \in \{0, \dots, N\}$  be such that  $t \leq t'$ ,  $t \simeq t'$ ,  $t \in [t_p, t_{p+1}]$  and  $t' \in [t_q, t_{q+1}]$ . We write

$$x(t_q) - x(t_p) = \sum_{n=p}^{q-1} (x(t_{n+1}) - x(t_n)) = \sum_{n=p}^{q-1} (t_{n+1} - t_n)[F(t_n, x(t_n)) + \eta_n], \quad (3.2)$$

where  $\eta_n \simeq 0$  for all  $n \in \{p, \dots, q-1\}$ . Denote

$$\eta = \max_{p \leq n \leq q-1} |\eta_n|, \quad m = \max_{p \leq n \leq q-1} |F(t_n, x(t_n))|.$$

We have  $\eta \simeq 0$  and  $m = |F(t_s, x(t_s))|$  for some  $s \in \{p, \dots, q-1\}$ . Since  $(t_s, x(t_s))$  is limited, it is nearstandard. Since the function  $F$  is standard and continuous,  $F(t_s, x(t_s))$  is nearstandard. So is  $m$ . Hence (3.2) leads to the approximation

$$|x(t') - x(t)| \simeq |x(t_q) - x(t_p)| \leq (m + \eta)(t_q - t_p) \simeq 0$$

which proves the S-continuity of  $x$  at  $t$  and completes the proof.  $\square$

When instead of  $L$  limited we suppose  $L$  standard, Lemma 3.4 transforms to the following result with more properties about the function  $x$ .

**Lemma 3.5.** *Let  $L > 0$  be standard such that  $[0, L] \subset I$ . Suppose that conditions (i) and (ii) in Lemma 3.4 hold. Then the standard function  $y : [0, L] \rightarrow \mathbb{R}^d$  defined, for all standard  $t \in [0, L]$ , by  $y(t) = {}^o(x(t))$ , is a solution of (3.1). Moreover, the approximation  $x(t) \simeq y(t)$  holds for all  $t \in [0, L]$ .*

*Proof.* To prove the lemma we proceed in two steps.

*Step 1.* We claim that the function  $y$  is continuous on  $[0, L]$ . Indeed, by Lemma 3.4 the function  $x$  is S-continuous on  $[0, L]$ . Taking hypothesis (i) into account, the claim follows from Theorem 3.1. We have moreover

$$y(t) \simeq y({}^o t) \simeq x({}^o t) \simeq x(t), \quad \forall t \in [0, L].$$

The first part of the proof is complete.

*Step 2.* To show that the function  $y$  satisfies, for all  $t \in [0, L]$ ,

$$y(t) = x_0 + \int_0^t F(s, y(s)) ds,$$

let  $t \in [0, L]$  and  $n \in \{0, \dots, N\}$  be such that  $t \in [t_n, t_{n+1}]$  with  $t$  standard. Then

$$\begin{aligned} y(t) - x_0 \simeq x(t_n) - x(0) &= \sum_{k=0}^{n-1} (x(t_{k+1}) - x(t_k)) \\ &= \sum_{k=0}^{n-1} (t_{k+1} - t_k)[F(t_k, x(t_k)) + \eta_k], \end{aligned} \quad (3.3)$$

where  $\eta_k \simeq 0$  for all  $k \in \{0, \dots, n-1\}$ . As  $F$  is standard and continuous, and by Step 1 above  $x(t_k) \simeq y(t_k)$  with  $x(t_k)$  nearstandard, we have  $F(t_k, x(t_k)) \simeq F(t_k, y(t_k))$

so that (3.3) gives

$$y(t) - x_0 \simeq \sum_{k=0}^{n-1} (t_{k+1} - t_k) [F(t_k, y(t_k)) + \beta_k + \eta_k] \simeq \int_0^t F(s, y(s)) ds,$$

where  $\beta_k \simeq 0$  for all  $k \in \{0, \dots, n-1\}$ . Thus the approximation

$$y(t) \simeq x_0 + \int_0^t F(s, y(s)) ds \quad (3.4)$$

holds for all standard  $t \in [0, L]$ . Actually (3.4) is an equality since both sides of which are standard. We have thus, for all standard  $t \in [0, L]$ ,

$$y(t) = x_0 + \int_0^t F(s, y(s)) ds \quad (3.5)$$

and by transfer (3.5) holds for all  $t \in [0, L]$ . The proof is complete.  $\square$

The following statement is a consequence of Lemma 3.5.

**Lemma 3.6.** *Let  $L > 0$  be standard such that  $[0, L] \subset I$ . Suppose that*

- (i) *The function  $x$  is limited on  $[0, L]$ .*
- (ii) *There exists  $\mu > 0$  such that, for all  $t \in [0, L]$ , there exists  $t' \in I$  such that  $\mu < t' - t \simeq 0$ ,  $[t, t'] \subset I$ ,  $x(s) \simeq x(t)$  for all  $s \in [t, t']$ , and  $\frac{x(t') - x(t)}{t' - t} \simeq F(t, x(t))$ .*  
*Then the function  $x$  is  $S$ -continuous on  $[0, L]$  and its shadow is a solution  $y$  of (3.1). So, we have  $x(t) \simeq y(t)$  for all  $t \in [0, L]$*

*Proof.* First of all we have  $\lambda \in A_\mu$  for all standard real number  $\lambda > 0$ , where  $A_\mu$  is the subset of  $\mathbb{R}$  defined by

$$A_\mu = \{\lambda \in \mathbb{R} / \forall t \in [0, L] \exists t' \in I : \mathcal{P}_\mu(t, t', \lambda)\}$$

and  $\mathcal{P}_\mu(t, t', \lambda)$  is the property

$$\mu < t' - t < \lambda, [t, t'] \subset I, \forall s \in [t, t'] |x(s) - x(t)| < \lambda, \left| \frac{x(t') - x(t)}{t' - t} - F(t, x(t)) \right| < \lambda.$$

By overspill there exists also  $\lambda_0 \in A_\mu$  with  $0 < \lambda_0 \simeq 0$ . Thus, for all  $t \in [0, L]$ , there is  $t' \in I$  such that  $\mathcal{P}_\mu(t, t', \lambda_0)$  holds. Applying now the axiom of choice to obtain a function  $c : [0, L] \rightarrow I$  such that  $c(t) = t'$ , that is,  $\mathcal{P}_\mu(t, c(t), \lambda_0)$  holds for all  $t \in [0, L]$ . Since  $c(t) - t > \mu$  for all  $t \in [0, L]$ , there are a positive integer  $N$  and an infinitesimal partition  $\{t_n : n = 0, \dots, N+1\}$  of  $[0, L]$  such that  $t_0 = 0$ ,  $t_N \leq L < t_{N+1}$  and  $t_{n+1} = c(t_n)$ . Finally, the conclusion follows from Lemma 3.5.  $\square$

**3.2.2. Proof of Theorem 3.3.** Let  $L > 0$  be standard in  $J$ . Fix  $\rho > 0$  to be standard and let  $W$  be the (standard) tubular neighborhood around  $\Gamma = \{y(t) : t \in [0, L]\}$  given by

$$W = \{z \in \mathbb{R}^d / \exists t \in [0, L] : |z - y(t)| \leq \rho\}.$$

Let  $A$  be the nonempty ( $0 \in A$ ) subset of  $[0, L]$  defined by

$$A = \{L_1 \in [0, L] / [0, L_1] \subset I \text{ and } \{x(t) : t \in [0, L_1]\} \subset W\}.$$

The set  $A$  is bounded above by  $L$ . Let  $L_0$  be the upper bound of  $A$  and let  $L_1 \in A$  be such that  $L_0 - \mu < L_1 \leq L_0$ . Since  $\{x(t) : t \in [0, L_1]\} \subset W$ , the function  $x$  is limited on  $[0, L_1]$ .

Taking hypothesis (b) into account, we now apply Lemma 3.6 to obtain, for any standard real number  $T$  such that  $0 < T \leq L_1$ ,

$$x(t) \simeq y(t), \quad \forall t \in [0, T]. \quad (3.6)$$

By overspill approximation (3.6) still holds for some  $T \simeq L_1$ . Next, by Lemma 3.4 and the continuity of  $y$  we have

$$x(t) \simeq x(T) \quad \text{and} \quad y(t) \simeq y(T), \quad \forall t \in [T, L_1].$$

Combining this with (3.6) yields

$$x(t) \simeq y(t), \quad \forall t \in [0, L_1]. \quad (3.7)$$

Moreover, by hypothesis (a) there exists  $L'_1 > L_1 + \mu$  such that  $[L_1, L'_1] \subset I$  and

$$x(t) \simeq y(t), \quad \forall t \in [L_1, L'_1]. \quad (3.8)$$

Together (3.7) and (3.8) show that  $x(t) \simeq y(t)$  for all  $t \in [0, L'_1]$ .

It remains to verify that  $L \leq L'_1$ . If this is not true, then  $[0, L'_1] \subset I$  and  $\{x(t) : t \in [0, L'_1]\} \subset W$  imply  $L'_1 \in A$ . This contradicts the fact that  $L'_1 > L_0$ . So the proof is complete.

**3.3. The Stroboscopic Method for RFDEs.** Let  $\phi \in \mathcal{C}$  be standard and let  $F : \mathbb{R}_+ \times \mathcal{C} \rightarrow \mathbb{R}^d$  be a standard and continuous function. Let  $I$  be some subset of  $\mathbb{R}$  and let  $x : I \rightarrow \mathbb{R}^d$  be a function such that  $[-r, 0] \subset I$  and  $x_0 \simeq \phi$ .

**Definition 3.7** (*F*-Stroboscopic property). The function  $x$  is said to satisfy the *F*-Stroboscopic property if there exists  $\mu > 0$  such that, for all limited  $t \geq 0$  in  $I$  satisfying  $[0, t] \subset I$  and  $x(s)$  and  $F(s, x_s)$  are limited for all  $s \in [0, t]$ , there exists  $t' \in I$  such that  $\mu < t' - t \simeq 0$ ,  $[t, t'] \subset I$ ,  $x(s) \simeq x(t)$  for all  $s \in [t, t']$  and  $\frac{x(t') - x(t)}{t' - t} \simeq F(t, x_t)$ .

In the same manner as in Section 2, for  $r = 0$  we identify the Banach space  $\mathcal{C}$  with  $\mathbb{R}^d$  and  $x_t$  with  $x(t)$ . By continuity property of  $F$ , we have then  $x(s)$  is limited for all  $s \in [0, t]$  implies that  $F(s, x(s))$  is limited for all  $s \in [0, t]$ . So, Definition 3.2 is a particular case of Definition 3.7. In the following result we assert that a function which satisfies the *F*-stroboscopic property can be approximated by a solution of the RFDE

$$\dot{y}(t) = F(t, y_t), \quad y_0 = \phi. \quad (3.9)$$

**Theorem 3.8** (Stroboscopic Lemma for RFDEs). *Suppose that*

- (a) *The function  $x$  satisfies the  $F$ -stroboscopic property.*
- (b) *The initial value problem (3.9) has a unique solution  $y$ . Let  $J = [-r, \omega)$ ,  $0 < \omega \leq \infty$ , be its maximal interval of definition.*

*Then, for every standard and positive  $L \in J$ ,  $[-r, L] \subset I$  and the approximation  $x(t) \simeq y(t)$  holds for all  $t \in [-r, L]$ .*

To prove Stroboscopic Lemma for RFDEs we need first to establish the following preliminary lemmas.

## 3.3.1. Preliminaries.

**Lemma 3.9.** *Let  $L > 0$  be limited such that  $[0, L] \subset I$ . Suppose that*

- (i)  $x(t)$  and  $F(t, x_t)$  are limited for all  $t \in [0, L]$ .  
(ii) There exist some positive integer  $N$  and some infinitesimal partition  $\{t_n : n = 0, \dots, N+1\}$  of  $[0, L]$  such that  $t_0 = 0$ ,  $t_N \leq L < t_{N+1}$  and, for  $n = 0, \dots, N$ ,  $t_{n+1} \simeq t_n$ ,  $x(t) \simeq x(t_n)$  for all  $t \in [t_n, t_{n+1}]$ , and  $\frac{x(t_{n+1}) - x(t_n)}{t_{n+1} - t_n} \simeq F(t_n, x_{t_n})$ .

Then the function  $x$  is S-continuous at each point in  $[0, L]$ .

*Proof.* Let  $t, t' \in [0, L]$  with  $t \leq t'$  and  $t \simeq t'$ . Let  $p, q \in \{0, \dots, N\}$  be such that  $t \in [t_p, t_{p+1}]$  and  $t' \in [t_q, t_{q+1}]$  with  $t_p \simeq t_q$ . Define

$$\eta_n = \frac{x(t_{n+1}) - x(t_n)}{t_{n+1} - t_n} - F(t_n, x_{t_n}), \quad \forall n \in \{p, \dots, q-1\}.$$

If  $\eta$  and  $m$  are the respective maximum values of  $|\eta_n|$  and  $|F(t_n, x_{t_n})|$  for  $n = p, \dots, q-1$ , then we have

$$x(t_q) - x(t_p) = \sum_{n=p}^{q-1} (x(t_{n+1}) - x(t_n)) = \sum_{n=p}^{q-1} (t_{n+1} - t_n) [F(t_n, x_{t_n}) + \eta_n] \quad (3.10)$$

Now,  $\eta_n \simeq 0$  for all  $n \in \{p, \dots, q-1\}$  implies  $\eta \simeq 0$ . On the other hand, in view of hypothesis (i),  $m = |F(t_s, x_{t_s})|$  for some  $s \in \{p, \dots, q-1\}$ , is limited. Then (3.10) yields

$$|x(t') - x(t)| \simeq |x(t_q) - x(t_p)| \leq (m + \eta)(t_q - t_p) \simeq 0$$

which shows the S-continuity of  $x$  at each point in  $[0, L]$ . The proof is complete.  $\square$

If the real number  $L$  in Lemma 3.9 is standard one obtains more precise information about the function  $x$ .

**Lemma 3.10.** *Let  $L > 0$  be standard such that  $[0, L] \subset I$ . Suppose that conditions (i) and (ii) in Lemma 3.9 are satisfied. Then the standard function  $y : [-r, L] \rightarrow \mathbb{R}^d$  defined by*

$$y_0 = \phi, \quad \text{and} \quad y(t) = {}^o(x(t)) \text{ for all standard } t \in [0, L],$$

is a solution of (3.9) and satisfies

$$x(t) \simeq y(t), \quad \forall t \in [0, L]. \quad (3.11)$$

*Proof.* To prove the lemma one first note that the continuity of  $y$  on  $[0, L]$  follows by the same argument as in Lemma 3.5. As a consequence we get the approximation (3.11) and then, for all  $t \in [0, L]$ ,  $x_t$  is nearstandard and  $x_t \simeq y_t$ . Now it remains to show that  $y$  satisfies the integral equation

$$y(t) = \phi(0) + \int_0^t F(s, y_s) ds, \quad \forall t \in [0, L].$$

Since the proof does not differ from this one in Step 2 of the proof of Lemma 3.5, it is omitted.  $\square$

From Lemma 3.10 we deduce the result below.

**Lemma 3.11.** *Let  $L > 0$  be standard such that  $[0, L] \subset I$ . Suppose that*

(i)  *$x(t)$  and  $F(t, x_t)$  are limited for all  $t \in [0, L]$ .*

(ii) *There exists  $\mu > 0$  such that, for all  $t \in [0, L]$ , there exists  $t' \in I$  such that  $\mu < t' - t \simeq 0$ ,  $[t, t'] \subset I$ ,  $x(s) \simeq x(t)$  for all  $s \in [t, t']$ , and  $\frac{x(t') - x(t)}{t' - t} \simeq F(t, x_t)$ .*

*Then the function  $x$  is S-continuous on  $[0, L]$  and its shadow is a solution of (3.9) and satisfies approximation (3.11).*

*Proof.* As in the proof of Lemma 3.6, we obtain a function  $c : [0, L] \rightarrow I$  satisfying, for all  $t \in [0, L]$ ,

$$\mu < c(t) - t \simeq 0, \quad [t, c(t)] \subset I, \quad \forall s \in [t, c(t)] \quad x(s) \simeq x(t), \quad \frac{x(c(t)) - x(t)}{c(t) - t} \simeq F(t, x_t).$$

If we let  $t_0 = 0$  and  $t_{n+1} = c(t_n)$  for  $n = 0, \dots, N$ , where the integer  $N$  is such that  $t_N \leq L < t_{N+1}$ , the conclusion follows by applying Lemma 3.10.  $\square$

**3.3.2. Proof of Theorem 3.8.** Let  $L > 0$  be standard in  $J$  and  $\Gamma = \{y_t : t \in [0, L]\}$ . Since  $F$  is standard and continuous, and  $[0, L] \times \Gamma$  is a standard compact subset of  $\mathbb{R}^d \times \mathcal{C}$ , there exists  $\rho > 0$  and standard such that  $F$  is limited on  $[0, L] \times W$ , where  $W$  is the standard tubular neighborhood around  $\Gamma$  defined by

$$W = \{z \in \mathcal{C} / \exists t \in [0, L] : |z - y_t| \leq \rho\}.$$

Now, since the set

$$A = \{L_1 \in [0, L] / [0, L_1] \subset I \text{ and } \{x_t : t \in [0, L_1]\} \subset W\}$$

is nonempty ( $0 \in A$ ) and bounded above by  $L$ , there exists  $L_1 \in A$  such that  $L_0 - \mu < L_1 \leq L_0$ , where  $L_0 = \sup A$ . However  $L_1 \in A$  implies

$$[0, L_1] \times \{x_t : t \in [0, L_1]\} \subset [0, L] \times W$$

and then  $x(t)$  and  $F(t, x_t)$  are limited for all  $t \in [0, L_1]$ .

If  $T$  is standard and  $0 < T \leq L_1$ , according to Lemma 3.11 the shadow on  $[0, T]$  of the function  $x$  is a solution of (3.9). In view of hypothesis (b), this shadow coincides with  $y$  on  $[-r, T]$ . Also, we have

$$x(t) \simeq y(t), \quad \forall t \in [0, T].$$

By overspill the property above holds for some  $T \simeq L_1$ . On the other hand, due to the S-continuity of  $x$  at each point in  $([T, L_1] \subset [0, L_1])$  (see Lemma 3.9) and the continuity of  $y$ , we have

$$x(t) \simeq x(T) \quad \text{and} \quad y(t) \simeq y(T), \quad \forall t \in [T, L_1]$$

which achieves to prove that

$$x(t) \simeq y(t), \quad \forall t \in [0, L_1]. \tag{3.12}$$

By hypothesis (a)

$$x(t) \simeq y(t), \quad \forall t \in [L_1, L'_1] \tag{3.13}$$

for some  $L'_1$  such that  $L'_1 > L_1 + \mu$  and  $[L_1, L'_1] \subset I$ . Taking into account that  $x_0 \simeq \phi = y_0$  and combining (3.12) and (3.13), we conclude that  $x_t \simeq y_t$  for all  $t \in [0, L'_1]$ . Assume that  $L'_1 \leq L$ . Then  $[0, L'_1] \subset I$  and  $\{x_t : t \in [0, L'_1]\} \subset W$  imply  $L'_1 \in A$ , which is absurd since  $L'_1 > L_0$ . Thus  $L'_1 > L$ . Finally, for any standard  $L \in J$  we have shown that  $x(t) \simeq y(t)$  for all  $t \in [0, L] \subset [0, L'_1]$ . This completes the proof of the theorem.

## 4. PROOFS OF THE RESULTS

We prove Theorems 2.1, 2.4 and 2.8 within IST. By transfer it suffices to prove those results for *standard data*  $f$ ,  $x_0$  and  $\phi$ . We will do this by applying Stroboscopic Lemma for ODEs (Theorem 3.3) in both cases of Theorems 2.1 and 2.4, and Stroboscopic Lemma for RFDEs (Theorem 3.8) in case of Theorem 2.8. For this purpose we need first to translate all conditions (C1) and (C2) in Section 2.1, and (H1), (H2) and (H3) in Section 2.2 into their external forms and then prove some technical lemmas. In the external formulas, we use the following abbreviations [21] :

$$\forall^{\text{st}} A \text{ for } \forall x(\text{st}x \Rightarrow A) \quad \text{and} \quad \exists^{\text{st}} A \text{ for } \exists x(\text{st}x \ \&A).$$

Let  $f : \mathbb{R}_+ \times \mathcal{C} \rightarrow \mathbb{R}^d$  be a standard and continuous function, where  $\mathcal{C} = \mathcal{C}([-r, 0], \mathbb{R}^d)$  and  $r \geq 0$ . We recall that when  $r = 0$ ,  $\mathcal{C}$  is identified with  $\mathbb{R}^d$ . The external formulations of conditions (C1) and (C2) are:

$$(C1') \quad \forall^{\text{st}} x \in \mathbb{R}^d \forall x' \in \mathbb{R}^d \forall t \in \mathbb{R}_+ (x' \simeq x \Rightarrow f(t, x') \simeq f(t, x)).$$

$$(C2') \quad \exists^{\text{st}} F : \mathbb{R}^d \rightarrow \mathbb{R}^d \forall^{\text{st}} x \in \mathbb{R}^d \forall R \simeq +\infty F(x) \simeq \frac{1}{R} \int_0^R f(t, x) dt.$$

The external formulation of conditions (H1), (H2) and (H3) are, respectively:

$$(H1') \quad \forall^{\text{st}} x \in \mathcal{C} \forall x' \in \mathcal{C} \forall t \in \mathbb{R}_+ (x' \simeq x \Rightarrow f(t, x') \simeq f(t, x)).$$

$$(H2') \quad \forall x \in \mathcal{C} \text{ and limited } \forall t \in \mathbb{R}_+, f(t, x) \text{ is limited in } \mathbb{R}^d.$$

$$(H3') \quad \exists^{\text{st}} F : \mathcal{C} \rightarrow \mathbb{R}^d \forall^{\text{st}} x \in \mathcal{C} \forall R \simeq +\infty F(x) \simeq \frac{1}{R} \int_0^R f(t, x) dt.$$

**4.1. Technical Lemmas.** In Lemmas 4.1 and 4.2 below we formulate some properties of the average of the function  $f$  (i.e. the function  $F$  defined in (C2) and (H3)).

**Lemma 4.1.** *Suppose that the function  $f$  satisfies conditions (C1) and (C2) when  $r = 0$  and conditions (H1) and (H3) when  $r > 0$ . Then the function  $F$  in (C2) or (H3) is continuous and satisfies*

$$F(x) \simeq \frac{1}{R} \int_0^R f(t, x) dt$$

for all nearstandard  $x \in \mathcal{C}$  and all  $R \simeq +\infty$ .

*Proof.* The proof is the same in both cases  $r = 0$  and  $r > 0$ . So, there is no restriction to suppose that  $r = 0$ . Let  $x, {}^o x \in \mathbb{R}^d$  be such that  ${}^o x$  is standard and  ${}^o x \simeq x$ . Fix  $\delta > 0$  to be infinitesimal. By condition (C2)

$$\left| F(x) - \frac{1}{T} \int_0^T f(t, x) dt \right| < \delta, \quad \forall T > T_0$$

for some  $T_0 > 0$ . Hence there exists  $T \simeq +\infty$  such that

$$F(x) \simeq \frac{1}{T} \int_0^T f(t, x) dt.$$

By condition (C1') we have  $f(t, x) \simeq f(t, {}^o x)$  for all  $t \in \mathbb{R}_+$ . Therefore

$$F(x) \simeq \frac{1}{T} \int_0^T f(t, {}^o x) dt.$$

By condition (C2') we deduce that  $F(x) \simeq F({}^o x)$ . Thus  $F$  is continuous. Moreover, for all  $T \simeq +\infty$ , we have

$$F(x) \simeq F({}^o x) \simeq \frac{1}{T} \int_0^T f(t, {}^o x) dt \simeq \frac{1}{T} \int_0^T f(t, x) dt.$$

So, the proof is complete.  $\square$

**Lemma 4.2.** *Suppose that the function  $f$  satisfies conditions (C1) and (C2) when  $r = 0$  and conditions (H1) and (H3) when  $r > 0$ . Let  $F$  be as in (C2) or (H3). Let  $\varepsilon > 0$  be infinitesimal. Then, for all limited  $t \in \mathbb{R}_+$  and all nearstandard  $x \in \mathcal{C}$ , there exists  $\alpha = \alpha(\varepsilon, t, x)$  such that  $0 < \alpha \simeq 0$ ,  $\varepsilon/\alpha \simeq 0$  and*

$$\frac{\varepsilon}{\alpha} \int_{t/\varepsilon}^{t/\varepsilon + T\alpha/\varepsilon} f(\tau, x) d\tau \simeq TF(x), \quad \forall T \in [0, 1].$$

*Proof.* The proof is the same in both cases  $r = 0$  and  $r > 0$ . Let  $t$  be limited in  $\mathbb{R}_+$  and let  $x$  be nearstandard in  $\mathcal{C}$ . We denote for short  $g(r) = f(r, x)$ . Let  $T \in [0, 1]$ . We consider the following two cases.

Case 1:  $t/\varepsilon$  is limited. Let  $\alpha > 0$  be such that  $\varepsilon/\alpha \simeq 0$ . If  $T\alpha/\varepsilon$  is limited then we have  $T \simeq 0$  and

$$\frac{\varepsilon}{\alpha} \int_{t/\varepsilon}^{t/\varepsilon + T\alpha/\varepsilon} g(r) dr \simeq 0 \simeq TF(x).$$

If  $T\alpha/\varepsilon \simeq +\infty$  we write

$$\frac{\varepsilon}{\alpha} \int_{t/\varepsilon}^{t/\varepsilon + T\alpha/\varepsilon} g(r) dr = \left(T + \frac{t}{\alpha}\right) \frac{1}{t/\varepsilon + T\alpha/\varepsilon} \int_0^{t/\varepsilon + T\alpha/\varepsilon} g(r) dr - \frac{\varepsilon}{\alpha} \int_0^{t/\varepsilon} g(r) dr.$$

By Lemma 4.1 we have

$$\frac{1}{t/\varepsilon + T\alpha/\varepsilon} \int_0^{t/\varepsilon + T\alpha/\varepsilon} g(r) dr \simeq F(x).$$

Since  $\frac{\varepsilon}{\alpha} \int_0^{t/\varepsilon} g(r) dr \simeq 0$  and  $t/\alpha \simeq 0$ , we have

$$\frac{\varepsilon}{\alpha} \int_{t/\varepsilon}^{t/\varepsilon + T\alpha/\varepsilon} g(r) dr \simeq TF(x).$$

This approximation is satisfied for all  $\alpha > 0$  such that  $\varepsilon/\alpha \simeq 0$ . Choosing then  $\alpha$  such that  $0 < \alpha \simeq 0$  and  $\varepsilon/\alpha \simeq 0$  gives the desired result.

Case 2:  $t/\varepsilon$  is unlimited. Let  $\alpha > 0$ . We have

$$\frac{\varepsilon}{\alpha} \int_{t/\varepsilon}^{t/\varepsilon + T\alpha/\varepsilon} g(r) dr = T\eta(\alpha) + \frac{t}{\alpha} [\eta(\alpha) - \eta(0)], \quad (4.1)$$

where

$$\eta(\alpha) = \frac{1}{t/\varepsilon + T\alpha/\varepsilon} \int_0^{t/\varepsilon + T\alpha/\varepsilon} g(r) dr.$$

By Lemma 4.1 we have  $\eta(\alpha) \simeq F(x)$  for all  $\alpha \geq 0$ . Return to (4.1) and assume that  $\alpha$  is not infinitesimal. Then

$$\frac{\varepsilon}{\alpha} \int_{t/\varepsilon}^{t/\varepsilon + T\alpha/\varepsilon} g(r) dr \simeq TF(x), \quad (4.2)$$

By overspill (4.2) holds for some  $\alpha \simeq 0$  which can be chosen such that  $\varepsilon/\alpha \simeq 0$ . This completes the proof of the lemma.  $\square$



**4.2. Proof of Theorem 2.1.** Assume that  $x_0$  and  $L$  are standard. To prove Theorem 2.1 is equivalent to show that, for every infinitesimal  $\varepsilon > 0$ , every solution  $x$  of (2.1) is defined at least on  $[0, L]$  and satisfies  $x(t) \simeq y(t)$  for all  $t \in [0, L]$ . We need first to prove the following result which discuss some properties of solutions of a certain ODE needed in the sequel.

**Lemma 4.3.** *Let  $g : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  and  $h : \mathbb{R}_+ \rightarrow \mathbb{R}^d$  be continuous functions. Let  $x_0$  be limited in  $\mathbb{R}^d$ . Suppose that*

- (i)  $g(t, x) \simeq h(t)$  holds for all  $t \in [0, 1]$  and all limited  $x \in \mathbb{R}^d$ .
- (ii)  $\int_0^t h(s)ds$  is limited for all  $t \in [0, 1]$ .

Then any solution  $x$  of the initial value problem

$$\dot{x} = g(t, x), \quad t \in [0, 1]; \quad x(0) = x_0$$

is defined and limited on  $[0, 1]$  and satisfies

$$x(t) \simeq x_0 + \int_0^t h(s)ds, \quad \forall t \in [0, 1].$$

*Proof.* By overspill there exists  $\omega \simeq +\infty$  such that the approximation in hypothesis (i) holds for all  $t \in [0, 1]$  and all  $x \in B(0, \omega)$ , where  $B(0, \omega) \subset \mathbb{R}^d$  is the ball of center 0 and radius  $\omega$ . Assume that  $x(t') \simeq \infty$  for some  $t' \in [0, 1]$ . Let  $t \in [0, 1]$  be such that  $t \leq t'$ ,  $x(t) \simeq \infty$  and  $x(s) \in B(0, \omega)$  for all  $s \in [0, t]$ . Then we have

$$x(t) = x_0 + \int_0^t g(s, x(s))ds \simeq x_0 + \int_0^t h(s)ds$$

whence, in view of hypothesis (ii),  $x(t)$  is limited; this is a contradiction. Therefore  $x(t)$  is defined and limited for all  $t \in [0, 1]$ .  $\square$

Let us now prove Theorem 2.1. Fix  $\varepsilon > 0$  to be infinitesimal and let  $x : I \rightarrow \mathbb{R}^d$  be a maximal solution of (2.1). We claim that  $x$  satisfies the  $F$ -stroboscopic property. To see this, let  $t_0 \in I$  such that  $t_0 \geq 0$  and limited, and  $x(t)$  is limited for all  $t \in [0, t_0]$ . By Lemma 4.2 there exists  $\alpha = \alpha(\varepsilon, t_0, x(t_0))$  such that  $0 < \alpha \simeq 0$ ,  $\varepsilon/\alpha \simeq 0$  and

$$\frac{\varepsilon}{\alpha} \int_{t_0/\varepsilon}^{t_0/\varepsilon + T\alpha/\varepsilon} f(t, x(t_0))dt \simeq TF(x(t_0)), \quad \forall T \in [0, 1]. \quad (4.3)$$

Introduce the function

$$X(T) = \frac{x(t_0 + \alpha T) - x(t_0)}{\alpha}, \quad T \in [0, 1].$$

Differentiating and substituting the above into (2.1) gives, for  $T \in [0, 1]$ ,

$$\frac{dX}{dT}(T) = f\left(\frac{t_0}{\varepsilon} + \frac{\alpha}{\varepsilon}T, x(t_0) + \alpha X(T)\right). \quad (4.4)$$

By (C1') and Lemma 4.3 the function  $X$ , as a solution of (4.4), is defined and limited on  $[0, 1]$  and, for  $T \in [0, 1]$ ,

$$X(T) \simeq \int_0^T f\left(\frac{t_0}{\varepsilon} + \frac{\alpha}{\varepsilon}t, x(t_0)\right) dt = \frac{\varepsilon}{\alpha} \int_{t_0/\varepsilon}^{t_0/\varepsilon + T\alpha/\varepsilon} f(t, x(t_0))dt.$$

Using now (4.3) this leads to the approximation

$$X(T) \simeq TF(x(t_0)), \quad \forall T \in [0, 1].$$

Define  $t_1 = t_0 + \alpha$  and set  $\mu = \varepsilon$ . Then  $\mu < \alpha = t_1 - t_0 \simeq 0$ ,  $[t_0, t_1] \subset I$  and  $x(t_0 + \alpha T) = x(t_0) + \alpha X(T) \simeq x(t_0)$  for all  $T \in [0, 1]$ , that is,  $x(t) \simeq x(t_0)$  for all  $t \in [t_0, t_1]$ , whereas

$$\frac{x(t_1) - x(t_0)}{t_1 - t_0} = X(1) \simeq F(x(t_0)),$$

which shows the claim. Finally, by (C3) and Theorem 3.3, the solution  $x$  is defined at least on  $[0, L]$  and satisfies  $x(t) \simeq y(t)$  for all  $t \in [0, L]$ . The proof of Theorem 2.1 is complete.

**4.3. Proof of Theorem 2.4.** Assume that  $\phi$  and  $L$  are standard. All we have to prove is that, when  $\varepsilon > 0$  is infinitesimal, every solution  $x$  of (2.5) is defined at least on  $[-\varepsilon r, L]$  and satisfies  $x(t) \simeq y(t)$  for all  $t \in [0, L]$ . We start by showing some auxiliary result which is needed in the proof of Theorem 2.4.

**Lemma 4.4.** *Let  $g : \mathbb{R}_+ \times \mathcal{C} \rightarrow \mathbb{R}^d$  be a continuous function. Suppose that, for all  $t \in \mathbb{R}_+$  and all  $x \in \mathcal{C}$ ,  $t$  and  $x$  limited imply that  $g(t, x)$  is limited. Let  $\phi \in \mathcal{C}$  be standard. Let  $x : I \rightarrow \mathbb{R}^d$  be a maximal solution of the initial value problem*

$$\dot{x}(t) = g(t, x_{t,\varepsilon}), \quad x(t) = \phi(t/\varepsilon), \quad t \in [-\varepsilon r, 0]. \quad (4.5)$$

Let  $t_0 \geq 0$  be limited in  $I$  such that  $x$  is limited on  $[0, t_0]$ . Then the solution  $x$  is

- (i)  $S$ -continuous at each  $t \in [0, t_0]$ .
- (ii) defined and limited at each  $t \geq t_0$  such that  $t \simeq t_0$ .

*Proof.* (i) Let  $t \in [0, t_0]$ . We will show that  $x$  is  $S$ -continuous at  $t$ . If  $t' \in [0, t_0]$  is such that  $t \leq t'$  and  $t \simeq t'$ , the integral equation for the solutions of (4.5) implies

$$|x(t') - x(t)| \leq \int_t^{t'} |g(s, x_{s,\varepsilon})| ds \leq (t' - t) \sup_{s \in [t, t']} |g(s, x_{s,\varepsilon})|.$$

Since  $x_{s,\varepsilon}(\theta) = x(s + \varepsilon\theta)$  is limited for all  $\theta \in [-r, 0]$  and all  $s \in [t, t']$ , we have  $x_{s,\varepsilon}$  is limited for all  $s \in [t, t']$ . Now in view of assumptions on  $x$  and  $g$ ,  $\sup_{s \in [t, t']} |g(s, x_{s,\varepsilon})|$  is limited so that  $x(t') \simeq x(t)$ , which is the desired result.

(ii) Let  $I = [-\varepsilon r, b)$ ,  $0 < b \leq \infty$ . We assume by contradiction that  $x$  is not defined for all  $t \geq t_0$  such that  $t \simeq t_0$ . Then  $b \simeq t_0$ . We have  $x(t') \simeq \infty$  for some  $t' \in [t_0, b)$ . Indeed, taking into account that  $x$  is limited on  $[-\varepsilon r, 0]$  since  $x([-\varepsilon r, 0]) = \phi([-r, 0])$ , if it was also limited on  $[t_0, b)$  then  $\lim_{t \rightarrow b} x(t)$  exists and  $x$  can be extended to a continuous function on  $[-\varepsilon r, b]$  by setting  $x(b) = \lim_{t \rightarrow b} x(t)$ . Consequently,  $x_{b,\varepsilon} \in \mathcal{C}$  and then one can find a solution of (4.5) through the point  $(b, x_{b,\varepsilon})$  to the right of  $b$ , which contradicts the noncontinuability hypothesis on  $x$ . Now, by the continuity of  $x$  there exists  $t \in [t_0, b)$  such that  $x$  is limited on  $[t_0, t]$  and  $x(t) \not\simeq x(t_0)$ . Using Part (i) of the lemma,  $x$  is  $S$ -continuous at each point in  $[0, t]$  and  $x(t) \simeq x(t_0)$  since  $t \simeq t_0$ . This is a contradiction and proves the first claim of Part (ii) of the lemma. If the second claim of Part (ii) is not true, then  $x(t') \simeq \infty$  for some  $t' \in [t_0, b)$  with  $t' \simeq t_0$ . Again using continuity,  $x$  is limited on  $[t_0, t]$  and  $x(t) \not\simeq x(t_0)$  for some  $t \in [t_0, b)$ . The same reasoning as above gives a contradiction and proves that  $x(t)$  is limited for all  $t \geq t_0$  such that  $t \simeq t_0$ . This completes the proof of the lemma.  $\square$

The proof of Theorem 2.4 is as follows. Let  $\varepsilon > 0$  be infinitesimal. Let  $x$  be a maximal solution of (2.5) defined on  $I$ , an interval of  $\mathbb{R}$ . Let  $t_0 \in I$  such that  $t_0 \geq 0$  and limited, and  $x(t)$  is limited for all  $t \in [0, t_0]$ . Since  $x(t_0)$  is limited,

it is nearstandard and so is  $\tilde{x}^{t_0}$  where  $\tilde{x}^{t_0} \in \mathcal{C}$  is defined by  $\tilde{x}^{t_0}(\theta) = x(t_0)$  for all  $\theta \in [-r, 0]$ . Now we apply Lemma 4.2 to obtain some constant  $\alpha = \alpha(\varepsilon, t_0, \tilde{x}^{t_0})$  such that  $0 < \alpha \simeq 0$ ,  $\varepsilon/\alpha \simeq 0$  and

$$\frac{\varepsilon}{\alpha} \int_{t_0/\varepsilon}^{t_0/\varepsilon + T\alpha/\varepsilon} f(t, \tilde{x}^{t_0}) dt \simeq TF(\tilde{x}^{t_0}) = TG(x(t_0)), \quad \forall T \in [0, 1]. \quad (4.6)$$

By Lemma 4.4 the solution  $x$  is defined for all  $t \geq t_0$  and  $t \simeq t_0$ . Hence one can consider the function

$$X(\theta, T) = \frac{x(t_0 + \alpha T + \varepsilon\theta) - x(t_0)}{\alpha}, \quad \theta \in [-r, 0], \quad T \in [0, 1].$$

We have, for  $T \in [0, 1]$ ,

$$X(0, T) = \frac{x(t_0 + \alpha T) - x(t_0)}{\alpha}, \quad x_{t_0 + \alpha T, \varepsilon} = \tilde{x}^{t_0} + \alpha X(\cdot, T)$$

and therefore

$$\frac{\partial X}{\partial T}(0, T) = f\left(\frac{t_0}{\varepsilon} + \frac{\alpha}{\varepsilon}T, \tilde{x}^{t_0} + \alpha X(\cdot, T)\right).$$

Integration between 0 and  $T$ , for  $T \in [0, 1]$ , yields

$$X(0, T) = \int_0^T f\left(\frac{t_0}{\varepsilon} + \frac{\alpha}{\varepsilon}t, \tilde{x}^{t_0} + \alpha X(\cdot, t)\right) dt. \quad (4.7)$$

We now consider the following two cases:

Case 1:  $T \in [0, \varepsilon r/\alpha]$ . Note that  $\tilde{x}^{t_0} + \alpha X(\cdot, T)$  is limited for all  $T \in [0, 1]$  (in particular for all  $T \in [0, \varepsilon r/\alpha]$ ). Using (H3') and taking into account that  $\varepsilon r/\alpha \simeq 0$ , (4.7) leads to the approximation

$$X(0, T) \simeq 0. \quad (4.8)$$

Case 2:  $T \in [\varepsilon r/\alpha, 1]$ . By Lemma 4.4 the solution  $x$  is S-continuous at each point in  $[0, t_0 + \alpha]$  so that, for  $\theta \in [-r, 0]$ ,

$$\alpha X(\theta, T) = x(t_0 + \alpha T + \varepsilon\theta) - x(t_0) \simeq 0,$$

since  $t_0 + \alpha T + \varepsilon\theta \in [t_0, t_0 + \alpha] \subset [0, t_0 + \alpha]$  and  $t_0 + \alpha T + \varepsilon\theta \simeq t_0$ .

Return now to (4.7). For  $T \in [0, 1]$ , we write

$$X(0, T) = \left( \int_0^{\varepsilon r/\alpha} + \int_{\varepsilon r/\alpha}^T \right) f\left(\frac{t_0}{\varepsilon} + \frac{\alpha}{\varepsilon}t, \tilde{x}^{t_0} + \alpha X(\cdot, t)\right) dt.$$

Note that  $x(t_0)$  is nearstandard implies that, for all  $T \in [0, 1]$ ,  $\tilde{x}^{t_0} + \alpha X(\cdot, T)$  is nearstandard. Using (4.8), (H1') and (H2'), we thus get, for  $T \in [0, 1]$ ,

$$\begin{aligned} X(0, T) &\simeq \int_{\varepsilon r/\alpha}^T f\left(\frac{t_0}{\varepsilon} + \frac{\alpha}{\varepsilon}t, \tilde{x}^{t_0}\right) dt \simeq \int_0^T f\left(\frac{t_0}{\varepsilon} + \frac{\alpha}{\varepsilon}t, \tilde{x}^{t_0}\right) dt \\ &= \frac{\varepsilon}{\alpha} \int_{t_0/\varepsilon}^{t_0/\varepsilon + T\alpha/\varepsilon} f(t, \tilde{x}^{t_0}) dt \end{aligned}$$

whence, in view of (4.6),

$$X(0, T) \simeq TG(x(t_0)).$$

Defining  $t_1 = t_0 + \alpha$  and setting  $\mu = \varepsilon$ , the following properties are true:  $\mu < \alpha = t_1 - t_0 \simeq 0$ ,  $[t_0, t_1] \subset I$ ,  $x(t_0 + \alpha T) = x(t_0) + \alpha X(0, T) \simeq x(t_0)$  for all  $T \in [0, 1]$ , that is,  $x(t) \simeq x(t_0)$  for all  $t \in [t_0, t_1]$ , and

$$\frac{x(t_1) - x(t_0)}{t_1 - t_0} = X(0, 1) \simeq G(x(t_0)).$$

This proves that  $x$  satisfies the  $F$ -stroboscopic property. Taking (H4) into account, we finally apply Theorem 3.3 (Stroboscopic Lemma for ODEs) to obtain the desired result, that is, the solution  $x$  is defined at least on  $[-\varepsilon r, L]$  and satisfies  $x(t) \simeq y(t)$  for all  $t \in [0, L]$ . The theorem is proved.

**4.4. Proof of Theorem 2.8.** Let  $\phi$  and  $L$  be standard. To prove Theorem 2.8 is equivalent to show that for every infinitesimal  $\varepsilon > 0$ , every solution  $x$  of (2.8) is defined at least on  $[-r, L]$  and  $x(t) \simeq y(t)$  holds for all  $t \in [0, L]$ . Before this, we first prove the following result.

**Lemma 4.5.** *Let  $g : \mathbb{R}_+ \times \mathcal{C} \rightarrow \mathbb{R}^d$  be a continuous function. Suppose that, for all  $t \in \mathbb{R}_+$  and all  $x \in \mathcal{C}$ ,  $t$  and  $x$  limited imply that  $g(t, x)$  is limited. Let  $\phi \in \mathcal{C}$  be standard. Let  $x : I \rightarrow \mathbb{R}^d$  be a maximal solution of the initial value problem*

$$\dot{x}(t) = g(t, x_t), \quad x_0 = \phi.$$

*Let  $t_0 \geq 0$  be limited in  $I$  such that  $x$  is limited on  $[0, t_0]$ . Then*

- (i)  *$x$  is S-continuous at each  $t \in [-r, t_0]$  and  $x_t$  is nearstandard for all  $t \in [0, t_0]$ .*
- (ii)  *$x$  is defined and limited at each  $t \simeq t_0$ .*

*Proof.* To prove (i) we first note that  $x$  is S-continuous at each  $t \in [-r, 0]$ , since it coincides with the standard and continuous function  $\phi$  on the (standard) interval  $[-r, 0]$ . Now consider the interval  $[0, t_0]$ . Let  $t \in [0, t_0]$ . If  $t' \in [0, t_0]$  is such that  $t \leq t'$  and  $t \simeq t'$ , then

$$|x(t') - x(t)| \leq \int_t^{t'} |g(s, x_s)| ds \leq (t' - t) \sup_{s \in [t, t']} |g(s, x_s)|.$$

In view of assumptions on  $x$  and  $g$ , the quantity  $\sup_{s \in [t, t']} |g(s, x_s)|$  is limited so that  $x(t') \simeq x(t)$ . This shows the S-continuity of  $x$  at  $t$ .

It remains to prove that  $x_t$  is nearstandard for all  $t \in [0, t_0]$ . We have,  $x$  is limited and S-continuous at each  $t \in [-r, t_0]$  implies that  $x_t$  is limited and S-continuous for all  $t \in [0, t_0]$ . So, the desired result follows from Theorem 3.1.

To prove (ii) we let  $I = [-r, b)$ ,  $0 < b \leq \infty$ , and suppose that  $x$  is not defined for all  $t \simeq t_0$ , that is,  $b \simeq t_0$ . Then there exists  $t' \in [t_0, b)$  such that  $x(t') \simeq \infty$ . Otherwise,  $\lim_{t \rightarrow b} x(t)$  exists and  $x$  can be continued through the point  $(b, x_b)$  to the right of  $b$ , which contradicts the noncontinuability hypothesis on  $x$ . Now, by the continuity of  $x$  there exists  $t \in [t_0, b)$  such that  $x$  is limited on  $[t_0, t]$  and  $x(t) \not\simeq x(t_0)$ . On the other hand, by Part (i) of the lemma  $x$  is S-continuous at each point in  $[-r, t]$ . Since  $t \simeq t_0$ , it follows that  $x(t) \simeq x(t_0)$ , which is absurd. This proves that  $x(t)$  is defined for all  $t \simeq t_0$ .

Suppose now that  $x(t)$  is not limited for all  $t \simeq t_0$ , that is,  $x(t') \simeq \infty$  for some  $t' \in [t_0, b)$  with  $t' \simeq t_0$ . Again, by the continuity of  $x$  there exists  $t \in [t_0, b)$  such that  $x$  is limited on  $[t_0, t]$  and  $x(t) \not\simeq x(t_0)$ . The same argument as above leads to a contradiction. This proves that  $x(t)$  is limited for all  $t \simeq t_0$ . Lemma 4.5 is proved.  $\square$

For the proof of Theorem 2.8 we fix  $\varepsilon > 0$  to be infinitesimal and we let  $x : I \rightarrow \mathbb{R}^d$  to be a maximal solution of (2.8). We will first show that  $x$  satisfies the  $F$ -stroboscopic property. Let  $t_0 \in I$  such that  $t_0 \geq 0$  and limited, and  $x(t)$  and  $F(x_t)$  are limited for all  $t \in [0, t_0]$ . According to (H2') Lemma 4.5 applies. Thus  $x_t$  is nearstandard for all  $t \in [0, t_0]$ .

Now, applied to  $t_0$  and  $x_{t_0}$ , Lemma 4.2 gives

$$\frac{\varepsilon}{\alpha} \int_{t_0/\varepsilon}^{t_0/\varepsilon + T\alpha/\varepsilon} f(t, x_{t_0}) dt \simeq TF(x_{t_0}), \quad \forall T \in [0, 1] \quad (4.9)$$

for some  $\alpha = \alpha(\varepsilon, t_0, x_{t_0})$  such that  $0 < \alpha \simeq 0$  and  $\varepsilon/\alpha \simeq 0$ .

Let  $X : [-r, 0] \times [0, 1] \rightarrow \mathbb{R}^d$  be the function given by

$$X(\theta, T) = \frac{x(t_0 + \alpha T + \theta) - x(t_0 + \theta)}{\alpha}, \quad \theta \in [-r, 0], \quad T \in [0, 1].$$

By Lemma 4.5 the function  $X$  is well defined. It satisfies, for  $T \in [0, 1]$ ,

$$X(0, T) = \frac{x(t_0 + \alpha T) - x(t_0)}{\alpha}, \quad x_{t_0 + \alpha T} = x_{t_0} + \alpha X(\cdot, T).$$

Hence, for  $T \in [0, 1]$ ,

$$\frac{\partial X}{\partial T}(0, T) = f\left(\frac{t_0}{\varepsilon} + \frac{\alpha}{\varepsilon}T, x_{t_0} + \alpha X(\cdot, T)\right).$$

Solving this equation gives, for  $T \in [0, 1]$ ,

$$X(0, T) = \int_0^T f\left(\frac{t_0}{\varepsilon} + \frac{\alpha}{\varepsilon}t, x_{t_0} + \alpha X(\cdot, t)\right) dt. \quad (4.10)$$

Now according to Lemma 4.5 the solution  $x$  is S-continuous at each point in  $[-r, t_0 + \alpha]$ . Therefore, for  $\theta \in [-r, 0]$  and  $T \in [0, 1]$ ,  $X(\theta, T)$  satisfies, since  $t_0 + \alpha T + \theta \simeq t_0 + \theta$ ,

$$\alpha X(\theta, T) = x(t_0 + \alpha T + \theta) - x(t_0 + \theta) \simeq 0. \quad (4.11)$$

By (H1') and taking (4.11) into account, (4.10) leads, for  $T \in [0, 1]$ , to the approximation

$$X(0, T) \simeq \int_0^T f\left(\frac{t_0}{\varepsilon} + \frac{\alpha}{\varepsilon}t, x_{t_0}\right) dt = \frac{\varepsilon}{\alpha} \int_{t_0/\varepsilon}^{t_0/\varepsilon + T\alpha/\varepsilon} f(t, x_{t_0}) dt.$$

From (4.9) we have

$$X(0, T) \simeq TF(x_{t_0}), \quad \forall T \in [0, 1].$$

Let  $t_1 = t_0 + \alpha$  and set  $\mu = \varepsilon$ . The instant  $t_1$  and the constant  $\mu$  are such that:  $\mu < \alpha = t_1 - t_0 \simeq 0$ ,  $[t_0, t_1] \subset I$ ,  $x(t_0 + \alpha T) = x(t_0) + \alpha X(0, T) \simeq x(t_0)$  for all  $T \in [0, 1]$ , that is,  $x(t) \simeq x(t_0)$  for all  $t \in [t_0, t_1]$  and

$$\frac{x(t_1) - x(t_0)}{t_1 - t_0} = X(0, 1) \simeq F(x_{t_0}),$$

which form the  $F$ -stroboscopic property. Finally, using (H6) we get, by means of Theorem 3.8 (Stroboscopic Lemma for RFDEs), the solution  $x$  is defined at least on  $[-r, L]$  and satisfies  $x(t) \simeq y(t)$  for all  $t \in [0, L]$ . So the proof is complete.

## REFERENCES

- [1] I. P. van den Berg, *Nonstandard Asymptotic Analysis*, Lecture Notes in Math. 1249, Springer Verlag, Berlin, 1987.
- [2] N. N. Bogolyubov and Yu. A. Mitropolsky, *Asymptotic Methods in the Theory of Nonlinear Oscillations*, Gordon and Breach, New York, 1961.
- [3] J. L. Callot and T. Sari, *Stroboscopie infinitésimale et moyennisation dans les systèmes d'équations différentielles à solutions rapidement oscillantes*, in Landau I. D., éditeur, Outils et modèles mathématiques pour l'automatique, l'analyse des systèmes et le traitement du signal, tome 3, Editions du CNRS (1983), 345-353.
- [4] F. Diener and M. Diener (Eds.), *Nonstandard Analysis in Practice*, Universitext, Springer Verlag, 1995.
- [5] J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields*, Applied Mathematical Sciences 42, Springer Verlag, New York, 1983.
- [6] A. Halanay, *On the method of averaging for differential equations with retarded argument*, J. Math. Anal. Appl. **14** (1966), 70-76.
- [7] J. K. Hale, *Averaging methods for differential equations with retarded arguments and a small parameter*, J. Differential Equations **2** (1966), 57-73.
- [8] J. K. Hale, *Ordinary Differential Equations*, Pure and Applied Mathematics XXI, Wiley-Interscience [John Wiley and Sons], New York, 1969.
- [9] J. K. Hale and S. M. Verduyn Lunel, *Averaging in infinite dimensions*, J. Integral Equations Appl. **2** (1990), No. 4, 463-494.
- [10] J. K. Hale and S. M. Verduyn Lunel, *Introduction to functional differential equations*, Applied Mathematical Sciences 99, Springer Verlag, New York, 1993.
- [11] M. Lakrib, *The method of averaging and functional differential equations with delay*, Int. J. Math. Math. Sci. **26** (2001), No. 8, 497-511.
- [12] M. Lakrib, *On the averaging method for differential equations with delay*, Electron. J. Diff. Eqns. **2002** (2002), No. 65, 1-16.
- [13] M. Lakrib, *Stroboscopie et moyennisation dans les équations différentielles fonctionnelles à retard*, Thèse de Doctorat en Mathématiques, Université de Haute Alsace, Mulhouse, 2004.
- [14] M. Lakrib and T. Sari, *Averaging results for functional differential equations*, Sibirsk. Mat. Zh. **45** (2004), No. 2, 375-386; *translation in Siberian Math. J.* **45** (2004), No. 2, 311-320.
- [15] B. Lehman, *The influence of delays when averaging slow and fast oscillating systems: overview*, IMA J. Math. Control Inform. **19** (2002), No. 1-2, 201-215.
- [16] B. Lehman and S. P. Weibel, *Fundamental theorems of averaging for functional differential equations*, J. Differential Equations **152** (1999), No. 1, 160-190.
- [17] C. Lobry, T. Sari and S. Touhami, *On Thkhonov's theorem for convergence of solutions of slow and fast systems*, Electron. J. Diff. Eqns. **1998** (1998), No. 19, 1-22.
- [18] R. Lutz, *L'intrusion de l'Analyse non standard dans l'étude des perturbations singulières*, in IIIème Rencontre de Géométrie du Schnepfenried, Vol. 2, Astérisque **107-108** (1983), 101-140.
- [19] R. Lutz and T. Sari, *Applications of Nonstandard Analysis in boundary value problems in singular perturbation theory*, Theory and Applications of Singularly Perturbations (Oberwolfach 1981), Lecture Notes in Math. 942, Springer Verlag, Berlin, 1982, pp. 113-135.
- [20] G. N. Medvedev, *Asymptotic solutions of some systems of differential equations with deviating argument*, Soviet Math. Dokl. **9** (1968), 85-87.
- [21] E. Nelson, *Internal Set Theory: a new approach to nonstandard analysis*, Bull. Amer. Math. Soc. **83** (1977), No. 6, 1165-1198.
- [22] A. Robinson, *Nonstandard Analysis*, American Elsevier, New York, 1974.
- [23] J. A. Sanders and F. Verhulst, *Averaging Methods in Nonlinear Dynamical Systems*, Applied Mathematical Sciences 59, Springer Verlag, New York, 1985.
- [24] T. Sari, *Sur la théorie asymptotique des oscillations non stationnaires*, in IIIème Rencontre de Géométrie du Schnepfenried, Vol. 2, Astérisque **109-110** (1983), 141-158.
- [25] T. Sari, *Stroboscopy and averaging*, in Colloque Trajectorien à la Mémoire de Georges Reeb et Jean-Louis Callot (Strasbourg-Obernai, 1995) (A. Fruchard et A. Troesch, eds.), Publication de l'Institut de Recherche Mathématique Avancée, Université Louis Pasteur, Strasbourg, 1995, pp. 95-124.

- [26] T. Sari and K. Yadi, *On Pontryagin-Rodygin's theorem for convergence of solutions of slow and fast systems*, Electron. J. Diff. Eqns. **2004** (2004), No. 139, 1-17.
- [27] V. M. Volosov, G. N. Medvedev and B. I. Morgunov, *On the applications of the averaging method for certain systems of differential equations with delay*, Vestnik M.G.U. Ser. III, Fizika, Astronomija (1968), 251-294.

LABORATOIRE DE MATHÉMATIQUES, UNIVERSITÉ DJILLALI LIABÈS, B.P. 89, 22000 SIDI BEL  
ABBÈS, ALGÉRIE

*E-mail address:* mlakrib@univ-sba.dz

LABORATOIRE DE MATHÉMATIQUES, INFORMATIQUE ET APPLICATIONS, 4, RUE DES FRÈRES LUMIÈRE,  
68093 MULHOUSE, FRANCE

*E-mail address:* Tewfik.Sari@uha.fr