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APPROXIMATION AND BILLIARDS

SERGE TROUBETZKOY

Abstract. This survey is based on a series of talks I gave at the conference "Dynamical systems and Diophantine approximation" at l'Instut Henri Poincaré in June 2003. I will present asymptotic results (transitivity, ergodicity, weak-mixing) for billiards based on the approximation technique developed by Katok and Zemlyakov. I will also present approximation techniques which allow to prove the abundance of periodic trajectories in certain irrational polygons.

1. Introduction.

The use of approximation techniques in dynamical systems was apparently started in 1941 by Oxtoby and Ulam [OxUl] who proved that for a finite-dimensional compact manifold with a non-atomic measure which is positive on open sets the set of ergodic measure-preserving homeomorphisms is generic in the strong topology. In 1967 Katok and Stepin [KaSt] proved the genericity of of ergodicity and weak-mixing for certain classes of interval exchange transformations. The approximation method was first applied to polygonal billiard by Katok and Zemlyakov who, using the fact that for rational polygons the directional billiard is minimal in almost all directions, proved that the typical irrational billiard is topological transitive (i.e. has dense orbits) [KaZe].

In this survey we will first reprove the Katok Zemlyakov result and then give further developments based on this idea. Then we will discuss another approximation result, based on inhomogeneous diophantine approximation, to conclude that there are many periodic billiard orbits in irrational parallelograms and related polygons.

2. Background.

In this section we give the necessary background on billiards, more details can be found in the surveys [Ta],[MaTa]. The *billiard flow* in a domain $P \subset \mathbb{R}^2$ is defined as follows: a point mass moves freely inside P and when it reaches the boundary it is reflected following the usual law of geometric optics, the angle of incidence equals the angle of reflection. The first return map to the boundary of P is called the

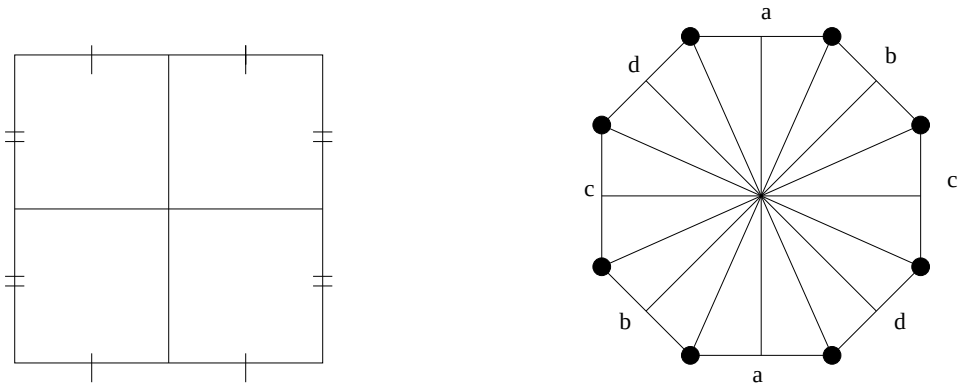

FIGURE 1. Invariant surfaces.

billiard map. The billiard map preserves a natural measure $\sin\theta\, ds\, d\theta$ where s is the arc length parameter on ∂P and $\theta \in \mathbf{S^1}$ is the angle between the direction of the billiard motion and the direction of the boundary.

A polygon is called *rational* is all the angles between sides are rational multiples of π. For rational polygons there is a well known construction of *invariant billiard surfaces*. Suppose the angles are of the form $m_i\pi/n_i$. Consider the group $G(P)$ generated by the reflections in the sides of P and let $O(P) \subset \mathbf{S^1}$ be the linearization of $G(P)$. The group $O(P)$ is simply the dihedral group D_N generated by the reflections in lines through the origin that meet at angles π/N. Here N is the least common multiple of the $n_i's$.

Consider the *phase space* of the billiard flow $P \times \mathbf{S^1}$, and let R_θ be its subset of points whose second coordinate belongs to the orbit of θ under D_N. Since every trajectory changes directions at reflections by elements of D_N we have that R_θ is an invariant set. In fact it is a flat surface (possibly) with conical singularities.

To construct R_θ (for $\theta \neq k\pi/N$) consider $2N$ disjoint parallel copies $P_1, P_2, \ldots, P_{2N}$ of P. Consider the $O(P)$ orbits $\{\theta_1, \theta_2, \ldots, \theta_{2N}\}$ of θ. Consider a pair (P_i, θ_i). Fix a side of P_i and reflect θ_i in this side, this yields a θ_j. Paste the given side of P_i and P_j. Doing this for each i yields the surface. For example the billiard in the square has a torus as invariant surface while the invariant surface for the billiard in the in the right triangle with angle $\pi/8$ is of genus two (see Figure 1).

3. Old approximation results for polygons.

It is not very difficult to prove that the billiard flow/map restricted to R_θ is minimal for all but countably many θ. We are now ready to state and prove Katok-Zemlyakov's result. Consider the space X of simply

connected n-gons. Note that this space is not compact since n-gons can degenerate into n' gons with $n' < n$. Thus we say that a property is *typical* if for any compact set $K \subset X$ the property holds for a dense G_δ subset of K.[1]

Theorem 3.1. [KaZe] *The billiard in a typical polygon is topologically transitive.*

Proof. Let $\mathcal{D} \subset \mathbb{R}^2$ be the disc. Identify the phase space of the billiard flow in each n-gon with $\mathcal{D} \times \mathbf{S^1}$ and assume that this identification depends continuously on the polygon. Let B_i be a countable basis for the topology of $\mathcal{D} \times \mathbf{S^1}$.

Fix an nonempty open set $U_k \subset \mathcal{D} \times \mathbf{S^1}$ (the choice of this set will be made precise below) and let $K_k \subset K$ be the set of n-gons P such that there exists a billiard trajectory starting in U_k that visits all (the images of) the sets $B_1, \ldots, B_k$ in the phase space of the billiard flow in P. Each K_k is open, and their intersection is a G_δ set.

To see denseness let Y_q be the set of rational n-gons with angles $\pi p_i/q_i$, with p_i, q_i co-prime and the least common multiple of the q_i at least q. For every $P \in Y_q$ each invariant surface M_θ is $1/q$ dense in the phase space. Therefore for every k there exists q such that for every $P \in Y_q$ the surface M_θ (for all θ) intersects U_k and all the (images of) the sets $B_1, \ldots, B_k$ in the phase space of the billiard flow in P. Since M_θ has a dense trajectory for all but countably many θ we have $Y_q \subset X_k$. Thus since Y_q is dense in the space of n-gons, the set K_k is as well. It follows that $\cap K_k$ is dense.

Let P be a polygon in $\cap K_k$. The choice of U_1 is arbitrary. Let $\tilde{U}_1 \subset U_1$ be a nonempty compact neighborhood. Suppose we have inductively chosen a nonempty compact neighborhood $\tilde{U}_{k-1} \subset U_{k-1}$. Let $U_k \subset \tilde{U}_{k-1}$ be open. Since $P \in K_k$ we can find a billiard trajectory starting in U_k that visits each $B_1, \ldots, B_k$. By continuity there is an compact neighborhood $U_k \subset \tilde{U}_{k-1}$ such that each trajectory in U_k visits $B_1, \ldots, B_k$ within a bounded time T_k. Thus any $x \in \cap U_k = \cap \tilde{U}_k$ has a dense trajectory, possibly singular.[2]

To see that there are non-singular trajectories it suffices to produce an uncountable collection of such dense trajectories. Since there are only countably many saddle connections, i.e. orbits segments starting and ending at a vertex, most of the dense trajectories are not saddle connections. If such a trajectory is singular, i.e. the nth iterate arrives at a vertex, then the (forward) orbit starting at time $n+1$ is dense and non-singular.

[1] This detail is often overlooked in the literature.

[2] This detail is also often overlooked in the literature.

Thus we modify the construction to produce a Cantor set (an uncountable set) of points with dense trajectories. To do this choose $\tilde{U}_k$ be the disjoint union of 2^k nonempty compact neighborhoods such that each neighborhood contains exactly two of the neighborhoods of $\tilde{U}_{k+1}$ and such that there is a billiard trajectory in each of the neighborhoods which visits the sets $B_1, \dots, B_k$. □

In 1986, Kerckhoff, Masur and Smillie, using Teichmüller theory, proved that the directional billiard flow is uniquely ergodic for almost every direction [KeMaSm]. The surfaces M_θ are not only $1/q$ dense in the phase space, but also approximately well distributed. Combining these two facts and no other new ideas one can conclude:

Theorem 3.2. [KeMaSm, Ka, PoSt] *The billiard in a typical polygon is ergodic.*

Let $\phi : \mathbb{N} \to \mathbb{R}^+$ such that $\lim_{q\to\infty} \phi(q) = 0$. Let P be a n-gon with angles $\alpha_1, \dots, \alpha_n$. We say that P admits approximation by rational polygons at rate $\phi(q)$ if for every $q_0 > 0$ there is $q > q_0$ and positive integers $p_1, \dots, p_n$ each co-prime with q such that $|\alpha_i - \pi p_i/q| < \phi(q)$ for all i.

Vorobets has proven the following constructive ergodicity result:

Theorem 3.3. [Vo] *Let P be a polygon that admits approximation by rational polygons at the rate*

$$\phi(q) = \left(2^{2^{2^{2^q}}}\right)^{-1},$$

then the billiard flow in P is ergodic.

I proved the following theorem:

Theorem 3.4. [Tr2] *The billiard map in a typical polygon is totally ergodic, i.e. T^n is ergodic for all $n \in \mathbb{N}$.*

Both the constructive ergodicity and the total ergodicity theorems are proven by using an alternative version of the proof of the result of Kerckhoff, Masur and Smillie based on a combination of ideas of Masur and Boshernitzan [Ma, Bo2].

In 1999 I applied the approximation method to infinite polygons [Tr1]. My result was improved by Degli Esposti, Del Magno and Lenci who showed:

Theorem 3.5. [DeDeLe] *Suppose p_n is a monotonically decreasing sequence of positive numbers satisfying $\sum p_n = 1$. Let*

$$P = \cup_{n\geq 0}[n, n+1] \times [0, p_n].$$

Then the billiard in a typical P is ergodic for a.e. direction.

Here the topology on the space of such P is given by the metric

$$d(P,Q) := \sum |p_n - q_n| = Area(P \triangle Q).$$

To prove this result approximate P by finite polygons from the class, i.e. $q_n \equiv 0$, for $n \geq N$.

4. Old approximation results for convex smooth tables.

Let C be a strictly convex billiard table. A compact convex set K is a *caustic* of C if the boundary of C is obtained by wrapping a string around K, pulling it tight at a point and moving the point around K while keeping the string tight. If a billiard orbit is tangent to a caustic once, then it is tangent to it in-between every pair of bounces.

In 1979 Lazutkin showed:

Theorem 4.1. [La] *If C is a sufficiently smooth strictly convex billiard table then the table contains "many" caustics.*

Many means the union of the caustics has positive area.

Corollary 4.2. *The billiard in a sufficiently smooth strictly convex table is not ergodic, not even topologically transitive.*

The proof of Lazutkin's result is based on KAM theory. Lazutkin's original proof need the table to be C^{553}. Rüssmann's version of KAM can be used to replace 553 by 8, and finally R. Douady's version of KAM shows that 7 is sufficient [Rü, Do].

In 1990 Gruber noticed that we can apply the approximation ideas of the previous section to prove contrasting results for the low smoothness case.

Theorem 4.3. [Gr] *The C^0-typical convex billiard table has the following properties:*

a) *it contains no caustics,*
b) *it is strictly convex,*
c) *it is of class C^1,*
d) *it is topologically transitive.*

To prove the topological transitivity we approximate a convex table by finite polygons with increasing number of sides.

In fact, using approximation teachings and the results of Kerckhoff, Masur and Smillie one can easily conclude that

Theorem 4.4. *C^1-typical C^1-convex billiard table is strictly convex and ergodic.*

5. New results on smooth tables.

With A. Stepin we have have improved Gruber's result. To describe our result we begin by a new characterization of weak mixing.

5.1. **Weak mixing.** Let (X, β, μ) be a probability space and $T : X \to X$ a measure preserving transformation. T is *weak mixing* iff

$$\forall A, B \in \beta : \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} |\mu(T^{-k}A \cap B) - \mu(A)\mu(B)| = 0$$

iff $\forall A, B \in \beta$: there exists $J = J(A, B) \subset \mathbb{N}$ of full density (i.e. $(\#n \in J : n \leq N)/N \to 1$) such that

$$\lim_{n\in J\to\infty} \mu(T^{-k}A \cap B) = \mu(A)\mu(B).$$

A set $J \subset \mathbb{N}$ is called *full* if J contains arbitrarily long runs, i.e. $\forall N, \exists k$ s.t. $\{k, k+1, \ldots, k+N\} \subset J$.

Theorem 5.1. [StTr] *T is weak-mixing iff $\forall A \in \beta$ there exists J full such that*

$$\lim_{n\in J\to\infty} \mu(T^{-k}A \cap A) = \mu(A)^2$$

Proof. First we claim that T is ergodic. Suppose $T^{-1}(A) = A$ mod 0. By invariance we have $\mu(T^{-n}A \cap A) = \mu(A)$, but by mixing along full sequences this quantity converges to $\mu(A)^2$ for $n \in J$. Thus $\mu(A) = 0$ or $\mu(A) = 1$.

Now we are ready to prove that T is weak-mixing. Suppose not, then there exists a Kronecker factor $(\hat{X}, \hat{\beta}, \hat{\mu}, \hat{T})$, i.e. a rotation $\hat{T}$ on a compact Abelian group $\hat{X}$ such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{T} & X \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ \hat{X} & \xrightarrow{\hat{T}} & \hat{X} \end{array}$$

Fix $\hat{A} \in \hat{\beta}$ such that $0 < \hat{\mu}(\hat{A}) < 1$ and let $A := \pi^{-1}(\hat{A})$.

Then since $\hat{T}$ is a rotation, there is a quasi-periodic sequence n_i such that $\hat{\mu}(\hat{T}^{-n_i}\hat{A} \cap \hat{A}) \approx \hat{\mu}(\hat{A})$ and thus

$$\mu(T^{-n_i}A \cap A) = \hat{\mu}(\hat{T}^{-n_i}\hat{A} \cap \hat{A}) \approx \hat{\mu}(\hat{A}) = \mu(A)$$

which contradicts fullness. □

5.2. **Hyperbolic billiards.** Instead of approximating a smooth table by polygons we approximate by certain hyperbolic tables. Namely consider a polygon P. Replace each vertex with a (small) circular arc such that 1) the resulting table is C^1 and 2) the focusing circles lie inside the table. We call such tables B-tables and if the original polygon is convex then CT-tables.

Theorem 5.2. *(Bunimovich* [Bun]*) For any B-tables each ergodic component is open (mod 0) and the billiard on each ergodic component is mixing, K-mixing, Bernoulli.*

Theorem 5.3. *(Chernov, Troubetzkoy* [ChTr]*) Any CT-table is ergodic, hence mixing, K-mixing, Bernoulli.*

5.3. **Back to C^1 billiards.**

Theorem 5.4. [StTr] *The billiard flow/map in a C^1-typical C^1 billiard table is weak-mixing.*

Sketch of the proof: If the table is convex we approximate by billiards from the class of CT-tables introduced above. Fix a CT-table and a finite collection of sets. For all tables sufficiently close to the fixed table we can find an arbitrarily long run of times such that these sets approximately mix along this run.

If the table is not convex then we do not have a nice class of ergodic tables like the CT-class. To prove the theorem we approximate the table by polygons to obtain generic ergodicity. Then we approximate by B-tables to get the weak-mixing. □

5.4. **Convex tables.** We can slightly improve the smoothness is the table is convex, or piecewise concave-convex. Let PCC be the class of C^1 piecewise convex-concave billiard domains. Because of the piecewise monotonicity of the derivative, locally the second derivative exists almost everywhere. Consider the Hausdorff distance ρ_H between the second derivative of the boundaries of such tables. Let B^2 be the closure of the class of B-tables in the topology induced by the metric $\rho = \rho_H + \rho_{C^1}$.

Theorem 5.5. [StTr] *Weak-mixing is typical in B^2*

We remark that $PCC \subset B^2 \subset C^1$ and that every table in PCC is C^2 except at a finite number of points. Furthermore the set of strictly convex tables is residual in B^2_{convex}, thus the generic table in B^2_{convex} is strictly convex and weak-mixing.

6. Back to polygonal billiards.

Recently Avila and Forni have shown that a.e. interval exchange not of rotation type is weak-mixing [AvFo]. Furthermore, for almost every translation surface they can show that the directional flow is weak-mixing. A priori, these results do not hold for polygonal billiards. Thus the results obtained by Stepin and myself stated here are conditional.

Let $WMix_c$ be the set of rational polygons for which at least $c\%$ of the ergodic components are weak-mixing. Note that the square and the equilateral triangle are not in $WMix_c$ for any $c > 0$. The result of Avila-Forni gives hope that all but finitely many rational polygons are in $WMix_1$.

Theorem 6.1. [StTr] *1) If there exists $c > 0$ such that $WMix_c$ is dense in the set of polygons then weak-mixing is generic for polygonal billiards.*
2) Either there exists m such that $WMix_{\frac{1}{m}}$ is somewhere dense in the set of polygons (and hence weak-mixing is on 2nd category for polygonal billiards) or $\cup_m WMix_{\frac{1}{m}}$ is nowhere dense.

7. Angular recurrence and periodic billiard orbits.

The billiard orbit of any point which begins perpendicular to a side of a polygon and at a later instance hits some side perpendicularly retraces its path infinitely often in both senses between the two perpendicular collisions and thus is periodic.

7.1. **History and notation.** In 1991 Ruijgrok conjectured based on numerical evidence that for any irrational triangle almost every orbit which starts perpendicular to a side is twice perpendicular and thus periodic[Ru]. Here almost every is with respect to the length measure on the side. In 1992 Boshernitzan [Bo1] and independently Galperin, Stepin and Vorobets [GaStVo] proved this conjecture for all rational polygons. In this case the set of non-periodic perpendicular orbits is a finite set. In 1995 Cipra, Kolan and Hansen proved this conjecture for all right triangles [CiHaKo]. Apparently these four articles are completely independent of each other.

Applying Theorem 3.4, I obtained the first approximation result in this direction.

Theorem 7.1. [Tr2] *The conjecture is true for dense G_δ of n–gons with $n-2$ angles fixed rational multiples of π.*

Billiards are not affected by similarity, thus it is convenient to fix two corners, say at the origin and the point $(1, 0)$. For triangles if we fix

one angle to be a rational multiple of π then the set of triangles is an interval parametrized by one of the other angles. This theorem says that for a dense G_δ subset of (any compact subset of) this interval the conjecture is true.

To discuss our results on billiards in parallelograms we must introduce some more notation. For the moment we assume that P is convex. We will describe the billiard map T as a transformation of the set X of rays which intersect P. Let θ be the angle between the billiard trajectory and the positive x–axis. Consider the perpendicular cross section X_θ to the set of rays whose angle is θ. The set X_θ is simply an interval. Let w be the arc-length on X_θ.

For a non-convex polygon we must differentiate the portion of rays which enter and leave the polygon several times. The above construction can be done locally, yielding the set X_θ which consists of a finite union of intervals. Let w be the unnormalized length measure on X_θ.

For every direction θ let F_θ be the set of $x \in X_\theta$ whose forward orbit never returns parallel to x. A polygon is called *angularly recurrent* if for every direction θ we have $w(F_\theta) = 0$. Angularly recurrent polygons satisfy the conclusion of Ruijgrok's conjecture, almost every perpendicular orbit is periodic! There is no reason to think that a polygon chosen at random should be angularly recurrent, this property is much stronger than Poincaré recurrence which only implies that almost every point comes back close to itself (angles close not equal!).

A polygon P is called a *generalized parallelogram* if all sides of P are parallel to two fixed vectors. Note that a right triangle unfolds to a rhombus and thus results for generalized parallelograms hold also for right triangles.

In 1996 Gutkin and I generalizing the result of Cipra, Kolan and Hansen proved that all right triangles and all generalized parallelograms are angularly recurrent [GuTr].

7.2. **Dimensions.** Let $Y \subset \mathbf{R^n}$. Let $N(\epsilon)$ denote the minimal number of ϵ balls needed to cover Y. The *lower box dimension* of Y, denoted by $\dim_{LB} Y$ is given by

$$\liminf_{\epsilon \to 0} \frac{\log N(\epsilon)}{\log 1/\epsilon} \ .$$

The *upper box dimension* $\dim_{UB}$ is defined similarly, replacing the $\liminf$ by $\limsup$. If $\dim_{UB} Y$ and $\dim_{LB} Y$ both exist and are equal, we define the box dimension of Y to be this value, and write $\dim_B Y = \dim_{UB} Y = \dim_{LB} Y$.

Let $s \in [0, \infty]$. The *s-dimensional Hausdorff measure* $\mathcal{H}^s(Y)$ of a subset Y of $\mathbf{R^n}$ is defined by the following limit of covering sums:

$$\begin{aligned}\mathcal{H}^s(Y) &= \lim_{\epsilon \to 0}(\inf\{\sum_{i=1}^{\infty}(\text{ diam } U_i)^s : \\ &\qquad Y \subset \bigcup_{i=1}^{\infty} U_i \text{ and } \sup_i \text{ diam } U_i \leq \epsilon\}).\end{aligned}$$

It is easy to see that there exists a unique $s_0 = s_0(Y)$ such that

$$\mathcal{H}^s(Y) = \infty \text{ for } s < s_0 \text{ and } 0 \text{ for } s > s_0 \ .$$

The number s_0 is called the *Hausdorff dimension* of Y and is denoted by $\dim_H Y$. Standard arguments give that

$$\dim_H Y \leq \dim_{LB} Y \leq \dim_{UB} Y$$

7.3. **Approximation questions.** With Jörg Schmeling we showed that for all generalized parallelograms $\dim_{LB}(F_\theta) \leq 1/2$ for all θ [ScTr].

The following theorem shows that we can do better that 1/2 for well approximable directions in a given parallelogram or for all directions in a parallelogram with well approximable angle.

Theorem 7.2. [ScTr] *Fix a generalized parallelogram with irrational angle α. For any θ such that $\|(\theta + p\alpha)/\pi\| < p^{-\mu}$ has infinitely many solutions $p \in \mathbb{N}$. Then*

$$\dim_{LB}(F_\theta \cap U_\theta) \leq \frac{1}{\mu + 1}.$$

Similarly $\dim_{LB}(F_\theta \cap D_\theta) \leq \frac{1}{\mu+1}$ *provided the angle $\theta - \alpha/2$ is μ-well approximable.*

Two questions naturally arise, *for a fixed generalized parallelogram how many directions have small F_θ* and *for a fixed direction θ_0 how many parallelograms have a small F_{θ_0}?*

Our answer to these question required new and recent results in diophantine approximation.

7.4. **New number theoretic results.** For $\mu \geq 1$ and $\omega \in \mathbf{S^1}$ let

$$\mathcal{A}_{\mu,\omega} := \{t \in \mathbf{S^1} : \quad \|t + p\omega\| < p^{-\mu} \text{ for infinitely many } p \in \mathbb{N}\}.$$

Minkowski's classical theorem states that if $\mu = 1$ then $\mathcal{A}_{\mu,\omega}$ consists of all $t \in \mathbf{S^1}$ such that t is not in the orbit of ω.

Using different techniques Schmeling and I and independently Y. Bugeaud have shown:

Theorem 7.3. [ScTr],[Bu] *For any $\mu > 1$ we have*

$$\dim_H(\mathcal{A}_{\mu,\omega}) = 1/\mu.$$

For $\mu \geq 1$ and $t \in \mathbf{S^1}$ let

$$\mathcal{B}_{\mu,t} := \{\omega \in \mathbf{S^1} : \|t + p\omega\| < p^{-\mu} \text{ for infinitely many } p \in \mathbb{N}\}.$$

The other result we need to apply is a recent one of Levesly:

Theorem 7.4. *(Levesly* [Le]*) For any $\mu > 1$ we have*

$$\dim_H(\mathcal{B}_{\mu,t}) = \frac{2}{1+\mu}.$$

Remark: this is a classical theorem of Jarnik for $t = 0$.

Besides the applications to billiards these results can be used to analyze the rotation driven dynamical variant of the classical Dvoretzky covering of the circle [FaSc]. More recently we have studied dynamical diophantine approximation for the circle doubling map [FaScTr].

7.5. **Application to billiards.** We reply to the first question, for a fixed generalized parallelogram how many directions have small F_θ? Let

$$\mathcal{C}_s := \{\theta : \dim_{LB}(F_\theta \cap U_\theta) \leq s\}.$$

Theorem 7.5. [ScTr] *For all $s \in [0, 1/2]$*

$$\dim_H(\mathcal{C}_s) \geq \frac{s}{1-s}.$$

The set $\mathcal{C}_s$ is residual with box dimension 1.

Proof. This follows immediately from Theorems 7.2 and 7.3. □

We turn to the second question, for a fixed direction θ_0 how many parallelograms have a small F_{θ_0}? Let

$$\begin{aligned}\mathcal{D}_s \quad := \quad & \{\alpha \in \mathbf{S^1} : \dim_{LB}(F_{\theta_0} \cap U_{\theta_0}) < s \text{ for all} \\ & \text{generalized parallelograms with angle } \alpha\}.\end{aligned}$$

Theorem 7.6. [ScTr] *For θ_0 fixed and $s \in [0, 1/2]$ we have*

$$\dim_H \mathcal{D}_s \geq 2s.$$

The set $\mathcal{D}_s$ is residual and has box dimension 1.

Proof. This follows immediately from Theorems 7.2 and 7.4. □

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