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# On homogenization of space-time dependent and degenerate random flows

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## Abstract

We study a diffusion with random, time dependent coefficients. Moreover, the diffusion coefficient is allowed to degenerate. We prove the invariance principle when this diffusion is supposed to be controlled by another one with time independent coefficients.

## 1 Introduction

We want to prove the invariance principle for a diffusive particle in a random flow described by the following stochastic differential equation

$$X_t^\omega = x + \int_0^t b(r, X_r^\omega, \omega) ds + \int_0^t \sigma(r, X_r^\omega, \omega) dB_s \quad (1)$$

where  $B$  is a  $d$ -dimensional Brownian motion and  $\sigma, b$  are stationary random fields such that the generator of the diffusion can be rewritten in divergence form

$$L^\omega = \frac{e^{2V(x,\omega)}}{2} \operatorname{div}_x \left( [a(t, x, \omega) e^{-2V(x,\omega)} + H(t, x, \omega)] \nabla_x \right).$$

Here  $a(t, x, \omega)$  is equal to  $\sigma \sigma^*(t, x, \omega)$ ,  $V$  and  $H$  are still random fields such that  $V$  is bounded and  $H$  antisymmetric.

This will allow us to describe the limit as  $\varepsilon \rightarrow 0$  of  $z(x/\varepsilon, t/\varepsilon^2, \omega)$  where  $z$  is the solution of the parabolic PDE:

$$\begin{cases} \frac{\partial}{\partial t} z(x, t, \omega) &= \frac{1}{2} \sum_{i,j=1}^d a_{ij}(t, x, \omega) \frac{\partial^2}{\partial x_i \partial x_j} z(x, t, \omega) + \sum_i^d b_i(t, x, \omega) \frac{\partial}{\partial x_i} z(x, t, \omega) \\ z(x, 0, \omega) &= f(x) \end{cases} \quad (2)$$

We will prove that, in probability with respect to  $\omega$ ,

$$\lim_{\varepsilon \rightarrow 0} z(x/\varepsilon, t/\varepsilon^2, \omega) = \bar{z}(t, x) \quad (3)$$

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where  $\bar{z}$  is the solution of the following deterministic equation of the type

$$\frac{\partial \bar{z}}{\partial t} = \sum_{i,j=1}^d A_{ij} \frac{\partial^2 \bar{z}}{\partial x_i \partial x_j}$$

where  $A$  is a constant matrix - the matrix of so-called effective coefficients. We can here mention that central limit theorems valid for almost every environment have already been obtained in the time independent case (see [11] for instance).

The homogenization problems have been extensively studied in the case of periodic flows (cf. [1], [12], [13], and many others). The study of random flows (see [15], [3], [7], [10], and many other) spread rapidly thanks to the techniques of the *environment as seen from the particle* introduced by Kipnis and Varadhan in [5], at least in the case of time independent random flows. Recently, there have been results going beyond these techniques in the case of isotropic coefficients which are small perturbations of a Brownian motion (see [16]). But there are only a few works in the case of space-time dependent random flows ([8] or [9] for instance). In these, the diffusion matrix  $\sigma$  is supposed to be constant. The novelty of this work is that the diffusion matrix can depend of the space and time variables. The only restriction about the time dependence is the control of our diffusion process with an ergodic and time independent one. As a consequence, this work includes the static case where all the coefficients do not depend on time. Moreover, these assumptions allow the diffusion matrix to degenerate under certain conditions. Typically it can degenerate in certain directions or vanish on subsets of null measure but can not totally vanish on an open subset of  $\mathbb{R}^d$ , like in [12] in the periodic case.

We will outline now the main ideas of the proof. Our goal is to show that the rescaled process

$$\varepsilon X_{t/\varepsilon^2}^\omega = \varepsilon \int_0^{t/\varepsilon^2} b(r, X_r^\omega, \omega) ds + \varepsilon \int_0^{t/\varepsilon^2} \sigma(r, X_r^\omega, \omega) dB_s$$

converges in law to a Brownian motion with a certain positive covariance matrix. The general strategy (see [6]) consists in finding an approximation of the first term on the right-hand side by a family of martingales and then apply the central limit theorem for martingales. In order to find such an approximation, we look at the environment as seen from the particle

$$Y_t = \tau_{t, X_t^\omega} \omega$$

where  $\{\tau_{t,x}\}$  is a group of measure preserving transformation on the medium  $\Omega$ . Thanks to the particular choice of the drift, we can explicitly find an invariant measure for this Markov process. Provided that the degeneracy of the diffusion is not too strong (see hypothesis 2.4), this measure is ergodic. The approximation that we want to find leads to study the equation ( $\lambda > 0$ )

$$\lambda \mathbf{u}_\lambda - (\mathbf{L} + D_t) \mathbf{u}_\lambda = \mathbf{b}$$

where  $\mathbf{L} + D_t$  is the generator of the process  $Y$  (the term  $D_t$  is due to the time translations). Here are arising the difficulties due to the time dependence. Indeed, this introduces additional difficulties and the usual techniques that are used in the static case fall short of establishing the so-called sublinear growth of the correctors  $\mathbf{u}_\lambda$ . Essentially, they are the result of the time degeneracy of the operator  $\mathbf{L} + D_t$ . To get round this difficulty, the regularity properties of the heat kernel are used in [8] or [9]. Here the degeneracy of the diffusion coefficient prevents us to use these arguments. That is why arises the last main assumption of this paper (see hypothesis 2.2). We introduce a new operator  $\tilde{\mathbf{S}}$

whose coefficients do not depend on time. Then the spectral calculus linked to the normal operator  $\widetilde{S} + D_t$  will be useful to establish the desired estimates for the solution  $v_\lambda$  of the equation

$$\lambda v_\lambda - \widetilde{S}v_\lambda - D_t v_\lambda = b.$$

Finally, with perturbation methods, we show that these estimates remain valid for the correctors  $v_\lambda$ .

## 2 Notations and setup

Let us first introduce a random medium with the next definition

**Definition 2.1.** *Let  $(\Omega, \mathcal{G}, \mu)$  be a probability space and  $\{\tau_{t,x}; (t, x) \in \mathbb{R} \times \mathbb{R}^d\}$  a stochastically continuous, measurable group of measure preserving transformations acting ergodically on  $\Omega$ . This means that*

- $\forall A \in \mathcal{G}, \forall (t, x) \in \mathbb{R} \times \mathbb{R}^d, \mu(\tau_{t,x}A) = \mu(A)$ ,
- *If for all  $(t, x) \in \mathbb{R} \times \mathbb{R}^d, \tau_{t,x}A = A$  then  $\mu(A) = 0$  or  $1$ ,*
- *For any measurable function  $f$  on  $(\Omega, \mathcal{G}, \mu)$ , the function  $(t, x, \omega) \mapsto f(t, x, \omega)$  is measurable on  $(\mathbb{R} \times \mathbb{R}^d \times \Omega, \mathcal{B}(\mathbb{R} \times \mathbb{R}^d) \otimes \mathcal{G})$ .*
- *For every  $f \in L^2(\Omega, \mu)$  and for every  $\delta > 0$ :*

$$\mu \{ |f(\tau_{t,x}\omega) - f(\omega)| \geq \delta \} \xrightarrow{(t,x) \rightarrow 0} 0.$$

We denote by  $L^2(\Omega)$  the space of square integrable functions on  $(\Omega, \mathcal{G}, \mu)$  equipped with its usual norm  $|\cdot|_2$  and  $(\cdot, \cdot)_2$  will denote the corresponding scalar product. Then the operators on  $L^2(\Omega)$  defined by

$$T_{(t,x)}f(\omega) = f(\tau_{t,x}\omega) \tag{4}$$

form a strongly continuous group of unitary maps in  $L^2(\Omega)$ . For every function  $f \in L^2(\Omega)$ , let  $f(t, x, \omega) = f(\tau_{t,x}\omega)$ . Each function  $f$  in  $L^2(\Omega)$  defines in this way a stationary ergodic random field on  $\mathbb{R}^{d+1}$ . Reciprocally, for each stationary ergodic random field one can always find a probability space where such a representation is possible. In what follows we will use the bold type to denote an element  $f \in L^2(\Omega)$  and the normal type  $f(t, x, \omega)$  to distinguish from the associated stationary field. The group possesses  $d + 1$  generators defined by (with the notation  $x_0 = t$ )

$$D_i f = \frac{\partial}{\partial x_i} T_x f|_{x=0}, \quad i = 0, \dots, d \tag{5}$$

which are closed and densely defined. In what follows we will also denote by  $D_t$  the generator  $D_0$ . We denote by  $\mathcal{C}$  the dense subset of  $L^2(\Omega)$  defined by

$$\mathcal{C} = \left\{ f * \varphi; f \in L^2(\Omega), \varphi \in C_c^\infty(\mathbb{R}^{d+1}) \right\} \tag{6}$$

where the convolution operator is defined by

$$f * \varphi(\omega) = \int_{\mathbb{R}^{d+1}} f(\tau_{t,x}\omega) \varphi(t, x) dt dx \tag{7}$$

We can remark that  $\mathcal{C} \subset \text{Dom}(D_i)$  and  $D_i(\mathbf{f} * \varphi) = -\mathbf{f} * \frac{\partial \varphi}{\partial x_i}$  and this last quantity is also equal to  $D_i \mathbf{f} * \varphi$  if  $\mathbf{f} \in \text{Dom}(D_i)$ .

We now consider a standard  $d$ -dimensional Brownian motion defined on a probability space  $(\Omega', \mathcal{F}, \mathbb{P})$  and the diffusion in random medium given by the following Itô equation

$$X_t^\omega = x + \int_0^t b(r, X_r^\omega, \omega) dr + \int_0^t \sigma(r, X_r^\omega, \omega) dB_r \quad (8)$$

The coefficients  $b(t, x, \omega)$  and  $\sigma(t, x, \omega)$  are stationary random fields defined on  $(\Omega; \mathcal{G}; \mu)$  such that  $\{X_t^{\varepsilon, \omega}; t \geq 0\}$  is finally defined on the product space  $(\Omega \times \Omega'; \mathcal{G} \otimes \mathcal{F}; \mu \times \mathbb{P})$  (the medium and the Brownian motion are mutually independent).

We are going to give more details on the coefficients.

### Hypothesis 2.2. Control of the coefficients

- We suppose that there exists a coefficient  $\tilde{\sigma}$  which does not depend on time, ie

$$\forall t \in \mathbb{R}, \quad \tilde{\sigma}(\tau_{t,0}\omega) = \tilde{\sigma}(\omega),$$

and two constants  $m$  and  $M$  such that  $\forall \omega \in \Omega$

$$m\tilde{\mathbf{a}}(\omega) \leq \mathbf{a}(\omega) \leq M\tilde{\mathbf{a}}(\omega) \quad (9)$$

where the symmetric, positive  $d \times d$  matrix  $\tilde{\mathbf{a}}$  is defined by  $\tilde{\mathbf{a}} = \tilde{\sigma}\tilde{\sigma}^*$ , which doesn't depend on time either.

- Let  $\mathbf{H}$  be a  $d \times d$  antisymmetric and stationary matrix (the so-called stream matrix) such that there exists a constant  $C_1$  such that

$$\mu \text{ as, } |\mathbf{H}| \leq C_1^H \tilde{\mathbf{a}}(\omega), \quad (10)$$

where  $|\mathbf{H}|$  is the symmetric positive matrix  $\sqrt{-\mathbf{H}^2}$ . Moreover we suppose that  $\mathbf{H}, \mathbf{a} \in \text{Dom}(D_t)$  and

$$\mu \text{ as, } |D_t \mathbf{H}|(\omega) \leq C_2^H \tilde{\mathbf{a}}(\omega), \quad (11)$$

$$\mu \text{ as, } |D_t \mathbf{a}|(\omega) \leq \tilde{C} \tilde{\mathbf{a}}(\omega) \quad (12)$$

For example, if the matrix  $\mathbf{a}$  is uniformly elliptic and bounded, we can choose  $\tilde{\sigma} = \text{Id}$  and then

$$(10), (11) \text{ and } (12) \Leftrightarrow \mathbf{H}, D_t \mathbf{H} \text{ and } D_t \mathbf{a} \in L^\infty(\Omega).$$

We consider moreover a stationary, real valued function  $\mathbf{V}$  which doesn't depend on time. We assume that

$$\forall i, j, k, l = 1, \dots, d, \quad \mathbf{a}_{ij}, \tilde{\mathbf{a}}_{ij}, \mathbf{V}, \mathbf{H}_{ij}, D_l \mathbf{a}_{ij} \text{ and } D_l \tilde{\mathbf{a}}_{ij} \in \text{Dom}(D_k).$$

Now we are able to define the coefficient  $\mathbf{b}$ . We assume that its entries are given by the following expression, for  $i = 1, \dots, d$

$$\mathbf{b}_i(\omega) = \sum_{j=1}^d \left( \frac{1}{2} D_j \mathbf{a}_{ij}(\omega) - \mathbf{a}_{ij} D_j \mathbf{V}(\omega) + \frac{1}{2} e^{2\mathbf{V}(\omega)} D_j \mathbf{H}_{ij}(\omega) \right). \quad (13)$$

In the same way, we define for  $i = 1, \dots, d$

$$\tilde{\mathbf{b}}_i(\omega) = \sum_{j=1}^d \left( \frac{1}{2} D_j \tilde{\mathbf{a}}_{ij}(\omega) - \tilde{\mathbf{a}}_{ij} D_j \mathbf{V}(\omega)(\omega) \right). \quad (14)$$

For the sake of clarity, we sum up the regularity assumptions of the coefficients as follows

**Hypothesis 2.3. Regularity of the coefficients**

- We assume that  $\forall i, j, k, l = 1, \dots, d$ ,

$$\mathbf{a}_{ij}, \tilde{\mathbf{a}}_{ij}, \mathbf{V}, \mathbf{H}_{ij}, D_l \mathbf{a}_{ij} \text{ and } D_l \tilde{\mathbf{a}}_{ij} \in \text{Dom}(D_k).$$

- In order to prove the existence of global solutions of the SDE (8), we assume that the applications  $x \mapsto b(t, x, \omega)$  and  $x \mapsto \sigma(t, x, \omega)$ ,  $x \mapsto \tilde{b}(t, x, \omega)$  and  $x \mapsto \tilde{\sigma}(t, x, \omega)$  are locally lipschitz and that the coefficients  $\sigma, \mathbf{a}, \mathbf{b}, \tilde{\sigma}, \tilde{\mathbf{b}}, \mathbf{V}, \mathbf{H}$  are uniformly bounded by a constant  $K$ .

In order to guarantee the homogenization property, we suppose the following

**Hypothesis 2.4. Ergodicity** Let  $\mathbf{L}$  be the unbounded operator on  $L^2(\Omega)$  defined by

$$\mathbf{L} = \frac{1}{2} \sum_{i,j=1}^d \mathbf{a}_{ij} D_i D_j + \sum_{i=1}^d \mathbf{b}_i D_i, \quad (15)$$

which can be rewritten in divergence form as  $\mathbf{L} = \frac{e^{2\mathbf{V}}}{2} \text{div}_x \left( (\mathbf{a} e^{-2\mathbf{V}} + \mathbf{H}) D_x \right)$ . We suppose that each  $\mathbf{L} + D_t$  harmonic function must be constant  $\mu$  almost surely.

Even if it means adding to  $\mathbf{V}$  a constant (and this will not change the drift  $\mathbf{b}$  if we divide the matrix  $\mathbf{H}$  by the exponential of twice the same constant), we make the assumption that  $\int e^{-2\mathbf{V}} d\mu = 1$ . This enable us to consider a new probability measure on  $\Omega$  given by

$$d\pi(\omega) = e^{-2\mathbf{V}(\omega)} d\mu(\omega). \quad (16)$$

Finally, let us define on  $\text{Dom}(\mathbf{L}) \cap \text{Dom}(\mathbf{L}^*)$  the operators  $\mathbf{S} = \frac{\mathbf{L} + \mathbf{L}^*}{2}$  and  $\mathbf{A} = \frac{\mathbf{L} - \mathbf{L}^*}{2}$ , respectively equal to the symmetric and antisymmetric parts of the operator  $\mathbf{L}$  in  $L^2(\Omega, \pi)$ , where  $\mathbf{L}^*$  stands for the adjoint operator of  $\mathbf{L}$  in  $L^2(\Omega, \pi)$ . These operators could be given more explicitly on  $\mathcal{C}$  by the following formulas

$$\mathbf{S} = \frac{e^{2\mathbf{V}}}{2} \text{div}_x \left( (\mathbf{a} e^{-2\mathbf{V}}) D_x \right), \quad \text{and} \quad \mathbf{A} = \frac{e^{2\mathbf{V}}}{2} \text{div}_x \left( \mathbf{H} D_x \right).$$

Let us now state a result which shows the importance of the symmetric operator

$$\tilde{\mathbf{S}} = \frac{e^{2\mathbf{V}}}{2} \text{div}_x \left( (\tilde{\mathbf{a}} e^{-2\mathbf{V}}) D_x \right)$$

linked to the process  $\tilde{X}$ .

**Lemma 2.5.** *We suppose that  $\tilde{\mathcal{S}}$  satisfies the following condition*

$$\forall \mathbf{f} \in \text{Dom}(\tilde{\mathcal{S}}), \quad \tilde{\mathcal{S}}\mathbf{f} = 0 \quad \Rightarrow \quad \mathbf{f} \text{ is equal to a function which is constant with respect to the space translations } \mu \text{ almost surely.}$$

*Then the hypothesis 2.4 is satisfied.*

**Proof:** We use the notations given in the subsections 5.1 and 5.3, so the proof could be omitted in first reading. Let  $\mathbf{f} \in \text{Dom}(\mathbf{L} + D_t)$  which is  $\mathbf{L} + D_t$ -harmonic. Thus  $\lambda G_\lambda \mathbf{f} = \mathbf{f} \in \mathbb{H}_1$ . Moreover, we have for any  $\psi \in \mathbb{H}_1$

$$\lambda |G_\lambda \psi|_2^2 + \|G_\lambda \psi\|_1^2 = \int_\Omega \psi G_\lambda \psi \, d\pi$$

and this necessarily leads to  $\|\mathbf{f}\|_1 = 0$ .

We deduce  $\forall \varphi \in \mathbb{H}_1$  and  $\forall \alpha > 0$

$$\alpha(\mathbf{f}, \varphi)_{2+} < \mathbf{f}, \varphi >_1 = (\alpha \mathbf{f}, \varphi)_2.$$

Hence, if we denote by  $G_\alpha^{\tilde{\mathcal{S}}}$  the resolvent of the generator  $\tilde{\mathcal{S}}$ , we have  $\mathbf{f} = \alpha G_\alpha^{\tilde{\mathcal{S}}}(\mathbf{f})$ , so that  $\mathbf{f}$  is  $\tilde{\mathcal{S}}$ -harmonic and then constant with respect to the space translation, then  $\mathbf{f} \in \text{Dom}(\mathbf{L})$  and  $\mathbf{L}\mathbf{f} = 0$ . From  $(\mathbf{L} + D_t)\mathbf{f} = 0$ , we deduce that  $\mathbf{f} \in \text{Dom}(D_t)$  and  $D_t\mathbf{f} = 0$ . Finally  $\mathbf{f}$  is  $\mu$  almost surely equal to a function which is invariant with respect to the time and space translations. By the ergodicity of the measure  $\mu$ , this function is constant and hence  $\mathbf{f}$  is constant  $\mu$  almost surely  $\square$

This kind of hypothesis is automatically true when for almost all  $\omega \in \Omega$ , the  $\mathbb{R}^d$ -valued Markov process  $X^{\omega, \tilde{\mathcal{S}}}$  with generator

$$\tilde{\mathcal{S}}^\omega = \frac{e^{2V(x, \omega)}}{2} \text{Div}_x \left( \tilde{a}(x, \omega) e^{-2V(x, \omega)} \nabla_x \cdot \right)$$

is irreducible in the sense that starting from any point of  $\mathbb{R}^d$ , the process reaches each subset of  $\mathbb{R}^d$  of non-null Lebesgue's measure in finite time, ie :

**Proposition 2.6.** *We suppose that the diffusion process  $X^{\omega, \tilde{\mathcal{S}}}$  with generator  $\tilde{\mathcal{S}}^\omega$  verifies the following hypothesis: there exists a measurable subset  $N \subset \Omega$  with  $\mu(N) = 0$  such that  $\forall \omega \in \Omega \setminus N$ , for each measurable subset  $B$  of  $\mathbb{R}^d$  with  $\lambda_{\text{Leb}}(B) > 0$ ,  $\forall x \in \mathbb{R}^d$ ,  $\exists t > 0$ ,*

$$\mathbb{P}_x \left( X_t^{\omega, \tilde{\mathcal{S}}} \in B \right) > 0.$$

*Then the hypothesis of lemma 2.5 is satisfied.*

**Proof:** First we remark that the  $\Omega$ -valued process  $Y^{\tilde{\mathcal{S}}}$  defined by

$$\begin{cases} Y_t^{\tilde{\mathcal{S}}}(\omega) &= \tau_{0, X_t^{\omega, \tilde{\mathcal{S}}}} \omega \\ X_0^{\omega, \tilde{\mathcal{S}}} &= 0 \end{cases}$$

admits  $\tilde{\mathcal{S}}$  as generator and the measure  $\pi$  is invariant for this process. Then it is sufficient to prove that each invariant subset for this process is invariant under space translations and we can proceed with the same kind of arguments as in [9] section 3 in order to conclude  $\square$

**Corollary 2.7.** *If the matrix  $\mathbf{a}$  is uniformly elliptic then the hypothesis 2.4 is satisfied.*

**Proof:** By the Aronson estimates for the heat kernels, it is clear that the hypothesis of the proposition 2.6 is satisfied  $\square$

When studying examples we will see other hypotheses which imply that hypothesis 2.4 is true.

### 3 Examples

The most simple examples which are possible to construct are given by the processes with periodic coefficients. The considered medium is then the torus  $\mathbb{T}^{d+1}$  equipped with its uniform measure which is ergodic. There are many ways to ensure the ergodicity of the process as seen from the particle, like the uniform ellipticity of the matrix  $\mathbf{a}$  as we already saw in proposition 2.7. In this case we already pointed out that the boundedness of the matrices  $\mathbf{H}$  and  $D_t\mathbf{H}$  is sufficient to deduce that the hypotheses (10) and (11) are true.

#### 3.1 An example where the diffusion coefficient vanishes

We will construct a periodic, reversible and static example on the torus  $\mathbb{T}^2$  where the diffusion matrix vanishes. It turns out from our hypotheses that the diffusion matrix can vanish only on a subset of zero Lebesgue measure.

We let  $\sigma(x, y) = (1 - \cos(x))(1 - \cos(y)) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ . Then we can choose any bounded, periodic function  $V$  and we choose  $\mathbf{H}(x, y) = 0$ .

We have then to verify that hypothesis 2.4 is satisfied. Thanks to the ellipticity of the diffusion coefficient inside the square  $]0, 2\pi[ \times ]0, 2\pi[$ , it is not very difficult to see that each  $L$ -harmonic function must be constant on this square Lebesgue almost surely. Then the periodicity of such a function ensures the validity of hypothesis 2.4.

**Remark 3.2.** *Although we consider not necessarily symmetric processes, the hypothesis (10) shows that they are not too far from being symmetric. New difficulties arise when the antisymmetric part is not anymore controlled by the symmetric one and it is still an open problem for the stationary and ergodic random media. On the other hand, we don't have the same restriction in the case of periodic coefficients (see [12]).*

#### 3.3 Example in a stationary, ergodic random medium

We are going to construct a medium and a process in this medium which verifies all our assumptions. These types of examples are often given in the Gaussian random fields but we want here to construct our process as simply as possible. The natural way to obtain a discrete space-time, ergodic stationary random field is to consider a product of the type  $\zeta^{\otimes \mathbb{Z}^d}$  where  $\zeta$  is a given probability law. We are going to proceed in a similar way. We consider a sequence  $(\varepsilon_{(k_1, \dots, k_d)})_{(k_1, \dots, k_d) \in \mathbb{Z}^d}$  of independent Bernouilli random variables with parameter  $p \in ]0, 1[$ . We define a process  $\tilde{\eta}$  as follows : for each  $x \in \mathbb{R}^d$ , there exists a unique  $(k_1, \dots, k_d) \in \mathbb{Z}^d$  such that  $x$  belongs to the hypercube  $[k_1, k_1 + 1[ \times \dots \times [k_d, k_d + 1[$ . Then we define  $\tilde{\eta}_x = \varepsilon_{(k_1, \dots, k_d)}$ .

The law of our process is invariant with respect to translations in  $\mathbb{Z}^d$  but not in  $\mathbb{R}^d$ . To find a solution to this problem, we choose a uniform variable  $U$  on the cube  $[0, 1[^d$  which is independent

of the sequence  $(\varepsilon_{(k_1, \dots, k_d)})_{(k_1, \dots, k_d) \in \mathbb{Z}^d}$  and we define

$$\bar{\eta}_x = \tilde{\eta}_{x+U}.$$

We have then defined a stationary ergodic random field on  $\mathbb{R}^d$ . The last difficulty that will later arise is the regularity of the trajectories. To get around this, we consider a  $C^\infty(\mathbb{R}^d)$  function  $\varphi$  with compact support included in the ball  $B(0, 1/4)$  and define a new process  $\eta$

$$\eta_x = \int_{\mathbb{R}^d} \bar{\eta}_y \varphi(x - y) dy = \bar{\eta} * \varphi(x).$$

It is then a stationary ergodic random field with smooth trajectories.

Let us consider now the process

$$\omega_{(t,x)} = (\beta_t, \alpha_{x_1}^1, \alpha_{x_2}^2)_{t \in \mathbb{R}, x = (x_1, x_2) \in \mathbb{R}^2}$$

where the three processes  $\alpha^1, \alpha^2$  and  $\beta$  are mutually independent and constructed according to the previous method. Hence  $\{\omega_{(t,x)}; (t,x) \in \mathbb{R} \times \mathbb{R}^2\}$  is an ergodic stationary process. The considered medium  $(\Omega, \mu)$  will be the space  $C(\mathbb{R} \times \mathbb{R}^2; \mathbb{R}^3)$  equipped with the probability law of this process.

We define the matrix

$$\tilde{\sigma}(\omega) = \begin{bmatrix} 1 & 0 \\ 0 & \alpha_0^1 \end{bmatrix}$$

and  $V = 0$  (or any bounded function of the random field  $\alpha$ ). We can choose a time dependent matrix  $U$  such that  $UU^*$  is uniformly elliptic and bounded, and then we set

$$\sigma = \tilde{\sigma}U.$$

In this case, we can prove that proposition 2.6 is satisfied. We just outline the main ideas. Let  $(\varepsilon_k)_{k \in \mathbb{Z}}$  be the independent Bernoulli random variables of parameter  $p \in ]0; 1[$  used to construct the process  $\alpha^1$ . There exists  $N \subset \Omega$  such that  $\mu(N) = 0$  and  $\forall \omega \in \Omega \setminus N$ , there exists  $k \in \mathbb{Z}$  such that  $\varepsilon_k = 1$ . Hence, by the construction of  $\alpha^1$  (and more precisely the fact that  $\text{Supp}(\varphi) \subset B(0, 1/4)$ ), there exists  $z \in \mathbb{R}$  such that  $\alpha^1(x, \omega) = 1$  for  $x \in B_z = \{(x, y) \in \mathbb{R}^2; |x - z| < \frac{1}{4}\}$ .

Then, starting from any point of the plan, it is not difficult to see that the process  $X^{\omega, \tilde{S}}$  can reach the strip  $B_z$ , because of the non-degeneracy of the diffusion coefficient along the x-direction. Then the reversibility of this process ensures that, starting from  $B_z$ , we can reach each subset of  $\mathbb{R}^2$ . Finally, by the uniform ellipticity of the diffusion coefficient over  $B_z$  and the strong Markov property, we can prove that the assumption of proposition 2.6 is valid. The ergodicity is proved.

We can then consider a stream matrix  $H$ . For example we can choose

$$H(\eta) = \begin{bmatrix} 0 & (\alpha_0^1)^2 \beta_0 \\ -(\alpha_0^1)^2 \beta_0 & 0 \end{bmatrix},$$

and all our assumptions are verified.

We should also point out that for almost all the trajectories of the process  $\eta$ , the diffusion matrix  $\sigma$  is degenerated in the direction  $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$  on certain open regions of the space  $\mathbb{R}^2$ .

## 4 The environment as seen from the particle

We can now look at the *environment as seen from the particle* : this is a Markov process on the probability space  $\Omega$  defined by

$$Y_t(\omega) = \tau_{t, X_t^\omega} \omega. \quad (17)$$

where the process  $X^\omega$  is starting from the point  $0 \in \mathbb{R}^d$ . Its generator is equal to  $\mathbf{L} + D_t$  and it turns out that  $\pi$  is an invariant measure for the process  $Y_t$ . Moreover it is ergodic. Indeed, each  $\mathbf{L} + D_t$  harmonic function  $\varphi$  is constant  $\mu$  almost surely because of the hypothesis 2.4. As the two measures  $\mu$  and  $\pi$  are equivalent,  $\varphi$  is also constant  $\pi$  almost surely and this proves the ergodicity.

We should point out that there is no need for the invariant measure of the process to be unique.

## 5 The Poisson equation

The aim of this section is to solve the resolvent equation, for  $\lambda > 0$ , where  $\mathbf{u}_\lambda$  is our unknown :

$$\lambda \mathbf{u}_\lambda - (\mathbf{L} + D_t) \mathbf{u}_\lambda = \mathbf{b}. \quad (18)$$

In fact, the existence of such a solution is in general not very hard to obtain, at least in the  $L^2$  sense. But we choose the following approach in order to prove that the solution is in a good space which can guarantee a not so bad regularity of the solution and which will allow us to obtain an appropriate control of the solution as  $\lambda \rightarrow 0$ .

### 5.1 Setup

For each  $\omega \in \Omega$ , we define the diffusion process  $X^{\omega, \tilde{S}}$  by the following Itô equation

$$X_t^{\omega, \tilde{S}} = x + \int_0^t \tilde{b}(X_r^{\omega, \tilde{S}}, \omega) dr + \int_0^t \tilde{\sigma}(X_r^{\omega, \tilde{S}}, \omega) dB_r. \quad (19)$$

Then the process  $Y^{\tilde{S}}$  is defined by  $Y^{\tilde{S}} = \tau_{t, X_t^{\omega, \tilde{S}}} \omega$ , where  $X_0^{\omega, \tilde{S}} = 0$ , and the generator of this process is equal to  $\tilde{S} + D_t$  so that the measure  $\pi$  is again invariant for this process. Let us denote by  $P_t$  the semigroup on  $L^2(\Omega, \pi)$  generated by this process and by  $P_t^*$  its adjoint operator on this space. By the time independance of the coefficients in (19), the Markov property and the reversibility of the process, it is easy to prove that  $\forall \mathbf{f} \in L^2(\Omega, \pi)$

$$P_t \circ P_t^* \mathbf{f} = \mathbb{E}_0[f(0, X_{2t}^{\omega, \tilde{S}}, \omega)] = P_t^* \circ P_t \mathbf{f}.$$

Then the theorem 13.38 in [14] ensures us that the operator  $\tilde{S} + D_t$  is normal so that we can find a spectral resolution of the identity  $E$  on the Borelian subsets of  $\mathbb{R}_+ \times \mathbb{R}$  such that

$$-\tilde{S} - D_t = \int_{\mathbb{R}_+ \times \mathbb{R}} (x + iy) E(dx, dy).$$

For any  $\varphi, \psi \in L^2(\Omega)$ , we will denote by  $E_{\varphi, \psi}$  the measure defined by  $E_{\varphi, \psi} = (E\varphi, \psi)_2$ . By the symmetry (resp. antisymmetry) of the operator  $\tilde{S}$  (resp.  $D_t$ ), we have

$$-\tilde{S} = \int_{\mathbb{R}_+ \times \mathbb{R}} x E(dx, dy), \quad \text{and} \quad -D_t = \int_{\mathbb{R}_+ \times \mathbb{R}} iy E(dx, dy).$$

From now, we will denote by  $(\cdot, \cdot)$  the usual scalar product in  $L^2(\Omega)$ . For any  $\varphi, \psi \in \mathcal{C}$  we define

$$\langle \varphi, \psi \rangle_1 = - \int_{\mathbb{R}_+ \times \mathbb{R}} x E_{\varphi, \psi}(dx, dy) = -(\varphi, \tilde{\mathbf{S}}\psi) \quad (20)$$

and  $\|\varphi\|_1 = \sqrt{\langle \varphi, \varphi \rangle_1}$ . It is important to remark that, because of hypothesis (9), this semi-norm is equivalent on  $\mathcal{C}$  to the semi-norm defined by  $\sqrt{-(\varphi, \mathbf{S}\varphi)}$  with the constants

$$m\|\varphi\|_1^2 \leq -(\varphi, \mathbf{S}\varphi) \leq M\|\varphi\|_1^2.$$

We will denote by  $\mathbb{D}$  the closure in  $(L^2(\Omega), \|\cdot\|_2)$  of the subspace  $\{\sqrt{-\tilde{\mathbf{S}}}\varphi; \varphi \in \mathcal{C}\}$ .

Let  $\mathbb{F}$  be the Hilbert space equal to the closure of  $\mathcal{C}$  in  $L^2(\Omega)$  with respect to the scalar product  $\varepsilon$  defined on  $\mathcal{C}$  by

$$\varepsilon(\varphi, \psi) = (\varphi, \psi) + \langle \varphi, \psi \rangle_1 + (D_t\varphi, D_t\psi). \quad (21)$$

The application  $\Phi^{\mathbb{F}} : \mathcal{C} \subset \mathbb{F} \rightarrow L^2(\Omega) \times (L^2(\Omega))^d \times L^2(\Omega)$  is a linear isomorphism. So we

$$\varphi \mapsto \Phi^{\mathbb{F}}(\varphi) = (\varphi, \sigma^* D_x \varphi, D_t \varphi)$$

can extend it to  $\mathbb{F}$  and we will note for any  $\mathbf{u} \in \mathbb{F}$ ,  $(\mathbf{u}, \nabla^\sigma \mathbf{u}, D_t \mathbf{u}) \stackrel{\text{def}}{=} \Phi^{\mathbb{F}}(\mathbf{u})$ .

In the same way, Let  $\mathbb{H}_1$  be the Hilbert space generated by the closure of  $\mathcal{C}$  in  $L^2(\Omega)$  with respect to the scalar product  $\kappa$  defined on  $\mathcal{C}$  by

$$\kappa(\varphi, \psi) = (\varphi, \psi) + \langle \varphi, \psi \rangle_1. \quad (22)$$

The application  $\Phi^{\mathbb{H}_1} : \mathcal{C} \subset \mathbb{H}_1 \rightarrow L^2(\Omega) \times (L^2(\Omega))^d$  is again a linear isomorphism. So

$$\varphi \mapsto \Phi^{\mathbb{H}_1}(\varphi) = (\varphi, \sigma^* D_x \varphi)$$

we can extend it to  $\mathbb{H}_1$  and we will note for any  $\mathbf{u} \in \mathbb{H}_1$ ,  $(\mathbf{u}, \nabla^\sigma \mathbf{u}) \stackrel{\text{def}}{=} \Phi^{\mathbb{H}_1}(\mathbf{u})$ .

**Remark 5.2.** For a sequence  $(\varphi_n)_n \in \mathbb{H}_1$ , we have

$$m|\sqrt{-\tilde{\mathbf{S}}}\varphi_n|_2^2 \leq \frac{1}{2}\|\nabla^\sigma \varphi_n\|_2^2 \leq M|\sqrt{-\tilde{\mathbf{S}}}\varphi_n|_2^2,$$

so that convergence of the sequence  $(\sqrt{-\tilde{\mathbf{S}}}\varphi_n)_n$  in  $L^2(\Omega)$  is equivalent to convergence of the sequence  $(\nabla^\sigma \varphi_n)_n$  in  $(L^2(\Omega))^d$ .

If for a certain function  $\mathbf{f} \in L^2(\Omega)$  we have  $\int_{\mathbb{R}_+ \times \mathbb{R}} \frac{1}{x} E_{\mathbf{f}, \mathbf{f}}(dx, dy) < \infty$ , we will define

$$\|\mathbf{f}\|_{-1}^2 = \int_{\mathbb{R}_+ \times \mathbb{R}} \frac{1}{x} E_{\mathbf{f}, \mathbf{f}}(dx, dy). \quad (23)$$

We point out that  $\|\mathbf{f}\|_{-1} < \infty$  if and only if there exists  $C \in \mathbb{R}$  such that for any  $\varphi \in \mathcal{C}$ ,  $\int_{\Omega} \mathbf{f}\varphi d\pi \leq C\|\varphi\|_1$ . In this case we have

$$\inf \left\{ C \in \mathbb{R}; \forall \varphi \in \mathcal{C}, \int_{\Omega} \mathbf{f}\varphi d\pi \leq C\|\varphi\|_1 \right\} = \|\mathbf{f}\|_{-1}.$$

We denote by  $\mathbb{H}_{-1}$  the closure in  $\mathbb{H}_1^*$  of the space  $L^2(\Omega)$  for the norme  $\|\cdot\|_{-1}$ .

Now we introduce some notations and definitions about the antisymmetric part  $\mathbf{H}$ . We have

$$\begin{aligned} |(u, \mathbf{H}v)| &\leq (u, |\mathbf{H}|u)^{1/2} (v, |\mathbf{H}|v)^{1/2} \\ &\leq C_1^{\mathbf{H}} (u, \tilde{\mathbf{a}}u)^{1/2} (v, \tilde{\mathbf{a}}v)^{1/2} \end{aligned} \quad (24)$$

where the second inequality follows from (10), while the first inequality in (24) is a general fact of linear algebra. Hence we deduce that

$$\forall \varphi, \psi \in \mathcal{C}, \quad \left| \frac{1}{2} \int_{\Omega} \mathbf{H} D_x \varphi \cdot D_x \psi \, d\mu \right| \leq C_1^H e^{2K} \|\psi\|_1 \|\varphi\|_1.$$

Thus there exists an antisymmetric and continuous bilinear form  $\mathbf{T}$  on  $\mathbb{D} \times \mathbb{D} \subset L^2(\Omega, \pi) \times L^2(\Omega, \pi)$  such that

$$\forall \varphi, \psi \in \mathcal{C}, \quad \frac{1}{2} \int_{\Omega} \mathbf{H} \nabla_x \varphi \cdot \nabla_x \psi \, d\mu = \mathbf{T}(\sqrt{-\tilde{S}}\varphi, \sqrt{-\tilde{S}}\psi). \quad (25)$$

Similarly, thanks to the hypothesis (11) we know that there exists an other antisymmetric and continuous bilinear form  $\mathbf{T}'$  on  $\mathbb{D} \times \mathbb{D} \subset L^2(\Omega, \pi) \times L^2(\Omega, \pi)$  such that

$$\forall \varphi, \psi \in \mathcal{C}, \quad \frac{1}{2} \int_{\Omega} D_t \mathbf{H} \nabla_x \varphi \cdot \nabla_x \psi \, d\mu = \mathbf{T}'(\sqrt{-\tilde{S}}\varphi, \sqrt{-\tilde{S}}\psi). \quad (26)$$

### 5.3 Existence of a solution :

Our goal in this subsection is to prove the following proposition

**Proposition 5.4.** *Suppose that  $\mathbf{h} \in \mathbb{H}_{-1} \cap L^2(\Omega)$  is such that there exists a constant  $C_h$  satisfying  $\forall s > 0, \forall \varphi \in \mathcal{C}$*

$$\left\langle \mathbf{h}, \frac{T_{(-s,0)}\varphi - \varphi}{s} \right\rangle_{-1,1} \leq C_h \|\varphi\|_1. \quad (27)$$

*Then, for any  $\lambda > 0$ , there exists an unique solution  $\mathbf{u}_\lambda \in \mathbb{F}$  of the equation*

$$\lambda \mathbf{u}_\lambda - \mathbf{L} \mathbf{u}_\lambda - D_t \mathbf{u}_\lambda = \mathbf{h} \quad (28)$$

*Moreover  $D_t \mathbf{u}_\lambda \in \mathbb{H}_1$  and*

$$\kappa(D_t \mathbf{u}_\lambda, D_t \mathbf{u}_\lambda) \leq \left( C_h + C_2^H e^{2K} \|\mathbf{h}\|_{-1} + \tilde{C} \|\mathbf{h}\|_{-1} \right)^2.$$

Before proving this result we first introduce a result in the case of time independent coefficients. On the first side, this will be a good starting point for the proof in the time dependent case and this will bring out the difficulties arising with the dependency on time. On the other side, we will need this result in the last section of this paper in order to prove the tightness of some process.

**Proposition 5.5.** *Suppose that  $\mathbf{h} \in \mathbb{H}_{-1} \cap L^2(\Omega)$  Then, for any  $\lambda > 0$ , there exists an unique solution  $\mathbf{w}_\lambda \in \mathbb{H}_1$  of the equation*

$$\lambda \mathbf{w}_\lambda - \mathbf{S} \mathbf{w}_\lambda = \mathbf{h} \quad (29)$$

*Moreover we have  $\lambda \|\mathbf{w}_\lambda\|_2^2 + m \|\mathbf{w}_\lambda\|_1^2 \leq \frac{\|\mathbf{h}\|_{-1}^2}{m}$ .*

**Proof :** The main tool of this proof will be the Lax-Milgram theorem. Let  $\lambda > 0$  be fixed. For any  $\varphi, \psi \in \mathcal{C}$  we consider the following bilinear form on  $\mathcal{C} \times \mathcal{C}$  defined by

$$D_\lambda(\varphi, \psi) = \lambda(\varphi, \psi)_2 - (\varphi, \mathbf{S}\psi)_2.$$

Thanks to hypothesis 2.2, it is easy to verify that this form is coercive and continuous on  $\mathcal{C} \times \mathcal{C}$  so that it can be extended to the whole space  $\mathbb{H}_1 \times \mathbb{H}_1$ . This extension is also coercive and continuous. Then it is sufficient to see that the application  $\varphi \mapsto \int_{\Omega} \mathbf{h} \varphi \, d\pi$  is continuous in  $\mathbb{H}_1$  in order to apply the Lax-Milgram theorem and to obtain a solution  $\mathbf{w}_\lambda$ .

Moreover, we have

$$\begin{aligned} \lambda \|\mathbf{w}_\lambda\|_2^2 + m \|\mathbf{w}_\lambda\|_1^2 &\leq D_\lambda(\mathbf{w}_\lambda, \mathbf{w}_\lambda) = \int_{\Omega} \mathbf{h} \mathbf{w}_\lambda \, d\pi \leq \|\mathbf{h}\|_{-1} \|\mathbf{w}_\lambda\|_1 \\ &\leq \frac{\|\mathbf{h}\|_{-1}^2}{m} \end{aligned}$$

and this concludes the proof  $\square$

**Proof of the proposition 5.4:** In order to prove the existence of a solution to the resolvent equation we want to apply the Lax-Milgram theorem to the following bilinear form which is defined for  $\lambda > 0$  and  $\varphi, \psi \in \mathcal{C}$

$$B_\lambda(\varphi, \psi) = \lambda \int_{\Omega} \varphi \psi \, d\pi + \frac{1}{2} \int_{\Omega} (\mathbf{a} + \mathbf{H} e^{2\mathbf{V}}) D_x \varphi \cdot D_x \psi \, d\pi - \int_{\Omega} D_t \varphi \psi \, d\pi \quad (30)$$

But the problem is the degeneracy with respect to the time variable. This bilinear form is not coercive on  $\mathbb{F}$ . So we add some ellipticity with respect to the time variable by adding a term as follows. We define for any  $\lambda, \delta > 0$  and  $\varphi, \psi \in \mathcal{C}$

$$B_{\lambda, \delta}(\varphi, \psi) = \lambda \int_{\Omega} \varphi \psi \, d\pi + \frac{1}{2} \int_{\Omega} (\mathbf{a} + \mathbf{H} e^{2\mathbf{V}}) D_x \varphi \cdot D_x \psi \, d\pi - \int_{\Omega} D_t \varphi \psi \, d\pi + \frac{\delta}{2} \int_{\Omega} D_t \varphi \cdot D_t \psi \, d\pi. \quad (31)$$

Notice that for  $\varphi, \psi \in \mathcal{C}$  we have  $\int_{\Omega} (\lambda - \mathbf{L} - \frac{\delta}{2} D_t^2 - D_t) \varphi \psi \, d\pi = B_{\lambda, \delta}(\varphi, \psi)$ . With the help of the inequality (24) and the boundedness of  $\mathbf{V}$  we obtain that  $|B_{\lambda, \delta}(\varphi, \psi)| \leq C \sqrt{\varepsilon(\varphi, \varphi) \times \varepsilon(\psi, \psi)}$ . As a result of the time-independence of  $\mathbf{V}$  we have for any  $\varphi \in \mathcal{C}$ ,  $\int_{\Omega} \varphi D_t \varphi \, d\pi = 0$  and as a consequence

$$\min(\lambda, \delta/2, m) \varepsilon(\varphi, \varphi) \leq B_{\lambda, \delta}(\varphi, \varphi). \quad (32)$$

We can then say that  $B_{\lambda, \delta}$  defines a continuous and coercive bilinear form on  $\mathbb{F} \times \mathbb{F}$ . The Lax-Milgram theorem allows us to solve the equation ( $\mathbf{u}$  stands for the unknown):

$$\forall \varphi \in \mathbb{F}, \quad B_{\lambda, \delta}(\mathbf{u}, \varphi) = \mathbf{F}(\varphi)$$

for each continuous linear form  $\mathbf{F}$  on  $\mathbb{F}$ . Moreover we can construct for any  $\delta > 0$  a strongly continuous resolvent  $G_{\lambda, \delta}$  on  $L^2(\Omega)$ , with generator  $\mathbf{L}^\delta = \mathbf{L} + \frac{\delta}{2} D_t^2 + D_t$ , and we define for  $\mathbf{h} \in L^2(\Omega)$ ,  $\mathbf{u}_{\lambda, \delta} = G_{\lambda, \delta} \mathbf{h}$ , which solves the equation, for  $\mathbf{h} \in L^2(\Omega)$

$$\forall \varphi \in \mathcal{C}, \quad B_{\lambda, \delta}(\mathbf{u}_{\lambda, \delta}, \varphi) = (\mathbf{h}, \varphi). \quad (33)$$

We want now to pass to the limit as  $\delta$  goes to 0 in order to obtain a solution  $\mathbf{u}_\lambda$  to the equation

$$\forall \varphi \in \mathcal{C}, \quad B_\lambda(\mathbf{u}_\lambda, \varphi) = (\mathbf{h}, \varphi). \quad (34)$$

But we are faced with the following problem: we cannot control the term  $D_t \mathbf{u}_{\lambda, \delta}$  because we lose our control of this term as  $\delta \rightarrow 0$ . The idea is to differentiate equation (33) with respect to the time variable, in order to obtain an equation verified by  $D_t \mathbf{u}_{\lambda, \delta}$  and this will allow us to control this

term. So we let for each fixed  $\lambda, \delta$ ,  $\mathbf{v}_s = \frac{T_{(s,0)}\mathbf{u}_{\lambda,\delta} - \mathbf{u}_{\lambda,\delta}}{s}$ , and we remark that  $\mathbf{v}_s$  solves the following equation

$$\forall \varphi \in \mathbb{F}, B_{\lambda,\delta}(\mathbf{v}_s, \varphi) = \mathbf{F}_s(\varphi) \quad (35)$$

where  $\mathbf{F}_s$  is a continuous linear form on  $\mathbb{H}_1$  defined by,  $\forall \varphi \in \mathbb{F}$ ,

$$\begin{aligned} \mathbf{F}_s(\varphi) = & \left\langle \mathbf{h}, \frac{T_{(-s,0)}\varphi - \varphi}{s} \right\rangle_{-1,1} - \frac{\mathbf{T}(\sqrt{-\tilde{\mathbf{S}}}\mathbf{u}_{\lambda,\delta}, \sqrt{-\tilde{\mathbf{S}}}T_{(-s,0)}\varphi) - \mathbf{T}(\sqrt{-\tilde{\mathbf{S}}}T_{(s,0)}\mathbf{u}_{\lambda,\delta}, \sqrt{-\tilde{\mathbf{S}}}\varphi)}{s} \\ & - \frac{(\sqrt{-\tilde{\mathbf{S}}}\mathbf{u}_{\lambda,\delta}, \sqrt{-\tilde{\mathbf{S}}}T_{(-s,0)}\varphi) - (\sqrt{-\tilde{\mathbf{S}}}T_{(s,0)}\mathbf{u}_{\lambda,\delta}, \sqrt{-\tilde{\mathbf{S}}}\varphi)}{s} \end{aligned}$$

For the first term, by assumption we have  $\forall \varphi \in \mathbb{F}$ ,  $\left\langle \mathbf{h}, \frac{T_{(-s,0)}\varphi - \varphi}{s} \right\rangle_{-1,1} \leq C_h \|\varphi\|_1$ . For the other terms we need the following lemma

**Lemma 5.6.** *For any  $\mathbf{g} \in \mathbb{F}$  and for any  $\varphi \in \mathbb{F}$  we have*

$$\left| \frac{\mathbf{T}(\sqrt{-\tilde{\mathbf{S}}}\mathbf{g}, \sqrt{-\tilde{\mathbf{S}}}T_{(-s,0)}\varphi) - \mathbf{T}(\sqrt{-\tilde{\mathbf{S}}}T_{(s,0)}\mathbf{g}, \sqrt{-\tilde{\mathbf{S}}}\varphi)}{s} \right| \leq C_2^H e^{2K} \|\mathbf{g}\|_1 \|\varphi\|_1,$$

and

$$\left| \frac{(\sqrt{-\tilde{\mathbf{S}}}\mathbf{g}, \sqrt{-\tilde{\mathbf{S}}}T_{(-s,0)}\varphi) - (\sqrt{-\tilde{\mathbf{S}}}T_{(s,0)}\mathbf{g}, \sqrt{-\tilde{\mathbf{S}}}\varphi)}{s} \right| \leq \tilde{C} \|\mathbf{g}\|_1 \|\varphi\|_1$$

**Proof :** For any  $\varphi, \psi \in \mathcal{C}$  we define

$$2\mathbf{T}_s(\varphi, \psi) = \frac{1}{s} \int_{\Omega} [\mathbf{H}(\tau_s\omega) D_x \varphi(\tau_s\omega) \cdot D_x \psi(\omega) - \mathbf{H}(\omega) D_x \varphi(\tau_s\omega) \cdot D_x \psi(\omega)] d\mu(\omega),$$

and we have

$$\begin{aligned} 2\mathbf{T}_s(\varphi, \psi) &= \int_{\Omega} \frac{\mathbf{H}(\tau_s\omega) - \mathbf{H}(\omega)}{s} D_x \varphi(\tau_s\omega) \cdot D_x \psi(\omega) d\mu(\omega) \\ &= \int_{\Omega} \int_0^1 D_t \mathbf{H}(\tau_u\omega) du D_x \varphi(\tau_s\omega) \cdot D_x \psi(\omega) d\mu(\omega) \\ &\leq \int_0^1 \int_{\Omega} |D_t \mathbf{H}(\tau_u\omega) D_x \varphi(\tau_s\omega) \cdot D_x \psi(\omega)| d\mu(\omega) du \\ &\stackrel{(11)}{\leq} 2C_2^H e^{2K} \|T_{(s,0)}\varphi\|_1 \|\psi\|_1 \\ &= 2C_2^H e^{2K} \|\varphi\|_1 \|\psi\|_1 \end{aligned}$$

Hence  $\mathbf{T}_s$  is a continuous bilinear form and can be extended to  $\mathbb{H}_1 \times \mathbb{H}_1$ . Let us consider now  $\varphi \in \mathcal{C}$ , we have

$$\begin{aligned} \frac{\mathbf{T}(\sqrt{-\tilde{\mathbf{S}}}\mathbf{g}, \sqrt{-\tilde{\mathbf{S}}}T_{(-s,0)}\varphi) - \mathbf{T}(\sqrt{-\tilde{\mathbf{S}}}T_{(s,0)}\mathbf{g}, \sqrt{-\tilde{\mathbf{S}}}\varphi)}{s} &= \frac{1}{s} \int_{\Omega} -\mathbf{g} \mathbf{A} T_{(-s,0)}\varphi + T_{(s,0)}\mathbf{g} \mathbf{A} \varphi d\pi \\ &= \frac{1}{s} \left( - \int_{\Omega} \mathbf{g}(\tau_s\omega) \frac{1}{2} \operatorname{div}(\mathbf{H}(\tau_s\omega) D_x \varphi) d\pi \right. \\ &\quad \left. + \int_{\Omega} \mathbf{g}(\tau_s\omega) \frac{1}{2} \operatorname{div}(\mathbf{H}(\omega) D_x \varphi) d\pi \right) \\ &= \mathbf{T}_s(\mathbf{g}, \varphi) \end{aligned}$$

and the first inequality of the lemma follows from the fact that  $\mathcal{C}$  is dense in  $\mathbb{H}_1$ . The second inequality that we have to prove requires the same arguments  $\square$

So we have  $\forall s \geq 0$ ,  $\mathbf{F}_s(\varphi) \leq (C_h + C_2^H e^{2K} \|\mathbf{u}_{\lambda,\delta}\|_1 + \tilde{C} \|\mathbf{u}_{\lambda,\delta}\|_1) \|\varphi\|_1$ , and therefore

$$B_{\lambda,\delta}(\mathbf{v}_s, \mathbf{v}_s) = \mathbf{F}_s(\mathbf{v}_s) \leq (C_h + C_2^H e^{2K} \|\mathbf{u}_{\lambda,\delta}\|_1 + \tilde{C} \|\mathbf{u}_{\lambda,\delta}\|_1) \|\mathbf{v}_s\|_1,$$

which implies  $\lambda \|\mathbf{v}_s\|_2^2 + \|\mathbf{v}_s\|_1^2 \leq \left( C_h + C_2^H e^{2K} \|\mathbf{u}_{\lambda,\delta}\|_1 + \tilde{C} \|\mathbf{u}_{\lambda,\delta}\|_1 \right)^2$ , and  $\|\mathbf{v}_s\|_1$  is bounded by a constant which doesn't depend on  $s$ .

Since there exists a constant  $A$  such that  $\forall s > 0, \forall \varphi \in \mathcal{C}$ ,  $\mathbf{F}_s(\varphi) \leq A \|\varphi\|_1$ , there is a linear form  $\mathbf{F}_0$  on  $\mathbb{H}_1$  such that  $\mathbf{F}_s$  converges weakly to  $\mathbf{F}_0$  in  $\mathbb{H}_1^*$  along a subsequence  $(s_n)_n$  which converges to 0. With the help of (32), we have

$$\min(\lambda, \delta/2, 1) \varepsilon(\mathbf{v}_{s_n}, \mathbf{v}_{s_n}) \leq B_{\lambda,\delta}(\mathbf{v}_{s_n}, \mathbf{v}_{s_n}) = \mathbf{F}_{s_n}(\mathbf{v}_{s_n}) \leq A \|\mathbf{v}_{s_n}\|_1.$$

We deduce that the sequence  $(\mathbf{v}_{s_n})_n$  is bounded in the space  $\mathbb{F}$  and we can extract a subsequence which is still denoted by  $(\mathbf{v}_{s_n})_n$  and converges weakly in  $\mathbb{F}$  towards some function  $\mathbf{v}_0 \in \mathbb{F}$ . But, because of the weak convergences, we have  $\forall \varphi \in \mathcal{C}$ ,

$$B_{\lambda,\delta}(\mathbf{v}_0, \varphi) = \lim_{n \rightarrow \infty} B_{\lambda,\delta}(\mathbf{v}_{s_n}, \varphi) = \lim_{n \rightarrow \infty} \mathbf{F}_{s_n}(\varphi) = \mathbf{F}_0(\varphi).$$

Hence we deduce that  $\mathbf{v}_0$  is the solution of the equation

$$\forall \varphi \in \mathbb{F}, B_{\lambda,\delta}(\mathbf{v}_0, \varphi) = \mathbf{F}_0(\varphi). \quad (36)$$

On the other hand, we know that  $\mathbf{v}_s = \frac{T_{(s,0)} \mathbf{u}_{\lambda,\delta} - \mathbf{u}_{\lambda,\delta}}{s}$  converges strongly in  $L^2(\Omega)$  to  $D_t \mathbf{u}_{\lambda,\delta}$ , so we deduce that  $D_t \mathbf{u}_{\lambda,\delta} \in \mathbb{F}$  and solves the equation (36). To sum up, we found a family of functions  $(\mathbf{u}_{\lambda,\delta})_{\lambda,\delta}$  such that  $\mathbf{u}_{\lambda,\delta} \in \mathbb{F}$ ,  $D_t \mathbf{u}_{\lambda,\delta} \in \mathbb{F}$  and

$$\forall \varphi \in \mathcal{F}, B_{\lambda,\delta}(\mathbf{u}_{\lambda,\delta}, \varphi) = \langle \mathbf{h}, \varphi \rangle_{-1,1}, \quad (37)$$

$$\forall \varphi \in \mathcal{F}, B_{\lambda,\delta}(D_t \mathbf{u}_{\lambda,\delta}, \varphi) = \mathbf{F}_0(\varphi). \quad (38)$$

After choosing  $\varphi = \mathbf{u}_{\lambda,\delta}$  in (37) and  $\varphi = D_t \mathbf{u}_{\lambda,\delta}$  in (38), we have

$$\lambda \|\mathbf{u}_{\lambda,\delta}\|_2^2 + \|\mathbf{u}_{\lambda,\delta}\|_1^2 + \frac{\delta}{2} |D_t \mathbf{u}_{\lambda,\delta}|_2^2 = \langle \mathbf{h}, \mathbf{u}_{\lambda,\delta} \rangle_{-1,1} \leq \|\mathbf{h}\|_{-1}^2, \quad (39)$$

$$\lambda |D_t \mathbf{u}_{\lambda,\delta}|_2^2 + \|D_t \mathbf{u}_{\lambda,\delta}\|_1^2 + \frac{\delta}{2} |D_t^2 \mathbf{u}_{\lambda,\delta}|_2^2 \leq A \|D_t \mathbf{u}_{\lambda,\delta}\|_1 \quad (40)$$

By (39) and (40), the sequence  $(\mathbf{u}_{\lambda,\delta})_{\delta>0}$  is bounded in the space  $\mathbb{F}$  as well as the sequence  $(D_t \mathbf{u}_{\lambda,\delta})_{\delta>0}$  is bounded in  $\mathbb{H}_1$ , so there is a subsequence  $(\mathbf{u}_{\lambda,\delta})_{\delta>0} \in \mathbb{F}$  with  $D_t \mathbf{u}_{\lambda,\delta} \in \mathbb{H}_1$  such that  $(\mathbf{u}_{\lambda,\delta})_{\delta>0}$  converges weakly in  $\mathbb{F}$  to  $\mathbf{u}_\lambda$  and  $D_t \mathbf{u}_{\lambda,\delta}$  converges weakly in  $\mathbb{H}_1$  to  $D_t \mathbf{u}_\lambda$  as  $\delta \rightarrow 0$ . Moreover with the help of (39) we know that  $\delta |D_t \mathbf{u}_{\lambda,\delta}|_2^2 \leq 2 \|\mathbf{h}\|_{-1}^2$  which means in particular that  $\delta D_t \mathbf{u}_{\lambda,\delta} \xrightarrow[\delta \rightarrow 0]{L^2} 0$ . This allows us to pass to the limit as  $\delta$  goes to 0 in the equation (33) and to prove that

$$\forall \varphi \in \mathbb{F}, B_\lambda(\mathbf{u}_\lambda, \varphi) = \langle \mathbf{h}, \varphi \rangle_{-1,1} \quad (41)$$

Now we just have to prove the uniqueness. But this is simple because if we have two solutions  $\mathbf{u}_\lambda$  and  $\mathbf{v}_\lambda$  then  $\forall \varphi \in \mathbb{F}$

$$B_\lambda(\mathbf{u}_\lambda - \mathbf{v}_\lambda, \varphi) = 0.$$

By choosing  $\varphi = \mathbf{u}_\lambda - \mathbf{v}_\lambda$  we obtain  $\mathbf{u}_\lambda = \mathbf{v}_\lambda$ ,  $\mu$  almost surely  $\square$

## 5.7 Control of the solution

Like for the existence of the solution, we begin with the time independent case by way of introduction.

**Proposition 5.8.** *Let  $\mathbf{h}$  be in  $\mathbb{H}_{-1} \cap L^2(\Omega)$  and for any  $\lambda > 0$ , let  $\mathbf{w}_\lambda$  be defined as the unique solution in  $\mathbb{H}_1$  to the equation*

$$\lambda \mathbf{w}_\lambda - \mathbf{S} \mathbf{w}_\lambda = \mathbf{h}$$

Then  $\lambda |\mathbf{w}_\lambda|_2^2 \xrightarrow{\lambda \rightarrow 0} 0$  and there exists  $\boldsymbol{\zeta} \in L^2(\Omega)^d$  such that  $|\nabla^\sigma \mathbf{w}_\lambda - \boldsymbol{\zeta}|_2 \xrightarrow{\lambda \rightarrow 0} 0$ .

**Proof :** We know (see proposition 5.5) that  $\lambda |\mathbf{w}_\lambda|_2^2 + m \|\mathbf{w}_\lambda\|_1^2 \leq \frac{\|\mathbf{h}\|_{-1}^2}{m}$ . Thus even if it means extracting a subsequence, we can find  $\mathbf{g} \in L^2(\Omega)$  such that  $(\sqrt{-\mathbf{S}} \mathbf{w}_\lambda)_\lambda$  converges weakly in  $L^2(\Omega)$  towards  $\mathbf{g}$ . Moreover we have  $\lambda \mathbf{w}_\lambda \xrightarrow[\lambda \rightarrow 0]{L^2(\Omega)} 0$ . Since for any  $\boldsymbol{\varphi} \in \mathbb{H}$  we have (see proposition 5.5)

$$\lambda (\mathbf{w}_\lambda, \boldsymbol{\varphi})_2 + (\sqrt{-\mathbf{S}} \mathbf{w}_\lambda, \sqrt{-\mathbf{S}} \boldsymbol{\varphi})_2 = D_\lambda(\mathbf{w}_\lambda, \boldsymbol{\varphi}) = (\mathbf{h}, \boldsymbol{\varphi})_2,$$

we can pass to the limit when  $\lambda$  goes to 0 and we obtain

$$(\mathbf{g}, \sqrt{-\mathbf{S}} \boldsymbol{\varphi})_2 = (\mathbf{h}, \boldsymbol{\varphi})_2. \quad (42)$$

Hence we deduce

$$\begin{aligned} \limsup_{\lambda \rightarrow 0} |\mathbf{g}|_2 |\sqrt{-\mathbf{S}} \mathbf{w}_\lambda|_2 &\geq \limsup_{\lambda \rightarrow 0} (\mathbf{g}, \sqrt{-\mathbf{S}} \mathbf{w}_\lambda)_2 \\ &= \limsup_{\lambda \rightarrow 0} (\mathbf{h}, \mathbf{w}_\lambda)_2 \\ &= \limsup_{\lambda \rightarrow 0} \lambda |\mathbf{w}_\lambda|_2^2 + |\sqrt{-\mathbf{S}} \mathbf{w}_\lambda|_2^2 \\ &\geq \limsup_{\lambda \rightarrow 0} |\sqrt{-\mathbf{S}} \mathbf{w}_\lambda|_2^2 \end{aligned}$$

So we obtain  $|\mathbf{g}|_2 \geq \limsup_{\lambda \rightarrow 0} |\sqrt{-\mathbf{S}} \mathbf{w}_\lambda|_2$ . Moreover, by the weak convergence, we have  $|\mathbf{g}|_2 \geq \liminf_{\lambda \rightarrow 0} |\sqrt{-\mathbf{S}} \mathbf{w}_\lambda|_2$ . Finally

$$|\mathbf{g}|_2 = \lim_{\lambda \rightarrow 0} |\sqrt{-\mathbf{S}} \mathbf{w}_\lambda|_2$$

and the strong convergence holds for the family  $(\sqrt{-\mathbf{S}} \mathbf{w}_\lambda)_\lambda$  towards  $\mathbf{g}$ . In this case, there must be equality at each line of the previous calculus so that  $\lambda |\mathbf{w}_\lambda|_2^2 \xrightarrow{\lambda \rightarrow 0} 0$ . Because of equation (42), it is easy to prove the uniqueness of the weak limit  $\square$

Our goal is now to obtain a good control of the solution  $\mathbf{u}_\lambda^i$  of the equation

$$\lambda \mathbf{u}_\lambda^i - \mathbf{L} \mathbf{u}_\lambda^i - D_t \mathbf{u}_\lambda^i = \mathbf{b}_i, \quad (43)$$

this means we want to show that  $\lambda |\mathbf{u}_\lambda^i|_2^2 \xrightarrow{\lambda \rightarrow 0} 0$  and that  $(\nabla^\sigma \mathbf{u}_\lambda^i)_\lambda$  converges in  $(L^2(\Omega))^d$  as  $\lambda$  goes to 0. Our strategy consists in showing that the operator  $\lambda - \mathbf{L} - D_t$  is just a perturbation of the operator  $\lambda - \tilde{\mathbf{S}} - D_t$  so that our study can be reduced to the study of the solution of the following equation

$$\lambda \mathbf{v}_\lambda - \tilde{\mathbf{S}} \mathbf{v}_\lambda - D_t \mathbf{v}_\lambda = \mathbf{b}_\lambda,$$

where  $\mathbf{b}_\lambda$  will be defined below and possesses a strong limit in  $\mathbb{H}_{-1}$ . This kind of equation is more convenient to study because the operators  $-\tilde{\mathbf{S}}$  and  $D_t$  commute and hence we can use a common spectral decomposition of these operators. In our proof, this will be of the utmost importance and will allow us to prove that the antisymmetric term  $D_t$  does not play an essential role in the control of the solution. So we begin by proving the second point:

**Proposition 5.9.** *Let  $(\mathbf{b}_\lambda)_{\lambda>0}$  be a family of functions in  $L^2(\Omega) \cap \mathbb{H}_{-1}$  which is strongly convergent in  $\mathbb{H}_{-1}$  to  $\mathbf{b}_0 \in \mathbb{H}_{-1}$ . Suppose that there is a family of functions in  $\mathbb{F} \cap \text{Dom}(\tilde{\mathbf{S}})$  denoted by  $(\mathbf{v}_\lambda)_{\lambda>0}$  such that*

$$\forall \lambda > 0, \quad \lambda \mathbf{v}_\lambda - \tilde{\mathbf{S}}\mathbf{v}_\lambda - D_t \mathbf{v}_\lambda = \mathbf{b}_\lambda. \quad (44)$$

Then there exists  $\boldsymbol{\eta} \in (L^2(\Omega))^d$  such that

$$\lambda \|\mathbf{v}_\lambda\|_2^2 \xrightarrow{\lambda \rightarrow 0} 0 \text{ and } \|\nabla^\sigma \mathbf{v}_\lambda - \boldsymbol{\eta}\|_2 \xrightarrow{\lambda \rightarrow 0} 0$$

**Proof :** We remind that we have

$$-\tilde{\mathbf{S}} = \int_{\mathbb{R}_+ \times \mathbb{R}} x E(dx, dy) \text{ and } -D_t = \int_{\mathbb{R}_+ \times \mathbb{R}} iy E(dx, dy).$$

After multiplying (44) by  $\mathbf{v}_\lambda$  and integrating with respect to the measure  $\pi$ , we have

$$\lambda \|\mathbf{v}_\lambda\|_2^2 + \|\mathbf{v}_\lambda\|_1^2 = \int_{\Omega} \mathbf{b}_\lambda \mathbf{v}_\lambda d\pi \leq C \|\mathbf{v}_\lambda\|_1 \leq C^2 \quad (45)$$

where  $C = \sup_{\lambda>0} \|\mathbf{b}_\lambda\|_{-1}$ . Hence we know that there is  $\mathbf{h} \in L^2(\Omega)$  and a subsequence still denoted by  $(\mathbf{v}_\lambda)_\lambda$  such that  $(\sqrt{-\tilde{\mathbf{S}}}\mathbf{v}_\lambda)_\lambda$  converges weakly in  $L^2(\Omega)$  to  $\mathbf{h}$ .

• We have  $\sup_{\lambda>0} \|\lambda \mathbf{v}_\lambda\|_{-1} < \infty$  and  $\sup_{\lambda>0} \|D_t \mathbf{v}_\lambda\|_{-1} < \infty$ :

$$\begin{aligned} \left| \int_{\Omega} \lambda \mathbf{v}_\lambda \varphi d\pi \right| &= \left| \int_{\mathbb{R}_+ \times \mathbb{R}} \frac{\lambda}{(\lambda + x) + iy} dE_{\mathbf{b}_\lambda, \varphi} \right| \\ &\leq \sqrt{\int_{\mathbb{R}_+ \times \mathbb{R}} \frac{\lambda^2}{x[(\lambda + x)^2 + y^2]} dE_{\mathbf{b}_\lambda, \mathbf{b}_\lambda}} \sqrt{\int_{\mathbb{R}_+ \times \mathbb{R}} x dE_{\varphi, \varphi}} \\ &\leq \left( \sup_{\lambda>0} \sqrt{\int_{\mathbb{R}_+ \times \mathbb{R}} \frac{1}{x} dE_{\mathbf{b}_\lambda, \mathbf{b}_\lambda}} \right) \|\varphi\|_1 \\ &= \sup_{\lambda>0} \|\mathbf{b}_\lambda\|_{-1} \|\varphi\|_1 \\ &\leq \bar{C} \|\varphi\|_1 \end{aligned}$$

where  $\bar{C}$  does not depend on  $\lambda$ . Hence we deduce from  $D_t \mathbf{v}_\lambda = \lambda \mathbf{v}_\lambda - \tilde{\mathbf{S}}\mathbf{v}_\lambda - \mathbf{b}_\lambda$  that there is a constant  $A$  independent of  $\lambda$  such that  $\forall \varphi \in \mathcal{C}$ ,  $\int_{\Omega} D_t \mathbf{v}_\lambda \varphi d\pi \leq A \|\varphi\|_1$ , so that, even if it means extracting a subsequence, there is a family  $(\mathbf{F}_\lambda)_{\lambda \geq 0}$  of linear forms on  $\text{ID} \subset L^2(\Omega)$  such that  $\forall \lambda > 0, \forall \varphi \in \mathcal{C}$ ,  $\mathbf{F}_\lambda(\sqrt{-\tilde{\mathbf{S}}}\varphi) = \int_{\Omega} D_t \mathbf{v}_\lambda \varphi d\pi$ , and  $(\mathbf{F}_\lambda)_{\lambda>0}$  is weakly convergent to  $\mathbf{F}_0$ :

$$\forall \varphi \in \mathcal{C}, \quad \mathbf{F}_\lambda(\sqrt{-\tilde{\mathbf{S}}}\varphi) \xrightarrow{\lambda \rightarrow 0} \mathbf{F}_0(\sqrt{-\tilde{\mathbf{S}}}\varphi).$$

• **We have  $F_0(\mathbf{h}) = 0$  :**

Since

$$F_\lambda(\sqrt{-\tilde{\mathbf{S}}}\mathbf{v}_\mu) = \int_{\Omega} D_t \mathbf{v}_\lambda \mathbf{v}_\mu d\pi = - \int_{\Omega} D_t \mathbf{v}_\mu \mathbf{v}_\lambda d\pi = -F_\mu(\sqrt{-\tilde{\mathbf{S}}}\mathbf{v}_\lambda)$$

we deduce by passing to the limit as  $\lambda$  goes to 0 that  $F_0(\sqrt{-\tilde{\mathbf{S}}}\mathbf{v}_\mu) = -F_\mu(\mathbf{h})$  and then by passing to the limit as  $\mu$  goes to 0 we have  $F_0(\mathbf{h}) = -F_0(\mathbf{h}) = 0$ .

• **The limit equation :**

By (45), we know that  $\lambda|\mathbf{v}_\lambda|_2^2 \leq C^2$  so that  $\lambda\mathbf{v}_\lambda \xrightarrow[\lambda \rightarrow 0]{L^2(\Omega)} 0$ . Then, if we multiply the equation (44) by  $\varphi \in \mathcal{C}$  we have  $\lambda(\mathbf{v}_\lambda, \varphi) + \langle \mathbf{v}_\lambda, \varphi \rangle_1 - (D_t \mathbf{v}_\lambda, \varphi) = (\mathbf{b}_\lambda, \varphi)$  and if we pass to the limit as  $\lambda \rightarrow 0$  we obtain

$$(\mathbf{h}, \sqrt{-\tilde{\mathbf{S}}}\varphi) - F_0(\sqrt{-\tilde{\mathbf{S}}}\varphi) = \langle \mathbf{b}_0, \varphi \rangle_{-1,1}. \quad (46)$$

Because of the density of  $\mathcal{C}$  in  $\mathbb{F}$ , this equality remains valid for any  $\varphi \in \mathbb{F}$ .

• **Final step for the convergence:**

If we use the fact that  $F_0(\mathbf{h}) = 0$  we have

$$\lim_{\lambda \rightarrow 0} (\mathbf{h}, \sqrt{-\tilde{\mathbf{S}}}\mathbf{v}_\lambda) = \lim_{\lambda \rightarrow 0} \left( (\mathbf{h}, \sqrt{-\tilde{\mathbf{S}}}\mathbf{v}_\lambda) - F_0(\sqrt{-\tilde{\mathbf{S}}}\mathbf{v}_\lambda) \right) \quad (47)$$

Hence we deduce

$$\begin{aligned} \limsup_{\lambda \rightarrow 0} |\mathbf{h}|_2 \|\mathbf{v}_\lambda\|_1 &\geq \limsup_{\lambda \rightarrow 0} (\mathbf{h}, \sqrt{-\tilde{\mathbf{S}}}\mathbf{v}_\lambda) \\ &= \limsup_{\lambda \rightarrow 0} \left( (\mathbf{h}, \sqrt{-\tilde{\mathbf{S}}}\mathbf{v}_\lambda) - F_0(\sqrt{-\tilde{\mathbf{S}}}\mathbf{v}_\lambda) \right) \\ &\stackrel{(46)}{=} \limsup_{\lambda \rightarrow 0} \langle \mathbf{b}_0, \mathbf{v}_\lambda \rangle_{-1,1} \\ &= \limsup_{\lambda \rightarrow 0} (\mathbf{b}_\lambda, \mathbf{v}_\lambda) \\ &\stackrel{(45)}{=} \limsup_{\lambda \rightarrow 0} (\lambda|\mathbf{v}_\lambda|_2^2 + \|\mathbf{v}_\lambda\|_1^2) \\ &\geq \limsup_{\lambda \rightarrow 0} \|\mathbf{v}_\lambda\|_1^2 \end{aligned}$$

Hence, thanks to the same arguments as in the static case, we have

$$|\mathbf{h}|_2 = \lim_{\lambda \rightarrow 0} \|\mathbf{v}_\lambda\|_1.$$

Thus we obtain the strong convergence of  $(\sqrt{-\tilde{\mathbf{S}}}\mathbf{v}_\lambda)_\lambda$  towards  $\mathbf{h}$  along a subsequence. But there must be equality at each line of the previous calculation so that we also have  $\lambda|\mathbf{v}_\lambda|_2^2 \xrightarrow[\lambda \rightarrow 0]{} 0$ .

• **Uniqueness of the weak limit :**

This convergence holds for the whole family if there is uniqueness for the weak limit. That's what we are going to prove. Let  $\mathbf{h}$  and  $\mathbf{h}'$  be two weak limits of two subsequences of  $(\mathbf{v}_\lambda)_\lambda$ , and  $F_0, F'_0$  the corresponding linear forms defined as described above. We still have the two limit equations for any  $\varphi \in \mathbb{F}$

$$(\mathbf{h}, \sqrt{-\tilde{\mathbf{S}}}\varphi) - F_0(\sqrt{-\tilde{\mathbf{S}}}\varphi) = \langle \mathbf{b}_0, \varphi \rangle_{-1,1}, \quad (48)$$

and similarly

$$(\mathbf{h}', \sqrt{-\tilde{\mathbf{S}}}\boldsymbol{\varphi}) - \mathbf{F}'_0(\sqrt{-\tilde{\mathbf{S}}}\boldsymbol{\varphi}) = \langle \mathbf{b}_0, \boldsymbol{\varphi} \rangle_{-1,1}. \quad (49)$$

In a similar way we did previously we have

$$\mathbf{F}_\lambda(\sqrt{-\tilde{\mathbf{S}}}\mathbf{v}_\mu) = \int D_t \mathbf{v}_\lambda \mathbf{v}_\mu d\pi = - \int D_t \mathbf{v}_\mu \mathbf{v}_\lambda d\pi = -\mathbf{F}_\mu(\sqrt{-\tilde{\mathbf{S}}}\mathbf{v}_\lambda)$$

We can pass to the limit as  $\lambda$  goes to 0 along the first subsequence and then to the limit as  $\mu$  goes to 0 along the second subsequence and we obtain

$$\mathbf{F}_0(\mathbf{h}') = -\mathbf{F}'_0(\mathbf{h}). \quad (50)$$

By choosing  $\boldsymbol{\varphi} = \mathbf{v}_\mu$  in (48) and (49) and taking the limit along the second subsequence we obtain :

$$\begin{aligned} (\mathbf{h}, \mathbf{h}') - \mathbf{F}_0(\mathbf{h}') &= \langle \mathbf{b}_0, \mathbf{h}' \rangle_{-1,1}, \\ (\mathbf{h}', \mathbf{h}') &= (\mathbf{h}', \mathbf{h}') - \mathbf{F}'_0(\mathbf{h}') = \langle \mathbf{b}_0, \mathbf{h}' \rangle_{-1,1}, \end{aligned}$$

In the same way, by choosing  $\boldsymbol{\varphi} = \mathbf{v}_\lambda$  in (48) and (49) and taking the limit along the first subsequence we obtain

$$\begin{aligned} (\mathbf{h}, \mathbf{h}) &= (\mathbf{h}, \mathbf{h}) - \mathbf{F}_0(\mathbf{h}) = \langle \mathbf{b}_0, \mathbf{h} \rangle_{-1,1} \\ (\mathbf{h}', \mathbf{h}) - \mathbf{F}'_0(\mathbf{h}) &= \langle \mathbf{b}_0, \mathbf{h} \rangle_{-1,1} \end{aligned}$$

From these 4 last equalities we deduce

$$\begin{aligned} \|\mathbf{h} - \mathbf{h}'\|_2^2 &= (\mathbf{h}, \mathbf{h}) - 2(\mathbf{h}', \mathbf{h}) + (\mathbf{h}', \mathbf{h}') \\ &= -\mathbf{F}'_0(\mathbf{h}) - \mathbf{F}_0(\mathbf{h}') \\ &\stackrel{(50)}{=} 0 \end{aligned}$$

and this allows us to deduce  $\mathbf{h} = \mathbf{h}'$  □

We want now to treat the general case (with the symmetric part  $\mathbf{S}$  and the antisymmetric part). First we explain the idea of the proof. Formally we have

$$\begin{aligned} \lambda - \mathbf{S} - \mathbf{A} - D_t &= \lambda - \tilde{\mathbf{S}} - D_t - (\mathbf{S} - \tilde{\mathbf{S}}) - \mathbf{A} \\ &= \left( \mathbf{I} - [(\mathbf{S} - \tilde{\mathbf{S}}) + \mathbf{A}](\lambda - \tilde{\mathbf{S}} - D_t)^{-1} \right) (\lambda - \tilde{\mathbf{S}} - D_t) \end{aligned}$$

If we can prove that the operator

$$[(\mathbf{S} - \tilde{\mathbf{S}}) + \mathbf{A}](\lambda - \tilde{\mathbf{S}} - D_t)^{-1}$$

is bounded with a norm strictly less than 1, then we will be able to inverse this operator. In general, its norm is bigger than 1 but we will bring back our study to this case by introducing a small parameter  $\delta$ . Then the norm of the operator

$$\delta[(\mathbf{S} - \tilde{\mathbf{S}}) + \mathbf{A}](\lambda - \tilde{\mathbf{S}} - D_t)^{-1}$$

will be strictly less than 1. We will deduce the general case from this situation.

**Lemma 5.10.** Let  $\mathbf{u} \in \mathbb{F}$  be such that  $D_t \mathbf{u} \in \mathbb{H}_1$ . Then  $\forall \varphi \in \mathcal{C}$  we have,

$$\int \mathbf{A} \mathbf{u} \frac{T_{(-s,0)} \varphi - \varphi}{s} d\pi \leq 2e^{2K} (C_2^H \|\mathbf{u}\|_1 + C_1^H \|D_t \mathbf{u}\|_1) \|\varphi\|_1,$$

and

$$\int \mathbf{S} \mathbf{u} \frac{T_{(-s,0)} \varphi - \varphi}{s} d\pi \leq 2 \left( \tilde{C} \|\mathbf{u}\|_1 + M \|D_t \mathbf{u}\|_1 \right) \|\varphi\|_1.$$

**Proof :** For any  $\varphi, \psi \in \mathcal{C}$  we define

$$2\mathbf{Q}_s(\varphi, \psi) = \frac{1}{s} \int_{\Omega} \mathbf{H}(\tau_s \omega) D_x \varphi(\tau_s \omega) \cdot D_x \psi(\omega) - \mathbf{H}(\omega) D_x \varphi(\omega) \cdot D_x \psi(\omega) d\mu,$$

and we have

$$\begin{aligned} 2\mathbf{Q}_s(\varphi, \psi) &= \int_{\Omega} \frac{\mathbf{H}(\tau_s \omega) - \mathbf{H}(\omega)}{s} D_x \varphi(\tau_s \omega) \cdot D_x \psi(\omega) d\mu \\ &\quad + \int_{\Omega} \mathbf{H}(\omega) \frac{D_x \varphi(\tau_s \omega) - D_x \varphi(\omega)}{s} \cdot D_x \psi(\omega) d\mu \\ &= 2\mathbf{Q}_s^1(\varphi, \psi) + 2\mathbf{Q}_s^2(\varphi, \psi) \end{aligned}$$

Then

$$\begin{aligned} 2|\mathbf{Q}_s^1(\varphi, \psi)| &= \left| \int_{\Omega} \int_0^1 D_t \mathbf{H}(\tau_u \omega) du D_x \varphi(\tau_s \omega) \cdot D_x \psi(\omega) d\mu \right| \\ &\leq \int_0^1 \int_{\Omega} |D_t \mathbf{H}(\tau_u \omega) D_x \varphi(\tau_s \omega) \cdot D_x \psi(\omega)| d\mu du \\ &\stackrel{(11)}{\leq} 2C_2^H e^{2K} \|T_{(s,0)} \varphi\|_1 \|\psi\|_1 \\ &= 2C_2^H e^{2K} \|\varphi\|_1 \|\psi\|_1 \end{aligned}$$

and

$$\begin{aligned} 2|\mathbf{Q}_s^2(\varphi, \psi)| &= \left| \int_{\Omega} \mathbf{H}(\omega) D_x \left( \frac{\varphi(\tau_s \omega) - \varphi(\omega)}{s} \right) \cdot D_x \psi d\mu \right| \\ &\stackrel{(10)}{\leq} 2C_1^H e^{2K} \left\| \frac{\varphi(\tau_s \omega) - \varphi(\omega)}{s} \right\|_1 \|\psi\|_1 \\ &= 2C_1^H e^{2K} \left\| \int_0^1 D_t \varphi(\tau_{su} \omega) du \right\|_1 \|\psi\|_1 \\ &\leq 2C_1^H e^{2K} \left( \int_0^1 \|D_t \varphi(\tau_{su} \omega)\|_1 du \right) \|\psi\|_1 \\ &= 2C_1^H e^{2K} \|D_t \varphi\|_1 \|\psi\|_1 \end{aligned}$$

Hence  $\mathbf{Q}_s$  is a continuous bilinear form and can be extend to  $\mathbb{H}_1 \times \mathbb{H}_1$ . Let us consider now  $\varphi \in \mathcal{C}$ , we have

$$\begin{aligned} \frac{1}{s} \int_{\Omega} \mathbf{A} \mathbf{u} T_{(-s,0)} \varphi - \mathbf{A} \mathbf{u} \varphi d\pi &= \frac{1}{s} \left( - \int_{\Omega} \mathbf{u}(\tau_s \omega) \frac{1}{2} \operatorname{div}(\mathbf{H}(\tau_s \omega) D_x \varphi) d\pi \right. \\ &\quad \left. + \int_{\Omega} \mathbf{u}(\omega) \frac{1}{2} \operatorname{div}(\mathbf{H}(\omega) D_x \varphi), d\pi \right) \\ &= -2\mathbf{Q}_s(\mathbf{u}, \varphi) \end{aligned}$$

and the lemma follows from the fact that  $\mathcal{C}$  is dense in  $\mathbb{H}_1$ . We do not prove the second inequality because the proof is very similar to the first one  $\square$

Equipped with the above lemma we are able to prove the first step of our demonstration.

**Proposition 5.11.** *Let  $(\mathbf{b}_\lambda)_{\lambda>0}$  be a family of functions in  $\mathbb{H}_{-1} \cap L^2(\Omega)$  which is strongly convergent in  $\mathbb{H}_{-1}$  to some  $\mathbf{b}_0 \in \mathbb{H}_{-1}$ . We also suppose that  $\mathbf{b}_\lambda$  verifies the assumption of proposition 5.4, ie there exists a constant  $C$  (which does not depend on  $\lambda$ ) such that  $\forall s > 0$  and  $\forall \varphi \in \mathcal{C}$*

$$\left\langle \mathbf{b}_\lambda, \frac{T_{(-s,0)}\varphi - \varphi}{s} \right\rangle_{-1,1} \leq C \|\varphi\|_1.$$

There exists  $\delta > 0$  such that: for each  $\lambda > 0$ , if we denote by  $\mathbf{u}_\lambda \in \mathbb{F}$  with  $D_t \mathbf{u}_\lambda \in \mathbb{H}_1$  the unique solution (see proposition 5.4) of the equation

$$\lambda \mathbf{u}_\lambda - \delta \mathbf{S} \mathbf{u}_\lambda - (1 - \delta) \tilde{\mathbf{S}} \mathbf{u}_\lambda - \delta \mathbf{A} \mathbf{u}_\lambda - D_t \mathbf{u}_\lambda = \mathbf{b}_\lambda,$$

then there exists  $\boldsymbol{\eta} \in L^2(\Omega)$  such that  $\sqrt{-\tilde{\mathbf{S}}} \mathbf{u}_\lambda \xrightarrow[\lambda \rightarrow 0]{L^2(\Omega)} \boldsymbol{\eta}$  and  $\lambda \|\mathbf{u}_\lambda\|_2^2 \xrightarrow[\lambda \rightarrow 0]{} 0$ .

**Proof:** We denote by  $\mathcal{H}$  the subspace of  $\mathbb{H}_{-1}$  whose elements satisfy the condition:  $\forall s > 0$  and  $\forall \varphi \in \mathcal{C}$

$$\left\langle \mathbf{h}, \frac{T_{(-s,0)}\varphi - \varphi}{s} \right\rangle_{-1,1} \leq C \|\varphi\|_1, \quad (51)$$

where the constant might be different for different  $\mathbf{h} \in \mathcal{H}$ . For any  $\mathbf{h} \in \mathcal{H}$ , the smallest  $C$  which satisfies this condition will be denoted  $\|\mathbf{h}\|_{\mathcal{H}}$ . Then  $\mathcal{H}$  is closed for the norm  $\|\cdot\|_{\mathcal{H}} = \|\cdot\|_{-1} + \|\cdot\|_T$ . We consider now the operator  $T_\lambda : \mathcal{H} \rightarrow \mathcal{H}$  defined by  $T_\lambda(\mathbf{b}) = \delta(\mathbf{S} - \tilde{\mathbf{S}} + \mathbf{A})(\lambda - \tilde{\mathbf{S}} - D_t)^{-1}(\mathbf{b})$ . We have already seen that  $\|(\lambda - \tilde{\mathbf{S}} - D_t)^{-1} \mathbf{b}\|_1^2 \leq \|\mathbf{b}\|_{-1}^2$ , so that

$$\begin{aligned} \|\delta(\mathbf{S} - \tilde{\mathbf{S}} + \mathbf{A})(\lambda - \tilde{\mathbf{S}} - D_t)^{-1}(\mathbf{b})\|_{-1} &\leq \delta(1 + M + e^{2K} C_1^H) \|(\lambda - \tilde{\mathbf{S}} - D_t)^{-1}(\mathbf{b})\|_1 \\ &\leq \delta(1 + M + e^{2K} C_1^H) \|\mathbf{b}\|_{-1} \end{aligned} \quad (52)$$

Moreover, thanks to proposition 5.4, we know that  $D_t(\lambda - \tilde{\mathbf{S}} - D_t)^{-1}(\mathbf{b}) \in \mathbb{H}_1$  and

$$\|D_t(\lambda - \tilde{\mathbf{S}} - D_t)^{-1}(\mathbf{b})\|_1 \leq \|\mathbf{b}\|_T + (C_2^H + \tilde{C}) \|\mathbf{b}\|_{-1},$$

and thanks to lemma 5.10, for any  $\mathbf{u} \in \mathbb{F}$  with  $D_t \mathbf{u} \in \mathbb{H}_1$ ,

$$\|\delta(\mathbf{S} - \tilde{\mathbf{S}} + \mathbf{A})(\mathbf{u})\|_T \leq 2(e^{2K} C_2^H + \tilde{C}) \|\mathbf{u}\|_1 + (2e^{2K} C_1^H + M + 1) \|D_t \mathbf{u}\|_1,$$

so that

$$\begin{aligned} \|\delta(\mathbf{S} - \tilde{\mathbf{S}} + \mathbf{A})(\lambda - \tilde{\mathbf{S}} - D_t)^{-1}(\mathbf{b})\|_T &\leq \delta \left( 2e^{2K} C_2^H + 2\tilde{C} + (\tilde{C} + C_2^H)(2e^{2K} C_1^H + M + 1) \right) \|\mathbf{b}\|_{-1} \\ &\quad + \delta(2e^{2K} C_1^H + M + 1) \|\mathbf{b}\|_T. \end{aligned} \quad (53)$$

Hence we can choose  $\delta$  small enough in order that  $\forall \lambda > 0$ ,  $\|T_\lambda\|_{\mathcal{H} \rightarrow \mathcal{H}} < d$  and  $\|T_\lambda\|_{\mathbb{H}_1 \rightarrow \mathbb{H}_1} < d$  with  $d < 1$ . This implies that we can define the operator  $[\mathbf{I} - T_\lambda]^{-1} : \mathcal{H} \rightarrow \mathcal{H}$ . Moreover it is

sufficient to prove that  $T_\lambda(\mathbf{b}_\lambda)$  converges in  $\mathbb{H}_{-1}$  in order to prove the convergence of  $[\mathbb{I} - T_\lambda]^{-1}(\mathbf{b}_\lambda)$  in  $\mathbb{H}_{-1}$ . But, for any  $\lambda, \mu > 0$ , we have

$$\|T_\lambda(\mathbf{b}_\lambda) - T_\mu(\mathbf{b}_\mu)\|_{-1} \leq \delta(1 + M + e^{2K} C_1^H) \|(\lambda - \tilde{\mathbf{S}} - D_t)^{-1}(\mathbf{b}_\lambda) - (\mu - \tilde{\mathbf{S}} - D_t)^{-1}(\mathbf{b}_\mu)\|_1$$

and this last term converges to 0 according to proposition 5.9.

Finally, it is sufficient to remark that

$$(\lambda - \delta \mathbf{S} - (1 - \delta) \tilde{\mathbf{S}} - \delta \mathbf{A} - D_t)^{-1} = (\lambda - \tilde{\mathbf{S}} - D_t)^{-1} [\mathbb{I} - T_\lambda]^{-1}$$

and to apply proposition 5.9 in order to conclude the proof  $\square$

So we have almost proved the desired result but the coefficient  $\delta$  is in general strictly less than 1. In order to get round this difficulty, we put  $\delta_1 = \delta$  and we choose  $\delta_2 > 0$ , and then we write

$$\begin{aligned} & \lambda - (\delta_1 + \delta_2) \mathbf{S} - (1 - \delta_1 - \delta_2) \tilde{\mathbf{S}} - (\delta_1 + \delta_2) \mathbf{A} - D_t \\ = & [\mathbb{I} - \delta_2(\mathbf{S} - \tilde{\mathbf{S}} + \mathbf{A})][\lambda - \delta_1 \mathbf{S} - (1 - \delta_1) \tilde{\mathbf{S}} - \delta_1 \mathbf{A} - D_t]^{-1} (\lambda - \delta_1 \mathbf{S} - (1 - \delta_1) \tilde{\mathbf{S}} - \delta_1 \mathbf{A} - D_t), \end{aligned}$$

so that we want exactly repeat the proof of proposition 5.11 except that the operator  $\lambda - (1 - \delta_1) \tilde{\mathbf{S}} - \delta_1 \mathbf{S} - \delta_1 \mathbf{A} - D_t$  is replaced by the operator  $\lambda - (1 - \delta_1 - \delta_2) \tilde{\mathbf{S}} - (\delta_1 + \delta_2) \mathbf{S} - (\delta_1 + \delta_2) \mathbf{A} - D_t$  and the use of proposition 5.9 with the operator  $\lambda - \tilde{\mathbf{S}} - D_t$  is everywhere replaced by the use of the proposition 5.11 with the operator  $\lambda - (1 - \delta_1) \tilde{\mathbf{S}} - \delta_1 \mathbf{S} - \delta_1 \mathbf{A} - D_t$ , provided that  $\delta_2$  is chosen small enough. Even if it means substituting  $\tilde{a}$  with  $m\tilde{a}$ , we can suppose without loss of generality that  $m = 1$ . Hence, because of the inequality

$$\tilde{a} \leq a \leq M\tilde{a},$$

the inequalities (52) and (53) remains valid where  $\delta$  is replaced by  $\delta_2$  so that the choice of  $\delta_2$  is independent on  $\delta_1$ . This remark is important because this means that we can iterate these arguments until we find  $\delta_n$  such that  $\delta_1 + \delta_2 + \dots + \delta_n = 1$  and such that proposition 5.11 remains valid if we relace everywhere  $\delta$  by  $\delta_1 + \dots + \delta_n = 1$ .

For the sake of clarity, we sum up our discussion in the following proposition:

**Proposition 5.12.** *Let  $(\mathbf{b}_\lambda)_{\lambda > 0}$  be a family of functions in  $\mathbb{H}_{-1} \cap L^2(\Omega)$  which is strongly convergent in  $\mathbb{H}_{-1}$  to some  $\mathbf{b}_0 \in \mathbb{H}_{-1}$ . We suppose that there exists a constant  $C$  (which does not depend on  $\lambda$ ) such that  $\forall s > 0$  and  $\forall \varphi \in \mathcal{C}$*

$$\left\langle \mathbf{b}_\lambda, \frac{T_{(-s,0)}\varphi - \varphi}{s} \right\rangle_{-1,1} \leq C \|\varphi\|_1.$$

Then solution  $\mathbf{u}_\lambda \in \mathbb{F}$  of the equation

$$\lambda \mathbf{u}_\lambda - \mathbf{S} \mathbf{u}_\lambda - \mathbf{A} \mathbf{u}_\lambda - D_t \mathbf{u}_\lambda = \mathbf{b}_\lambda$$

satisfies:

- there exists  $\boldsymbol{\eta} \in L^2(\Omega)$  such that  $\sqrt{-\tilde{\mathbf{S}}} \mathbf{u}_\lambda \xrightarrow[\lambda \rightarrow 0]{L^2(\Omega)} \boldsymbol{\eta}$ ,
- $\lambda |\mathbf{u}_\lambda|_2^2 \xrightarrow[\lambda \rightarrow 0]{} 0$ .

Now we prove that we can apply this result if we choose  $\mathbf{b}_\lambda$  equal to the drift  $\mathbf{b}$  of our diffusion process. That is why we need the following lemma

**Lemma 5.13.** *For each  $i \in \{1, \dots, d\}$ , we have  $\|\mathbf{b}_i\|_{-1} < \infty$  and  $\forall s > 0, \forall \varphi \in \mathcal{C}$*

$$\int \mathbf{b}_i \frac{T_{(-s,0)}\varphi - \varphi}{s} d\pi \leq \|D_t \mathbf{b}_i\|_2 \|\varphi\|_1.$$

**Remark 5.14.** *If  $\|\mathbf{g}\|_{-1} < \infty$  then we have  $\int_\Omega \mathbf{g} d\pi = 0$ .*

**Proof of the lemma:** Let  $(E_1, \dots, E_d)$  be the canonical basis of  $\mathbb{R}^d$ . Then we have

$$\begin{aligned} \left| \int_\Omega \mathbf{b}_i \varphi d\pi \right| &= \left| \int_\Omega \sum_j \left( \frac{1}{2} D_j \mathbf{a}_{ij} - \mathbf{a}_{ij} D_j \mathbf{V} + \frac{e^{2\mathbf{V}}}{2} D_j \mathbf{H}_{ij} \right) \varphi d\pi \right| \\ &= \left| \frac{1}{2} \int_\Omega (\mathbf{a} - e^{2\mathbf{V}} \mathbf{H}) D\varphi \cdot E_i d\pi \right| \\ &\leq \left| \frac{1}{2} \int_\Omega \mathbf{a} D\varphi \cdot E_i d\pi \right| + \left| \frac{1}{2} \int_\Omega \mathbf{H} D\varphi \cdot E_i d\mu \right| \\ &\stackrel{C-S}{\leq} M \|\varphi\|_1 \left| \int (\mathbf{a} E_i \cdot E_i) d\pi \right|^{1/2} + C_1^H \int_\Omega (\tilde{\mathbf{a}} E_i \cdot E_i)^{1/2} (\tilde{\mathbf{a}} D\varphi \cdot D\varphi)^{1/2} d\mu \\ &\stackrel{C-S}{\leq} C \|\varphi\|_1 \sqrt{\int \tilde{\mathbf{a}}_{ii} d\pi} \end{aligned}$$

and this proves the first point. For the second point we have  $\forall t > 0, \forall \varphi \in \mathcal{C}$

$$\begin{aligned} \int \mathbf{b}_i \frac{T_{(-s,0)}\varphi - \varphi}{s} d\pi &= - \int (\mathbf{a} e^{-2\mathbf{V}} + \mathbf{H}) E_i \cdot \frac{T_{(-s,0)} D\varphi - D\varphi}{2s} d\mu \\ &= - \int \frac{T_{(s,0)} (\mathbf{a} e^{-2\mathbf{V}} + \mathbf{H}) - (\mathbf{a} e^{-2\mathbf{V}} + \mathbf{H})}{2s} E_i \cdot D\varphi d\mu \\ &\leq (\tilde{C} + e^{2K} C_2^H) \sqrt{\int \tilde{\mathbf{a}}_{ii} d\pi} \|\varphi\|_1 \end{aligned}$$

□

## 6 Itô's formula

Since the matrix  $\mathbf{a}$  is not uniformly elliptic, we can't prove the regularity of the functions  $\mathbf{u}_\lambda^i$  in order to apply Itô's formula. Here we will use the fact that the process  $X$  is almost symmetric in the sense that we can control the antisymmetric part with the symmetric one. The aim of this section is to prove the following theorem:

**Proposition 6.1.** *For each function  $\mathbf{f} \in \text{Dom}(L + D_t) \cap \mathbb{F}$ , we have*

$$\mathbb{P}_\pi \text{ ps, } \mathbf{f}(Y_t) = \mathbf{f}(Y_0) + \int_0^t (\mathbf{L}\mathbf{f} + D_t \mathbf{f})(Y_r) dr + \int_0^t \nabla^\sigma \mathbf{f}^*(Y_r) dB_r \quad (54)$$

where  $\mathbb{P}_\pi$  is the law of the process  $Y$  starting with the distribution  $\pi$  on  $\Omega$ .

**Proof:** Consider  $\mathbf{f} \in \text{Dom}(L + D_t) \subset \mathbb{F}$  and a sequence  $(\varphi_n)_n \in (\mathcal{C})^{\mathbb{N}}$  such that  $\varphi_n \xrightarrow[n \rightarrow \infty]{\mathbb{F}} \mathbf{f}$  and  $L\varphi_n \xrightarrow[n \rightarrow \infty]{L^2(\Omega)} L\mathbf{f}$ . Thanks to the regularity of the functions  $\varphi_n$ , we can apply the classical Itô formula:

$$\varphi_n(Y_t) = \varphi_n(Y_0) + \int_0^t (\mathbf{L}\varphi_n + D_t\varphi)_n(Y_r) dr + \int_0^t \nabla^\sigma \varphi_n^*(Y_r) dB_r. \quad (55)$$

Thanks to the invariance of the measure  $\pi$  for the process  $Y$  and the Cauchy-Schwarz inequality, it is not difficult to see that

$$\mathbb{E}_\pi \left[ |\varphi_n(Y_t) - \mathbf{f}(Y_t)|^2 + |\varphi_n(Y_0) - \mathbf{f}(Y_0)|^2 + \left( \int_0^t (\mathbf{L}\varphi_n + D_t\varphi)_n(Y_r) dr - \int_0^t (\mathbf{L}\mathbf{f} + D_t\mathbf{f})(Y_r) dr \right)^2 \right]$$

converges to 0 as  $n$  goes to infinity. Moreover

$$\begin{aligned} \mathbb{E}_\pi \left[ \sup_{0 \leq s \leq t} \left( \int_0^s (\nabla^\sigma \varphi_n^* - \nabla^\sigma \mathbf{f}^*)(Y_r) dB_r \right)^2 \right] &\leq \mathbb{E}_\pi \left[ \int_0^t \text{Tr}[(\nabla^\sigma \varphi_n^* - \nabla^\sigma \mathbf{f}^*)(\nabla^\sigma \varphi_n^* - \nabla^\sigma \mathbf{f}^*)^*](Y_r) dr \right] \\ &\leq 2Mt \|\mathbf{f} - \varphi_n\|_1^2 \end{aligned}$$

converges also to 0 as  $n$  tends to infinity. The above last inequality results from the definition of  $\nabla^\sigma$  and inequality (9). It is now sufficient to take the limit in equation (55) in order to prove the proposition  $\square$

## 7 The invariance principle

**Notation :**

Up to the end of this paper, for  $i \in \{1, \dots, d\}$  we will denote by  $\mathbf{u}_\lambda^i$  the solution of the equation

$$\lambda \mathbf{u}_\lambda^i - \mathbf{L}\mathbf{u}_\lambda^i - D_t \mathbf{u}_\lambda^i = \mathbf{b}_i.$$

From proposition 5.12, we have  $\lambda |\mathbf{u}_\lambda^i|_2^2 \xrightarrow[\lambda \rightarrow 0]{} 0$  and we can define  $\boldsymbol{\xi}_i = \lim_{\lambda \rightarrow 0} \nabla^\sigma \mathbf{u}_\lambda^i$  where the limit is taken in the space  $L^2(\Omega)^d$   $\square$

By applying the Itô formula (see section 6) to the function  $\mathbf{u}_{\varepsilon^2}$ , we obtain

$$\varepsilon X_{t/\varepsilon^2}^\omega = H_t^{\varepsilon, \omega} + \varepsilon \int_0^{t/\varepsilon^2} (\sigma + \nabla^\sigma u_\lambda^*) (r, X_r^\omega, \omega) dB_r,$$

where

$$H_t^{\varepsilon, \omega} = \varepsilon^3 \int_0^{t/\varepsilon^2} u_{\varepsilon^2}(r, X_r^{1, \omega}, \omega) dr - \varepsilon u_{\varepsilon^2}(t/\varepsilon^2, X_{t/\varepsilon^2}^{1, \omega}, \omega) + \varepsilon u_{\varepsilon^2}(0, 0, \omega).$$

We still denote by  $Y_t = \tau_{t, X_t^\omega}$  and  $\mathbb{P}_\pi$  the law of the process  $Y_t$  with initial distribution  $\pi$ . We want to show that the finite dimensional distributions of the process  $H^{\varepsilon, \omega}$  converges in  $\mathbb{P}_\pi$ -probability to 0. Using the Cauchy-Schwarz inequality and the invariance of the measure  $\pi$ , we get the estimate

$$\mathbb{E}_\pi[(H_t^{\varepsilon, \omega})^2] \leq 3(2 + t^2)\varepsilon^2 |u_{\varepsilon^2}|_2^2$$

and this last term converge to 0 as  $\varepsilon$  goes to 0.

Now we have to study the limit of the process  $t \mapsto \varepsilon \int_0^{t/\varepsilon^2} (\sigma + \nabla^\sigma u_\lambda^*)(r, X_r^\omega, \omega) dB_r$  whose quadratic variations are given by

$$\begin{aligned} \varepsilon^2 \int_0^{t/\varepsilon^2} (\sigma + \nabla^\sigma u_{\varepsilon^2}^*)(\sigma + \nabla^\sigma u_{\varepsilon^2}^*)^*(Y_r) dr &= \varepsilon^2 \int_0^{t/\varepsilon^2} (\sigma + \xi^*)(\sigma + \xi^*)^*(Y_r) dr \\ &+ \left( \varepsilon^2 \int_0^{t/\varepsilon^2} (\sigma + \nabla^\sigma u_{\varepsilon^2}^*)(\sigma + \nabla^\sigma u_{\varepsilon^2}^*)^*(Y_r) dr - \varepsilon^2 \int_0^{t/\varepsilon^2} (\sigma + \xi^*)(\sigma + \xi^*)^*(Y_r) dr \right). \end{aligned}$$

By the ergodic theorem, the first of the two last terms converges  $\pi$  almost surely to the process  $t \mapsto At$  where the matrix  $A$  is given by

$$A = \int_{\Omega} (\sigma + \xi^*)(\sigma + \xi^*)^* d\pi,$$

and the second one converges in  $L^1$  to 0. Indeed, after integrating with respect to the probability measure  $\mathbb{P}_\pi$ , it is bounded by  $Ct|\nabla^\sigma u_{\varepsilon^2} - \xi|_2^2$ . Hence we conclude by applying the central limit theorem for martingales that the finite dimensional distributions of the process  $\varepsilon X_{t/\varepsilon^2}^\omega$  converge in law to the process  $A^{1/2}B_t$ .

**Proposition 7.1.** *The process  $\varepsilon X_{t/\varepsilon^2}^\omega$  is tight in the space  $C([0, T]; \mathbb{R}^d)$ . Hence it converges in law in the space  $C([0, T]; \mathbb{R}^d)$  towards the process  $A^{1/2}B_t$ .*

**Proof :** The next section will be devoted to the proof of the tightness □

We have now to determine the limit when the starting point is not 0 anymore but  $x/\varepsilon$ .

$$\begin{aligned} \mathbb{E}_{x/\varepsilon} \left[ f(\varepsilon X_{t/\varepsilon^2}^\omega) \right] &= \mathbb{E}_0 \left[ f(x + \varepsilon X_{t/\varepsilon^2}^{\tau(0, x/\varepsilon)\omega}) \right] \\ &\stackrel{\text{in law with respect to } \mu}{=} \mathbb{E}_0 \left[ f(x + \varepsilon X_{t/\varepsilon^2}^\omega) \right] \\ &\xrightarrow[\varepsilon \rightarrow 0]{\pi \text{ prob}} \mathbb{E} \left[ f(x + A^{1/2}B_t) \right] \end{aligned}$$

For the first above equality we used that if

$$X_t = x + \int_0^t b(r, X_r, \omega) dr + \int_0^t \sigma(r, X_r, \omega) dB_r$$

and  $Z_t \triangleq X_t - x$  then  $Z_t$  solves the SDE

$$Z_t = \int_0^t b(r, Z_r, \tau(0, x)\omega) dr + \int_0^t \sigma(r, Z_r, \tau(0, x)\omega) dB_r.$$

Hence we have proved the following result

**Theorem 7.2.** *Let  $f$  be a continuous, bounded and  $\mathbb{R}^d$  valued function. Then the solution  $z(x, t, \omega)$  of the partial differential equation*

$$\begin{cases} \frac{d}{dt} z(x, t, \omega) &= \frac{1}{2} \sum_{i,j=1}^d a_{ij}(t, x, \omega) \frac{\partial^2}{\partial x_i \partial x_j} z(x, t, \omega) + \sum_i^d b_i(t, x, \omega) \frac{\partial}{\partial x_i} z(x, t, \omega) \\ z(x, 0, \omega) &= f(x) \end{cases}$$

satisfies the property:  $z(x/\varepsilon, t/\varepsilon^2, \omega)$  converges in  $\pi$ -probability as  $\varepsilon \rightarrow 0$  to  $\mathbb{E} [f(x + A^{1/2}B_t)]$  which is the viscosity solution of the deterministic equation

$$\begin{cases} \frac{\partial u}{\partial t}(t, x) = \frac{1}{2} \sum_{i,j} A_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j}(t, x) \\ u(0, x) = f(x). \end{cases},$$

with

$$A = \int_{\Omega} (\boldsymbol{\sigma} + \boldsymbol{\xi}^*)(\boldsymbol{\sigma} + \boldsymbol{\xi}^*)^* d\pi.$$

## 8 Tightness

In order to obtain the tightness in the space  $C([0, T]; \mathbb{R}^d)$  of the process

$$\varepsilon X_{t/\varepsilon^2}^{1,\omega} = \varepsilon \int_0^{t/\varepsilon^2} b(r, X_r^{1,\omega}, \omega) dr + \varepsilon \int_0^{t/\varepsilon^2} \sigma(r, X_r^{1,\omega}, \omega) dB_r,$$

we are faced with the two terms of the right side of the above identity. The tightness of the second term in  $C([0, T]; \mathbb{R}^d)$  is not very difficult to prove by using the Burkholder-Davis-Gundy inequality and the boundedness of the coefficient  $\sigma$ . For the first term, we are going to exploit ideas of [15] or [17].

**Proposition 8.1.** *Let  $\mathbf{g} \in \mathbb{H}_{-1} \cap L^2(\Omega)$ . Then for any  $T > 0$ , the family of processes indexed by  $\varepsilon > 0$*

$$\left( \left( \varepsilon \int_0^{t/\varepsilon^2} \mathbf{g}(Y_r) dr \right)_{0 \leq t \leq T} \right)_{\varepsilon > 0}$$

is tight in the space  $C([0, T]; \mathbb{R})$ .

**Proof :** For any  $\lambda > 0$ , we put  $\mathbf{w}_\lambda = (\lambda - \mathbf{S})^{-1} \mathbf{g} \in \mathbb{H}_1 \cap \text{Dom}(\mathbf{S})$  (see proposition 5.5). Moreover we know (see proposition 5.8) that there exists  $\boldsymbol{\zeta} \in (L^2(\Omega))^d$  such that

$$\lambda |\mathbf{w}_\lambda|_2^2 \xrightarrow{\lambda \rightarrow 0} 0 \quad \text{and} \quad \nabla^\sigma \mathbf{w}_\lambda \xrightarrow{\lambda \rightarrow 0} \boldsymbol{\zeta}.$$

Then by choosing  $\lambda = \varepsilon^2$ , we obtain

$$\varepsilon \int_0^{t/\varepsilon^2} \mathbf{g}(Y_r) dr = \varepsilon^3 \int_0^{t/\varepsilon^2} \mathbf{w}_{\varepsilon^2}(Y_r) dr - \varepsilon \int_0^{t/\varepsilon^2} \mathbf{S} \mathbf{w}_{\varepsilon^2}(Y_r) dr$$

By using the Cauchy-Schwarz inequality and the invariance of the measure  $\pi$ , we have

$$\begin{aligned} \mathbb{E}_\pi \left[ \sup_{0 \leq t \leq T} \left| \varepsilon^3 \int_0^{t/\varepsilon^2} \mathbf{w}_{\varepsilon^2}(Y_r) dr \right|^2 \right] &\leq \mathbb{E}_\pi \left[ \varepsilon^6 \left( \int_0^{T/\varepsilon^2} |\mathbf{w}_{\varepsilon^2}(Y_r)|^2 dr \right)^2 \right] \\ &\leq T^2 \varepsilon^2 |\mathbf{w}_{\varepsilon^2}|_2^2 \end{aligned}$$

and this converges to 0 as  $\varepsilon \rightarrow 0$ . Hence we only have to show the tightness of the family  $\left(\varepsilon \int_0^{\cdot/\varepsilon^2} \mathbf{S} \mathbf{w}_{\varepsilon^2}(Y_r) dr\right)_{\varepsilon > 0}$ . Let  $\{\varphi_\lambda, \lambda > 0\} \subset \mathcal{C}$  be a family such that  $|\varphi_\lambda - \mathbf{w}_\lambda|_2 \rightarrow 0$  and  $|\mathbf{S}\varphi_\lambda - \mathbf{S}\mathbf{w}_\lambda|_2^2 \leq \lambda^{3/2}$  when  $\lambda \rightarrow 0$ . We have

$$\varepsilon \int_0^{t/\varepsilon^2} \mathbf{S} \mathbf{w}_{\varepsilon^2}(Y_r) dr = \varepsilon \int_0^{t/\varepsilon^2} \mathbf{S}(\mathbf{w}_{\varepsilon^2} - \varphi_{\varepsilon^2})(Y_r) dr + \varepsilon \int_0^{t/\varepsilon^2} \mathbf{S}\varphi_{\varepsilon^2}(Y_r) dr,$$

and as previously

$$\mathbb{E}_\pi \left[ \sup_{0 \leq t \leq T} \left| \varepsilon \int_0^{t/\varepsilon^2} \mathbf{S}(\mathbf{w}_{\varepsilon^2} - \varphi_{\varepsilon^2})(Y_r) dr \right|^2 \right] \leq T^2 \frac{|\mathbf{S}(\mathbf{w}_{\varepsilon^2} - \varphi_{\varepsilon^2})|_2^2}{\varepsilon^2} \xrightarrow{\varepsilon \rightarrow 0} 0.$$

Finally we just have to study the tightness of the family  $\left(\varepsilon \int_0^{\cdot/\varepsilon^2} \mathbf{S}\varphi_{\varepsilon^2}(Y_r) dr\right)_{\varepsilon > 0}$ . First, we remark that  $\lim_{\lambda \rightarrow 0} |\nabla^\sigma \varphi_\lambda - \nabla^\sigma \mathbf{w}_\lambda|_2^2 = -2 \lim_{\lambda \rightarrow 0} (\varphi_\lambda - \mathbf{w}_\lambda, \mathbf{S}\varphi_\lambda - \mathbf{S}\mathbf{w}_\lambda)_2 \xrightarrow{\lambda \rightarrow 0} 0$  and this implies that  $\lim_{\lambda \rightarrow 0} |\nabla^\sigma \varphi_\lambda - \zeta|_2^2 \xrightarrow{\lambda \rightarrow 0} 0$ . Then, we remind that the operator  $\mathbf{L}^*$  is the adjoint operator of  $\mathbf{L}$ . It is easy to check that  $\mathcal{C} \subset \text{Dom}(\mathbf{L}^*)$ . Then we observe that  $\forall 0 \leq t \leq T$  and  $\varepsilon > 0$

$$\varphi_{\varepsilon^2}(Y_{t/\varepsilon^2}) - \varphi_{\varepsilon^2}(Y_0) - \int_0^{t/\varepsilon^2} [\mathbf{L}\varphi_{\varepsilon^2} + D_0\varphi_{\varepsilon^2}](Y_r) dr = \mathcal{M}_{t/\varepsilon^2}^\varepsilon - \mathcal{M}_0^\varepsilon,$$

where  $\mathcal{M}^\varepsilon$  is a martingale with respect to the forward filtration  $(\mathcal{F}_t^\varepsilon)_{0 \leq t \leq T}$ , where  $\mathcal{F}_t^\varepsilon$  is the  $\sigma$ -algebra on  $\Omega$  generated by  $\{Y_r; 0 \leq r \leq t/\varepsilon^2\}$ . In the same way,

$$\varphi_{\varepsilon^2}(Y_0) - \varphi_{\varepsilon^2}(Y_{t/\varepsilon^2}) - \int_0^{t/\varepsilon^2} [\mathbf{L}^*\varphi_{\varepsilon^2} - D_0\varphi_{\varepsilon^2}](Y_r) dr = \mathcal{M}_0^{*,\varepsilon} - \mathcal{M}_{t/\varepsilon^2}^{*,\varepsilon},$$

where  $\mathcal{M}^{*,\varepsilon}$  is a martingale with respect to the backward filtration  $(\mathcal{G}_t^\varepsilon)_{0 \leq t \leq T}$ , where  $\mathcal{G}_t^\varepsilon$  is the  $\sigma$ -algebra on  $\Omega$  generated by  $\{Y_r; t/\varepsilon^2 \leq r \leq T/\varepsilon^2\}$ . We deduce from these two expressions:  $\forall 0 \leq t \leq T$

$$-\varepsilon \int_0^{t/\varepsilon^2} \mathbf{S}\varphi_{\varepsilon^2}(Y_r) dr = \frac{\varepsilon \mathcal{M}_{t/\varepsilon^2}^\varepsilon - \varepsilon \mathcal{M}_0^\varepsilon}{2} - \frac{\varepsilon \mathcal{M}_{t/\varepsilon^2}^{*,\varepsilon} - \varepsilon \mathcal{M}_0^{*,\varepsilon}}{2}.$$

By proving the tightness of these two last terms, we will conclude our proof. In order to prove the tightness of these martingales, it is sufficient to prove the tightness of their brackets (see [4] theorem 4.13). But for  $0 \leq t \leq T$ , their brackets are equal to

$$\varepsilon^2 \int_0^{t/\varepsilon^2} \nabla^\sigma \varphi_{\varepsilon^2} \nabla^\sigma \varphi_{\varepsilon^2}^*(Y_r) dr.$$

Moreover we have

$$\begin{aligned} \varepsilon^2 \int_0^{t/\varepsilon^2} \nabla^\sigma \varphi_{\varepsilon^2} \nabla^\sigma \varphi_{\varepsilon^2}^*(Y_r) dr &= \varepsilon^2 \int_0^{t/\varepsilon^2} [\nabla^\sigma \varphi_{\varepsilon^2} \nabla^\sigma \varphi_{\varepsilon^2}^* - \zeta \zeta^*](Y_r) dr \\ &\quad + \varepsilon^2 \int_0^{t/\varepsilon^2} \zeta \zeta^*(Y_r) dr \end{aligned}$$

We can easily see that

$$\mathbb{E}_\pi \left[ \sup_{0 \leq t \leq T} \left| \varepsilon^2 \int_0^{t/\varepsilon^2} [\nabla^\sigma \varphi_{\varepsilon^2} \nabla^\sigma \varphi_{\varepsilon^2}^* - \zeta \zeta^*](Y_r) dr \right| \right] \leq T |\nabla^\sigma \varphi_{\varepsilon^2} - \zeta|_2 \xrightarrow{\varepsilon \rightarrow 0} 0.$$

Moreover for each fixed  $t$ , the Birkhoff ergodic theorem proves that the term  $\varepsilon^2 \int_0^{t/\varepsilon^2} \zeta \zeta^*(Y_r) dr$  converges  $\pi$  almost surely to  $t \int_\Omega \zeta \zeta^*(\omega) d\pi$ . Then theorem 3.37 in [4] says that the brackets are tight in  $D([0, T]; \mathbb{R}^{d \times d})$   $\square$

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