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Bernard Brighi, Augustin Fruchard, Tewfik Sari

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ON THE BLASIUS PROBLEM

Bernard BRIGHI, Augustin FRUCHARD and Tewfik SARI

Dedicated to the centenary of Blasius' Thesis

Abstract. The Blasius problem $f''' + ff'' = 0$, $f(0) = -a$, $f'(0) = b$, $f'(+\infty) = \lambda$ is exhaustively investigated. In particular the difficult and scarcely studied case $b < 0 \leq \lambda$ is analyzed in details, in which the shape and the number of solutions is determined. The method is first to reduce to the Crocco equation $uu'' + s = 0$ and then to use an associated autonomous planar vector field. The most useful properties of Crocco solutions appear to be related to canard solutions of a slow fast vector field.

KEYWORDS Blasius equation, Crocco equation, boundary value problem on infinite interval, canard solution

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AMS Subject Classifications: 34B15, 34B40; 34C11, 76D10.

1. INTRODUCTION

Given $a, b, \lambda \in \mathbb{R}$, the boundary value problem

$$f''' + ff'' = 0 \quad \text{on } [0, +\infty), \quad (1.1)$$

$$f(0) = -a, \quad f'(0) = b, \quad \lim_{t \rightarrow +\infty} f'(t) = \lambda, \quad (1.2)$$

arises for the first time, with $a = b = 0$, $\lambda = 2$, in 1907 in the PhD thesis of Blasius [7], and plays a central role in fluid mechanics. Equation (1.1) was obtained using a similarity transform and enabled successful treatment of the laminar boundary layer on a flat plate. In this way, Blasius accomplished one of the most significant developments in fluid mechanics in the twentieth century.

More general boundary conditions are physically relevant. For the laminar flow over a flat plate with suction or blowing, the problem to be solved is (1.1-1.2) with $a \in \mathbb{R}^*$, $b = 0$ and $\lambda = 1$ (see for example [33], [34] and [39]).

With $a \in \mathbb{R}$, $b = 1$ and $\lambda = 0$, the Blasius problem (1.1-1.2) appears in the context of free convection about a vertical flat plate embedded in a porous medium. With the same boundary conditions, it also appears within the framework of boundary layer flow adjacent to a stretching wall. For a survey and some generalizations, see [11, 24] and the references therein.

In the study of mixed convection in porous media, the Blasius problem (1.1-1.2) is considered with $a = 0$, $b \in \mathbb{R}$ and $\lambda = 1$ (see [1]). The case $b < 0$ arises also in the boundary layer problem for a flat plate moving at steady speed opposite in direction to that of a uniform mainstream, with $a = 0$ and $\lambda = 1$ (see [53]).

Due to the following classical similarity property of Blasius equation (1.1)

$$\text{if } f \text{ is a solution, so is } t \mapsto \sigma f(\sigma t) \text{ for all } \sigma \in \mathbb{R}, \quad (1.3)$$

we see that, in the case $a = b = 0$, the value $\lambda = 2$ can be replaced by any positive number. In that case, H. Weyl [55] proves that (1.1-1.2) has one and only one solution. The method, following an argument first advanced by C. Töpfer [46], is very elementary but uses strongly the fact that $a = b = 0$. See also B. Brighi [10], P. Hartman [27].

In the general case, the approach is different according to the sign of $\lambda - b$. If f is a solution of (1.1) on some interval J , since this equation can be seen as a linear homogeneous first order ODE for f'' , we deduce that f'' cannot vanish without being identically equal to 0 on J , see (2.1). Therefore, the study of the problem (1.1-1.2) naturally splits into three cases: affine, concave or convex, depending on $\lambda = b$, $\lambda < b$ or $\lambda > b$. The affine case is quickly solved: if $\lambda = b$, then (1.1-1.2) has one and only one solution given by $f(t) = bt - a$. The concave case is well known. In fact, by a direct approach, Z. Belhachmi, B. Brighi, K. Taous [4] proved that the Blasius boundary value problem (1.1-1.2) has exactly one (concave) solution if $0 \leq \lambda < b$, and no solution if $\lambda < 0$. For the sake of completeness, we provide an alternative proof of this result in Section 8.

In the convex case, the situation is quite different, and the proofs of uniqueness of the solution of (1.1-1.2) depend on the introduction of suitable changes of variables, see Section 9.5. The most powerful among them is the so-called *Crocco transformation*, see L. Crocco [15]. This change of variables, detailed in Section 2.1, consists of choosing $s = f'$ as independent variable and expressing $u = f''$ as a function of s . This yields the *Crocco equation*

$$\frac{d^2u}{ds^2} = -\frac{s}{u}. \quad (1.4)$$

As we will see in Section 3.5, the Crocco change of variables provides an alternative, elementary and very short uniqueness proof of the solution of (1.1-1.2), for $b \geq 0$ and all $a \in \mathbb{R}$. See also A.J. Callegari, M.B. Friedman [13] and K. Vajravelu, E. Soewono, R.N. Mohapatra [49].

In the case $b < 0$, the Crocco transformation is still valid and, using it, non-uniqueness for the problem (1.1-1.2) is mentioned for the first time by M.Y. Hussaini, W.D. Laikin [31], but only supported by numerical investigations. Some partial proofs are then given by M.Y. Hussaini, W.D. Laikin, A. Nachman [32] and by E. Soewono, K. Vajravelu, R.N. Mohapatra [43].

In this work, we will focus on the case $b < 0$ and $\lambda > b$. In order to study the solutions of (1.1-1.2), we use the shooting method: let $f(\cdot; a, b, c)$ denote the solution of the following initial value problem with $c > 0$

$$f''' + ff'' = 0, \quad f(0) = -a, \quad f'(0) = b, \quad f''(0) = c. \quad (1.5)$$

In Proposition 3.1, we prove that $f(\cdot; a, b, c)$ is defined at least for $t \in [0, +\infty)$ and that its derivative has a finite and non-negative limit as $t \rightarrow +\infty$. In the sequel, $\Lambda(a, b, c)$ denotes this limit

$$\Lambda(a, b, c) := \lim_{t \rightarrow +\infty} f'(t; a, b, c) \in [0, +\infty).$$

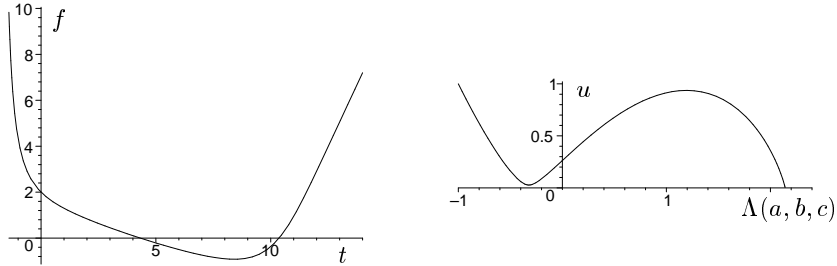


FIGURE 1. On the left, the Blasius solution $f(\cdot; -2, -1, 1)$; on the right, the corresponding Crocco solution.

As we will see later (*cf.* Proposition 2.1), $\Lambda(a, b, c)$ is also the right bound of existence of the solution of (1.4) with initial conditions $u(b) = c$, $u'(b) = a$; see Figure 1 for a comparison between a Blasius solution and the corresponding Crocco solution.

Our strategy will be, for $b < 0$ and for any value of a , to count the number of values of c for which $\Lambda(a, b, c)$ equals some given value λ . Due to (1.3), we can assume, without restriction, that $b = -1$, and we use the notation

$$\tilde{\Lambda} : \mathbb{R} \times (0, +\infty) \rightarrow [0, +\infty), \quad (a, c) \mapsto \Lambda(a, -1, c).$$

Some useful properties of this function $\tilde{\Lambda}$ are stated in Section 2.3. In particular, we show that $\tilde{\Lambda}$ is continuous on the upper half-plane, except on a spiraling curve Γ_∞ ; see Figure 2 for an illustration of this discontinuity. We also prove that Γ_∞ is of class C^∞ without inflexion point, see Theorem 2.4-7.

The statement of our main result requires to introduce some new functions. For that purpose we use the Crocco equation (1.4). In addition to its explicit solution $u_*(s) = \frac{2}{\sqrt{3}}(-s)^{3/2}$, two other solutions, denoted by u_- and u_+ , will play an important role in our study. The solution u_- is the unique solution of (1.4) with boundary conditions $u_-(0^-) = 0$, $u'_-(0^-) = -1$. Here and in the sequel, the notation $\varphi(0^-)$, resp. $\varphi(0^+)$, stands for the limit of $\varphi(s)$ as $s \rightarrow 0$, $s < 0$, resp. $s > 0$. The maximal interval of definition of u_- is $(-\infty, 0)$; see Figure 3. The solution u_+ is the unique solution of (1.4) with initial conditions $u_+(0^+) = 0$, $u'_+(0^+) = 1$. The maximal interval of definition of u_+ is $(0, \lambda_+)$; numerical computations give $\lambda_+ \approx 1.303918$; see Figure 3.

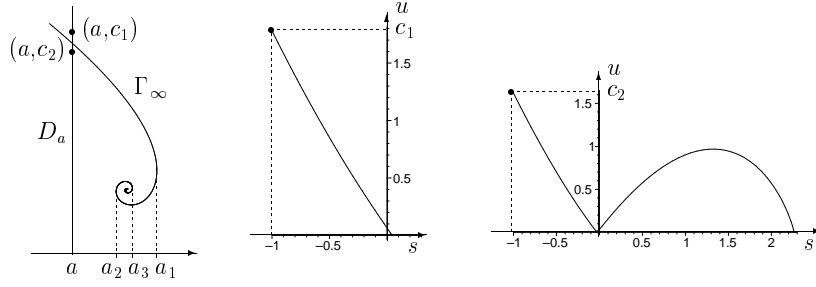


FIGURE 2. A sketch of the spiral Γ_∞ and two numerical Crocco solutions with initial conditions $u_1(-1) = c_1, u_1'(-1) = a$ and $u_2(-1) = c_2, u_2'(-1) = a$, where (a, c_1) and (a, c_2) are on the convex and on the concave sides of Γ_∞ respectively. The numerical values are $a = -2, c_1 = 1.78$ and $c_2 = 1.62$. The sequence (a_n) and the ray D_a in the sketch on the left are defined in Proposition 1.1 and in (1.9).

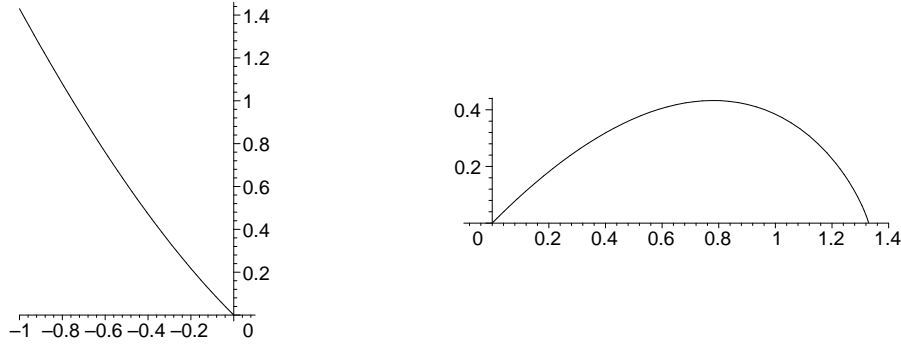


FIGURE 3. The graphs of u_- on the left and of u_+ on the right.

We now give more details about the discontinuity of $\tilde{\Lambda}$ on Γ_∞ . First, we have the following parametrization of Γ_∞

$$\Gamma_\infty = \left\{ \left((-s)^{-1/2} u'_-(s), (-s)^{-3/2} u_-(s) \right) ; s < 0 \right\}.$$

Secondly, the discontinuity of $\tilde{\Lambda}$ at a point $((-s)^{-1/2} u'_-(s), (-s)^{-3/2} u_-(s))$ on Γ_∞ is as follows (see Theorem 2.5): on the convex side of Γ_∞ , $\tilde{\Lambda}$ tends to 0, whereas on the concave side, $\tilde{\Lambda}$ tends to $-\frac{\lambda_+}{s}$. It follows that for all $\lambda > 0$ there is a unique point on Γ_∞ , namely with $s = -\frac{\lambda_+}{\lambda}$, where $\tilde{\Lambda}$ takes values respectively 0 and λ on each side of Γ_∞ . Let $A(\lambda)$ denote the abscissa of this point. In other words, we have

$$A : (0, +\infty) \rightarrow (-\infty, 0), \quad \lambda \mapsto \sqrt{\frac{\lambda}{\lambda_+}} u'_- \left(-\frac{\lambda_+}{\lambda} \right). \quad (1.6)$$

See Figure 4 for a numerical graph of A and Figure 7 for a sketch showing the oscillations near $\lambda = 0$. This function A satisfies the following properties.

Proposition 1.1. 1. *The function A is C^∞ and has an infinite sequence of extremal points $(\lambda_n)_{n \geq 1}$ decreasing to 0: local minima at λ_{2n} and local maxima at λ_{2n+1} .*

2. *Let $a_n = A(\lambda_n)$ denote these extremal values. Sequences (a_{2n}) and (a_{2n+1}) are adjacent, i.e., (a_{2n}) increases, (a_{2n+1}) decreases and they have the same limit.*

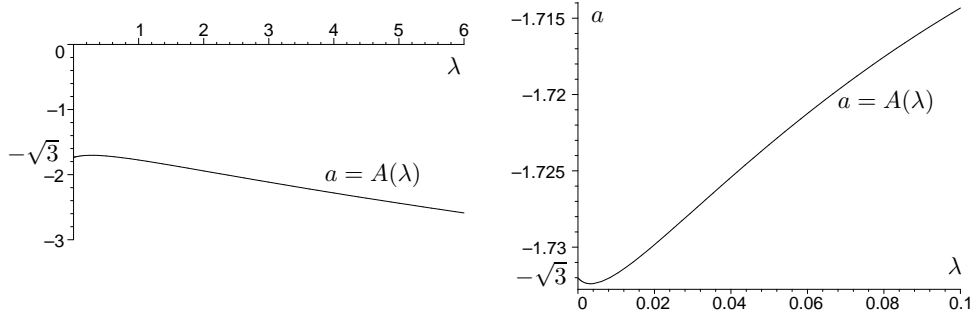


FIGURE 4. On the left: the graph of A . On the right: enlargement near $(0, -\sqrt{3})$.

3. The asymptotic behavior of A is described as follows:

$$A(\lambda) \sim -\sqrt{\frac{\lambda}{\lambda_+}} \quad \text{as } \lambda \rightarrow +\infty,$$

and there exist $\alpha, \beta \in \mathbb{R}$ such that

$$A(\lambda) = -\sqrt{3} + \lambda \left(\alpha \cos \frac{\ln \lambda}{\sqrt{2}} + \beta \sin \frac{\ln \lambda}{\sqrt{2}} + o(1) \right) \quad \text{as } \lambda \rightarrow 0. \quad (1.7)$$

The proof is given in Section 2.4. Formula (1.7) is illustrated in Figure 5. As a consequence of (1.7), the common limit of (a_{2n}) and (a_{2n+1}) is $a_\infty := -\sqrt{3}$, and sequences (λ_n) and $(a_n + \sqrt{3})$ are asymptotically geometric:

$$\lim_{n \rightarrow +\infty} \frac{\lambda_{n+1}}{\lambda_n} = e^{-\pi\sqrt{2}}, \quad \lim_{n \rightarrow +\infty} \frac{a_{n+1} + \sqrt{3}}{a_n + \sqrt{3}} = -e^{-\pi\sqrt{2}}. \quad (1.8)$$

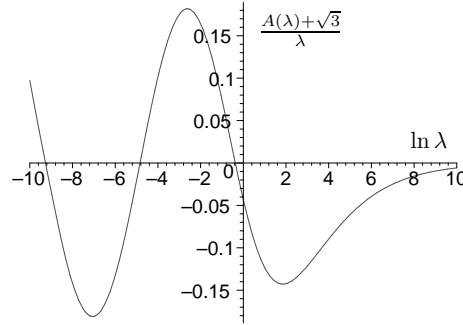


FIGURE 5. The graph of $\frac{A(\lambda)+\sqrt{3}}{\lambda}$ as a function of $\ln \lambda$, illustrating the asymptotic formula (1.7).

Given $a \in \mathbb{R}$, $\lambda > 0$ and $b = -1$, counting the number of solutions of the Blasius Problem (1.1 - 1.2) amounts to counting the number of times the function $\tilde{\Lambda}$ takes the value λ on a vertical ray

$$D_a := \{a\} \times (0, +\infty). \quad (1.9)$$

For that purpose, we introduce the function

$$\tilde{\Lambda}_a : (0, +\infty) \rightarrow [0, +\infty), \quad c \mapsto \tilde{\Lambda}(a, c) = \Lambda(a, -1, c).$$

Figure 6 shows some graphs of $\tilde{\Lambda}_a$ for miscellaneous values of a .

The description below is succinct. We refer to Section 2.4 for proofs, additional details and explanatory figures.

Let $n \geq 1$ be such that a is between a_{n-2} and a_n , possibly $a = a_n$ (with the convention $a_{-1} = +\infty$, $a_0 = -\infty$). Then, the ray D_a crosses $n - 1$ times the spiral Γ_∞ (if $a = a_n$, there is an n -th point of contact but without crossing, hence without creating any discontinuity for $\tilde{\Lambda}_a$).

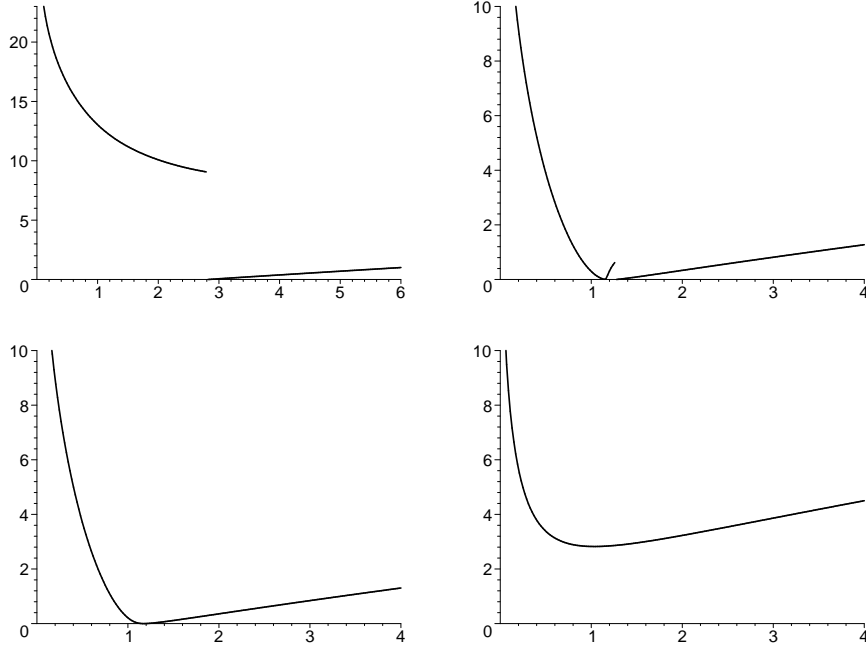


FIGURE 6. Numerical graphs of $\tilde{\Lambda}_a$. Top: on the left $a = -3$, on the right $a = -1.7324 \approx a_2$. Bottom: on the left $a = -1.7027 \approx a_1$, on the right $a = 0$.

First, assume that n is even. As we will see in Lemmata 2.10 and 2.12, the graph of $\tilde{\Lambda}_a$ consists of n pieces: $\frac{n}{2} - 1$ on the left, one “central” and $\frac{n}{2}$ on the right. On the central part, it turns out that, if a is close to a_n , then $\tilde{\Lambda}_a$ has a minimum close to 0. Therefore, we consider $d_n \in [a_{n-2}, a_n)$ as close to a_{n-2} as possible such that, for any $a \in (d_n, a_n]$, $\tilde{\Lambda}_a$ attains its infimum on this central part, at some (possibly non-unique) abscissa $c = C_n(a)$. Let $\mu_n \in (\lambda_{n-1}, \lambda_{n-2}]$ (with $\lambda_1 < \mu_2 \leq +\infty$) be such that $d_n = A(\mu_n)$. For $a \in (d_n, a_n]$, we define $\Lambda_n(a)$ as the minimum of $\tilde{\Lambda}_a$ on the central part. This yields a continuous map $\Lambda_n : (d_n, a_n] \rightarrow [0, \mu_n)$, satisfying $\Lambda_n(a_n) = 0$ and $\Lambda_n(a) \rightarrow \mu_n$ as $a \rightarrow d_n^+$.

If n is odd (Lemmata 2.9 and 2.11), then the graph of $\tilde{\Lambda}_a$ still consists of n pieces: $\frac{n-1}{2}$ on the left, one central and $\frac{n-1}{2}$ on the right. In the same manner, we consider $d_n \in (a_n, a_{n-2}]$ as close to a_{n-2} as possible such that, for any $a \in [a_n, d_n)$, $\tilde{\Lambda}_a$ attains its infimum on its central part. In the case $n = 1$, we choose $d_1 = +\infty$. For $n \geq 3$, let $\mu_n \in (\lambda_{n-1}, \lambda_{n-2}]$ be such that $d_n = A(\mu_n)$. We define $\Lambda_n(a)$ as the minimum of $\tilde{\Lambda}_a$ on its central part for $a \in [a_n, d_n)$. With the convention $\mu_1 = +\infty$, this yields a continuous map $\Lambda_n : [a_n, d_n) \rightarrow [0, \mu_n)$, still satisfying $\Lambda_n(a_n) = 0$ and $\Lambda_n(a) \rightarrow \mu_n$ as $a \rightarrow d_n^-$. See Figure 7 for sketches of graphs of these Λ_n . We now present our main result.

Theorem 1.2. Consider the Blasius Problem (1.1-1.2) in the convex case $b < \lambda$, and in the case $b = -1$, i.e., the boundary value problem

$$f''' + ff'' = 0, \quad f(0) = -a, \quad f'(0) = -1, \quad f'(+\infty) = \lambda > -1. \quad (1.10)$$

This problem has no solution if and only if:

$$\text{either } -1 < \lambda < 0 \quad \text{or} \quad a > a_1 \quad \text{and} \quad 0 \leq \lambda < \Lambda_1(a).$$

Let $n \in \mathbb{N} \setminus \{0\}$. Problem (1.10) has at least n solutions if (λ, a) belongs to one of the regions marked n in Figure 7 right, in other words, if:

- either $a = A(\lambda)$ with $\mu_{n+1} \leq \lambda < \mu_n$,
- or $\lambda = \Lambda_n(a)$ with $a \in [a_n, d_n)$, if n is odd, $a \in (d_n, a_n]$ if n is even,
- or (λ, a) is in the region below the graphs of Λ_2 and A in the case $n = 1$, and in the region between the graphs of $\Lambda_{n-1}, \Lambda_{n+1}$ and A in the case $n \geq 2$.

If $\lambda = 0$ and $a = -\sqrt{3}$, then Problem (1.10) has infinitely many solutions.

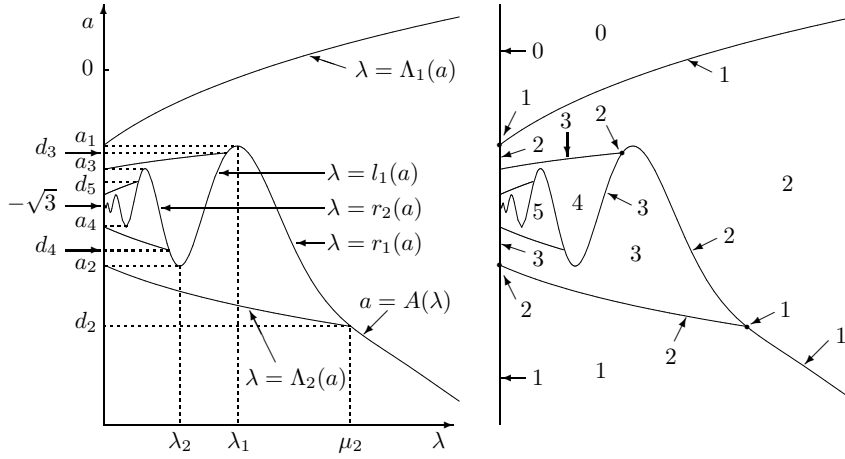


FIGURE 7. In the (λ, a) plane, a lower bound of the number of solutions of (1.10). On the left, a sketch of the graphs of the functions A and Λ_n , on the right the conjectured number of solutions of (1.10). We stress that the distances are not respected: due to (1.8) with $e^{\pi\sqrt{2}} \approx 85$, on the true graph of A no more than one extremal point is visible, see Figure 4.

The proof is given in Section 2.4. In the affine case $\lambda = -1$, there is a unique solution $f : t \mapsto -t - a$. By Corollary 8.6 there is no solution in the case $\lambda < -1$. At the end of Section 2.4, we will also comment the following conjecture.

Conjecture 1.3. The lower bounds above are sharp. In other words, the expression “at least” in Theorem 1.2 can be replaced by “exactly”.

In addition, we formulate two other conjectures. The first one is motivated by numerical experiments.

Conjecture 1.4. For all $n \in \mathbb{N}$ we have $d_{n+2} \neq a_n$; in particular, the constants d_2 and μ_2 are finite. For any $n \geq 2$ the function Λ_n is monotonous, increasing if n is odd, decreasing if n is even.

Conjecture 1.5. The function $\tilde{\Lambda}$ is of class C^∞ outside $\Gamma_\infty \cup \{S^*\}$ and its minima Λ_n are non-degenerate, i.e., $\frac{\partial^2 \tilde{\Lambda}}{\partial c^2}(a, C_n(a)) \neq 0$, so that the functions C_n and Λ_n are C^∞ , too.

We will see (statement 2 of Lemma 2.9) that Λ_1 is increasing at least on $[0, +\infty)$. Observe, also, that our Conjecture 1.5 would already imply that $d_{n+2} \neq a_n$ for all $n \geq 1$. Indeed, if $\tilde{\Lambda}$ is of class C^1 and a is between a_n and a_{n+2} , a close to a_n , then a computation shows that the central part of $\tilde{\Lambda}_a$ would be C^1 -close to a small segment of slope $\frac{\lambda_n a_n}{a_n^2 - 3}$, thus monotonous. See Figures 12, 13 and 14 for an illustration.

The proofs of Proposition 1.1 and Theorem 1.2, given in Section 2.4, need several intermediate results stated in Sections 2.1 to 2.3 whose proofs are postponed to Sections 3 to 5. We first use the Crocco change of variables, then another change of variables leads to an autonomous planar vector field. In Section 3, are collected results for the Blasius Problem in the already known cases, as well as some useful preliminaries. Section 4 is devoted to a thorough study of the vector field and Section 5 deals with some further deeper results on the Crocco equation (1.4). In Section 6, we present an alternative proof of these results, using properties of *canard solutions* of a related singularly perturbed differential system in \mathbb{R}^3 . The precise link between the Blasius and Crocco problem is explained in Section 7. In Section 8, we treat the already known concave case for the sake of completeness, and also because the use of the Crocco equation provides new shorter proofs. In Section 9, we conclude with additional results, alternative proofs and historical comments. These comments are not exhaustive. Actually, Blasius equation gave rise to a huge number of publications. Some of them treat more general equations, some others are incomplete or contain only numerical results.

Acknowledgements. The authors wish to thank Prof. Reinhard Schäfke for numerous and fruitful discussions about this work.

2. OUR STRATEGY OF PROOF

2.1. The Crocco change of variables. A basic property of the Blasius equation is that, besides affine solutions ($f'' = 0$), all other solutions f are such that f'' does not vanish. Indeed, (1.1) yields

$$f''(t) = f''(t_0) \exp \left\{ - \int_{t_0}^t f(\tau) d\tau \right\}, \quad (2.1)$$

which shows that f'' cannot vanish without being identically zero. It follows that $t \mapsto f'(t)$ is a diffeomorphism for non-affine solutions.

The Crocco change of variable consists of expressing f'' as a function of f' : if we put $u = f'' \circ (f')^{-1}$, then differentiating $u(f') = f''$ (the variable t is omitted for simplicity) we obtain $u'(f')f' = f''' = -f f''$ hence, $u'(f') = -f$. Differentiating once again we obtain $u''(f')f'' = -f'$, *i.e.*, equation (1.4), rewritten below for the reader's convenience (with the independent variable f' denoted by s)

$$u'' = -\frac{s}{u}. \quad (2.2)$$

We stress that, by construction, a Crocco solution cannot vanish. Actually we had to divide twice by $f'' = u$ to obtain (2.2). In Section 9.1, we study solutions of the almost equivalent equation $uu'' + s = 0$ that vanish somewhere.

Notice that any positive solution of (2.2) on an interval I is convex on $(-\infty, 0] \cap I$ and concave on $[0, +\infty) \cap I$.

The Crocco change of variable yields the following characterization of Λ , see Section 3.1 for the proof.

Proposition 2.1. *For all $a, b \in \mathbb{R}$ and $c > 0$, $[b, \Lambda(a, b, c))$ is the maximal right interval of existence of the solution $u := u(\cdot; a, b, c)$ of (2.2) with initial conditions*

$$u(b) = c, \quad u'(b) = a. \quad (2.3)$$

Moreover, $u(s)$ tends to 0 as $s \rightarrow \Lambda(a, b, c)$, $s < \Lambda(a, b, c)$. If $\Lambda(a, b, c) > 0$, then we also have $u'(s) \rightarrow -\infty$ as $s \rightarrow \Lambda(a, b, c)$, $s < \Lambda(a, b, c)$.

This means that, in terms of Crocco equation, boundary conditions (1.2) become

$$u'(b) = a, \quad \lim_{s \rightarrow \lambda} u(s) = 0.$$

The similarity property (1.3) implies

$$\text{if } u \text{ is a solution of (2.2), so is } u^\sigma : s \mapsto \sigma^3 u(\sigma^{-2}s) \text{ for all } \sigma \neq 0, \quad (2.4)$$

which can be rewritten as

$$u(\sigma^2 s; \sigma a, \sigma^2 b, \sigma^3 c) = \sigma^3 u(s; a, b, c). \quad (2.5)$$

It is easy to check that only one positive solution of (2.2), denoted by u_* , is self-similar by (2.4), i.e., $u_*(s) = \sigma^3 u_*(\sigma^{-2}s)$ for all s, σ , $s < 0 < \sigma$, namely

$$u_*(s) = \frac{2}{\sqrt{3}}(-s)^{3/2}. \quad (2.6)$$

We will see that u_* is also the unique solution of (2.2) on $(-\infty, 0)$ such that $u_*(0^-) = u_*'(0^-) = 0$, see item 4 (1) of Theorem 2.4, and Section 9.2.

We now present the main result of this section. The difficult and important statement 4 is the key to analyze the discontinuity of Λ , as described below in the second part of Theorem 2.5.

Theorem 2.2. 1. *Every solution of (2.2-2.3) with $b < 0$ and $c > 0$ is defined at least on $(-\infty, 0)$ and is asymptotic to u_* as $s \rightarrow -\infty$.*

2. *There is a unique solution of (2.2) on $(-\infty, 0)$, denoted by u_- , satisfying*

$$u_-(0^-) = 0, \quad u_-'(0^-) = -1. \quad (2.7)$$

Similarly, for any $\sigma > 0$ there is a unique solution of (2.2) on $(-\infty, 0)$ with $u(0^-) = 0$, $u'(0^-) = -\sigma$, namely $s \mapsto \sigma^3 u_-(\sigma^{-2}s)$.

3. *There is a unique solution u_+ of (2.2) satisfying*

$$u_+(0^+) = 0, \quad u_+'(0^+) = 1, \quad (2.8)$$

defined on some maximal interval $(0, \lambda_+)$, with $1 < \lambda_+ < s_0$, where $s_0 \approx 1.43$ is the positive root of equation $(2s - 4) \ln(1 - \frac{s}{2}) = s$.

4. *For every sequence $((\alpha_n, \gamma_n))_{n \in \mathbb{N}}$ which tends to $(u_-(-1), u_-(-1))$, the sequence $(u'(0^-; \alpha_n, -1, \gamma_n))_{n \in \mathbb{N}}$ is bounded and has at most two cluster points: 1 and -1 .*

Statements 1 and 2 are proved in Section 5.3, statement 3 is proved in 3.4, and statement 4 in 5.2. Besides, we have the following asymptotic formulae for Crocco solutions starting close to $u = 0$ at $s = -1$.

Proposition 2.3. *For any $a \in \mathbb{R}$ fixed, we have*

$$u'(s; a, -1, c) \sim \sqrt{2 \ln \frac{1}{c}} \operatorname{sgn}(s + 1) \quad \text{as } c \rightarrow 0^+, \quad (2.9)$$

$$u(s; a, -1, c) \sim \sqrt{2 \ln \frac{1}{c}} |s + 1| \quad \text{as } c \rightarrow 0^+, \quad (2.10)$$

uniformly for s in any compact subset of $(-\infty, -1) \cup (-1, 0]$.

The proof is in Section 5.1.

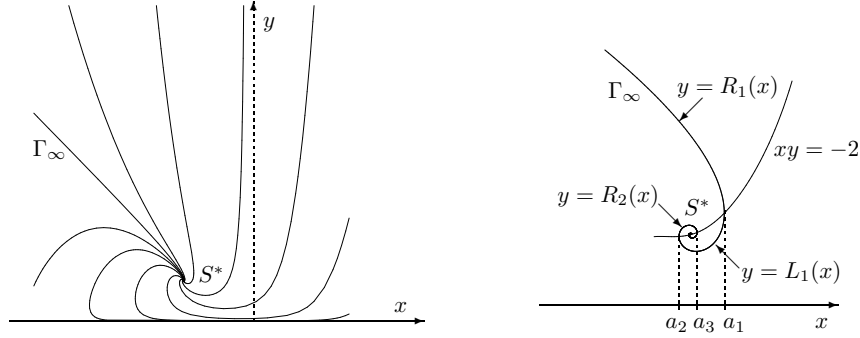


FIGURE 8. On the left: the phase portrait of (2.12). On the right: sketch of enlargement of Γ_∞ near S^* . The functions L_n and R_n are defined in Section 2.4.

2.2. The associated vector field. The similarity property (2.4) allows one to reduce the Crocco equation to a system of autonomous differential equations. The change of variables

$$x(t) = e^{t/2}u'(-e^{-t}), \quad y(t) = e^{3t/2}u(-e^{-t}), \quad (2.11)$$

leads to the system

$$\dot{x} = \frac{1}{2}x + \frac{1}{y}, \quad \dot{y} = x + \frac{3}{2}y. \quad (2.12)$$

The phase portrait of this system is depicted in Figure 8. Since this system is invariant by the change $(x, y) \mapsto (-x, -y)$, we will consider it only for $y > 0$; this corresponds to positive Crocco solutions and to convex Blasius solutions. The initial conditions (2.3) with $b = -1$ correspond to

$$x(0) = a, \quad y(0) = c. \quad (2.13)$$

Notice that this vector field describes the Crocco equation (2.2) only for $s < 0$. In Section 9.3, we introduce an analogous vector field for $s > 0$.

Because the transformation $u \mapsto u^\sigma$ given by (2.4) (for $\sigma > 0$) corresponds to a shift by $t \mapsto t - \ln \sigma$ in (2.11), to each orbit $\{(x(t), y(t)); t \in \mathbb{R}\}$ of some solution of (2.12) corresponds a whole family $(u^\sigma)_{\sigma > 0}$ of solutions of (2.2) connected by the similarity (2.4).

In particular, the unique stationary point $S^* = (-\sqrt{3}, \frac{2}{\sqrt{3}})$ corresponds to u_* given by (2.6), which is the unique positive Crocco solution invariant by (2.4). To the solution u_- given by (2.7) corresponds a solution of (2.12), denoted by (x_-, y_-) . Remark that, with $s = -e^{-t}$, we have

$$\frac{x_-(t)}{y_-(t)} = \frac{-su'_-(s)}{u_-(s)} \rightarrow -1 \text{ as } t \rightarrow +\infty,$$

hence (x_-, y_-) parametrizes the orbit Γ_∞ of Theorem 2.4, item 3 below. In Section 4, we will prove the following result.

Theorem 2.4. 1. All solutions of (2.12) are defined on \mathbb{R} and tend to S^* as $t \rightarrow -\infty$.

2. More precisely, for any solution (x, y) of (2.12) there exist $A, B \in \mathbb{R}$ such that

$$x(t) = -\sqrt{3} + e^t \left(A \cos \frac{t}{\sqrt{2}} + B \sin \frac{t}{\sqrt{2}} + o(1) \right) \text{ as } t \rightarrow -\infty. \quad (2.14)$$

3. There is one and only one orbit, denoted by Γ_∞ , with the property that any solution (x, y) parametrizing Γ_∞ is such that $\frac{x(t)}{y(t)}$ tends to -1 as $t \rightarrow +\infty$.

There is one and only one orbit, denoted by Γ_0 , with the property that any solution (x, y) parametrizing Γ_0 is such that $x(t)$ tends to 0 as $t \rightarrow +\infty$.

For all non-constant solutions (x, y) , $y(t)$ tends to $+\infty$ as $t \rightarrow +\infty$.

For all solutions (x, y) , except those on $\Gamma_\infty \cup \{S^*\}$, $\frac{x(t)}{y(t)}$ tends to 0 as $t \rightarrow +\infty$. In particular, all solutions (x, y) , except those on $\Gamma_\infty \cup \{S^*\}$, eventually leave the region $x + y \leq 0$.

4. In terms of positive Crocco solutions, this means that we have the following equivalences

- (1) $(a, c) = S^*$ if and only if $u(0^-; a, -1, c) = 0$ and $u'(0^-; a, -1, c) = 0$,
- (2) $(a, c) \in \Gamma_\infty$ if and only if $u(0^-; a, -1, c) = 0$ and $u'(0^-; a, -1, c) < 0$,
- (3) $(a, c) \notin \Gamma_\infty \cup \{S^*\}$ if and only if $u(0; a, -1, c) > 0$.

5. For all solutions (x, y) except those on Γ_∞ , $\frac{x(t)^3}{y(t)}$ has a limit $k \in \mathbb{R}$ as $t \rightarrow +\infty$.

6. Conversely, for any $k \in \mathbb{R} \setminus \{-\frac{9}{2}\}$ there is one and only one orbit, denoted by Γ_k , such that for all solutions (x, y) that parametrize Γ_k , we have

$$\lim_{t \rightarrow +\infty} \frac{x(t)^3}{y(t)} = k.$$

For $k = -\frac{9}{2}$ there are two orbits: S^* and a second one denoted by $\Gamma_{-9/2}$.

7. If $k \geq 0$ or $k = \infty$, then Γ_k has no inflexion point; if $k < 0$, then Γ_k has exactly one inflexion point.

2.3. Properties of the function Λ . In this section, we state the properties of Λ needed for the proof of Theorem 1.2 and for arguments justifying our Conjecture 1.3. First, the similarity property (1.3) implies

$$\forall \sigma > 0, \quad \Lambda(\sigma a, \sigma^2 b, \sigma^3 c) = \sigma^2 \Lambda(a, b, c). \quad (2.15)$$

Therefore, as far as possible, properties of Λ for $b < 0$ will be stated below using the function $\tilde{\Lambda} : (a, c) \mapsto \Lambda(a, -1, c)$, as said in the introduction.

Notice that, for any positive solution u of (2.2) defined on some interval I , we have

$$\forall s \in I, \quad \tilde{\Lambda}(u'(-1), u(-1)) = \Lambda(u'(-1), -1, u(-1)) = \Lambda(u'(s), s, u(s)).$$

As a consequence, (2.15) gives

$$\forall s < 0, \quad \tilde{\Lambda}(u'(-1), u(-1)) = -s \tilde{\Lambda}((-s)^{-1/2} u'(s), (-s)^{-3/2} u(s)).$$

In terms of the associated vector field, we deduce that for all (x, y) solution of (2.12)

$$\forall t \in \mathbb{R}, \quad \tilde{\Lambda}(x(t), y(t)) = e^t \tilde{\Lambda}(x(0), y(0)). \quad (2.16)$$

This formula shows how the flow of (2.12) transforms a level curve of $\tilde{\Lambda}$ into another level curve of $\tilde{\Lambda}$.

Theorem 2.5. *The function $\tilde{\Lambda}$ is continuous on $(\mathbb{R} \times (0, +\infty)) \setminus \Gamma_\infty$. On Γ_∞ , the discontinuity of $\tilde{\Lambda}$ is described as follows. If (a, c) belongs to Γ_∞ , then for all sequences $((\alpha_n, \gamma_n))_{n \in \mathbb{N}}$ which tend to (a, c) on the convex side of Γ_∞ , the sequence $(\tilde{\Lambda}(\alpha_n, \gamma_n))_{n \in \mathbb{N}}$ tends to 0, whereas for all sequences $((\alpha_n, \gamma_n))_{n \in \mathbb{N}}$ which tend to (a, c) on the concave side of Γ_∞ , the sequence $(\tilde{\Lambda}(\alpha_n, \gamma_n))_{n \in \mathbb{N}}$ tends to $\lambda_+ e^t$, where $t \in \mathbb{R}$ is such that $a = x_-(t)$, $c = y_-(t)$.*

As a consequence, for all $\lambda > 0$ there is a unique point on Γ_∞ , with abscissa equal to $A(\lambda)$, such that $\tilde{\Lambda}$ jumps from 0 on the convex side of Γ_∞ , to λ on its concave side.

The proof is in Section 5.3. Besides this important regularity property we have some monotony properties.

Proposition 2.6. 1. For non-negative b , the functions $a \mapsto \Lambda(a, b, c)$ and $c \mapsto \Lambda(a, b, c)$ are increasing.

2. In the region above both Γ_∞ and the hyperbola $ac + 2 = 0$, the function $c \mapsto \tilde{\Lambda}(a, c)$ is increasing.

3. In the region on the right of both Γ_∞ and the straight line $2a + 3c = 0$, $a \mapsto \tilde{\Lambda}(a, c)$ is increasing.

Statement 1 is proved in Section 3.3; statements 2 and 3 are proved in 5.3. At last, we present asymptotic properties of $\tilde{\Lambda}$.

Proposition 2.7. For all $a \in \mathbb{R}$ fixed, we have

$$\tilde{\Lambda}(a, c) \sim 2\lambda_+ \ln \frac{1}{c} \quad \text{as } c \rightarrow 0, \quad (2.17)$$

$$\tilde{\Lambda}(a, c) \sim c^{2/3} \Lambda(0, 0, 1) \quad \text{as } c \rightarrow +\infty. \quad (2.18)$$

The proof is given in Section 5.3. Numerical computations give $\Lambda(0, 0, 1) \approx 1.655193$.

2.4. Back to the Blasius problem. In this section, we first prove Proposition 1.1. Then we write Γ_∞ as a union of graphs. Next, we describe the graph of $\tilde{\Lambda}_a$ in accordance with the relative position of a and a_n and we give details about the functions Λ_n . These descriptions are then used to prove Theorem 1.2. At last, we explain which elements lead us to state our Conjecture 1.3. Of course the results stated in the Sections 2.1 to 2.3 will be used throughout this section.

Proof of Proposition 1.1. As already announced in the introduction, the last item of Theorem 2.5 shows that, for all $\lambda \in (0, +\infty)$, there is a unique point $(x_-(t), y_-(t))$ on Γ_∞ , namely with $t = \ln\left(\frac{\lambda}{\lambda_+}\right)$, such that $\tilde{\Lambda}$ takes values respectively 0 and λ on each side of Γ_∞ . We recall that, by definition, the abscissa of this point is

$$x_- \left(\ln \left(\frac{\lambda}{\lambda_+} \right) \right) = A(\lambda). \quad (2.19)$$

Then, using (2.14) with $e^t = \frac{\lambda}{\lambda_+}$, we obtain (1.7). The asymptotic behavior as $t \rightarrow +\infty$ simply follows from (1.6) and from $\lim_{s \rightarrow 0^-} u'_-(s) = -1$. Due to (2.19), the n th extremum a_n of the function A , counted from the right is also the n th extremum of the function x_- with time reversed, *i.e.*, with t from $+\infty$ to $-\infty$, see Figure 8. Because Γ_∞ has no inflexion point, we have

$$a_2 < a_4 < a_6 < \dots < -\sqrt{3} < \dots < a_3 < a_1,$$

and items 1 and 2 follow. \square

We now describe the curve Γ_∞ as an union of graphs. Since Γ_∞ has no inflexion point and S^* is a focus, for all $n \geq 1$, with the convention $a_0 = -\infty$, there exist functions $L_n : [a_{2n}, a_{2n-1}] \rightarrow \mathbb{R}$, $R_n : [a_{2n-2}, a_{2n-1}] \rightarrow \mathbb{R}$ such that (see Figure 8 right for the graphs of R_1 , R_2 and L_1)

- Γ_∞ is the union of the graphs of the mappings $x \mapsto L_n(x)$ and $x \mapsto R_n(x)$,
- the functions L_n are convex and the functions R_n are concave,
- for all $n \geq 1$, we have

$$L_{n-1}(x) \leq L_n(x) \leq -\frac{2}{x} \leq R_n(x) \leq R_{n-1}(x), \quad (2.20)$$

where each inequality holds for all x such that both functions are defined, and is strict if x is not an end point a_n ,

- at the end points of the intervals we have

$$\begin{aligned} R_n(a_{2n-1}) &= L_n(a_{2n-1}) = -\frac{2}{a_{2n-1}}, \\ R_{n+1}(a_{2n}) &= L_n(a_{2n}) = -\frac{2}{a_{2n}}. \end{aligned} \quad (2.21)$$

Since the function $t \mapsto x(t)$ is decreasing under the hyperbola $xy = -2$ and increasing above this hyperbola, from (2.19) we deduce that the map $\lambda \mapsto A(\lambda)$ is increasing on each interval $[\lambda_{2n}, \lambda_{2n-1}]$ and decreasing on each $[\lambda_{2n-1}, \lambda_{2n-2}]$. Hence, for all $n \geq 1$, with the convention $\lambda_0 = +\infty$, there exist one-to-one mappings

$$l_n : [a_{2n}, a_{2n-1}] \rightarrow [\lambda_{2n}, \lambda_{2n-1}], \quad r_n : [a_{2n-2}, a_{2n-1}] \rightarrow [\lambda_{2n-1}, \lambda_{2n-2}],$$

such that the graph of $\lambda \mapsto A(\lambda)$ is the union of the graphs of $a \mapsto l_n(a)$ and $a \mapsto r_n(a)$, see Figure 7 left for the graphs of r_1 , r_2 and l_1 .

For all $n \geq 1$, we have

$$r_{n+1}(a) \leq l_n(a) \leq r_n(a), \quad (2.22)$$

where each inequality holds for all a where both functions are defined, and is strict except at the end points of the intervals. At the end points we have

$$l_n(a_{2n}) = r_{n+1}(a_{2n}) = \lambda_{2n}, \quad l_n(a_{2n-1}) = r_n(a_{2n-1}) = \lambda_{2n-1},$$

An immediate consequence of Theorem 2.5 is the following.

Lemma 2.8. *At a point $(a, R_n(a))$ on Γ_∞ , $\tilde{\Lambda}$ jumps from 0 on the convex side of Γ_∞ to $r_n(a)$ on the concave side. At a point $(a, L_n(a))$, $\tilde{\Lambda}$ jumps from 0 on the convex side to $l_n(a)$ on the concave side.*

Let us now describe the functions Λ_n . We recall that, roughly speaking, $\Lambda_n(a)$ is the minimum of $\tilde{\Lambda}_a$ on its central part, when this minimum is reached.

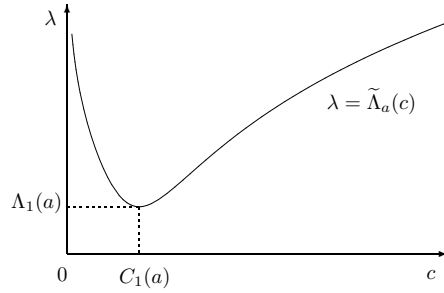


FIGURE 9. A schematic graph of $\tilde{\Lambda}_a : c \mapsto \Lambda(a, -1, c)$ in the case $a > a_1$. For convenience, explanatory figures 9, 10 and 11 are schematic. See Figure 6 and Figures 12 to 15 for numerical graphs; *e.g.* the present figure may be compared to Figure 6, bottom right.

Lemma 2.9. *Consider the case $a \geq a_1$.*

1. *The function $\tilde{\Lambda}_a$ is continuous on $(0, +\infty)$, tends to $+\infty$ as $c \rightarrow 0^+$ and as $c \rightarrow +\infty$. As a consequence, $\tilde{\Lambda}_a$ attains its minimum on $(0, +\infty)$, denoted by $\Lambda_1(a)$.*
2. *The function Λ_1 is continuous on $[a_1, +\infty)$ and increasing at least on $[0, +\infty)$.*
3. *We have $\Lambda_1(a_1) = 0$ and $\Lambda_1(a) \geq a^2 \lambda_+$ for all $a > 0$. In particular, $\Lambda_1(a)$ tends to $+\infty$ as $a \rightarrow +\infty$.*

Proof. 1. If $a > a_1$, then the ray D_a does not cross Γ_∞ and $\tilde{\Lambda}_a$ is continuous. If $a = a_1$, then the ray D_a touches Γ_∞ , without crossing it, at $(a_1, -\frac{2}{a_1})$, hence, without creating any discontinuity, and $\tilde{\Lambda}_a(-\frac{2}{a_1}) = 0$. Due to Proposition 2.7, $\tilde{\Lambda}_a(c)$ tends to $+\infty$ as $c \rightarrow 0^+$ and as $c \rightarrow +\infty$, hence $\tilde{\Lambda}_a$ has (at least) one global minimum $\Lambda_1(a)$, which is reached for some value $c = C_1(a)$, see Figure 9. The function Λ_1 is defined on $[a_1, +\infty)$ and satisfies $\Lambda_1(a_1) = 0$.

2. As for all fixed c the map $a \mapsto \tilde{\Lambda}(a, c)$ is continuous on $[a_1, +\infty)$, this is also the case for Λ_1 (even though C_1 may not be continuous). Furthermore, if $0 \leq a < a'$, then using item 3 of Proposition 2.6, we obtain

$$\tilde{\Lambda}_1(a) = \min_{c>0} \tilde{\Lambda}(a, c) \leq \tilde{\Lambda}(a, C_1(a')) < \tilde{\Lambda}(a', C_1(a')) = \min_{c>0} \tilde{\Lambda}(a', c) = \tilde{\Lambda}_1(a').$$

3. Set $u = u(\cdot; a, -1, C_1(a))$. Because u is convex on $[C_1(a), 0]$ and increasing, we have $u(0) \geq a$ and $u'(0) \geq a$. Therefore, by item 1 of Proposition 2.6, we obtain

$$\Lambda_1(a) = \Lambda(u'(0), 0, u(0)) \geq \Lambda(a, 0, a) = a^2 \Lambda(1, 0, a^{-2}) \geq a^2 \lambda_+. \quad \square$$

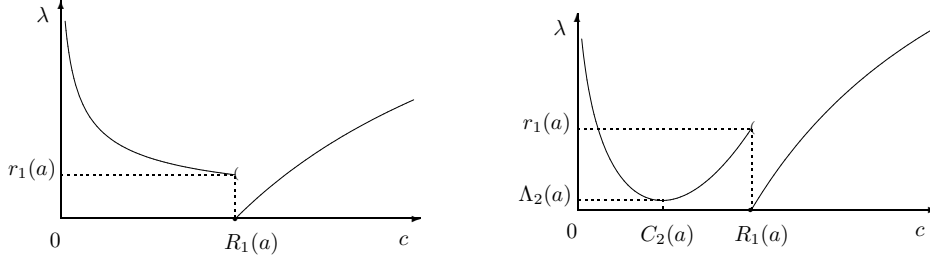


FIGURE 10. Sketches of graphs of $\tilde{\Lambda}_a$ in the case $a < a_2$. On the left: a close to $-\infty$, on the right: a close to a_2 . For numerical graphs, see the top of Figure 6.

Lemma 2.10. *Consider the case $a \leq a_2$.*

1. *The function $\tilde{\Lambda}_a$ has one discontinuity at $c = R_1(a)$.*
2. *As $c \rightarrow 0^+$, $\tilde{\Lambda}_a(c)$ tends to $+\infty$; as $c \rightarrow R_1(a)^-$, $\tilde{\Lambda}_a(c)$ tends to $r_1(a)$.*
3. *The function $\tilde{\Lambda}$ is increasing on $[R_1(a), +\infty)$ from $\tilde{\Lambda}_a(R_1(a)) = 0$ to $+\infty$.*
4. *There exists $d_2 \in [-\infty, a_2)$ such that*
 - *for all $a \in (d_2, a_2]$, $\tilde{\Lambda}_a$ reaches its minimum on $(0, R_1(a))$, denoted by $\Lambda_2(a)$,*
 - *for $a = d_2$ the infimum of $\tilde{\Lambda}_a$ on $(0, R_1(a))$ is equal to $\mu_2 := r_1(d_2)$ (with the convention $r_1(-\infty) = +\infty$).*
5. *The function $\Lambda_2 : (d_2, a_2] \rightarrow [0, \mu_2)$ is continuous.*

Proof. 1. If $a < a_2$, then the ray D_a crosses Γ_∞ once. By Lemma 2.8, $\tilde{\Lambda}_a$ has a discontinuity at $c = R_1(a)$ and jumps down from $r_1(a)$ to 0. If $a = a_2$, then the ray touches Γ_∞ also at $(a_2, -\frac{2}{a_2})$ but without discontinuity.

2. The property follows from (2.17) and item 1.

3. The property follows from (2.18) and Proposition 2.6.

4. If $a = a_2$, then $\tilde{\Lambda}_a$ has an isolated zero at $c = -\frac{2}{a_2}$. Thus, by continuity of $\tilde{\Lambda}$, for $a < a_2$, and a close to a_2 , $\tilde{\Lambda}_a$ has a local minimum close to 0 for some (possibly non-unique) value $c = C_2(a)$ close to $-\frac{2}{a_2}$ and $\Lambda_2(a) = \tilde{\Lambda}(C_2(a))$, see Figure 10. Now, let $d_2 \in [-\infty, a_2)$ be the infimum of those values d such that for all $a \in (d, a_2]$, $\tilde{\Lambda}_a$ reaches its minimum $\Lambda_2(a)$ on $(0, R_1(a))$. By construction, we have $\Lambda_2(d_2) = \mu_2 = r_1(d_2)$, see Figure 7.

5. As for all fixed c the map $a \mapsto \tilde{\Lambda}(a, c)$ is continuous on $[a_1, +\infty)$, this is also the case for Λ_2 . □

Lemma 2.11. *Consider the case $a \in [a_{2n+1}, a_{2n-1})$ with $n \geq 1$.*

1. The function $\tilde{\Lambda}_a$ has $2n$ discontinuities at $c = R_k(a)$ and $c = L_k(a)$, with $1 \leq k \leq n$, which satisfy

$$L_1(a) < \cdots < L_n(a) < -\frac{2}{a} < R_n(a) < \cdots < R_1(a).$$

2. As $c \rightarrow 0^+$, $\tilde{\Lambda}_a(c)$ tends to $+\infty$. On $[R_1(a), +\infty)$, $\tilde{\Lambda}_a$ is increasing from 0 to $+\infty$.
3. As $c \rightarrow R_k(a)^-$, $\tilde{\Lambda}_a(c)$ tends to $r_k(a)$; as $c \rightarrow L_k(a)^+$, $\tilde{\Lambda}_a(c)$ tends to $l_k(a)$.
4. We have $0 < l_n(a) < r_n(a) < \cdots < l_1(a) < r_1(a)$.
5. There exists $d_{2n+1} \in (a_{2n+1}, a_{2n-1}]$ such that
 - for all $a \in [a_{2n+1}, d_{2n+1})$, the minimum of $\tilde{\Lambda}_a$ on $(L_n(a), R_n(a))$ is reached and denoted by $\Lambda_{2n+1}(a)$, for some $c = C_{2n+1}(a)$ possibly non-unique,
 - for $a = d_{2n+1}$ the infimum of $\tilde{\Lambda}_a$ on $(L_n(a), R_n(a))$ is equal to $\mu_{2n+1} := l_n(d_{2n+1})$.
6. The function $\Lambda_{2n+1} : [a_{2n+1}, d_{2n+1}) \rightarrow [0, \mu_{2n+1})$ is continuous.

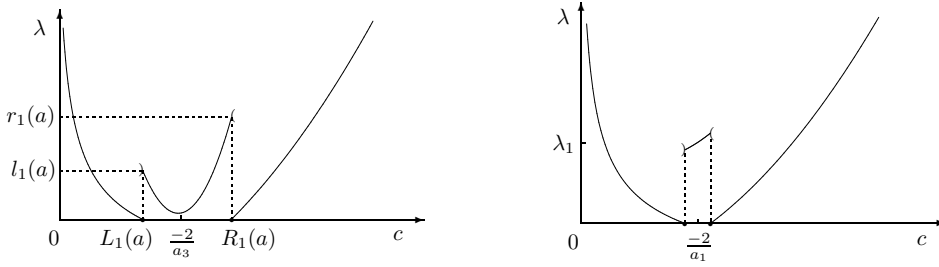


FIGURE 11. Sketches of graphs of $\tilde{\Lambda}_a$ in the case $a_3 \leq a < a_1$. On the left: a close to a_3 . On the right: a close to a_1 . Compare with Figure 14 top left and Figures 12 bottom, respectively.

Proof. 1. The ray D_a crosses Γ_∞ $2n$ times, hence $\tilde{\Lambda}_a$ has $2n$ discontinuities. The inequalities result from (2.20).

2. The property results from Propositions 2.6 and 2.7.

3. The property results from Lemma 2.8.

4. The property results from (2.22).

5. If $a = a_{2n+1}$, then we have $\tilde{\Lambda}_a(-\frac{2}{a_{2n+1}}) = 0$, but there is no discontinuity at this point.

Moreover, $\tilde{\Lambda}_a(c)$ tends to the value $r_n(a) > 0$ and $l_n(a) > 0$ as $c \rightarrow R_n(a)^-$ and $c \rightarrow L_n(a)^+$ respectively. Hence, for $a > a_{2n+1}$, a close to a_{2n+1} , $\tilde{\Lambda}_a$ reaches its minimum on $(L_n(a), R_n(a))$. This minimum $\Lambda_{2n+1}(a)$ is close to 0, attained for some $c = C_{2n+1}(a)$ (possibly non-unique, but necessarily close to $-\frac{2}{a_{2n+1}}$) see Figure 11 for the case $n = 1$. Let $d_{2n+1} \in (a_{2n+1}, a_{2n-1}]$, be the supremum of those values d such that for all $a \in [a_{2n+1}, d)$, $\Lambda_{2n+1}(a)$ achieves its minimum on $(L_n(a), R_n(a))$. By construction, $\Lambda_{2n+1}(a)$ tends to $\mu_{2n+1} = l_n(d_{2n+1})$ as $a \rightarrow d_{2n+1}$, see Figure 7.

6. As for all fixed c the map $a \mapsto \tilde{\Lambda}(a, c)$ is continuous on $[a_1, +\infty)$, this is also the case for Λ_{2n+1} . \square

Similarly, we have the following result, stated without proof; see Figure 13.

Lemma 2.12. Consider the case $a \in (a_{2n}, a_{2n+2}]$ with $n \geq 1$.

1. The function $\tilde{\Lambda}_a$ has $2n + 1$ discontinuities at $c = L_k(a)$, $1 \leq k \leq n$ and $c = R_k(a)$ $1 \leq k \leq n + 1$, such that

$$0 < L_1(a) < \cdots < L_n(a) < -\frac{2}{a} < R_{n+1}(a) < \cdots < R_1(a).$$

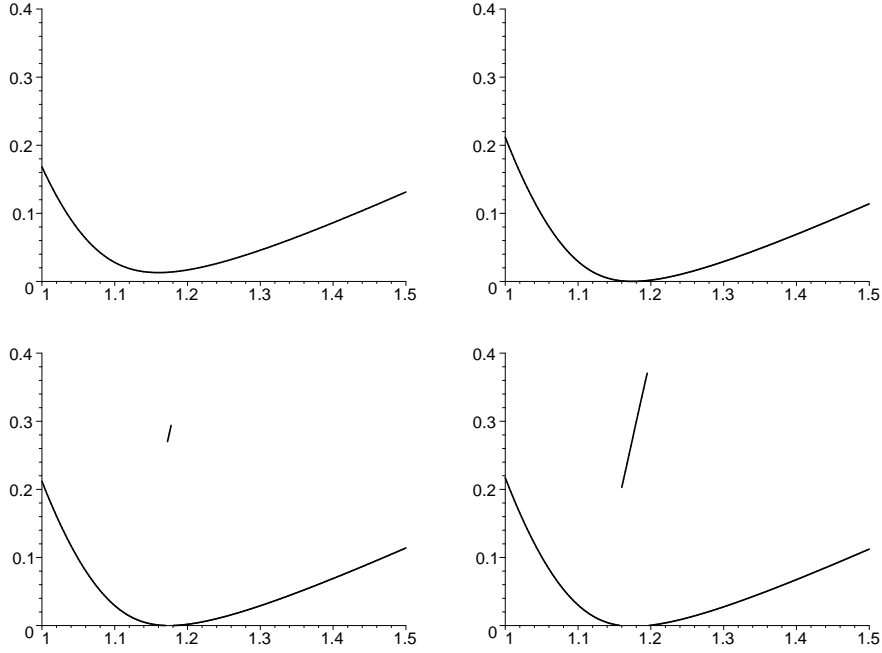


FIGURE 12. Scenario of bifurcation of the graph of $\tilde{\Lambda}_a$ near $a_1 \approx -1.702704$. Top: on the left $a = -1.68$, on the right $a = -1.7027$. Bottom: on the left $a = -1.7028$, on the right $a = -1.705$.

2. As $c \rightarrow 0^+$, $\tilde{\Lambda}_a(c)$ tends to $+\infty$. On $[R_1(a), +\infty)$, $\tilde{\Lambda}_a$ is increasing from 0 to $+\infty$.

3. At each $c = R_k(a)$ the function $\tilde{\Lambda}_a$ jumps from 0 to $l_k(a)$ and at each $c = L_k(a)$ it jumps down from $r_k(a)$ to 0, where l_k and r_k satisfy

$$0 < r_{n+1}(a) < l_n(a) < r_n(a) < \dots < l_1(a) < r_1(a).$$

4. Let $d_{2n+2} \in (a_{2n}, a_{2n+2}]$ be the infimum of d such that for all $a \in (d, a_{2n+2}]$, $\Lambda_{2n+1}(a)$ achieves its minimum on $(L_n(a), R_{n+1}(a))$, denoted by $\Lambda_{2n+2}(a)$.

This yields a continuous function $\Lambda_{2n+2} : (d_{2n+2}, a_{2n+2}] \rightarrow [0, \mu_{2n+2})$ such that $\Lambda_{2n+2}(a)$ tends to μ_{2n+2} as $a \rightarrow d_{2n+2}$, with $\mu_{2n+2} := r_{n+1}(d_{2n+2})$.

In the case $a = -\sqrt{3}$, the ray D_a crosses Γ_∞ infinitely times, and we have the following result, once again stated without proof; see Figure 15.

Lemma 2.13. Consider the case $a = -\sqrt{3}$. The function $\tilde{\Lambda}_a$ has infinitely many discontinuities at $c = L_n(a)$ and $c = R_n(a)$ with $n \geq 1$, such that

$$0 < L_1(a) < \dots < L_n(a) < \dots < \frac{2}{\sqrt{3}} < \dots < R_n(a) < \dots < R_1(a).$$

As $c \rightarrow 0^+$, $\tilde{\Lambda}_a(c)$ tends to $+\infty$. On $[R_1(a), +\infty)$, $\tilde{\Lambda}_a$ is increasing from 0 to $+\infty$. At each $c = R_n(a)$ the function $\tilde{\Lambda}_a$ jumps from 0 to $l_n(a)$ and at each $c = L_n(a)$ it jumps down from $r_n(a)$ to 0, where the sequences $(l_n)_{n \in \mathbb{N}}$ and $(r_n)_{n \in \mathbb{N}}$ satisfy

$$0 < \dots < l_n(a) < r_n(a) < \dots < l_1(a) < r_1(a).$$

From (2.21), we see that, as a increases and crosses the value a_{2n-1} , $R_n(a)$ and $L_n(a)$ collapse and two discontinuities disappear. This scenario of bifurcation is displayed on Figures 12 and

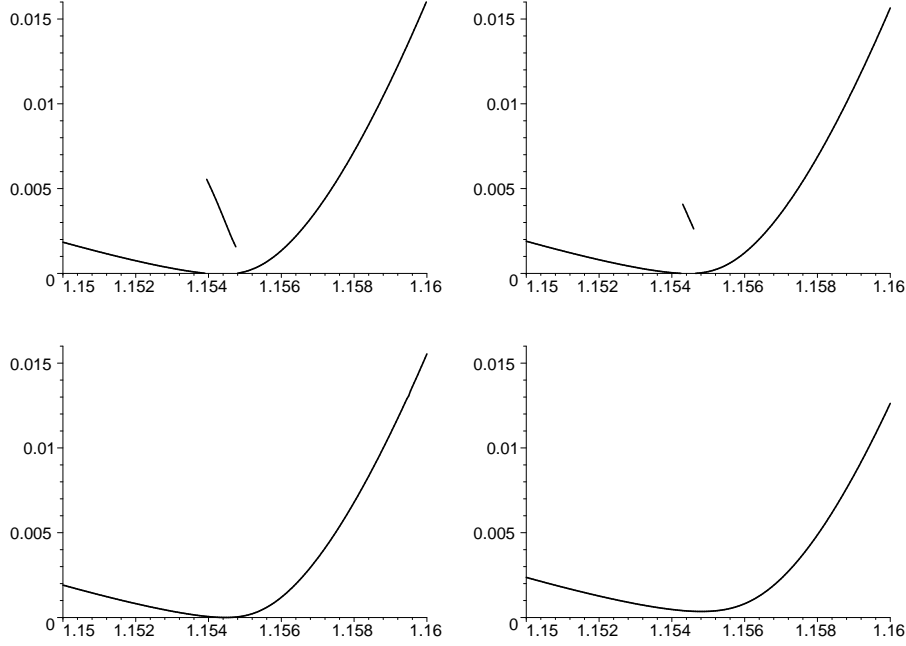


FIGURE 13. Scenario of bifurcation of the graph of $\tilde{\Lambda}_a$ near $a_2 \approx -1.7324$. Top: on the left $a = -1.7323$, on the right $a = -1.73238$. Bottom: on the left $a = -1.7324$, on the right $a = -1.733$.

14 for the cases $n = 1$ and $n = 2$. Similarly, as a decreases and crosses a_{2n} , $R_{n+1}(a)$ and $L_n(a)$ collapse and two discontinuities disappear. This scenario of bifurcation is displayed on Figure 13 for the case $n = 1$.

Proof of Theorem 1.2. If $-1 < \lambda < 0$, then there is no solution by Proposition 3.1. If $\lambda \geq 0$, we have to count the number of solutions of the equation $\tilde{\Lambda}_a(c) = \lambda$. A simple glance at the graph of $\tilde{\Lambda}_a$ shows the following.

The case $a \geq a_1$. From Lemma 2.9 we deduce that Problem (1.10) has no solution if and only if $0 \leq \lambda < \Lambda_1(a)$ and that this problem has at least one solution (satisfying $f''(0) = C_1(a)$) if $\lambda = \Lambda_1(a)$, and at least two solutions if $\lambda > \Lambda_1(a)$; see Figure 9.

The case $a \leq a_2$. From Lemma 2.10 we see that this case splits into two subcases. In the first case $d_2 < a \leq a_2$, we have the following:

- if $0 \leq \lambda < \Lambda_2(a)$, then Problem (1.10) has at least one solution;
- if $\lambda = \Lambda_2(a)$, then (1.10) has at least two solutions;
- if $\Lambda_2(a) < \lambda < r_1(a)$, then (1.10) has at least three solutions;
- if $r_1(a) \leq \lambda$, then (1.10) has at least two solutions; see Figure 10, right.

In the second case $a \leq d_2$:

- if $0 \leq \lambda \leq r_1(a)$, then (1.10) has at least one solution;
- if $r_1(a) < \lambda$, then (1.10) has at least two solutions, see Figure 10, left.

The case $a \in [a_{2n+1}, a_{2n-1})$, $n \geq 1$. From Lemma 2.10 we deduce that if $a_{2n+1} \leq a < d_{2n+1}$, then the number of solutions of Problem (1.10) is at least equal to

- $2n$, if $0 \leq \lambda < \Lambda_{2n+1}(a)$;
- $2n + 1$, if $\lambda = \Lambda_{2n+1}(a)$;

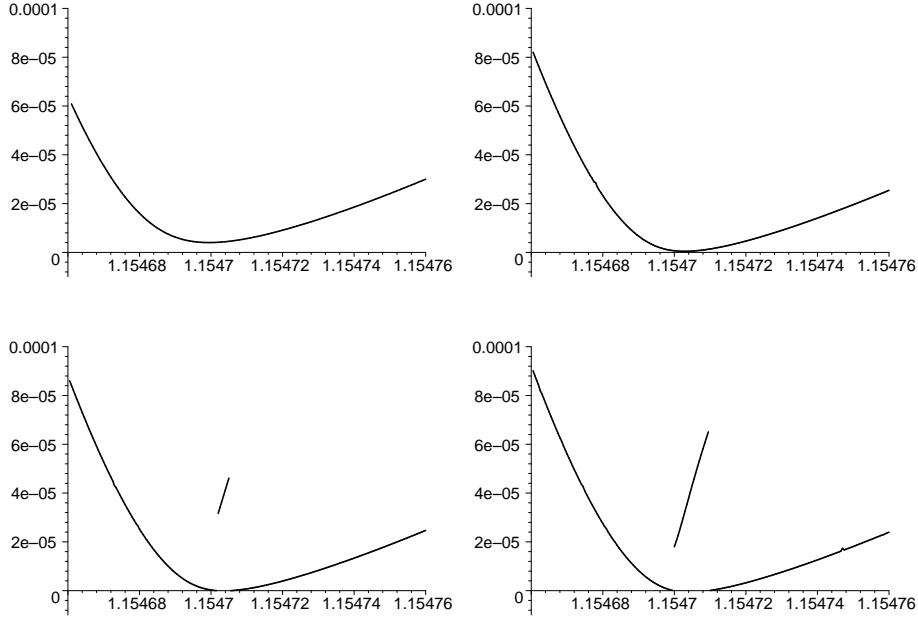


FIGURE 14. Scenario of bifurcation of the graph of $\tilde{\Lambda}_a$, near $a_3 \approx -1.7320463$. Top: on the left $a = -1.73204$, on the right $a = -1.732046$. Bottom: on the left $a = -1.732047$, on the right $a = -1.732048$.

- $2n + 2$, if $\Lambda_{2n+1}(a) < \lambda < l_n(a)$;
- $2n + 1$, if $l_n(a) \leq \lambda < r_n(a)$;
- $2k$, if $r_k(a) \leq \lambda < l_{n-1}(a)$, where $2 \geq k \geq n$;
- and 2 , if $r_1(a) \leq \lambda$; see Figure 11 left for the case $n = 1$.

If $d_{2n+1} \leq a < a_{2n-1}$, then the number of solutions of Problem (1.10) is at least equal to

- $2n$, if $0 \leq \lambda \leq l_n(a)$;
- $2n + 1$, if $l_n(a) \leq \lambda < r_n(a)$;
- $2k$, if $r_k(a) \leq \lambda < l_{n-1}(a)$, where $2 \geq k \geq n$;
- and 2 , if $r_1(a) \leq \lambda$; see Figure 11 right for the case $n = 1$.

The case $a \in (a_{2n}, a_{2n+2}]$, $n \geq 1$ is completely analogous to the former one.

The case $a = -\sqrt{3}$. By Lemma 2.13, Problem (1.10) admits as many solutions as possible for sufficiently small values of λ , and infinitely many solutions for $\lambda = 0$. \square

Comments on Conjecture 1.3. Some numerical experiments and a deeper study which are beyond the scope of the present article convinced us of the following assertion.

Conjecture 2.14. Let $n \geq 1$. If $d_{2n+1} \leq a < a_{2n-1}$, then the function $\tilde{\Lambda}_a$ is increasing on its central part $(L_n(a), R_n(a))$.

If $a_{2n+1} \leq a < d_{2n+1}$, then $\tilde{\Lambda}_a$ has a unique minimum on $(L_n(a), R_n(a))$ for $c = C_{2n+1}(a)$. Moreover, the function $\tilde{\Lambda}_a$ is decreasing on the interval $(L_n(a), C_{2n+1}(a)]$ and increasing on $[C_{2n+1}(a), R_n(a))$.

Similarly, if $a_{2n} < a \leq d_{2n+2}$, then $\tilde{\Lambda}_a$ is increasing on $(L_n(a), R_{n+1}(a))$ and if $d_{2n+2} < a \leq a_{2n+2}$, then $\tilde{\Lambda}_a$ has a unique minimum on $(L_n(a), R_{n+1}(a))$ for $c = C_{2n+2}(a)$, $\tilde{\Lambda}_a$ is decreasing on the interval $(L_n(a), C_{2n+2}(a)]$ and increasing on $[C_{2n+2}(a), R_{n+1}(a))$.

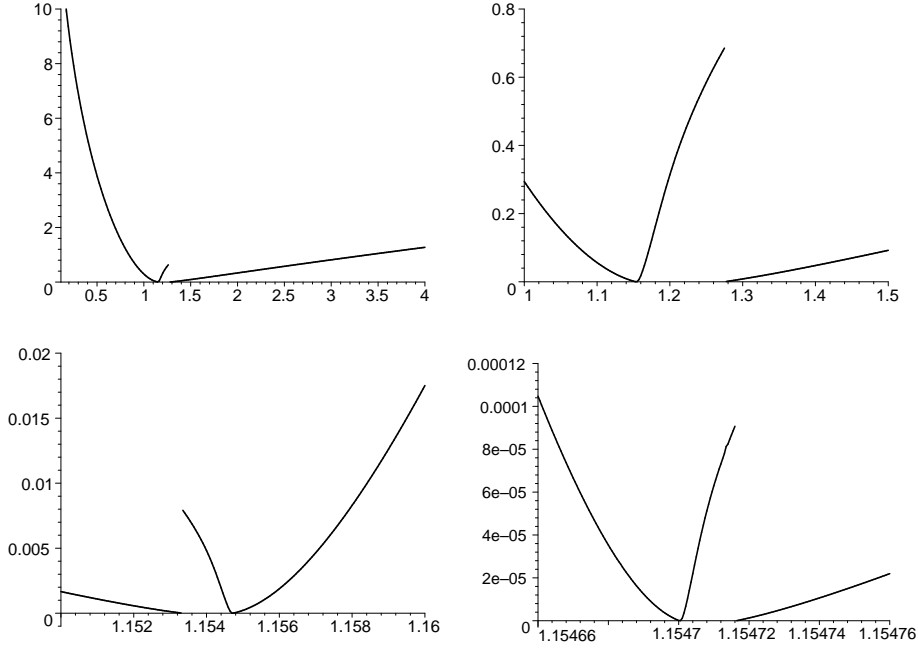


FIGURE 15. Numerical graph of $\tilde{\Lambda}_a$ for $a = -\sqrt{3}$, with successive enlargements.

For all $a \in \mathbb{R}$, on all parts of the graph of $\tilde{\Lambda}_a$, except the central one, the function $\tilde{\Lambda}_a$ is monotonous, decreasing on each part on the left of the central part, increasing on the right.

This would imply our Conjecture 1.3.

3. PRELIMINARY RESULTS

In this section, we present some basic results on the Blasius problem (1.1-1.2) in the convex case, *i.e.*, for $\lambda > b$. Most of these results are already known; however we tried to give, as far as possible, new shorter proofs. In particular in the case $b \geq 0$, we prove (*cf.* Corollary 3.6) that the Blasius problem has no solution if $\lambda < 0$ or if $b = 0 < a$ and $\lambda \leq a^2\lambda_+$ and a unique solution in the following cases:

- (i) $b > 0$ and $\lambda > b$,
- (ii) $b = 0$ and $a \leq 0 < \lambda$,
- (iii) $b = 0 < a$ and $\lambda > a^2\lambda_+$.

We first show the existence of our function Λ and we prove its continuity and monotony properties.

3.1. Definition of the function Λ and proof of Proposition 2.1. Let $a, b \in \mathbb{R}$ and $c \in (0, +\infty)$. We recall that $f(\cdot; a, b, c)$ is the solution of the initial value problem (1.5). Let $[0, T_{a,b,c})$ denote its right maximal interval of existence. We also have set

$$\Lambda(a, b, c) = \lim_{t \rightarrow T_{a,b,c}} f'(t; a, b, c).$$

Proposition 3.1. *If $c > 0$, then $T_{a,b,c} = +\infty$ and $\Lambda(a, b, c)$ exists, is finite and non-negative.*

Proof. The existence of $\Lambda(a, b, c) \in (b, +\infty]$ follows from the positivity of $f''(\cdot; a, b, c)$. We first show that $T_{a,b,c} = +\infty$. Since $f := f(\cdot; a, b, c)$ is convex on $[0, T_{a,b,c})$, we have $f(t) \geq$

$f(t_0) + f'(t_0)(t - t_0)$ for all t_0, t in $[0, T_{a,b,c})$. Hence (2.1) implies

$$0 < f''(t) \leq f''(t_0) \exp \left\{ -f(t_0)(t - t_0) - f'(t_0) \frac{(t - t_0)^2}{2} \right\}. \quad (3.1)$$

If $T_{a,b,c}$ were finite, then f, f' and f'' would have finite limits as $t \rightarrow T_{a,b,c}$, thus a contradiction. This proves $T_{a,b,c} = +\infty$.

If now $\Lambda(a, b, c) \in (b, 0)$, then $f(t) \rightarrow -\infty$ as $t \rightarrow +\infty$, hence there exists $t_1 \geq 0$ such that $f(t) \leq -1$ for $t \geq t_1$. From (2.1) we obtain $f''(t) \geq f''(t_1)e^{t-t_1}$, hence $f''(t) \rightarrow +\infty$, as $t \rightarrow +\infty$, which contradicts the fact that $f'(t)$ has a finite limit as $t \rightarrow +\infty$.

It remains to show that $\Lambda(a, b, c)$ is finite. We can assume that $\Lambda(a, b, c)$ is not equal to 0. Thus, $f(t) \rightarrow +\infty$ as $t \rightarrow +\infty$, hence there exists $t_2 \geq 0$ such that $f(t_2) = \max\{1, a + 1\}$. We have $f'(t_2) > 0$, hence writing (3.1) for $t_0 = t_2$ we obtain $0 < f''(t) \leq f''(t_2)e^{-(t-t_2)}$ for all $t \geq t_2$. Integrating, we obtain $0 \leq f'(t) - f'(t_2) \leq f''(t_2)(1 - e^{-(t-t_2)})$ for all $t \geq t_2$. This implies that $\Lambda(a, b, c)$ is finite. \square

Remark. Let $c > 0$. From its convexity, the function $f(\cdot; a, b, c)$ has a constant sign at infinity, hence it is so for $f'''(\cdot; a, b, c)$. Since $\Lambda(a, b, c)$ is finite, we obtain

$$\lim_{t \rightarrow +\infty} f''(t; a, b, c) = 0. \quad (3.2)$$

Proof of Proposition 2.1. Let $f = f(\cdot; a, b, c)$ be the solution of problem (1.5) with $c = f''(0) > 0$. From Proposition 3.1, we know that f is defined on $[0, +\infty)$ and that $f''(t) > 0$ for all t . Hence, the function $t \mapsto s = f'(t)$ is increasing and defines $t = (f')^{-1}(s)$ as a function of s .

As seen at the beginning of Section 2.1, the function

$$u : [b, \Lambda(a, b, c)) \rightarrow (0, +\infty), \quad s \mapsto f'' \circ (f')^{-1}(s),$$

is solution of problem (2.2-2.3). We also saw that $u' \circ f' = -f$. Moreover, from (3.2), we have

$$\lim_{s \rightarrow \Lambda(a, b, c)} u(s) = \lim_{t \rightarrow +\infty} f''(t) = 0.$$

This shows that u cannot be continued after $\Lambda(a, b, c)$. Finally, if $\Lambda(a, b, c)$ is positive, then $f(t) \rightarrow +\infty$ as $t \rightarrow +\infty$, hence

$$\lim_{s \rightarrow \Lambda(a, b, c)} u'(s) = \lim_{t \rightarrow +\infty} (-f(t)) = -\infty.$$

This completes the proof. \square

3.2. Useful identities. Below are some useful identities satisfied by any function u , solution of the Crocco equation (2.2) on some interval I . From this equation we immediately obtain that for all $s_0, s \in I$ we have

$$u'(s) - u'(s_0) = - \int_{s_0}^s \frac{\eta}{u(\eta)} d\eta, \quad (3.3)$$

and integrating once again

$$u(s) - u(s_0) - u'(s_0)(s - s_0) = - \int_{s_0}^s \frac{\eta(s - \eta)}{u(\eta)} d\eta. \quad (3.4)$$

Moreover, since $uu'' = (uu')' - u'^2$, it follows after integration, and using (2.2), that for all $s_0, s \in I$ we have

$$u(s)u'(s) - u(s_0)u'(s_0) = \int_{s_0}^s u'(\eta)^2 d\eta - \frac{1}{2}(s^2 - s_0^2),$$

and integrating again

$$\begin{aligned} u(s)^2 - u(s_0)^2 &= 2u(s_0)u'(s_0)(s - s_0) \\ &= 2 \int_{s_0}^s (s - \eta)u'(\eta)^2 d\eta - \frac{1}{3}(s - s_0)^2(s + 2s_0). \end{aligned} \quad (3.5)$$

Finally, multiplying equation (2.2) by $2u'$ and integrating, we obtain for all $s_0, s \in I$

$$u'(s)^2 - u'(s_0)^2 = -2s \ln |u(s)| + 2s_0 \ln |u(s_0)| + 2 \int_{s_0}^s \ln |u(\eta)| d\eta. \quad (3.6)$$

3.3. Further properties of the function Λ . Thanks to Proposition 2.1, we associate to $f(\cdot; a, b, c)$ the Crocco solution $u(\cdot; a, b, c)$ of (2.2-2.3). Precisely, $u := u(\cdot; a, b, c)$ satisfies

$$u'' = -\frac{s}{u} \text{ on } [b, \Lambda(a, b, c)), \quad u(b) = c, \quad u'(b) = a, \quad u > 0.$$

Proof of Proposition 2.6, Item 1. First, we prove that if $a \in \mathbb{R}$ and $b \in [0, +\infty)$, then the function $c \mapsto \Lambda(a, b, c)$ is increasing. Suppose $c_1 > c_2 > 0$ and $\Lambda(a, b, c_1) \leq \Lambda(a, b, c_2)$. For $i = 1, 2$ let us set $u_i = u(\cdot; a, b, c_i)$ and $w = u_1 - u_2$. We have $w(b) = c_1 - c_2 > 0$, $w'(b) = a - a = 0$, and

$$\forall s \in [b, \Lambda(a, b, c_1)), \quad w''(s) = u_1''(s) - u_2''(s) = \frac{-s}{u_1(s)} + \frac{s}{u_2(s)} = \frac{sw(s)}{u_1(s)u_2(s)}.$$

Therefore, as long as w is positive, w is convex, and so increasing in such a way that w remains positive on $[b, \Lambda(a, b, c_1))$. However, as $s \rightarrow \Lambda(a, b, c_1)$, $w(s)$ tends either to 0 if $\Lambda(a, b, c_1) = \Lambda(a, b, c_2)$, or to $-u_2(\Lambda(a, b, c_1)) < 0$ if $\Lambda(a, b, c_1) \neq \Lambda(a, b, c_2)$. In both cases, this is a contradiction.

Secondly, we prove that if $b \in [0, +\infty)$ and $c \in (0, +\infty)$, then the function $a \mapsto \Lambda(a, b, c)$ is increasing. With the notation $u_i = u(\cdot; a_i, b, c)$ (for $i = 1, 2$) and $w = u_1 - u_2$, we have $w(b) = c - c = 0$ and $w'(b) = a_1 - a_2 > 0$. If $\Lambda(a, b, c_1) \leq \Lambda(a, b, c_2)$, then

$$\forall s \in [b, \Lambda(a, b, c_1)), \quad w''(s) = \frac{sw(s)}{u_1(s)u_2(s)}.$$

and we conclude as in the previous case. \square

Remark. From this proof, we see that, if $0 \leq b \leq s$, then the function $c \mapsto u(s; a, b, c)$ is increasing (when defined).

Proposition 3.2. *If $(a, b, c) \in \mathbb{R} \times \mathbb{R} \times (0, +\infty)$ is such that $\Lambda(a, b, c) > 0$, then Λ is continuous at (a, b, c) .*

Proof. We have $\Lambda(a, b, c) = \sup\{f'(t; a, b, c) ; t \geq 0\}$. Since for every t the function $(a, b, c) \mapsto f'(t; a, b, c)$ is continuous, it follows that Λ is lower semicontinuous on $\mathbb{R} \times \mathbb{R} \times (0, +\infty)$.

Now we argue by contradiction by assuming that Λ is not upper semicontinuous at (a, b, c) . Then, there would exist a positive real number ε and a sequence (a_n, b_n, c_n) in $\mathbb{R} \times \mathbb{R} \times (0, +\infty)$ which converges to (a, b, c) such that, if we set $\lambda = \Lambda(a, b, c)$ and $\lambda_n = \Lambda(a_n, b_n, c_n)$, we have

$$\forall n \in \mathbb{N}, \quad \lambda + \varepsilon \leq \lambda_n. \quad (3.7)$$

Let $u = u(\cdot; a, b, c)$ and $u_n = u(\cdot; a_n, b_n, c_n)$. Since $\lambda > 0$ we deduce from Proposition 2.1 that $u(\eta) \rightarrow 0$ and $u'(\eta) \rightarrow -\infty$ as $\eta \rightarrow \lambda$, $\eta < \lambda$. Hence, there exists $s \in (0, \lambda)$ such that $u(s) < 1$ and $u'(s) < -\frac{3}{\varepsilon}$. By continuity with respect to the initial conditions, there exists $m \in \mathbb{N}$ such

that $u_m(s) < 2$ and $u'_m(s) < -\frac{2}{\varepsilon}$. Therefore, since u_m decreases and is concave on $[s, \lambda_m)$, we have $u_m(\lambda) < 2$ and $u'_m(\lambda) < -\frac{2}{\varepsilon}$. Hence,

$$\forall \eta \in [\lambda, \lambda_m), \quad u_m(\eta) \leq u_m(\lambda) + u'_m(\lambda)(\eta - \lambda) < 2 - \frac{2}{\varepsilon}(\eta - \lambda). \quad (3.8)$$

Since $u_m(\eta) \rightarrow 0$ as $\eta \rightarrow \lambda_m$, $\eta < \lambda_m$, then (3.7) and (3.8) give a contradiction. \square

3.4. Proof of Theorem 2.2, item 3. Existence. Let $u_n = u(\cdot; 1, 0, \frac{1}{n})$. For $s > 0$ small enough, we have $u_n(s) \geq \frac{s}{2}$. On any interval $[0, \alpha]$ where u_n is defined and satisfies $u_n(s) \geq \frac{s}{2}$, we have $u''_n(s) = \frac{-s}{u_n(s)} \in [-2, 0]$. Integrating twice, we obtain successively

$$1 - 2s \leq u'_n(s) \leq 1 \quad \text{and} \quad \frac{1}{n} + s - s^2 \leq u_n(s) \leq \frac{1}{n} + s. \quad (3.9)$$

Since $\frac{s}{2} < \frac{1}{n} + s - s^2$ for all $s \in [0, \frac{1}{2}]$, the a priori majorization principle implies that (3.9) is valid for all s in this interval. This shows that the sequence (u_n) is uniformly equicontinuous and bounded on $[0, \frac{1}{2}]$. Ascoli's Theorem implies that some subsequence $(u_{n_k})_{k \in \mathbb{N}}$ converges to some function u , which is automatically solution of (2.2) because

$$u''(s) = \lim_{k \rightarrow +\infty} u''_{n_k}(s) = \lim_{k \rightarrow +\infty} \frac{-s}{u_{n_k}(s)} = \frac{-s}{u(s)},$$

uniformly on each compact subset of $(0, \frac{1}{2}]$. Moreover, (3.9) implies that u satisfies $s - s^2 \leq u(s) \leq s$, hence u satisfies (2.8). This proves the existence of u_+ .

Remark. Since $c \mapsto u(s; a, b, c)$ increases on $(0, +\infty)$ if $0 \leq b \leq s$ (cf. the remark above Proposition 3.2), we also have

$$u_+(s) = \lim_{c \rightarrow 0} u(s; 1, 0, c). \quad (3.10)$$

Uniqueness. Suppose that u_1 and u_2 are two solutions of (2.2) on some interval $(0, T)$, satisfying $u(0^+) = 0$, $u'(0^+) = 1$. Setting $w = u_1 - u_2$, we obtain

$$w'' = \frac{sw}{u_1(s)u_2(s)} \quad \text{and} \quad w(0^+) = w'(0^+) = 0. \quad (3.11)$$

Assume first that w does not vanish on $(0, T)$. Without loss of generality we may assume that $w > 0$ on $(0, T)$. Then w is convex. This implies that $w(\eta) = w(\frac{\eta}{s}s) \leq \frac{\eta}{s}w(s)$ for all $\eta \in (0, s]$. Now (3.11) gives by integration, for all $s \in (0, T)$

$$\frac{w'(s)}{w(s)} = \frac{1}{w(s)} \int_0^s \frac{\eta w(\eta)}{u_1(\eta)u_2(\eta)} d\eta \leq \frac{1}{s} \int_0^s \frac{\eta^2}{u_1(\eta)u_2(\eta)} d\eta.$$

Since $u_1(\eta)u_2(\eta) \sim \eta^2$ as $\eta \rightarrow 0^+$, there exists ε such that for $0 \leq s < \varepsilon$ we have $w'(s) \leq 2w(s)$. By Gronwall's lemma, we deduce that $w' = 0$ on $[0, \varepsilon]$, hence $w = 0$ on $[0, \varepsilon]$, which contradicts our assumption.

Therefore, there is $s_0 \in (0, T)$ such that $w(s_0) = 0$. Then, $w = 0$ on $[0, s_0)$, because on the contrary it should exist $s_1 \in (0, s_0)$ such that, for example, $w(s_1) > 0$ and $w''(s_1) \leq 0$, which contradicts (3.11). It follows that $w = 0$ on $[0, T)$.

Remark. Concerning u_- , we give a similar proof in Section 9.2 and a shorter indirect proof in Section 5.3.

Bounds for λ_+ . Let $(0, \lambda_+)$ denote the interval of existence of u_+ . We have to prove that $1 < \lambda_+ < s_0$, where $s_0 > 0$ satisfies $(2s_0 - 4) \ln(1 - \frac{s_0}{2}) = s_0$, see Figure 16.

By concavity, we have $u_+(s) < s$ for all $s \in (0, \lambda_+)$. Thus, $u''_+(s) < -1$ for all $s \in (0, \lambda_+)$. Integrating twice we obtain $u_+(s) < s - \frac{s^2}{2}$ for all $s \in (0, \lambda_+)$. Hence, $u''_+(s) < -\frac{1}{1-\frac{s}{2}}$ for all

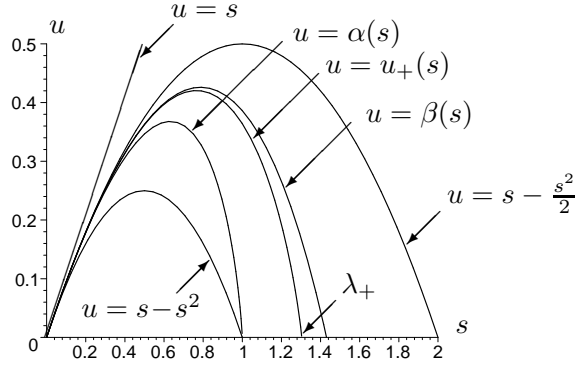


FIGURE 16. The solution u_+ of the Crocco equation. We have $\lambda_+ \approx 1.304$.

$s \in (0, \lambda_+)$. Integrating twice again, we obtain $u_+(s) < \beta(s) := (2s - 4) \ln(1 - \frac{s}{2}) - s$ for all $s \in (0, \lambda_+)$. Thus, $\lambda_+ < s_0$, where $s_0 \approx 1.43$ is the positive root of equation $\beta(s) = 0$.

On the other hand for $s > 0$ small enough we have $u_+(s) > \frac{s}{2}$. Thus, $u_+''(s) > -2$ as long as $s > 0$ and $u_+(s) > \frac{s}{2}$. Integrating twice we obtain $u_+(s) > s - s^2$ for all $s \in (0, \frac{1}{2})$. Thus, $u_+''(s) > -\frac{1}{1-s}$, as long as both $s \in (0, \lambda_+)$ and $u_+(s) > s - s^2$. Integrating twice again, we obtain $u_+(s) > \alpha(s) := (s - 1) \ln(1 - s)$ as long as $s \in (0, \lambda_+)$ and $u_+(s) > s - s^2$. Since $\alpha(s) > s - s^2$ for all $s \in (0, 1)$, we deduce $\lambda_+ > 1$. \square

3.5. The Blasius problem for $b \geq 0$.

Proposition 3.3. *We have*

$$\lambda_+ = \lim_{c \rightarrow 0^+} \Lambda(1, 0, c) = \inf_{c > 0} \Lambda(1, 0, c).$$

Proof. Since $c \mapsto \Lambda(1, 0, c)$ is increasing and positive, the infimum

$$\nu := \inf_{c > 0} \Lambda(1, 0, c),$$

is finite and equal to $\lim_{c \rightarrow 0^+} \Lambda(1, 0, c)$. Besides, the solution u_+ given by (3.10) is defined on $(0, \lambda_+)$ and, for $c > 0$, the solution $u(\cdot; 1, 0, c)$ is defined on $[0, \Lambda(1, 0, c)]$. By the lower semicontinuity of the positive maximal interval of existence of solutions of ODEs with respect to initial conditions, we obtain $\lambda_+ \leq \nu$.

By contradiction, assume that $\lambda_+ < \nu$. Notice that for all $c > 0$ we have $(0, \nu] \subset (0, \Lambda(1, 0, c))$. Consider $\varepsilon > 0$ and $B \in (0, \lambda_+)$, close to λ_+ . Then there exists $c_0 > 0$ such that for all $c \in (0, c_0)$

$$\forall s \in (0, B] \quad |u(s; 1, 0, c) - u_+(s)| < \varepsilon.$$

Since ε can be chosen as small as we want and B as close to λ_+ as we want, this would contradict the fact that $u(\cdot; 1, 0, c)$ is positive and concave on $[0, \nu]$. \square

We prove below the asymptotic formula (2.18) of Proposition 2.7, in a more general situation. This shows, in particular, that $\Lambda(a, b, c) \rightarrow +\infty$ as $c \rightarrow +\infty$.

Proposition 3.4. *For all $(a, b) \in \mathbb{R}^2$ the following holds*

$$\Lambda(a, b, c) \sim \Lambda(0, 0, 1)c^{2/3} \quad \text{as } c \rightarrow +\infty.$$

Proof. From (2.15), for $c > 0$, we have

$$c^{-2/3} \Lambda(a, b, c) = \Lambda\left(ac^{-1/3}, bc^{-2/3}, 1\right) \longrightarrow \Lambda(0, 0, 1) \quad \text{as } c \rightarrow +\infty,$$

since Λ is continuous at the point $(0, 0, 1)$. □

Remark. Section 9.5 contains an historical account about this constant $\Lambda(0, 0, 1)$.

Proposition 3.5. *Let $a \in \mathbb{R}$ and $b \geq 0$. Set $\mu_{a,b} = \inf\{\Lambda(a, b, c) ; c > 0\}$. The mapping $c \mapsto \Lambda(a, b, c)$ is increasing one-to-one from $(0, +\infty)$ onto $(\mu_{a,b}, +\infty)$. Furthermore, we have*

$$\mu_{a,b} = \begin{cases} b & \text{if } a \leq 0 \quad \text{or } b > 0, \\ a^2\lambda_+ & \text{if } a > 0 \quad \text{and } b = 0. \end{cases} \quad (3.12)$$

Proof. The first assertion follows from Proposition 2.6, item 1 and Propositions 3.2, 3.4. It remains to prove (3.12). Notice that we have

$$\mu_{a,b} = \lim_{c \rightarrow 0^+} \Lambda(a, b, c).$$

For the remainder of the proof, we set $u_c = u(\cdot ; a, b, c)$ and $\lambda_c = \Lambda(a, b, c)$.

- If $a \leq 0$, then u_c is decreasing on $[b, \lambda_c)$. Thanks to (3.4) with $s_0 = b$ and $s \rightarrow \lambda_c$, we obtain

$$\begin{aligned} c \geq c + a(\lambda_c - b) &= \int_b^{\lambda_c} \frac{\eta(\lambda_c - \eta)}{u_c(\eta)} d\eta \\ &\geq \frac{1}{c} \int_b^{\lambda_c} \eta(\lambda_c - \eta) d\eta = \frac{1}{6c} (\lambda_c - b)^2 (\lambda_c + 2b), \end{aligned}$$

hence $\lambda_c \rightarrow b$ as $c \rightarrow 0^+$.

- If $a > 0$ and $b > 0$, then there exists a unique $s_c \in (b, \lambda_c)$ in which u_c attains its maximum. Using identity (3.4) with $s_0 = s_c$, $s \rightarrow \lambda_c$ and the fact that u_c is decreasing on $[s_c, \lambda_c)$ we obtain as above

$$\begin{aligned} u_c(s_c) &= \int_{s_c}^{\lambda_c} \frac{\eta(\lambda_c - \eta)}{u_c(\eta)} d\eta \geq \frac{1}{u_c(s_c)} \int_{s_c}^{\lambda_c} \eta(\lambda_c - \eta) d\eta \\ &= \frac{1}{6u_c(s_c)} (\lambda_c - s_c)^2 (\lambda_c + 2s_c). \end{aligned}$$

This and the concavity of u_c yield

$$c + a(s_c - b) \geq u_c(s_c) \geq \frac{1}{\sqrt{6}} (\lambda_c - s_c) \sqrt{\lambda_c + 2s_c}. \quad (3.13)$$

In addition, from (3.3) and the concavity of u_c , we obtain

$$a = \int_b^{s_c} \frac{\eta}{u_c(\eta)} d\eta \geq b \int_b^{s_c} \frac{d\eta}{c + a(\eta - b)} = \frac{b}{a} \ln \left(1 + \frac{a}{c} (s_c - b) \right),$$

and since $b > 0$, this leads to

$$0 \leq s_c - b \leq \frac{c}{a} (e^{a^2/b} - 1),$$

hence, $s_c \rightarrow b$ as $c \rightarrow 0^+$. Coming back to (3.13) we obtain that $\lambda_c \rightarrow b$ as $c \rightarrow 0^+$.

- If $a > 0$ and $b = 0$, then from the similarity (2.15) and Proposition 3.3, we have

$$\mu_{a,0} = \lim_{c \rightarrow 0^+} \Lambda(a, 0, c) = a^2 \lim_{c \rightarrow 0^+} \Lambda(1, 0, ca^{-3}) = a^2\lambda_+. \quad \square$$

Corollary 3.6. *Let $a \in \mathbb{R}$ and $b \geq 0$. If $a \leq 0$ or $b > 0$, then the Blasius boundary value problem (1.1-1.2) has one and only one solution for all $\lambda \in (b, +\infty)$.*

If $a < 0$ and $b = 0$, then the Blasius problem (1.1-1.2) has one and only one solution for all $\lambda \in (a^2\lambda_+, +\infty)$ and no solution for all $\lambda \in (0, a^2\lambda_+]$.

Proof. This follows immediately from Proposition 3.5. □

4. PROPERTIES OF THE AUTONOMOUS VECTOR FIELD

This section is entirely devoted to the proof of Theorem 2.4.

 4.1. The behavior near S^* .

Proof of Theorem 2.4, item 1. Let (x, y) be a solution of (2.12-2.13) with $c > 0$. Its associated Crocco solution $u(\cdot; a, -1, c)$ is defined at least on $[-1, 0)$, hence (x, y) is defined at least on $[0, +\infty)$.

Besides, (2.12) has the same orbits as the polynomial one:

$$\dot{x} = 1 + \frac{1}{2}xy, \quad \dot{y} = xy + \frac{3}{2}y^2,$$

hence, no orbit of (2.12) can reach the horizontal axis $y = 0$.

The upper half plane ($y > 0$) is cut in four regions by the isoclines I_∞ ($\dot{x} = 0 \Leftrightarrow xy = -2$) and I_0 ($\dot{y} = 0 \Leftrightarrow y = -\frac{2}{3}x$). On the East, x and y increase with t , on the North, x decreases and y increases, etc., see Figure 17, left.

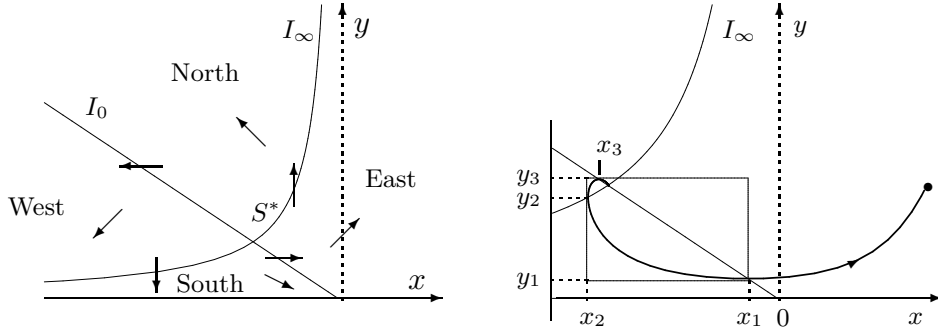


FIGURE 17. On the left, the four regions delimited by the isoclines I_0 and I_∞ ; on the right, a solution starting in the East region for negative time and its invariant box.

With the time t reversed, any solution starting in the East region must cross I_0 at some point (x_1, y_1) with $0 < y_1 < \frac{2}{\sqrt{3}}$ and $x_1 = -\frac{3}{2}y_1$, then crosses I_∞ at some (x_2, y_2) with $y_1 < y_2 < \frac{2}{\sqrt{3}}$ and $x_2 = -\frac{2}{y_2}$, then crosses I_0 once again at (x_3, y_3) with $x_2 < x_3 < -\sqrt{3}$, then I_∞ at (x_4, y_4) with $y_4 < y_3$, hence $x_4 \leq x_1$. This gives an invariant box containing the solution for $t \in (-\infty, t_1]$, see Figure 17, right. As a consequence the solution is defined for all $t < 0$. The divergence of (2.12) is given by

$$\frac{\partial}{\partial x} \left(\frac{1}{2}x + \frac{1}{y} \right) + \frac{\partial}{\partial y} \left(x + \frac{3}{2}y \right) = 2.$$

Because this divergence has a constant sign, there is no cycle by Dulac criterion, therefore any solution starting in the East region must tend to the unique stationary point S^* as $t \rightarrow -\infty$.

With the same arguments, any solution that starts from any other region eventually visits the East as t decreases, and we fall in the first case. \square

Proof of Theorem 2.4, item 2. The linear part of the vector field at the stationary point S^* has matrix $\begin{pmatrix} 1/2 & -3/4 \\ 1 & 3/2 \end{pmatrix}$, hence S^* is a source with simple eigenvalues $1 \pm \frac{i}{\sqrt{2}}$. Therefore, (2.14) means that x is approximated by a solution of the linear part of our system at S^* . This is proved in the following lemma.

Lemma 4.1. Consider a C^1 differential system in \mathbb{R}^n of the form

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{b}(\mathbf{x}). \quad (4.1)$$

Assume $\mathbf{b} : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfies $\mathbf{b}(\mathbf{x}) = O(|\mathbf{x}|^k)$ as $|\mathbf{x}| \rightarrow 0$ for some $k \in (1, +\infty)$, where $|\cdot|$ stands for some norm in \mathbb{R}^n and U is a neighborhood of $0 \in \mathbb{R}^n$.

Assume also that there is $\alpha > 0$ such that $-k\alpha < \operatorname{Re}(\lambda) < -\alpha$ for any eigenvalue λ of \mathbf{A} (with the same k).

Then, for any solution \mathbf{x} of (4.1) remaining in U and tending to $0 \in \mathbb{R}^n$, there exists $\bar{\mathbf{x}} \in \mathbb{R}^n$ such that

$$\mathbf{x}(t) = e^{t\mathbf{A}}(\bar{\mathbf{x}} + o(1)) \text{ as } t \rightarrow +\infty.$$

Proof. Although this result follows directly from the theory of normal forms (the assumptions imply that there is no resonance), we give here an elementary proof.

Since the linear part of the system at its stationary point $0 \in \mathbb{R}^n$ is \mathbf{A} , the assumption on the spectrum of \mathbf{A} already gives $\mathbf{x}(t) = O(e^{-\alpha t})$ as $t \rightarrow +\infty$, hence $\mathbf{b}(\mathbf{x}(t)) = O(e^{-k\alpha t})$ as $t \rightarrow +\infty$.

Let us view (4.1) as a linear system with $\mathbf{b}(\mathbf{x})$ considered as a “known” function. The variation of constant formula yields, for all $t > 0$:

$$\mathbf{x}(t) = e^{t\mathbf{A}}\left(\mathbf{x}(0) + \int_0^t e^{-s\mathbf{A}}\mathbf{b}(\mathbf{x}(s))ds\right).$$

By assumption, there exists $\delta > 0$ such that for all $s > 0$, $\|e^{-s\mathbf{A}}\| = O(e^{(k\alpha - \delta)s})$, where $\|\cdot\|$ stands for any matrix norm. Therefore, we obtain $e^{-s\mathbf{A}}\mathbf{b}(\mathbf{x}(s)) = O(e^{-\delta s})$ as $s \rightarrow +\infty$, which shows that the integral $\int_0^{+\infty} e^{-s\mathbf{A}}\mathbf{b}(\mathbf{x}(s))ds$ converges in \mathbb{R}^n . Now put:

$$\bar{\mathbf{x}} = \mathbf{x}(0) + \int_0^{+\infty} e^{-s\mathbf{A}}\mathbf{b}(\mathbf{x}(s))ds.$$

Then $\mathbf{x}(t) = e^{t\mathbf{A}}(\bar{\mathbf{x}} + r(t))$ with $r(t) = -\int_t^{+\infty} e^{-s\mathbf{A}}\mathbf{b}(\mathbf{x}(s))ds = o(1)$. □

Remarks. 1. We stated the result for the stationary point $0 \in \mathbb{R}^n$ but of course we have the same result at any stationary point of sink type of any nonlinear system. In the same way we have an analogous result as $t \rightarrow -\infty$ for stationary points of source type.

2. The assumption on the spectrum of \mathbf{A} is optimal: for example the system

$$x' = -kx + y^k, \quad y' = -y,$$

with initial conditions $x(0) = 0, y(0) = 1$ is solved by $x(t) = te^{-kt}, y(t) = e^{-t}$ which cannot be asymptotic to any solution of the linear part, of the form $(\bar{x}e^{-kt}, \bar{y}e^{-t})$.

Our system (2.12), being C^∞ near S^* , satisfies the assumptions of Lemma 4.1 (with time reversed and after translation $(x, y) \mapsto (x - \sqrt{3}, y + \frac{2}{\sqrt{3}})$, as explained in Remark 1 above) with $k = 2$ and any $\alpha \in (\frac{1}{2}, 1)$. This proves (2.14). □

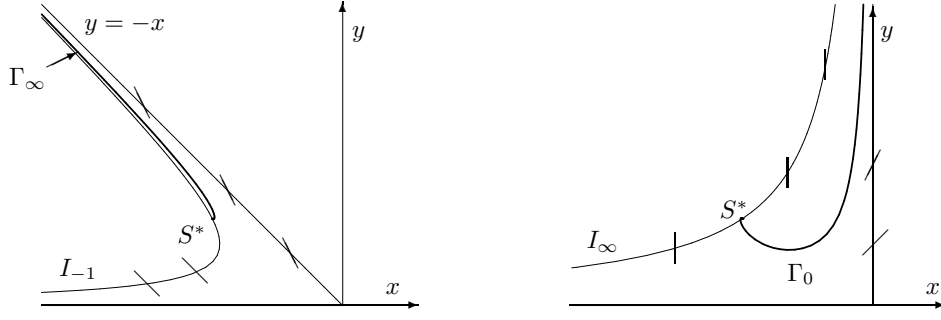
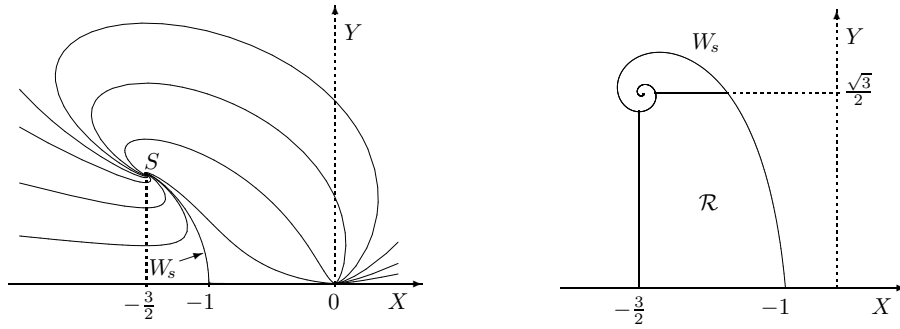
4.2. The orbits Γ_∞ and Γ_0 .

Proof of Theorem 2.4, item 3. The straight line $y = -x$ and the isocline I_{-1} given by $\dot{x} + \dot{y} = 0$, i.e., $x + y = -\frac{2}{3y}$, form a narrowing antifunnel (see Figure 18 left) with positive divergence, thus Theorem 4.10 of [30] applies. This proves the existence and uniqueness of Γ_∞ .

In the same manner as for Γ_∞ , there is a unique orbit Γ_0 in the narrowing antifunnel made of the axis $0y$ and the isocline I_∞ (see Figure 18 right).

Now the change of variables $X = \frac{x}{y}, Y = \frac{1}{y}$ transforms (2.12) into the polynomial system

$$\dot{X} = Y^2 - X^2 - X, \quad \dot{Y} = -Y\left(X + \frac{3}{2}\right). \quad (4.2)$$


 FIGURE 18. The orbits Γ_∞, Γ_0 and their antifunnels.

 FIGURE 19. On the left: the vector field (4.2). On the right: a sketch of the region \mathcal{R} .

We study this system in the half plane $Y \geq 0$, corresponding to $y > 0$. In addition to the stationary point $S = (-\frac{3}{2}, \frac{\sqrt{3}}{2})$ corresponding to the source S^* of (2.12), system (4.2) has two other stationary points, $(0, 0)$ and $(-1, 0)$, which lie on the axis $Y = 0$ and correspond to the asymptotic direction at infinity of solutions of system (2.12). The linear part of the vector field (4.2) at the stationary point $(-1, 0)$ has eigenvalues 1 and $-\frac{1}{2}$, hence $(-1, 0)$ is a saddle. The unstable separatrix of the saddle belongs to the axis $Y = 0$. The curve Γ_∞ corresponds to the stable separatrix W_s of this saddle. Using Taylor expansion for W_s , we get that, for y large enough, Γ_∞ is the graph of a function $x = x_\infty(y)$ that satisfies

$$x_\infty(y) = -y - \frac{1}{2y} - \frac{1}{12y^3} - \frac{1}{24y^5} - \frac{7}{240y^7} - \frac{13}{540y^9} + O(y^{-11}).$$

With the notation introduced in Section 2.4, x_∞ is the inverse function of R_1 .

The linear part of the vector field (4.2) at the stationary point $(0, 0)$ has eigenvalues -1 and $-\frac{3}{2}$. Hence $(0, 0)$ is a stable node. Let us show that all trajectories of system (4.2), except the part $(X \leq -1)$ of the axis $Y = 0$, the stationary point S , and the stable separatrix W , tend to $(0, 0)$ as $t \rightarrow +\infty$. Let \mathcal{R} denote the region enclosed by W_s and the lines $X = -\frac{3}{2}$ and $Y = \frac{\sqrt{3}}{2}$ as shown on Figure 19, right. Because \mathcal{R} contains neither limit cycles nor stationary points other than the saddle $(-1, 0)$, any trajectory $(X(t), Y(t))$ of system (4.2), starting in \mathcal{R} will cross the isocline $X = -\frac{3}{2}$ (see Figure 20 right). This means that the corresponding trajectory $(x(t), y(t))$ of system (2.12) will cross the isocline I_0 , given by $y = -\frac{2}{3}x$.

We claim that the corresponding orbit Γ is convex in the region $y < -\frac{2x}{3}$. For that purpose, we use the following result.

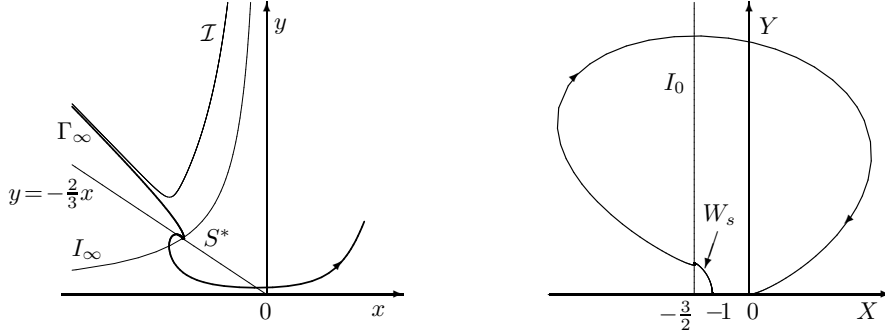


FIGURE 20. On the left: the spiral Γ_∞ and a trajectory of (2.12); on the right: the corresponding stable manifold W_s and the corresponding trajectory of (4.2). We also added the curve \mathcal{I} of inflexion points and I_∞ on the left figure, and isoclines 0 on each figure.

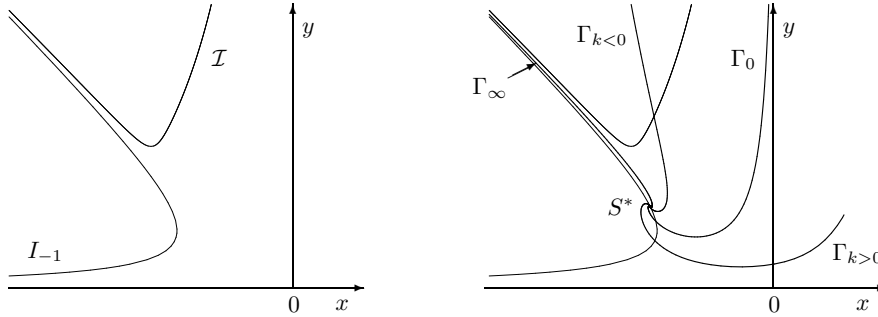


FIGURE 21. On the left: the curve of inflexion points \mathcal{I} of (2.12) given by (4.3) and the bounding curves I_{-1} and $0y$. On the right: the same ones with Γ_0 , Γ_∞ and two Γ_k with $k < 0$ and $k > 0$.

Lemma 4.2. *The set of inflexion points of the vector field (2.12) is the algebraic set of equation*

$$4 + 4x^2 + 20xy + 15y^2 + 3xy^3 + 3x^2y^2 = 0. \quad (4.3)$$

It consists of one isolated point $S^ = (-\sqrt{3}, \frac{2}{\sqrt{3}})$ and a curve \mathcal{I} with two branches $x = x_1(y)$ and $x = x_2(y)$, $y \in [2, +\infty)$, that satisfy*

$$\forall y \in (2, +\infty), \quad -y - \frac{1}{3y} < x_2(y) \leq x_1(y) < 0. \quad (4.4)$$

Moreover, we have $x_1(y) \sim -\frac{5}{y}$ and $x_2(y) + y \sim -\frac{1}{3y}$ as $y \rightarrow +\infty$. As a consequence, there is no inflexion point of (2.12) in the region $y < -\frac{2}{3}x$.

Proof. Inflexion points of a vector field

$$\dot{x} = h(x, y), \quad \dot{y} = g(x, y).$$

have to satisfy

$$h^2 g_x + hg(g_y - h_x) - g^2 h_y = 0,$$

where subscripts stand for the corresponding partial derivatives. To see this, if y can be written as a function of x (otherwise permute x and y), we have $y'(x) = \frac{g}{h}(x, y(x))$, thus we obtain:

$$y''(x) = 0 \Leftrightarrow h(x, y(x)) \frac{d}{dx} g(x, y(x)) - g(x, y(x)) \frac{d}{dx} h(x, y(x)) = 0$$

$$\Leftrightarrow h(g_x + g_y \cdot y') - g(h_x + h_y \cdot y') = 0,$$

then replace y' by $\frac{g}{h}$ and multiply by h .

In our case, we obtain that the inflexion points satisfy (4.3). This is a second degree algebraic equation in x , therefore its discriminant $\Delta = (y^2 - 4)(3y^2 - 4)^2$ has to be nonnegative. This implies $y = \frac{2}{\sqrt{3}}$ or $y \geq 2$.

For $y = \frac{2}{\sqrt{3}}$ we obtain $x = -\sqrt{3}$, *i.e.*, $(x, y) = S^*$. The case $y \geq 2$ gives the two branches

$$x_1(y) = \frac{-y(3y^2 + 20) + \sqrt{\Delta}}{2(4 + 3y^2)}, \quad x_2(y) = \frac{-y(3y^2 + 20) - \sqrt{\Delta}}{2(4 + 3y^2)}. \quad (4.5)$$

Let us now verify the inequalities and the asymptotics. Firstly, (4.3) has obviously no solution with both x and y nonnegative. The asymptotics of x_1 is found when one seeks for a solution of (4.3) with $y \rightarrow +\infty$ and $x \rightarrow 0$; thus two terms are in balance: $15y^2$ and $3xy^3$. We obtain successively $x_1(y) = O(y^{-1})$ and $5 + yx_1(y) = O(y^{-2})$, hence $x_1(y) = -\frac{5}{y} + O(y^{-3})$.

Concerning x_2 the most convenient is to put $x = z - y$ in (4.3). Then (4.3) gives $4 - y^2 + 12yz + 4z^2 - 3y^3z + 3y^2z^2 = 0$ or equivalently

$$(4 - y^2)(1 + 3yz) + (4 + 3y^2)z^2 = 0, \quad (4.6)$$

from which it is clear that $1 + 3yz$ has to be positive. This gives the left inequality of (4.4). The asymptotics of x_2 is found when one seeks for a solution of (4.6) with $y \rightarrow +\infty$ and $z \rightarrow 0$; thus two terms are in balance: $-y^2$ and $-3y^3z$. One obtains successively $z = O(y^{-1})$ and $1 + 3yz = O(y^{-2})$, hence $x_2(y) = -y - \frac{1}{3y} + O(y^{-3})$. \square

We now return to our proof of item 3. Since Γ is convex in the region $y < -\frac{2x}{3}$, the trajectory $(x(t), y(t))$ must cross the isocline I_∞ given by $xy = -2$ and enter in the South region (see Figure 20 left). Hence it must cross again the isocline I_0 . This means that the corresponding trajectory $(X(t), Y(t))$ must cross again the isocline $X = -\frac{3}{2}$ (see Figure 20 right) and enter in the region $X \geq -\frac{3}{2}$, in which Y is decreasing. Hence, it tends to the stationary point $(0, 0)$.

Consequently, any solution $(x(t), y(t))$ of system (2.12), except those corresponding to S^* and Γ_∞ , satisfies

$$\lim_{t \rightarrow +\infty} y(t) = +\infty, \quad \lim_{t \rightarrow +\infty} \frac{x(t)}{y(t)} = 0.$$

This proves item 3 of Theorem 2.4. \square

Before going further in the proof of the remaining items, we now prove the following.

Lemma 4.3. 1. *If u is a Crocco solution of (2.2) on $(-\infty, 0)$ satisfying $u(0^-) = 0$, then for all $s < 0$ we have $-\frac{su'(s)}{u(s)} \leq -1$.*

2. *Moreover, if $u'(0^-) = 0$, then for all $s < 0$ we have $-\frac{su'(s)}{u(s)} \leq -\frac{3}{2}$.*

Proof. 1. If u is a Crocco solution such that $u(0^-) = 0$, then by the finite-increment theorem, for all $s < 0$ there exists $\theta \in (0, 1)$ such that $u(s) = su'(\theta s)$. Since u' is negative increasing on $(-\infty, 0)$, we obtain

$$\forall s < 0, \quad -\frac{su'(s)}{u(s)} = -\frac{u'(s)}{u'(\theta s)} \leq -1.$$

2. For all $s < 0$ we have

$$2u'(s)u''(s) = -\frac{2su'(s)}{u(s)} \leq -2 \quad \text{and} \quad u'(0^-) = 0.$$

By integration we obtain $u'(s)^2 \geq -2s$, therefore $u'(s) \leq -\sqrt{-2s}$. Integrating once again with $u(0^-) = 0$ we obtain

$$\forall s < 0, \quad u(s) \geq \frac{2\sqrt{2}}{3}(-s)^{3/2},$$

hence, $-\frac{su'(s)}{u(s)} \leq -\frac{3}{2}$. \square

Proof of Theorem 2.4, item 4. Let us consider $(a, c) \in \mathbb{R} \times (0, +\infty)$, set $u = u(\cdot; a, -1, c)$ and let (x, y) denote the corresponding solution of (2.12) with initial conditions (2.13). Notice that, if s and t are linked by $s = -e^{-t}$, then we have

$$-\frac{su'(s)}{u(s)} = \frac{x(t)}{y(t)}.$$

Concerning (1), we have $(a, c) = S^* \Leftrightarrow u = u_*$, which satisfies the conditions $u(0^-) = u'(0^-) = 0$. Conversely, because S^* is the only trajectory that satisfies $\frac{x(t)}{y(t)} \leq -\frac{3}{2}$ for all $t \in \mathbb{R}$, by Lemma 4.3 above, only u_* can satisfy the required conditions. Incidentally, we proved that u_* is the unique Crocco solution with $u_*(0^-) = u_*'(0^-) = 0$, as claimed after (2.6).

For the proof of (3), if $(a, c) \notin \Gamma_\infty \cup \{S^*\}$, then from item 3 proven above, $\frac{x(t)}{y(t)}$ tends to 0 as $t \rightarrow +\infty$, i.e., $\frac{su'(s)}{u(s)} \rightarrow 0$ as $s \rightarrow 0^-$. By Lemma 4.3, $u(0^-) \neq 0$, i.e., u is defined (at least) until 0 and $u(0) \neq 0$. Conversely, if $u(0) \neq 0$, then $k := \frac{u'(0)^3}{u(0)^3}$ is finite, therefore (a, c) cannot be on Γ_∞ (because on Γ_∞ we have $\frac{u'(s)^3}{u(s)^3} = \frac{x(t)^3}{y(t)^3} \sim -x(t)^2 \rightarrow -\infty$) and of course $(a, c) \neq S^*$ since $u \neq u_*$. Now (2) follows by exclusion. \square

4.3. The orbits Γ_k .

Proof of Theorem 2.4, item 5. Along an orbit different from Γ_∞ and S^* , a corresponding Crocco solution has to satisfy $u(0) > 0$. Therefore, $\frac{x(t)^3}{y(t)^3} = \frac{u'(s)^3}{u(s)^3}$ has a limit k as $t \rightarrow +\infty$ i.e., $s \rightarrow 0^-$, namely $k = \frac{u'(0)^3}{u(0)^3}$. \square

Proof of item 6. Consider the trajectory of (2.12) corresponding to the Crocco solution with $u(0) = 1, u'(0) = k^{1/3}$. This shows the existence. For the uniqueness, if two Crocco solutions u_1, u_2 satisfy $\frac{u_i'(s)^3}{u_i(s)^3} \rightarrow k \notin \{-\frac{9}{2}, \infty\}$ as $s \rightarrow 0^-$, then the corresponding orbits of (2.12) are different from S^* and Γ_∞ , hence $u_i(0) > 0$ and $\frac{u_1'(0)^3}{u_1(0)^3} = \frac{u_2'(0)^3}{u_2(0)^3}$, therefore $u_2(s) = \sigma^3 u_1(s^{-2})$ with $\sigma = \frac{u_1'(0)}{u_2'(0)}$. In other words, the corresponding orbits of (2.12) coincide. \square

Proof of item 7. We recall that the inflexion points of system (2.12), already described in Lemma 4.2, are the point S^* and the two branches given for $y \geq 2$ by (4.5). Let us prove that the curve

$$\mathcal{I} = \{(x, y) : y \geq 2, x = x_1(y) \text{ or } x = x_2(y)\},$$

is a barrier. We have to show that for all $y > 2$, we have $p_1(y) > 0$ and $p_2(y) < 0$ where

$$p_1(y) = x_1'(y) - \frac{2 + yx_1(y)}{y(3y + 2x_1(y))}, \quad p_2(y) = x_2'(y) - \frac{2 + yx_2(y)}{y(3y + 2x_2(y))}.$$

By a tedious but straightforward computation we obtain

$$p_1(y) = \frac{2(3y^2 - 4) \left[2y(7y^2 - 12) + (13y^2 - 4)\sqrt{y^2 - 4} \right]}{y\sqrt{y^2 - 4}(2y + \sqrt{y^2 - 4})(4 + 3y^2)^2} > 0,$$

and

$$p_2(y) = \frac{-2(3y^2 - 4) \left[(2y(7y^2 - 12) - (13y^2 - 4)\sqrt{y^2 - 4}) \right]}{y\sqrt{y^2 - 4}(2y - \sqrt{y^2 - 4})(4 + 3y^2)^2} < 0.$$

The first inequality is obvious (recall that $y \geq 2$). The second inequality follows from the identity

$$2y(7y^2 - 12) - (13y^2 - 4)\sqrt{y^2 - 4} = \frac{27y^6 + 108y^4 + 144y^2 + 64}{2y(7y^2 - 12) + (13y^2 - 4)\sqrt{y^2 - 4}} > 0.$$

By Lemma 4.2, the curves $x = 0$ and $x = x_1(y)$ form a narrowing antifunnel which contains a unique orbit, namely the orbit Γ_0 . Similarly, the curves $x = -y - \frac{2}{3y}$ (i.e., the isocline I_{-1}) and $x = x_2(y)$ form a narrowing antifunnel which contains a unique orbit, namely the orbit Γ_∞ . Thus, the orbit Γ_0 remains on the right of the curve \mathcal{I} of inflexion points and the orbit Γ_∞ remains on the left of \mathcal{I} . Consequently, the orbit Γ_k has no inflexion point if $k \geq 0$ and the orbit Γ_k crosses the curve \mathcal{I} at one and only one point if $k < 0$ (see Figure 21). \square

5. FURTHER PROPERTIES OF CROCCO SOLUTIONS

We start with the following intermediate result.

Lemma 5.1. *Let $a \in \mathbb{R}$, $c > 0$ and $u = u(\cdot; a, -1, c)$.*

1. *For all $s_1 \in [-1, 0)$ such that $u'(s_1) > 0$, we have*

$$\forall s \in [s_1, 0], \int_{s_1}^s \ln u(\eta) d\eta \geq -\frac{1}{u'(s_1)}. \quad (5.1)$$

2. *For all $s_1 \leq -1$ such that $u'(s_1) < 0$, we have*

$$\forall s \leq s_1, \int_s^{s_1} \ln u(\eta) d\eta \geq \frac{1}{u'(s_1)}. \quad (5.2)$$

Proof. 1. Since u is convex on $[-1, 0]$, one has for all $\eta \in [-1, 0]$

$$u(\eta) \geq u'(s_1)(\eta - s_1) + u(s_1).$$

Denoting $\alpha_0 = u(s_1)$ and $\alpha_1 = u'(s_1)$ we then obtain

$$\begin{aligned} \int_{s_1}^s \ln u(\eta) d\eta &\geq \int_{s_1}^s \ln(\alpha_1(\eta - s_1) + \alpha_0) d\eta \\ &= \left[\left(\eta - s_1 + \frac{\alpha_0}{\alpha_1} \right) \ln(\alpha_1(\eta - s_1) + \alpha_0) - \eta \right]_{s_1}^s \\ &= \left(s - s_1 + \frac{\alpha_0}{\alpha_1} \right) \ln(\alpha_1(s - s_1) + \alpha_0) - \frac{\alpha_0}{\alpha_1} \ln \alpha_0 - (s - s_1) \\ &= (s - s_1) \ln(\alpha_1(s - s_1) + \alpha_0) \\ &\quad + \frac{\alpha_0}{\alpha_1} \ln \left(1 + \frac{\alpha_1}{\alpha_0} (s - s_1) \right) - (s - s_1) \\ &\geq (s - s_1) \ln(\alpha_1(s - s_1)) - (s - s_1) \geq -\frac{1}{\alpha_1}, \end{aligned}$$

since the function $t \mapsto t \ln(\alpha_1 t) - t$ defined for $t > 0$ is convex and achieves its minimum for $t = \frac{1}{\alpha_1}$. This completes the proof of item 1. The proof of item 2 follows the same way. \square

5.1. Proof of Proposition 2.3. Fix $a \in \mathbb{R}$ and set $u_c = u(\cdot; a, -1, c)$. Given a compact subset of $(-\infty, -1) \cup (-1, 0]$, we first choose $\delta \in (0, \frac{1}{4})$ such that this compact is included in

$$K = K_+ \cup K_- \text{ with } K_+ = [-\frac{1}{\delta}, -1 - 2\delta] \text{ and } K_- = [-1 + 2\delta, 0].$$

From (3.5) with $s_0 = -1$ we obtain for all $s \in K$:

$$u_c(s)^2 \geq c^2 + 2ac(s+1) + \frac{1}{3}(s+1)^2(2-s).$$

Since the constant term $\frac{1}{3}(s+1)^2(2-s)$ is bounded below by $\frac{2}{3}(2\delta)^2$, we have

$$\exists c_1 \in (0, \frac{\delta}{2}], \forall s \in K, \forall c \in (0, c_1), \quad u_c(s) > \delta. \quad (5.3)$$

Note also that, using the convexity of u_c on $(-\infty, 0]$, we have

$$\forall s \leq 0, \quad u_c(s) \leq c + (s+1)u'_c(s). \quad (5.4)$$

At this step we have to split the proof in two cases, whether $s > -1$ or $s < -1$.

Case 1. Assume first that $s \in K_+$. Using (3.6), the fact that $u_c(\eta) \leq u_c(s)$ for all $\eta \in [-1, s]$ and (5.4), we obtain for all $s \in K_+$ and all $c \in (0, c_1)$

$$u'_c(s)^2 = a^2 - 2s \ln u_c(s) - 2 \ln c + 2 \int_{-1}^s \ln u_c(\eta) d\eta \quad (5.5)$$

$$\begin{aligned} &\leq a^2 - 2 \ln c + 2 \ln u_c(s) \leq a^2 - 2 \ln c + 2 \ln(c + (s+1)u'_c(s)) \\ &\leq a^2 - 2 \ln c + 2 \ln(1 + u'_c(s)). \end{aligned} \quad (5.6)$$

In order to obtain (2.9) we need a converse inequality, and for that purpose we have to distinguish between the cases $a > 0$ and $a \leq 0$.

- If $a > 0$, then (5.1) for $s_1 = -1$ gives $\int_{-1}^s \ln u_c(\eta) d\eta \geq -\frac{1}{a}$ for all $s \in [-1, 0]$. Using (5.3), (5.5) and (5.6) we deduce that for all s in K_+ and all c in $(0, c_1)$

$$a^2 + 2 \ln \delta - \frac{2}{a} + 2 \ln \frac{1}{c} \leq u'_c(s)^2 \leq a^2 + 2 \ln \frac{1}{c} + 2 \ln(1 + u'_c(s)).$$

Since $u'_c(s) > 0$ we obtain

$$u'_c(s) \sim \sqrt{2 \ln \frac{1}{c}} \text{ as } c \rightarrow 0 \text{ uniformly for } s \in K_+. \quad (5.7)$$

- Let us assume now that $a \leq 0$. From (5.3), if $c \leq c_1$, then there exist $s_c \in (-1, -1+2\delta)$ and $\sigma_c \in [-1, s_c)$ such that $u_c(s_c) = 2c$ and $u'_c(\sigma_c) = 0$, see Figure 22. Of course, one has $u'_c(s_c) > 0$.

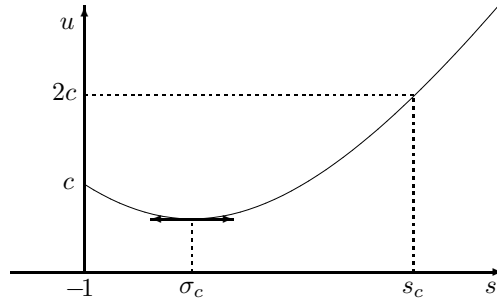


FIGURE 22. For $a < 0$, the graph of u_c near $s = -1$.

We claim that

$$\lim_{c \rightarrow 0} s_c = -1. \quad (5.8)$$

Indeed, from (3.4) with $s_0 = -1$, $s = s_c$, we obtain

$$\begin{aligned} c - a(s_c + 1) &= \int_{-1}^{s_c} \frac{-\eta(s_c - \eta)}{u_c(\eta)} d\eta \\ &\geq \frac{1}{2c} \int_{-1}^{s_c} (\eta^2 - \eta s_c) d\eta = \frac{1}{12c} (s_c + 1)^2 (2 - s_c) \geq \frac{1}{6c} (s_c + 1)^2. \end{aligned}$$

This gives (5.8). Moreover, on the one hand, using (3.6) with $s_0 = -1$ and $s = \sigma_c$, we obtain for all $c \in (0, c_1)$

$$\begin{aligned} -a^2 &= -2\sigma_c \ln u_c(\sigma_c) - 2 \ln c + 2 \int_{-1}^{\sigma_c} \ln u_c(\eta) d\eta \\ &\leq -2\sigma_c \ln u_c(\sigma_c) - 2 \ln c + 2(\sigma_c + 1) \ln c = -2\sigma_c (\ln u_c(\sigma_c) - \ln c), \end{aligned}$$

from which, using $\sigma_c \leq s_c \leq -\frac{1}{2}$, we derive that for $c < c_1$ we have

$$\ln u_c(\sigma_c) \geq \ln c - a^2. \quad (5.9)$$

On the other hand, for $c < c_1$, writing (3.6) with $s_0 = -1$, $s = s_c$, and using (5.9) we obtain

$$\begin{aligned} u'_c(s_c)^2 &= a^2 - 2s_c \ln 2c - 2 \ln c + 2 \int_{-1}^{s_c} \ln u_c(\eta) d\eta \\ &\geq a^2 - 2s_c \ln 2c - 2 \ln c + 2(s_c + 1) \ln u_c(\sigma_c) \\ &\geq -2s_c \ln 2 - (2s_c + 1)a^2 \geq \ln 2, \end{aligned} \quad (5.10)$$

since we have $s_c \leq -\frac{1}{2}$. Consequently, using (5.5), (5.1) and (5.10), we obtain, for $c < c_1$

$$\begin{aligned} u'_c(s)^2 &= a^2 - 2s \ln u_c(s) - 2 \ln c + 2 \int_{-1}^{s_c} \ln u_c(\eta) d\eta + 2 \int_{s_c}^s \ln u_c(\eta) d\eta \\ &\geq a^2 - 2s \ln u_c(s) - 2 \ln c + 2(s_c + 1) \ln u_c(\sigma_c) - \frac{2}{\sqrt{\ln 2}}. \end{aligned}$$

Using (5.9), (5.3) and $s_c \leq -\frac{1}{2}$ we deduce:

$$u'_c(s)^2 \geq 2s_c \ln c - 2s \ln u_c(s) - \frac{2}{\sqrt{\ln 2}} \geq 2s_c \ln c + 2 \ln \delta - \frac{2}{\sqrt{\ln 2}}.$$

Altogether with (5.6), we have for all $s \in K_+$ and all $c \in (0, c_1)$

$$-2s_c \ln \frac{1}{c} + 2 \ln \delta - \frac{2}{\sqrt{\ln 2}} \leq u'_c(s)^2 \leq a^2 + 2 \ln \frac{1}{c} + 2 \ln(1 + u'_c(s)).$$

Using (5.8), we see that (5.7) follows in this case too.

To obtain (2.10) for $s \in K_+$, let us set $v_c = \frac{u'_c}{\sqrt{2 \ln \frac{1}{c}}}$. For all $s \in K_+$ we have

$$\frac{u_c(s)}{\sqrt{2 \ln \frac{1}{c}}} = \frac{c}{\sqrt{2 \ln \frac{1}{c}}} + \int_{-1}^{-1+2\delta} v_c(\eta) d\eta + \int_{-1+2\delta}^s v_c(\eta) d\eta. \quad (5.11)$$

Since $v_c(\eta) \rightarrow 1$ as $c \rightarrow 0$, uniformly on K_+ , we have

$$\int_{-1+2\delta}^s v_c(\eta) d\eta \rightarrow s + 1 - 2\delta \quad \text{as } c \rightarrow 0 \quad \text{uniformly for } s \in K_+. \quad (5.12)$$

Moreover, for all fixed $\eta \in (-1, -1 + 2\delta]$, we have $v_c(\eta) \rightarrow 1$ as $c \rightarrow 0$, and since u_c is convex on $(-\infty, 0]$, the function v_c is increasing on $[-1, -1 + 2\delta]$ and thus

$$\forall \eta \in [-1, -1 + 2\delta], \quad -\infty < \inf_{c>0} v_c(-1) \leq v_c(\eta) \leq \sup_{c>0} v_c(-1 + 2\delta) < +\infty.$$

Therefore, applying the Lebesgue dominated convergence theorem, we obtain

$$\int_{-1}^{-1+2\delta} v_c(\eta) d\eta \rightarrow 2\delta \quad \text{as } c \rightarrow 0. \quad (5.13)$$

Combining (5.11), (5.12) and (5.13) we obtain $\frac{u_c(s)}{\sqrt{2 \ln \frac{1}{c}}} \rightarrow s+1$ as $c \rightarrow 0$, uniformly for $s \in K_+$.

Case 2. Assume now that $s \in K_- = [-\frac{1}{\delta}, -1 - 2\delta]$. Using (3.6), the fact that $u_c(\eta) \leq u_c(s)$ for any $\eta \in [s, -1]$, and (5.3), we obtain for all $s \in K_-$ and all $c \in (0, c_1)$

$$u'_c(s)^2 = a^2 - 2s \ln u_c(s) - 2 \ln c - 2 \int_s^{-1} \ln u_c(\eta) d\eta \quad (5.14)$$

$$\begin{aligned} &\geq a^2 - 2s \ln u_c(s) - 2 \ln c - 2(-1 - s) \ln u_c(s) \\ &= a^2 - 2 \ln c + 2 \ln u_c(s) \geq a^2 - 2 \ln c + 2 \ln \delta. \end{aligned} \quad (5.15)$$

To obtain a converse inequality, we distinguish between the cases $a < 0$ and $a \geq 0$.

- If $a < 0$, then (5.2) for $s_1 = -1$ implies that

$$\int_s^{-1} \ln u_c(\eta) d\eta \geq \frac{1}{a},$$

for all $s \leq -1$. Using (5.4) and (5.14) we obtain for all $s \in K_-$ and all $c \in (0, c_1)$

$$\begin{aligned} u'_c(s)^2 &\leq a^2 - 2s \ln u_c(s) - 2 \ln c - \frac{2}{a} \\ &\leq a^2 - 2s \ln(c + (s+1)u'_c(s)) - 2 \ln c - \frac{2}{a} \\ &\leq a^2 + \frac{2}{\delta} \ln(1 + \frac{1}{\delta}|u'_c(s)|) - 2 \ln c - \frac{2}{a}. \end{aligned} \quad (5.16)$$

Thanks to (5.15) and (5.16) we obtain for all $s \in K_-$ and all $c \in (0, c_1)$

$$a^2 + 2 \ln \delta + 2 \ln \frac{1}{c} \leq u'_c(s)^2 \leq a^2 - \frac{2}{a} + 2 \ln \frac{1}{c} + \frac{2}{\delta} \ln(1 + \frac{1}{\delta}|u'_c(s)|).$$

Since $u'_c(s) < 0$, we deduce

$$u'_c(s) \sim -\sqrt{2 \ln \frac{1}{c}} \quad \text{as } c \rightarrow 0 \quad \text{uniformly for } s \in K_-. \quad (5.17)$$

- Let us assume now that $a \geq 0$. From (5.3), if $c \leq c_1$, then there exist $s_c \in (-\frac{1}{\delta}, -1 - 2\delta)$ and $\sigma_c \in (s_c, -1]$ such that $u_c(s_c) = 2c$ and $u'_c(\sigma_c) = 0$. One has $u'_c(s_c) < 0$ and, exactly as in the previous case, we show

$$\lim_{c \rightarrow 0} s_c = -1. \quad (5.18)$$

Moreover, on the one hand, using (3.6) with $s_0 = -1$ and $s = \sigma_c$, we obtain for all $c \in (0, c_1)$

$$\begin{aligned} a^2 &= -2\sigma_c \ln u_c(\sigma_c) - 2 \ln c - 2 \int_{\sigma_c}^{-1} \ln u_c(\eta) d\eta \\ &\leq -2\sigma_c \ln u_c(\sigma_c) - 2 \ln c - 2(-1 - \sigma_c) \ln u_c(\sigma_c) = -2 \ln c + 2 \ln u_c(\sigma_c), \end{aligned}$$

from which, we derive that for $c < c_1$ we have

$$2 \ln u_c(\sigma_c) \geq 2 \ln c - a^2. \quad (5.19)$$

On the other hand, for $c < c_1$, using (3.6) we obtain

$$\begin{aligned} u'_c(s_c)^2 &= a^2 - 2s_c \ln 2c - 2 \ln c - 2 \int_{s_c}^{-1} \ln u_c(\eta) d\eta \\ &\geq a^2 - 2s_c \ln 2c - 2 \ln c - 2(-1 - s_c) \ln 2c = a^2 - 2 \ln 2, \end{aligned}$$

hence

$$u'_c(s_c) \leq -\sqrt{a^2 - 2 \ln 2}. \quad (5.20)$$

Consequently, using successively (5.14), (5.4), (5.2), (5.19) and (5.20), we obtain, for $c < c_1$ and $s \in K_-$

$$\begin{aligned} u'_c(s)^2 &= a^2 - 2s \ln u_c(s) - 2 \ln c - 2 \int_s^{s_c} \ln u_c(\eta) d\eta - 2 \int_{s_c}^{-1} \ln u_c(\eta) d\eta \\ &\leq a^2 - 2s \ln(c + (s+1)u'_c(s)) - 2 \ln c - \frac{2}{u'_c(s_c)} - 2(-1 - s_c) \ln u_c(s_c) \\ &\leq a^2 + \frac{2}{\delta} \ln(1 + \frac{1}{\delta}|u'_c(s)|) - 2 \ln c + \frac{2}{\sqrt{a^2 - 2 \ln 2}} + (1 + s_c)(2 \ln c - a^2) \\ &\leq -s_c a^2 + \frac{2}{\delta} \ln(1 + \frac{1}{\delta}|u'_c(s)|) + 2s_c \ln c + \frac{2}{\sqrt{a^2 - 2 \ln 2}}, \end{aligned}$$

and with (5.15) we obtain, for $c \in (0, c_1)$ and all $s \in K_-$

$$a^2 + 2 \ln \delta + 2 \ln \frac{1}{c} \leq u'_c(s)^2 \leq \frac{2}{\sqrt{a^2 - 2 \ln 2}} - s_c a^2 - 2s_c \ln \frac{1}{c} + \frac{2}{\delta} \ln(1 + \frac{1}{\delta}|u'_c(s)|).$$

Hence, using (5.18), we see that (5.17) holds in this case too. The relation (2.10) on $K_- = [-\frac{1}{\delta}, -1 - 2\delta]$ is obtained exactly as in the previous case. \square

5.2. Proof of Theorem 2.2, item 4. We need the following result whose proof is postponed to the end of this section.

Lemma 5.2. *Let $a \in \mathbb{R} \cup \{+\infty\}$ and let $(a_n)_{n \in \mathbb{N}}$ be a sequence such that $\lim_{n \rightarrow +\infty} a_n = a$. Let $(c_n)_{n \in \mathbb{N}}$ be another sequence of real number such that $c_n > 0$ for all $n \in \mathbb{N}$. Set $u_n = u(\cdot; a_n, 0, c_n)$. For all $s \in [-1, 0)$ fixed, the following holds:*

1. *If $a = 0$, $\lim_{n \rightarrow +\infty} c_n = 0$ and the sequence $(l_n)_{n \in \mathbb{N}}$ given by*

$$l_n := \sup_{\sigma \in [-1, 0)} |u'_n(\sigma)|,$$

is bounded, then $\lim_{n \rightarrow +\infty} u_n(s) = u_(s)$.*

2. *If $a \in \mathbb{R}^*$ and $\lim_{n \rightarrow +\infty} c_n = 0$, then $\lim_{n \rightarrow +\infty} u_n(s) = u_-^{|a|}(s) = |a|^3 u_-(sa^{-2})$.*
3. *If $a = +\infty$ and $\lim_{n \rightarrow +\infty} \frac{c_n}{a_n} = 0$, then $\lim_{n \rightarrow +\infty} u_n(s) = +\infty$.*

Proof of Theorem 2.2, item 4. Set $u_n = u(\cdot; \alpha_n, -1, \gamma_n)$, $a_n = u'_n(0^-)$ and $c_n = u_n(0^-)$. We define $u_n(0)$ by its limit c_n if necessary.

By the continuous dependence of solutions of ODEs with respect to initial conditions, we have for all $s < 0$ fixed, $u_n(s) \rightarrow u_-(s)$ and $u'_n(s) \rightarrow u'_-(s)$ as $n \rightarrow +\infty$. Since u_n is convex on $[-1, 0]$, u_n has a minimum at some $s_n \in (-1, 0]$ that satisfies $\lim_{n \rightarrow +\infty} s_n = \lim_{n \rightarrow +\infty} u_n(s_n) = 0$. Besides, integrating

$$\alpha_n \leq u'_n(s) \leq a_n, \quad (5.21)$$

on $[s_n, 0]$, we obtain

$$-\alpha_n s_n \leq c_n - u_n(s_n) \leq -a_n s_n. \quad (5.22)$$

Case 1. Suppose first that c_n is non zero for an infinite number of values of n . Considering a subsequence if necessary, we may assume without loss of generality that $c_n > 0$ for all n .

If (a_n) were unbounded, then considering once again a subsequence if necessary, we may assume without loss of generality that (a_n) is a positive sequence that tends to $+\infty$ as $n \rightarrow +\infty$, hence $\frac{c_n}{a_n} \rightarrow 0$ by (5.22). By Lemma 5.2, item 3, we would obtain a contradiction with $u_n(-1) = \gamma_n \rightarrow u_-(-1)$. This shows that the sequence (a_n) is bounded. From (5.22), the sequence (c_n) tends to 0. Let a be one of the cluster points of (a_n) .

• If $a = 0$, then because of (5.21), the assumptions of Lemma 5.2, item 1 would be satisfied. Since u_- and u_* cannot coincide on the whole $[-1, 0)$, we obtain once again a contradiction.

• If $a \neq 0$, then item 2 of Lemma 5.2 yields $u_- = u_-^{|a|}$ on $[-1, 0)$, hence $|a| = 1$.

Case 2. If c_n is non-zero only for a finite number of values of n , then considering once again a subsequence if necessary, we may assume without loss of generality that $c_n = 0$ for all n , therefore either u_n belongs to the family $(u_-^\sigma)_{\sigma>0}$ or $u_n = u_*$. Since $u_n \rightarrow u_-$ on $[-1, 0)$, it follows that for all n large enough, $u_n = u_-^{\sigma_n}$ for some $\sigma_n \in (0, +\infty)$ and necessarily $\sigma_n \rightarrow 1$ as $n \rightarrow +\infty$. Therefore, $a_n = (u_-^{\sigma_n})'(0^-) = -\sigma_n \rightarrow -1$. \square

The proof of Lemma 5.2 needs two technical results. The first one (Proposition 5.3) follows from elementary computations. The second one (Proposition 5.4) uses the deep asymptotic result given by Proposition 2.3.

Proposition 5.3. *Let $a < 0$. Let $(a_n)_{n \in \mathbb{N}}$, $(b_n)_{n \in \mathbb{N}}$, $(c_n)_{n \in \mathbb{N}}$ be sequences such that $a_n < 0$, $b_n \leq 0$, $c_n > 0$, $\lim_{n \rightarrow +\infty} a_n = a$, $\lim_{n \rightarrow +\infty} b_n = 0$, $\lim_{n \rightarrow +\infty} c_n = 0$ and $\lim_{n \rightarrow +\infty} b_n \ln c_n = 0$. Then, for all $s < 0$, we have*

$$\lim_{n \rightarrow +\infty} u(s; a_n, b_n, c_n) = -a^3 u_-(sa^{-2}).$$

Proof. The solution $u_n(s) := u(s; a_n, b_n, c_n)$ is defined for all $s < 0$ and satisfies $u_n(s) > 0$. Thus we have $u_n''(s) = -\frac{s}{u_n(s)} > 0$. Integrating twice we obtain

$$u_n(s) \geq c_n + a_n(s - b_n), \quad \text{for all } s < 0, \quad (5.23)$$

hence $0 < u_n''(s) \leq \frac{-s}{c_n + a_n(s - b_n)}$ for all $s < b_n$. Integrating once we obtain for all $s < b_n$

$$a_n - \frac{1}{a_n}(s - b_n) + \frac{c_n - a_n b_n}{a_n^2} \ln \frac{a_n(s - b_n) + c_n}{c_n} \leq u_n'(s) \leq a_n. \quad (5.24)$$

Integrating once again we obtain for all $s < b_n$

$$u_n(s) \leq c_n + a_n(s - b_n) - \frac{1}{2a_n}(s - b_n)^2 + \frac{c_n - a_n b_n}{a_n^3} \left((a_n(s - b_n) + c_n) \ln \frac{a_n(s - b_n) + c_n}{c_n} - a_n(s - b_n) \right). \quad (5.25)$$

Since $b_n \ln c_n \rightarrow 0$ as $n \rightarrow +\infty$, a close examination of (5.24) and (5.25) shows that the sequence (u_n) is uniformly equicontinuous and bounded on $[-1, -\delta]$ for all $\delta > 0$ and n large enough. Therefore, some subsequence u_{n_k} converges to some function u , which is a solution of (2.2), because

$$u''(s) = \lim_{k \rightarrow +\infty} u_{n_k}''(s) = \lim_{k \rightarrow +\infty} \left(-\frac{s}{u_{n_k}(s)} \right) = -\frac{s}{u(s)},$$

uniformly on each compact subset of $[-1, 0)$. Going to the limit in equations (5.23) and (5.25), we obtain $as \leq u(s) \leq as - \frac{1}{2a}s^2$ for all $s < 0$. We deduce that $u(0^-) = 0$ and $u'(0^-) = a$. By the uniqueness of $u_-^\sigma : s \mapsto \sigma^3 u_-(\sigma^{-2}s)$ as solution of (2.2) with condition $u_-^\sigma(0^-) = 0$, $(u_-^\sigma)'(0^-) = -\sigma$, we obtain that $u = u_-^\sigma$ with $\sigma = -a$.

We proved that any cluster point of the uniformly equicontinuous sequence (u_n) has to be equal to this u , hence (u_n) converges to u . \square

Proposition 5.4. *Assume that $a > 0$ and $\varepsilon > 0$. Then $u(s; a, 0, \varepsilon)$ reaches its minimum at $s = -\kappa(\varepsilon)$, where $\kappa(\varepsilon) > 0$ and, as $\varepsilon \rightarrow 0$, we have*

$$\kappa(\varepsilon) = \frac{\varepsilon}{a}(1 + o(1)) \text{ and } u(-\kappa(\varepsilon); a, 0, \varepsilon) = \exp\left(-\frac{a^3}{2\varepsilon}(1 + o(1))\right).$$

Moreover, we have

$$u(\varepsilon s; a, 0, \varepsilon) = \varepsilon(|as + 1| + o(1)) \text{ and } u'(\varepsilon s; a, 0, \varepsilon) = a \operatorname{sgn}(as + 1) + o(1)$$

as $\varepsilon \rightarrow 0$, uniformly on any compact subset of $(-\infty, -\frac{1}{a}) \cup (-\frac{1}{a}, 0]$.

Proof. Let us set $c(\varepsilon) = u(-\kappa(\varepsilon); a, 0, \varepsilon)$. For convenience, we omit the dependence in ε of κ and c and we write $o(1)$ for a function of ε tending to 0 with ε . By the similarity property (2.5), we have

$$u(\varepsilon s; a, 0, \varepsilon) = \kappa^{3/2} u\left(\frac{\varepsilon s}{\kappa}; \frac{a}{\kappa^{1/2}}, 0, \frac{\varepsilon}{\kappa^{3/2}}\right).$$

On the other hand, we have

$$u(\varepsilon s; a, 0, \varepsilon) = u(\varepsilon s; 0, -\kappa, c) = \kappa^{3/2} u\left(\frac{\varepsilon s}{\kappa}; 0, -1, \frac{c}{\kappa^{3/2}}\right). \quad (5.26)$$

Thus, $u'(0; 0, -1, \frac{c}{\kappa^{3/2}}) = \frac{a}{\kappa^{1/2}}$ and $u(0; 0, -1, \frac{c}{\kappa^{3/2}}) = \frac{\varepsilon}{\kappa^{3/2}}$. Formula (3.3) with $s_0 = -\kappa$ and $s = 0$ yields

$$a = \int_{-\kappa}^0 \frac{-\eta}{u(\eta; a, 0, \varepsilon)} d\eta.$$

Since $c \leq u(s; a, 0, \varepsilon) \leq \varepsilon$ for all $s \in [-\kappa, 0]$, we have $\frac{\kappa^2}{2\varepsilon} \leq a \leq \frac{\kappa^2}{2c}$. Hence, $0 < \kappa \leq \sqrt{2\varepsilon a}$ and $\frac{c}{\kappa^{3/2}} \leq \sqrt{\frac{\kappa}{2a}}$. Thus, $\frac{c}{\kappa^{3/2}} = o(1)$. By Proposition 2.3 we have

$$\frac{a}{\kappa^{1/2}} = \sqrt{2 \ln \frac{\kappa^{3/2}}{c}}(1 + o(1)) \text{ and } \frac{\varepsilon}{\kappa^{3/2}} = \sqrt{2 \ln \frac{\kappa^{3/2}}{c}}(1 + o(1)), \quad (5.27)$$

hence $\kappa = \frac{\varepsilon}{a}(1 + o(1))$. From (5.27) we deduce

$$\frac{a^2}{\kappa} = \left(2 \ln \frac{1}{c} + 3 \ln \kappa\right)(1 + o(1)) = \left(2 \ln \frac{1}{c}\right)(1 + o(1)),$$

thus,

$$c = \exp\left(-\frac{a^2}{2\kappa}(1 + o(1))\right) = \exp\left(-\frac{a^3}{2\varepsilon}(1 + o(1))\right).$$

From (5.26), (5.27), Proposition 2.3 and $\frac{\varepsilon}{\kappa} = a(1 + o(1))$, we have

$$\begin{aligned} u(\varepsilon s; a, 0, \varepsilon) &= \kappa^{3/2} u\left(as(1 + o(1)); 0, -1, \frac{c}{\kappa^{3/2}}\right) \\ &= \kappa^{3/2} \sqrt{2 \ln \frac{\kappa^{3/2}}{c}} |as(1 + o(1)) + 1|(1 + o(1)) = \varepsilon(|as + 1| + o(1)), \end{aligned}$$

uniformly on any compact subset of $(-\infty, -\frac{1}{a}) \cup (-\frac{1}{a}, 0]$. Similarly, from (5.27) and Proposition 2.3, we have

$$\begin{aligned} u'(\varepsilon s; a, 0, \varepsilon) &= \kappa^{1/2} u'\left(as(1 + o(1)); 0, -1, \frac{c}{\kappa^{3/2}}\right) \\ &= \kappa^{1/2} \sqrt{2 \ln \frac{\kappa^{3/2}}{c}} \operatorname{sgn}(as(1 + o(1)) + 1)(1 + o(1)) \\ &= a \operatorname{sgn}(as + 1) + o(1), \end{aligned}$$

uniformly on any compact subset of $(-\infty, -\frac{1}{a}) \cup (-\frac{1}{a}, 0]$. This completes the proof. \square

Proof of Lemma 5.2. Let $s \in [0, 1]$ be fixed.

1. By contradiction, if it were not the case, then there would exist a subsequence $(u_{n_k}(s))_{n \in \mathbb{N}}$ such that any possible cluster point would be different from $u_*(s)$. However, the sequence of functions $(u_{n_k})_{n \in \mathbb{N}}$ is uniformly equicontinuous and bounded on $[-1, 0]$, therefore has at least a cluster point u which is a Crocco solution and satisfies $u(0^-) = u'(0^-) = 0$. This would imply $u = u_*$ and the contradiction.

2. If $a < 0$, the result follows from Proposition 5.3 with $b_n = 0$. If $a > 0$, let $b_n = -2\frac{c_n}{a_n}$, $\tilde{a}_n = u'_n(b_n)$ and $\tilde{c}_n = u_n(b_n)$. Set $r_n = \frac{a_n}{a}$. From (2.5), we have

$$\tilde{c}_n = r_n^3 u(\varepsilon_n \sigma; a, 0, \varepsilon_n) \quad \text{with} \quad \sigma = -\frac{2}{a}, \quad \varepsilon_n = r_n^{-3} c_n \rightarrow 0.$$

By Proposition 5.4, we have

$$\tilde{c}_n = r_n^3 \varepsilon_n (|a\sigma + 1| + o(1)) = c_n (1 + o(1)).$$

Thus, $\tilde{c}_n \sim c_n \rightarrow 0$ and $b_n \ln \tilde{c}_n \rightarrow 0$. Similarly, we prove that $\tilde{a}_n \rightarrow -a$. Since $u_n(s) = u(s; \tilde{a}_n, b_n, \tilde{c}_n)$, the result follows from Proposition 5.3.

3. We have $u_n(s) = a_n^3 u(sa_n^{-2}; 1, 0, c_n a_n^{-3})$ and $c_n a_n^{-3} \rightarrow 0$ as $n \rightarrow +\infty$. Therefore, by Proposition 5.4, u_n has a minimum at some s_n satisfying

$$s_n a_n^{-2} = -c_n a_n^{-3} (1 + o(1)),$$

hence, $s_n = -c_n a_n^{-1} (1 + o(1))$. Again from Proposition 5.4, we obtain that u_n takes the value c_n at another point $\tilde{s}_n < 0$ satisfying

$$\tilde{s}_n = -2c_n a_n^{-1} (1 + o(1)) \quad \text{and} \quad \tilde{a}_n := u'_n(\tilde{s}_n) = -a_n (1 + o(1)).$$

Since, $u_n(s) \geq -\tilde{a}_n(s - \tilde{s}_n) + c_n$, we obtain $u_n(s) \rightarrow +\infty$ as $n \rightarrow +\infty$. \square

5.3. Last proofs.

Proof of Theorem 2.2, item 1. Let u denote the solution of (2.2-2.3) with $b < 0$ and $c > 0$. By Proposition 3.1, u is defined at least on $[b, 0)$. By (2.11) we have $u(s) = (-s)^{3/2} y(-\ln(-s))$, and by Theorem 2.4 item 1, $y(t) \rightarrow \frac{2}{\sqrt{3}}$ as $t \rightarrow -\infty$. It follows that u is defined at least on $(-\infty, b]$ and $u(s) \sim \frac{2}{\sqrt{3}} (-s)^{3/2}$ as $s \rightarrow -\infty$. Section 9.2 contains an alternative proof that uses only the Crocco equation. \square

Proof of Theorem 2.2, item 2. For an existence proof, let $(x, y) \in \Gamma_\infty$. Thanks to Theorem 2.4, item 4 (2), the Crocco solution $u_{x,y} := u(\cdot; x, -1, y)$ satisfies $u_{x,y}(0^-) = 0$ and $u'_{x,y}(0^-) = -\delta < 0$. Now, the solution $u_- : s \mapsto \delta^{-3} u_{x,y}(\delta^2 s)$ satisfies $u_-(0^-) = 0$ and $u'_-(0^-) = -1$. For any $\sigma > 0$, the Crocco solution $u_-^\sigma : s \mapsto \sigma^3 u_-(\sigma^{-2} s)$ satisfies $u_-^\sigma(0^-) = 0$ and $(u_-^\sigma)'(0^-) = -\sigma$. Uniqueness follows from the fact that Γ_∞ corresponds to a unique family $(u^\sigma)_{\sigma > 0}$ of solutions of the Crocco equation. \square

Proof of Theorem 2.5. The continuity outside of $\Gamma_\infty \cup \{S^*\}$ comes from the fact that Λ is non-zero on this set (see Theorem 2.4, item 4) and from Proposition 3.2. The continuity at S^* will be shown at the end of the proof.

We now study the discontinuity on Γ_∞ . Consider first the point (a, c) where $a = u_-(-1)$ and $c = u'_-(-1)$. Let $(\alpha_n, \gamma_n)_{n \in \mathbb{N}}$ be a sequence tending to (a, c) either from the convex side of Γ_∞ , or from its concave side (this means in particular $(\alpha_n, \gamma_n) \notin \Gamma_\infty$). Let us set $u_n = u(\cdot; \alpha_n, -1, \gamma_n)$, $a_n = u'_n(0)$ and $c_n = u_n(0) > 0$. Let us notice that

$$\tilde{\Lambda}(\alpha_n, \gamma_n) = \Lambda(\alpha_n, -1, \gamma_n) = \Lambda(a_n, 0, c_n).$$

Let $k_n = \frac{a_n^3}{c_n}$. By Theorem 2.4, items 5 and 6, we have $k_n \rightarrow -\infty$ on the convex side and $k_n \rightarrow +\infty$ on the concave side, as $n \rightarrow +\infty$. Therefore, $c_n = k_n^{-1} a_n^3 \rightarrow 0$ on both sides.

By Theorem 2.2, item 4, the sequence $(a_n)_{n \in \mathbb{N}}$ is bounded and has at most two cluster points: -1 and 1 . Without loss of generality, we can assume that $a_n \neq 0$. Since k_n is positive on the concave side and negative on the convex side of Γ_∞ , we obtain that:

- On the convex side, a_n has only -1 as a cluster point and is bounded, hence tends to -1 . Since the solution u_n is concave for $s > 0$ we have $\Lambda(a_n, 0, c_n) \leq -\frac{c_n}{a_n} \rightarrow 0$.
- On the concave side, $u'_n(0)$ tends to $+1$, hence from Proposition 3.3 we obtain

$$\Lambda(a_n, 0, c_n) = a_n^2 \Lambda(1, 0, k_n^{-1}) \rightarrow \lambda_+.$$

For another point (a, c) on Γ_∞ , let t be such that $a = x_-(t)$, $c = y_-(t)$. By (2.16), the discontinuity of $\tilde{\Lambda}$ at (a, c) is equal to $\lambda_+ e^t$.

As the point $(a, c) \rightarrow S^*$, then $t \rightarrow -\infty$ and the discontinuity tends to 0. This implies that $\tilde{\Lambda}$ is continuous at S^* . \square

Proof of Proposition 2.6, items 2. In this proof, the notation a_1 refers to Proposition 1.1, item 2, *i.e.*, a_1 is the first point counted from the right of intersection of Γ_∞ and I_∞ (defined by $y = -\frac{2}{x}$), the first branch of the orbit Γ_∞ included in the North region is the graph of the function $y = R_1(x)$ defined for $x \leq a_1$, see Figure 8, right. Let ψ be the function defined for $a < 0$ by

$$\psi(a) = -\frac{2}{a} \text{ if } a \in [a_1, 0), \quad \psi(a) = R_1(a) \text{ if } a \leq a_1.$$

We have to prove that for all $a < 0$, $\tilde{\Lambda}_a : c \mapsto \Lambda(a, -1, c)$ is increasing on $[\psi(a), +\infty)$.

Let $a < 0$ and $c_2 > c_1 > \psi(a)$. Let $u_i(s) = u(s; a, -1, c_i)$, $i = 1, 2$, and let $(x_i(t), y_i(t))$ denote the corresponding solutions of system (2.12). Since the orbit of (x_1, y_1) is on the left of the one of (x_2, y_2) , we deduce that $k_1 < k_2 (< 0)$, where $k_i = \frac{u'_i(0)^3}{u_i(0)}$. Thus we have

$$u'_1(0) < \sigma u'_2(0), \quad \text{where } \sigma := \left(\frac{u_1(0)}{u_2(0)} \right)^{1/3}. \quad (5.28)$$

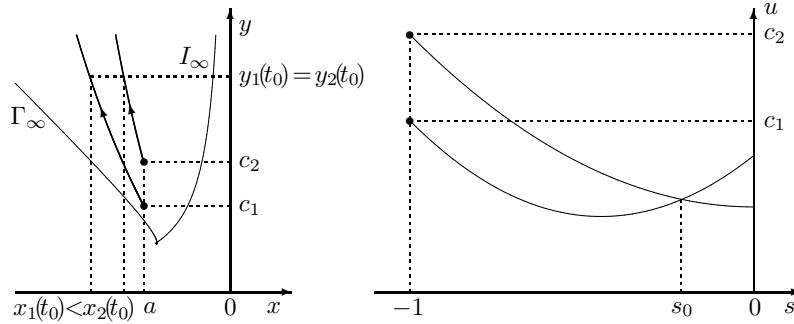


FIGURE 23. On the left: two solutions of (2.12) with initial conditions on the same vertical ray. If their associated Crocco solutions (on the right) were crossing, then this would yield a contradiction.

By contradiction, let us suppose that there exists $s \in [-1, 0]$ such that $u_1(s) \geq u_2(s)$ and consider

$$s_0 := \inf\{s > -1 : u_1(s) \geq u_2(s)\}.$$

Then, we have $u_1(s_0) = u_2(s_0)$ and $u_1(s) < u_2(s)$ for all $s \in [-1, s_0)$, hence $u'_2(s_0) \leq u'_1(s_0)$. This means that $y_1(t_0) = y_2(t_0)$ and $x_1(t_0) \geq x_2(t_0)$, where $t_0 = -\ln(-s_0)$. This is impossible

(see Figure 23) and thus $u_1(s) < u_2(s)$ for all $s \in [-1, 0]$ and then $u_1(0) < u_2(0)$. Consequently $\sigma < 1$, hence from item 1 of this Proposition 2.6 and (5.28) we obtain

$$\begin{aligned} \tilde{\Lambda}(a, c_1) &= \Lambda(a, -1, c_1) = \Lambda(u'_1(0), 0, u_1(0)) < \Lambda(\sigma u'_2(0), 0, u_1(0)) \\ &< \frac{1}{\sigma^2} \Lambda(\sigma u'_2(0), 0, u_1(0)) = \frac{1}{\sigma^2} \Lambda(\sigma u'_2(0), 0, \sigma^3 u_2(0)) \\ &= \Lambda(u'_2(0), 0, u_2(0)) = \Lambda(a, -1, c_2) = \tilde{\Lambda}(a, c_2). \end{aligned} \quad (5.29)$$

Proof of Proposition 2.6, item 3. The proof is very similar to that of item 2. Let c_1 be the first point (counted from below) of intersection of Γ_∞ and I_0 (defined by $y = -\frac{2}{3}x$), so that the first branch of the orbit Γ_∞ included in the East and in the North regions is the graph of the function $x = x_\infty(y)$ introduced in Section 4.2 and defined for $y \geq c_1$. Observe that x_∞ is the inverse function of R_1 introduced in Section 2.4 only on the interval $[-\frac{2}{a_1}, +\infty)$. Let ϕ be the function defined for $c > 0$ by

$$\phi(c) = -\frac{3c}{2} \quad \text{if } c \in (0, c_1], \quad \phi(c) = x_\infty(c) \quad \text{if } c \geq c_1.$$

Let $c > 0$ and $a_2 > a_1 > \phi(c)$. We have to prove that $\tilde{\Lambda}(a_1, c) < \tilde{\Lambda}(a_2, c)$.

Consider $u_i = u(\cdot; a_i, -1, c)$, $i = 1, 2$ and let $(x_i(t), y_i(t))$ denote the corresponding solutions of (2.12). Since the orbit of (x_1, y_1) is on the left of the one of (x_2, y_2) , we have $k_1 < k_2$, where $k_i = \frac{u'_i(0)^3}{u_i(0)}$. Thus we have

$$u'_1(0) < \sigma u'_2(0), \quad \text{with } \sigma := \left(\frac{u_1(0)}{u_2(0)} \right)^{1/3}. \quad (5.30)$$

As before, we prove by contradiction that $u_1(s) < u_2(s)$ for all $s \in [-1, 0]$. *A fortiori* we have $u_1(0) < u_2(0)$, and thus $\sigma < 1$. Item 1 and (5.30) yield $\tilde{\Lambda}(a_1, c) < \tilde{\Lambda}(a_2, c)$ similarly to (5.29). \square

Proof of Proposition 2.7. We use the notation $u = u(\cdot; a, -1, c)$. Concerning formula (2.17), by Proposition 2.3 we have

$$u(0) \sim \sqrt{2 \ln \frac{1}{c}} \quad \text{and} \quad u'(0) \sim \sqrt{2 \ln \frac{1}{c}} \quad \text{as } c \rightarrow 0,$$

hence $\frac{u(0)}{u'(0)^3} \rightarrow 0$ as $c \rightarrow 0$. Then by the similarity (2.15):

$$\begin{aligned} \tilde{\Lambda}(a, c) &= \Lambda(a, -1, c) = \Lambda(u'(0), 0, u(0)) \\ &= u'(0)^2 \Lambda\left(1, 0, \frac{u(0)}{u'(0)^3}\right) \sim u'(0)^2 \lambda_+ \sim 2\lambda_+ \ln \frac{1}{c}. \end{aligned}$$

Formula (2.18) results from Proposition 3.4 with $b = -1$. \square

6. CANARD SOLUTIONS

Two important properties of the Crocco equation are, on the one hand the behavior of solutions starting close to u_- at $t = -1$, as expressed in Theorem 2.2 item 4, and on the other hand the behavior of solutions starting close to 0 at $t = -1$, as described in Proposition 2.3. In the previous section, we first proved in details Proposition 2.3, then deduced Proposition 5.4, and finally deduced Theorem 2.2 item 4. As a consequence, both properties are closely related, and the link is the invariance property (1.3).

In this section, we present another proof of these results, based on the asymptotic behavior, as $\varepsilon \rightarrow 0^+$, of solutions of the ODE with small parameter $\varepsilon > 0$

$$U \frac{d^2 U}{dS^2} + \varepsilon S = 0. \quad (6.1)$$

We will describe the solutions of this equation and deduce Proposition 2.3 and Proposition 5.4 from this study. We hope that this alternative viewpoint will help the reader to have a new insight of the asymptotic properties of Crocco solutions.

Equation (6.1) can be transformed into a slow-fast system (see system (6.9) below) which has *canard solutions*. Canard solutions are special trajectories of slow-fast systems that first move near the stable part of the slow manifold, then move near the unstable part of it. These solutions were first discovered by E. Benoît, J.-L. Callot, F. Diener and M. Diener and studied in the framework of Non-standard Analysis, see [6, 16, 56] for historical comments and references. Related to canard solutions is also the important phenomenon of *stability loss delay in dynamical bifurcations*, see [2] pp. 179-192 and [5]. The study of canard solutions has also been made in the framework of classical asymptotic analysis [18], center manifold theory [17] and Gevrey complex asymptotics [20].

In the present context, the situation is particularly simple and does not need the whole theory of canards. Therefore, we provide a complete proof.

6.1. Canard solutions of equation (6.1). We have the following result.

Theorem 6.1. 1. *Let $A \in \mathbb{R}$, $B \leq 0$ and $C > 0$ be given and let $U(S, \varepsilon)$ denote the solution of (6.1) with initial conditions*

$$U(B) = C, \quad \frac{dU}{dS}(B) = A. \quad (6.2)$$

Then we have

$$U(S, \varepsilon) = |A(S - B) + C| + o(1) \quad \text{and} \quad \frac{dU}{dS}(S, \varepsilon) = A \operatorname{sgn}(A(S - B) + C) + o(1),$$

as $\varepsilon \rightarrow 0$, uniformly for S in any compact subset of $(-\infty, 0] \setminus \{B - C/A\}$.

If $A < 0$ and $AB - C > 0$ (or $A > 0$), then $U(S, \varepsilon)$ reaches its minimum at $K(\varepsilon) < 0$ satisfying

$$K(\varepsilon) = B - \frac{C}{A} + o(1) \quad \text{and} \quad U(K(\varepsilon), \varepsilon) = \exp\left(\frac{A^3 + o(1)}{2(AB - C)\varepsilon}\right), \quad (6.3)$$

as $\varepsilon \rightarrow 0$.

2. *Let $A \in \mathbb{R}$ and $K < 0$ be given and let $U(S, \varepsilon)$ denote a solution of (6.1) with initial conditions satisfying*

$$U(K) = \exp\left(\frac{A^2 + o(1)}{2K\varepsilon}\right), \quad \frac{dU}{dS}(K) = o(1). \quad (6.4)$$

Then we have

$$U(S, \varepsilon) = |A(S - K)| + o(1) \quad \text{and} \quad \frac{dU}{dS}(S, \varepsilon) = |A| \operatorname{sgn}(S - K) + o(1),$$

as $\varepsilon \rightarrow 0$, uniformly for S in any compact subset of $(-\infty, K) \cup (K, 0]$.

A complete proof of this result will be given in Section 6.2 ; we give here an idea of proof of item 1. Except near the axis $U = 0$, and for bounded values of S , U'' is close to 0, *i.e.*, the solutions are almost affine. Therefore, as long as $A(S - B) + C$ is positive, the approximation (6.3) is valid and quite natural indeed. What is less obvious is that the solution satisfies the same approximation after its passage near the axis, like a light ray reflecting on a mirror.

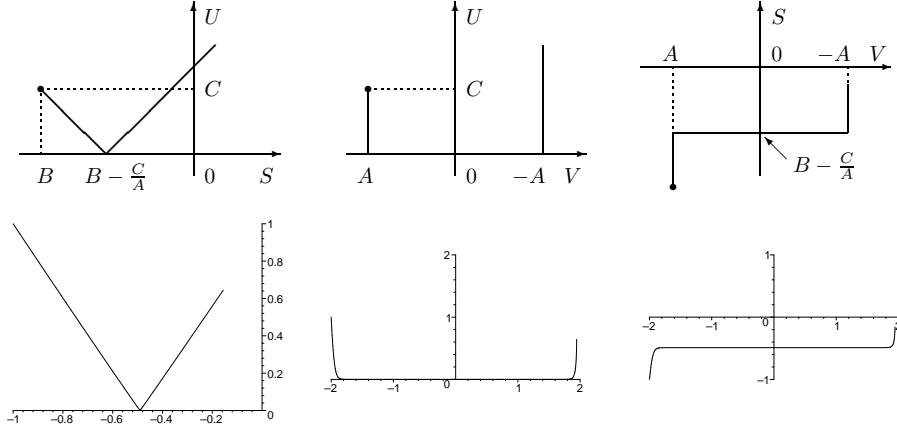


FIGURE 24. Above: schematic graphs of the solution of (6.1-6.2) in the limit $\varepsilon \rightarrow 0$, respectively in the variables (S, U) , the variables (U, V) and (S, V) . Below: the numerical solution corresponding to $\varepsilon = 0.1$, $A = -2$, $B = -1$ and $C = 1$.

To see what happens near $U = 0$, the best way is first to rewrite (6.1-6.2) as a first order initial value problem

$$\frac{dU}{dS} = V, \quad \frac{dV}{dS} = -\varepsilon \frac{S}{U}, \quad U(B) = C, \quad V(B) = A, \quad (6.5)$$

and then to choose V as independent variable, *i.e.*,

$$\varepsilon \frac{dS}{dV} = -\frac{U}{S}, \quad \varepsilon \frac{dU}{dV} = -\frac{VU}{S}, \quad S(A) = B, \quad U(A) = C. \quad (6.6)$$

Consider for instance the case $A, B < 0$ and $C < AB$. Then, in the U, V variables, V remains close to A until U is close to 0, where V suddenly changes its value from its *entry value* A to some *exit value*, meanwhile U and S remain almost constant, respectively close to 0, and close to $S_0 = B - \frac{C}{A} < 0$, see Figure 24. With this approximation for $S = S(V)$, the second equation of (6.6) appears as a singularly perturbed ODE exhibiting canards with a symmetric entry-exit relationship. Actually, this second equation can be rewritten in the integral form

$$U(V) = U(0) \exp\left(-\frac{1}{\varepsilon} \int_0^V \frac{v dv}{S(v)}\right) = U(0) \exp\left(-\frac{V^2(1 + o(1))}{2(B - \frac{C}{A})\varepsilon}\right),$$

as long as $U(V)$ is small. In fact, this approximation remains valid in the boundary layers located near $V = \pm A$. Using $V(A) = C = \exp(\frac{o(1)}{\varepsilon})$, we deduce that $U(0)$ is exponentially small compared to ε . Precisely we have

$$U(0) = \exp\left(\frac{A^2(1 + o(1))}{2(B - \frac{C}{A})\varepsilon}\right).$$

It follows that $U(V)$ is exponentially small for all fixed $V \in (A, -A)$ and that U is bounded below for some V close to $-A$, *i.e.*, with a reflection angle opposite to the incident angle. We refer the reader to the next section for a more precise and complete proof.

6.2. Proof of Theorem 6.1. For the sake of simplicity, the notation $o(1)$ implicitly means ‘as $\varepsilon \rightarrow 0$ ’.

Proof of item 1. The solution U of (6.1-6.2) is defined for all $S < 0$, and satisfies $U(S, \varepsilon) > 0$. If both $A < 0$ and $AB - C < 0$, then U is decreasing. If $A < 0$ and $AB - C > 0$ (or $A > 0$),

then there exists $K(\varepsilon) < 0$ such that U is decreasing on $(-\infty, K(\varepsilon)]$ and increasing on $[K(\varepsilon), 0)$. Assume that $A < 0$ and $AB - C > 0$ (the other cases are similar). By the continuous dependence of the solutions with respect to the parameters, for all L and B_0 , such that $L < B_0 < B - \frac{C}{A}$, we have

$$\frac{dU}{dS}(S, \varepsilon) = A + o(1) \quad \text{and} \quad U(S, \varepsilon) = A(S - B) + C + o(1), \quad (6.7)$$

uniformly for $S \in [L, B_0]$. Notice that L and B_0 are fixed but may be chosen as close to $-\infty$, resp. $B - \frac{C}{A}$, as we want. The problem now is to determine the asymptotic behavior of $U(S, \varepsilon)$ for $S \geq B_0$. Consider (6.5). Since $\frac{dV}{dS} > 0$ for all $S < 0$ and $U > 0$, we can use V as an independent variable. Hence, for all $S < 0$ and $U > 0$, the functions $S(V, \varepsilon)$ and $U(V, \varepsilon)$ are solutions of (6.6). Notice that

$$S(V) = \text{constant}, \quad U(V) = 0, \quad (6.8)$$

are solutions of the system in (6.6) but these solutions do not correspond to actual solutions of the system in (6.5). Let us study the asymptotic behavior of the solutions which satisfy $U > 0$. In order to transform system (6.6) into a slow-fast system in \mathbb{R}^3 with two slow variables and one fast variable, we use the variable $T = VS - U$ instead of U . Then (6.6) becomes

$$\varepsilon \frac{dS}{dV} = \frac{T - VS}{S}, \quad \frac{dT}{dV} = S, \quad S(A) = B, \quad T(A) = AB - C. \quad (6.9)$$

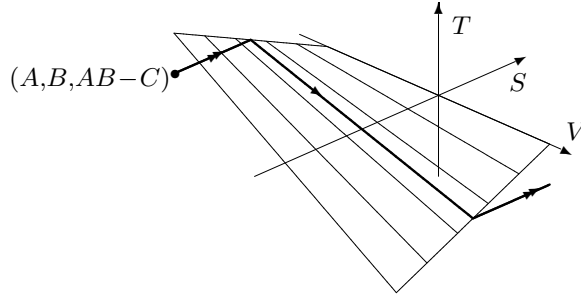


FIGURE 25. The approximation of the solution of problem (6.9) given by Tikhonov's theory.

This is a singularly perturbed system whose slow manifold is the surface defined by $T = VS$. This slow manifold is attracting for $V < 0$ and repulsive for $V > 0$. Tikhonov's theorem (see [45, 38] and [52] Section 39) describes the behavior of the solution $(S(V, \varepsilon), T(V, \varepsilon))$ of (6.9) in the following manner. There is a fast transition (see Figure 25) taking the trajectory $(V, S(V, \varepsilon), T(V, \varepsilon))$, from its initial point $(A, B, AB - C)$, to a $o(1)$ -neighborhood of the point $(A, B - \frac{C}{A}, AB - C)$ of the slow manifold, followed by a slow transition near the solution $S_0(V) = B - \frac{C}{A}$, $T_0(V) = (B - \frac{C}{A})V$ of the reduced problem

$$S = \frac{T}{V}, \quad \frac{dT}{dV} = S, \quad T(A) = AB - C.$$

More precisely, for any A_0 and A_1 , such that $A < A_0 < A_1 < 0$, we have

$$\begin{aligned} S(V, \varepsilon) &= B - \frac{C}{A} + o(1) \quad \text{uniformly for } V \in [A_0, A_1], \\ T(V, \varepsilon) &= \left(B - \frac{C}{A} \right) V + o(1) \quad \text{uniformly for } V \in [A, A_1]. \end{aligned} \quad (6.10)$$

Notice that A_0 (resp. A_1) is fixed but may be chosen as close to A (resp. 0) as we want. The approximation for S does not hold near A because of the boundary layer at A . We deduce that

$$U(V, \varepsilon) = VS(V, \varepsilon) - T(V, \varepsilon) = o(1) \quad \text{uniformly for } V \in [A_0, A_1]. \quad (6.11)$$

A priori, Tikhonov's theorem applies only for $V < 0$, because if $V > 0$, then the slow manifold becomes repulsive. However, we will see that (6.11) still holds for positive values of V . This is the so-called *bifurcation delay* [5]. The slow manifold is foliated by the explicit solutions

$$S(V) = S_0 = \text{constant}, \quad T(V) = VS_0,$$

corresponding to the solutions (6.8). These solutions are *canard solutions*: they follow the attracting part, then the repulsive part, of the slow manifold. Knowing the entry value $V = A$ of the solution $T(V, \varepsilon)$ in a small neighborhood of the slow manifold, we want to compute the exit value for which the solution is again far from the slow manifold. Since $U = VS - T > 0$, we use the new variable $W = \varepsilon \ln U$. Then (6.6) becomes

$$\frac{dS}{dV} = -\frac{e^{W/\varepsilon}}{\varepsilon S}, \quad \frac{dW}{dV} = -\frac{V}{S}, \quad (6.12)$$

with initial conditions

$$S(A, \varepsilon) = B, \quad W(A, \varepsilon) = \varepsilon \ln(C). \quad (6.13)$$

Since $W < 0$ and $S < 0$, we have

$$\lim_{\varepsilon \rightarrow 0} \frac{e^{W/\varepsilon}}{\varepsilon S} = 0. \quad (6.14)$$

Thus (6.12) is a regular perturbation of

$$\frac{dS}{dV} = 0, \quad \frac{dW}{dV} = -\frac{V}{S}. \quad (6.15)$$

Let $S_0, W_0 < 0$ be fixed. By the continuous dependence of the solutions with respect to

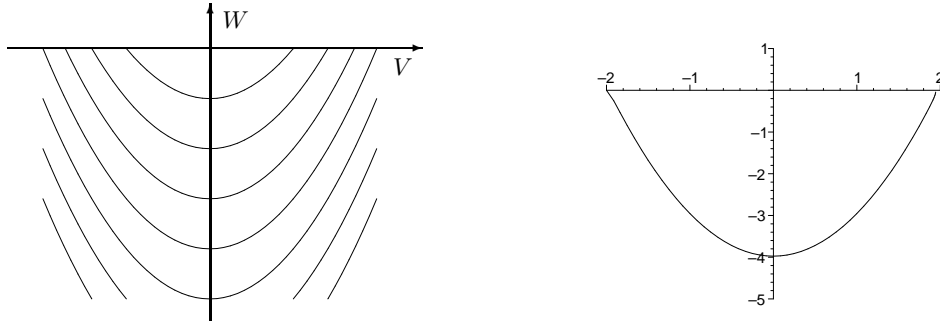


FIGURE 26. On the left: the vector field (6.15) in restriction to the invariant plane $S = S_0$. On the right: the numerical solution of (6.12) corresponding to $\varepsilon = 0.1$, $A = -2$, $B = -1$ and $C = 1$ in the variables V, W .

parameters, any solution $(S(V, \varepsilon), W(V, \varepsilon))$ of (6.12) such that $S(0, \varepsilon) = S_0 + o(1)$ and $W(0, \varepsilon) = W_0 + o(1)$, satisfies

$$S(V, \varepsilon) = S_0 + o(1) \quad \text{and} \quad W(V, \varepsilon) = \frac{-V^2}{2S_0} + W_0 + o(1), \quad (6.16)$$

uniformly for V in any compact subset of $(-V_0, V_0)$, where $V_0 = \sqrt{2S_0W_0}$, see Figure 26. If in addition the solution satisfies the initial condition (6.13), then the values of S_0 and W_0 are obtained from (6.10) and from $\varepsilon \ln(C) = o(1)$. As a consequence, we have

$$S(V, \varepsilon) = B - \frac{C}{A} + o(1), \quad W(V, \varepsilon) = \frac{A(A^2 - V^2)}{2(AB - C)} + o(1),$$

uniformly for $V \in [A_0, -A_0]$. Hence, we have $U(V, \varepsilon) = o(1)$ uniformly for $V \in [A_0, -A_0]$. Recall that $U = VS - T$ represents the ‘vertical distance’ between the solution of (6.9) and the slow manifold $T = VS$. Since A_0 may be chosen as close to A as we want, we conclude that the exit of the solution from the neighborhood of the slow manifold holds asymptotically for $V = -A$. In other words, $U(S, \varepsilon)$ is far away from 0 for $S \geq B_1 > B - \frac{C}{A}$, where B_1 is as close to $B - \frac{C}{A}$ as we want and we have

$$V(S, \varepsilon) = -A + o(1) \quad \text{uniformly for } S \in [B_1, 0]. \quad (6.17)$$

Since $U(B - \frac{C}{A}, \varepsilon) = o(1)$, we have

$$U(S, \varepsilon) = -A \left(S - B + \frac{C}{A} \right) + o(1) \quad \text{uniformly for } S \in [B_1, 0]. \quad (6.18)$$

Using (6.7), (6.17) and (6.18) we conclude that

$$U(S, \varepsilon) = |A(S - B) + C| + o(1), \quad \frac{dU}{dS}(S, \varepsilon) = A \operatorname{sgn}(A(S - B) + C) + o(1),$$

uniformly for $S \in [L, B_0] \cup [B_1, 0]$, see Figure 24. The minimum of $U(S, \varepsilon)$ is reached for $S = K(\varepsilon)$ which corresponds to $V = 0$. Hence,

$$K(\varepsilon) = B - \frac{C}{A} + o(1), \quad U(K(\varepsilon), \varepsilon) = \exp\left(\frac{W(0, \varepsilon)}{\varepsilon}\right) = \exp\left(\frac{A^3 + o(1)}{2(AB - C)\varepsilon}\right).$$

Proof of item 2. Let $U = U(S, \varepsilon)$ be a solution of (6.1) with initial conditions satisfying (6.4). In the variables (V, S, W) , we have

$$\frac{dS}{dV} = -\frac{e^{W/\varepsilon}}{\varepsilon S}, \quad \frac{dW}{dV} = -\frac{V}{S}, \quad S(V_0) = K, \quad W(V_0) = \frac{A^2 + o(1)}{2K},$$

where $V_0 := \frac{dU}{dS}(K) = o(1)$. Using (6.14) and the continuous dependence of the solutions with respect to parameters and initial conditions, we deduce that, for any A_0 such that $0 < A_0 < |A|$, we have

$$S(V, \varepsilon) = K + o(1) \quad \text{and} \quad W(V, \varepsilon) = \frac{A^2 - V^2}{2K} + o(1),$$

uniformly for $V \in [A_0, -A_0]$. It follows that $U(V, \varepsilon) = o(1)$ uniformly for $V \in [A_0, -A_0]$. Since A_0 may be chosen as close to $|A|$ as we want, we deduce that the entry of the corresponding solution $(V, S(V, \varepsilon), T(V, \varepsilon))$ near the slow manifold $T = SV$ holds for $V = -|A|$ and its exit holds asymptotically for $V = |A|$. In other words, $U(S, \varepsilon)$ is far away from 0 for $S \leq B_0 < K$ and $S \geq B_1 > K$, where B_0 and B_1 are as close to K as we want. Thus, we have

$$V(S, \varepsilon) = |A| \operatorname{sgn}(S - K) + o(1), \quad U(S, \varepsilon) = |A(S - K)| + o(1),$$

uniformly for $S \in [L, B_0] \cap [B_1, 0]$. □

6.3. From Theorem 6.1 to Propositions 5.4 and 2.3.

Proof of Proposition 5.4. The solution $u(s; a, 0, \varepsilon)$ is defined for all $s \leq 0$ and is positive. The function $U(S, \varepsilon)$, defined by

$$U(S, \varepsilon) = \frac{1}{\varepsilon} u(\varepsilon S; a, 0, \varepsilon),$$

is the solution of the initial value problem

$$U \frac{d^2 U}{dS^2} + \varepsilon S = 0, \quad U(0) = 1, \quad \frac{dU}{dS}(0) = a.$$

By Theorem 6.1, item 1, $U(S, \varepsilon)$ reaches its minimum at $K(\varepsilon)$, with $K(\varepsilon) < 0$ and we have

$$K(\varepsilon) = -\frac{1}{a} + o(1), \quad U(K(\varepsilon), \varepsilon) = \exp \frac{-a^3 + o(1)}{2\varepsilon} \quad \text{as } \varepsilon \rightarrow 0.$$

Hence, as $\varepsilon \rightarrow 0$, we have $\kappa(\varepsilon) = -\varepsilon K(\varepsilon) = \varepsilon(1/a + o(1))$ and, using

$$\varepsilon = \exp \frac{\varepsilon \ln \varepsilon}{\varepsilon} = \exp \frac{o(1)}{\varepsilon},$$

we obtain

$$c(\varepsilon) := u(-\kappa(\varepsilon), \varepsilon) = \varepsilon U(K(\varepsilon), \varepsilon) = \varepsilon \exp \frac{-a^3 + o(1)}{2\varepsilon} = \exp \frac{-a^3 + o(1)}{2\varepsilon}.$$

Given $L < 0$, we have

$$U(S, \varepsilon) = |aS + 1| + o(1) \quad \text{and} \quad \frac{dU}{dS}(S, \varepsilon) = a \operatorname{sgn}(aS + 1) + o(1),$$

uniformly for S in any compact subset of $[L, -1/a] \cup (-1/a, 0]$. As a consequence, we have

$$u(\varepsilon S; a, 0, \varepsilon) = \varepsilon (|aS + 1| + o(1)) \quad \text{and} \quad u'(\varepsilon S; a, 0, \varepsilon) = a \operatorname{sgn}(aS + 1) + o(1),$$

uniformly for S in any compact subset of $[L, -1/a] \cup (-1/a, 0]$. \square

Proof of Proposition 2.3. The solution $u(s; a, -1, c)$ is defined for all $s < 0$ and is positive. The function $U(S, \varepsilon)$, defined by

$$U(S, \varepsilon) = \sqrt{\varepsilon} u(S; a, -1, c), \quad \text{where} \quad \frac{1}{\varepsilon} = 2 \ln \frac{1}{c},$$

is the solution of the initial value problem

$$U \frac{d^2 U}{dS^2} + \varepsilon S = 0, \quad U(-1) = \sqrt{\varepsilon} \exp \frac{-1}{2\varepsilon}, \quad \frac{dU}{dS}(-1) = \sqrt{\varepsilon} a.$$

By Theorem 6.1, item 2, we have

$$U(S, \varepsilon) = |S + 1| + o(1) \quad \text{and} \quad \frac{dU}{dS}(S, \varepsilon) = \operatorname{sgn}(S + 1) + o(1),$$

uniformly for S in any compact subset of $(-\infty, -1) \cup (-1, 0]$, as $\varepsilon \rightarrow 0$. Hence, as $c \rightarrow 0$, we have

$$u(s; a, -1, c) = \frac{1}{\sqrt{\varepsilon}} (|s + 1| + o(1)),$$

and

$$u'(s; a, -1, c) = \frac{1}{\sqrt{\varepsilon}} (\operatorname{sgn}(s + 1) + o(1)),$$

uniformly for s in any compact subset of $(-\infty, -1) \cup (-1, 0]$. \square

7. FROM CROCCO TO BLASIUS

In this section, we investigate the way to recover a solution of the Blasius initial value problem (1.5) from a Crocco solution, and also to obtain the asymptotic behavior of Blasius solutions from the one of Crocco solutions. Our point of view, here, is to forget what we know about Blasius solutions, and to show that the properties of these solutions can be deduced from the properties of Crocco solutions. Some of the results of this section were obtained by J. Wang, W. Gao and Z. Zhang [51] in the more general case of the Falkner-Skan equation but with restrictive boundary conditions.

Proposition 7.1. *Let $c > 0$ and $u = u(\cdot; a, b, c)$ be the solution of*

$$u'' = -\frac{s}{u}, \quad u(b) = c, \quad u'(b) = a, \quad (7.1)$$

and let $[b, L(a, b, c))$ denote its right maximal interval of existence.

1. *Set $\lambda = L(a, b, c)$. Then $\lambda \in [0, +\infty)$ and $u(s) \rightarrow 0$ as $s \rightarrow \lambda$, $s < \lambda$. In addition, if $\lambda > 0$, then $u'(s) \rightarrow -\infty$ as $s \rightarrow \lambda$, $s < \lambda$, and we have*

$$u'(s) \sim -\sqrt{-2\lambda \ln(\lambda - s)} \quad \text{and} \quad u(s) \sim (\lambda - s)\sqrt{-2\lambda \ln(\lambda - s)}, \quad (7.2)$$

as $s \rightarrow \lambda$, $s < \lambda$.

2. *The solution f of the second order initial value problem*

$$f'' = u(f'), \quad f(0) = -a, \quad f'(0) = b, \quad (7.3)$$

is the solution of (1.5). The function f is defined on $[0, +\infty)$ and is given by

$$f(t) = -a + \int_0^t g(\tau) d\tau, \quad (7.4)$$

where g is implicitly defined by

$$t = \int_b^{g(t)} \frac{d\eta}{u(\eta)}. \quad (7.5)$$

Moreover, $f'(t) \rightarrow L(a, b, c)$ as $t \rightarrow +\infty$.

Remark. The interesting property of Crocco solutions (7.2) is somewhat surprising insofar as the asymptotic behavior is not sufficient to separate different solutions with the same λ , see Figure 27.

Proof. Let us notice that u is convex on $[b, \lambda) \cap (-\infty, 0]$ and concave on $[b, \lambda) \cap [0, +\infty)$.

1. First, we show that λ is finite. For contradiction, suppose that $\lambda = +\infty$. Since u is positive, and concave on $[b, +\infty) \cap [0, +\infty)$, we necessarily have $u' > 0$ on this interval. Let $s_0 \in [b, +\infty) \cap [0, +\infty)$. Using (3.3) and the concavity of u we obtain

$$-u'(s_0) < u'(s) - u'(s_0) = -\int_{s_0}^s \frac{\eta}{u(\eta)} d\eta < -\int_{s_0}^s \frac{\eta}{u(s_0) + u'(s_0)(\eta - s_0)} d\eta,$$

for all $s > s_0$, hence, with $\alpha_0 = s_0 - \frac{u(s_0)}{u'(s_0)}$, we derive

$$u'(s_0)^2 > \int_{s_0}^s \frac{\eta}{\eta - \alpha_0} d\eta = s - s_0 + \alpha_0 \ln \frac{s - \alpha_0}{s_0 - \alpha_0},$$

for all $s > s_0$. Since the right hand side of this inequality tends to $+\infty$ as $s \rightarrow +\infty$, this gives a contradiction. Hence, λ is finite and necessarily, by (7.1), we have $u(s) \rightarrow 0$ as $s \rightarrow \lambda$, $s < \lambda$.

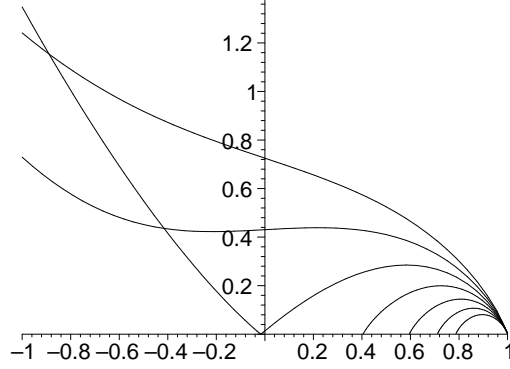


FIGURE 27. Several Crocco solutions with the same maximal right boundary $\lambda = 1$. All of them have the same asymptotic behavior near λ , given by (7.2).

Next, we show that $\lambda \geq 0$. On the contrary, choosing $s_0 \in [b, \lambda)$ such that $u(s) < 1$ for $s \in [s_0, \lambda)$, and using (3.6) we obtain for such a s

$$\begin{aligned} -u'(s_0)^2 &\leq u'(s)^2 - u'(s_0)^2 = -2s \ln u(s) + 2s_0 \ln u(s_0) + 2 \int_{s_0}^s \ln u(\eta) d\eta \\ &\leq -2s \ln u(s) + 2s_0 \ln u(s_0), \end{aligned}$$

which gives a contradiction since $-2s \ln u(s) \rightarrow -\infty$ as $s \rightarrow \lambda$.

Moreover, for $\lambda > 0$, the concavity of u close to λ shows that $u'(s) \rightarrow \mu \in [-\infty, 0)$ as $s \rightarrow \lambda$. If μ is finite, then $u(s) \sim \mu(\lambda - s)$ as $s \rightarrow \lambda$, hence identity (3.3) gives a contradiction. Thus $\mu = -\infty$. It remains to prove (7.2). Since $u(s) \rightarrow 0$ and $u'(s) \rightarrow -\infty$ as $s \rightarrow \lambda$, the integral $\int^\lambda \ln u(\eta) d\eta$ converges, and then from (3.6) we easily obtain that $u'(s)^2 \sim -2s \ln u(s)$ as $s \rightarrow \lambda$, hence

$$u'(s) \sim -\sqrt{2\lambda} \sqrt{-\ln u(s)} \quad \text{as } s \rightarrow \lambda. \quad (7.6)$$

On the other hand, for s close enough to λ , integration by parts gives

$$\int_s^\lambda \frac{u'(\eta)}{\sqrt{-\ln u(\eta)}} d\eta = -\frac{u(s)}{\sqrt{-\ln u(s)}} + \frac{1}{2} \int_s^\lambda \frac{u'(\eta)}{(-\ln u(\eta))^{3/2}} d\eta,$$

from which together with (7.6) we obtain

$$\frac{u(s)}{\sqrt{-\ln u(s)}} \sim \sqrt{2\lambda}(\lambda - s) \quad \text{as } s \rightarrow \lambda. \quad (7.7)$$

Taking the logarithm of each side of (7.7) we arrive to

$$\ln u(s) \sim \ln(\lambda - s) \quad \text{as } s \rightarrow \lambda. \quad (7.8)$$

Combining (7.6), (7.7) and (7.8) we obtain (7.2).

2. Let f be the solution of problem (7.3). Differentiating with respect to t we obtain $f'''(t) = u'(f'(t))f''(t)$. Using (3.3) we can write

$$u'(f'(t)) = a - \int_b^{f'(t)} \frac{\eta}{u(\eta)} d\eta$$

$$= a - \int_0^t \frac{f'(\tau)}{u(f'(\tau))} f''(\tau) d\tau = a - \int_0^t f'(\tau) d\tau = -f(t).$$

Thus, f is a solution of the Blasius equation $f''' = -ff''$. From (7.3) we obtain $f''(0) = c$. Hence f is the solution of (1.5). Denote by $[0, T)$ its right maximal interval of existence.

Since (7.3) is equivalent to integrate successively $g' = u(g)$ and $f' = g$ with the initial conditions $g(0) = b$ and $f(0) = -a$, we obtain (7.5) (recall that $u > 0$, so that, formula (7.5) defines $g(t)$ implicitly) and (7.4). Moreover, g is the inverse function of

$$v : s \mapsto \int_b^s \frac{d\eta}{u(\eta)}.$$

The function v is defined on $[b, \lambda)$ and $v(s) \rightarrow +\infty$ as $s \rightarrow \lambda$. Indeed, in the case $\lambda > 0$, thanks to (7.2) we have $u(s) \sim (\lambda - s)\sqrt{-2\lambda \ln(\lambda - s)}$ as $s \rightarrow \lambda$ and the integral $\int_0^{\frac{dx}{x\sqrt{\ln x}}}$ diverges. If $\lambda = 0$, then either $u(s) \sim \mu s$ (for some $\mu < 0$) as $s \rightarrow 0$, or $u(s) = \frac{2}{\sqrt{3}}(-s)^{3/2}$, and in both cases, we obtain $v(s) \rightarrow +\infty$ as $s \rightarrow 0$. Therefore, g and f are defined on $[0, +\infty)$ and $g(t) = f'(t)$ tends to λ as $t \rightarrow +\infty$. \square

The following result, already given by P. Hartman [26, 27] for $\lambda = 1$ (in the more general case of the Falkner-Skan equation) concerns the asymptotic behavior of the solutions of (1.1-1.2). It can be very easily proved from the behavior near λ of the solution u of (2.2), as described by (7.2).

Proposition 7.2. *Let $a, b \in \mathbb{R}$, $\lambda > 0$ and f be a solution of the Blasius problem (1.1-1.2). There exist constants κ_1 and κ_2 in \mathbb{R} such that*

$$f''(t) \sim \lambda t(\lambda - f'(t)) \quad \text{and} \quad \lambda - f'(t) \sim \frac{\kappa_1}{t} \exp\left\{-\frac{\lambda t^2}{2} + \kappa_2 t\right\}, \quad (7.9)$$

as $t \rightarrow +\infty$.

Proof. Using (7.2) and the relations $u(f'(t)) = f''(t)$ and $u'(f'(t)) = -f(t)$ we obtain

$$f''(t) = u(f'(t)) \sim (\lambda - f'(t))\sqrt{-2\lambda \ln(\lambda - f'(t))} \quad \text{as } t \rightarrow +\infty, \quad (7.10)$$

and

$$f(t) = -u'(f'(t)) \sim \sqrt{-2\lambda \ln(\lambda - f'(t))} \quad \text{as } t \rightarrow +\infty, \quad (7.11)$$

Combining (7.10), (7.11) and the fact that $f(t) \sim \lambda t$ as $t \rightarrow +\infty$, we obtain the first part of (7.9), and also

$$\ln(\lambda - f'(t)) \sim -\frac{\lambda t^2}{2} \quad \text{as } t \rightarrow +\infty. \quad (7.12)$$

In other words,

$$\lambda - f'(t) = \exp\left\{-\frac{\lambda t^2}{2}(1 + o(1))\right\} \quad \text{as } t \rightarrow +\infty. \quad (7.13)$$

By successive integration, we deduce from (7.13) that there exist constants $\mu < -a$ and $\nu \in \mathbb{R}$ such that

$$f(t) = \lambda t + \mu + O\left(\exp\left\{-\frac{\lambda t^2}{2}(1 + o(1))\right\}\right) \quad \text{as } t \rightarrow +\infty,$$

and

$$F(t) = \frac{\lambda t^2}{2} + \mu t + \nu + o(1) \quad \text{as } t \rightarrow +\infty,$$

where F is the anti-derivative of f such that $F(0) = 0$. To conclude, we use (2.1) with $\tau = 0$ to obtain

$$f''(t) \sim f''(0) \exp\left\{-\frac{\lambda t^2}{2} - \mu t - \nu\right\} \quad \text{as } t \rightarrow +\infty.$$

Hence, with $\kappa_1 = \frac{1}{\lambda} f''(0)e^{-\nu}$ and $\kappa_2 = -\mu$, the second part of (7.9) follows from the first one. \square

8. THE CONCAVE CASE

Here, the word ‘‘concave’’ refers to Blasius solutions and concerns the boundary value problem (1.1-1.2) with $\lambda < b$. Notice that, if f is a concave Blasius solution with associated Crocco solution u , then $t \mapsto -f(-t)$ is a convex Blasius solution with associated Crocco solution $-u$. However the boundary value problem has changed and needs a separate treatment.

Let $a, b \in \mathbb{R}$, $c \in (-\infty, 0)$ and $f(\cdot; a, b, c)$ be the solution of the initial value problem (1.5) rewritten below

$$f''' + ff'' = 0, \quad f(0) = -a, \quad f'(0) = b, \quad f''(0) = c.$$

Denote by $[0, T_{a,b,c})$ the right maximal interval of existence of $f(\cdot; a, b, c)$. Since $f''(\cdot; a, b, c) < 0$ if $c < 0$ and $f'''(\cdot; a, b, c) = 0$ the following limit exists

$$\Lambda(a, b, c) := \lim_{t \rightarrow T_{a,b,c}} f'(t; a, b, c) \in [-\infty, b).$$

For $c < 0$, we associate to $f(\cdot; a, b, c)$ the Crocco solution $u(\cdot; a, b, c)$ of (2.2-2.3). Precisely, if $c < 0$, then the Crocco changes of variable $f'' = u(f')$ yields $u := u(\cdot; a, b, c)$ satisfying $u'(f') = -f$ and

$$u'' = -\frac{s}{u} \quad \text{on} \quad (\Lambda(a, b, c), b], \quad u(b) = c, \quad u'(b) = a, \quad u < 0.$$

Lemma 8.1. *Let $a, b \in \mathbb{R}$ and $c \in (-\infty, 0)$. If $\Lambda(a, b, c)$ is finite, then we have $T_{a,b,c} = +\infty$, $\Lambda(a, b, c) \geq 0$ and $u(s; a, b, c) \rightarrow 0$ as $s \rightarrow \Lambda(a, b, c)$, $s > \Lambda(a, b, c)$. If in addition $\Lambda(a, b, c) > 0$, then $u'(s; a, b, c) \rightarrow -\infty$ as $s \rightarrow \Lambda(a, b, c)$, $s > \Lambda(a, b, c)$.*

Proof. Let us suppose that $\Lambda(a, b, c)$ and $T_{a,b,c}$ are finite. Then $f(t; a, b, c)$ has a limit as $t \rightarrow T_{a,b,c}$, and due to (2.1), it is so for $f''(t; a, b, c)$. This contradicts the fact that $f(\cdot; a, b, c)$ cannot be extended after $T_{a,b,c}$. Therefore, $T_{a,b,c} = +\infty$. If now $\Lambda(a, b, c) < 0$, then there exists $t_0 \geq 0$ such that $f(t; a, b, c) \leq -1$ for $t \geq t_0$. From (2.1) we obtain $|f'''(t; a, b, c)| \geq |f''(t_0; a, b, c)|e^{t-t_0}$, hence $f'''(t; a, b, c) \rightarrow -\infty$, as $t \rightarrow +\infty$, which contradicts the fact that $f'(t; a, b, c)$ has a finite limit as $t \rightarrow +\infty$. Finally, we have

$$\lim_{s \rightarrow \Lambda(a,b,c)} u(s; a, b, c) = \lim_{t \rightarrow +\infty} f''(t; a, b, c) = 0,$$

and, if $\Lambda(a, b, c) > 0$, then

$$\lim_{s \rightarrow \Lambda(a,b,c)} u'(s; a, b, c) = \lim_{t \rightarrow +\infty} (-f(t; a, b, c)) = -\infty.$$

This completes the proof. \square

Remark. From Lemma 8.1, we see that indeed $(\Lambda(a, b, c), b]$ is the left maximal interval of existence of $u(\cdot; a, b, c)$.

Lemma 8.2. *Let $a \in \mathbb{R}$, $b \in [0, +\infty)$ and $c_1, c_2 \in (-\infty, 0)$ such that one at least among $\Lambda(a, b, c_1)$ and $\Lambda(a, b, c_2)$ is finite. If $c_1 < c_2$, then we have $\Lambda(a, b, c_1) < \Lambda(a, b, c_2)$.*

Proof. Suppose $c_1 < c_2 < 0$ and $\Lambda(a, b, c_1) \geq \Lambda(a, b, c_2)$. Hence, $\Lambda(a, b, c_1)$ is finite and nonnegative by Lemma 8.1. For $i = 1, 2$ set $u_i = u(\cdot; a, b, c_i)$, and $w = u_2 - u_1$. We have $w(b) = c_2 - c_1 > 0$, $w'(b) = a - a = 0$, and

$$\forall s \in (\Lambda(a, b, c_1), b], \quad w''(s) = u_2''(s) - u_1''(s) = \frac{-s}{u_2(s)} + \frac{s}{u_1(s)} = \frac{sw(s)}{u_1(s)u_2(s)}.$$

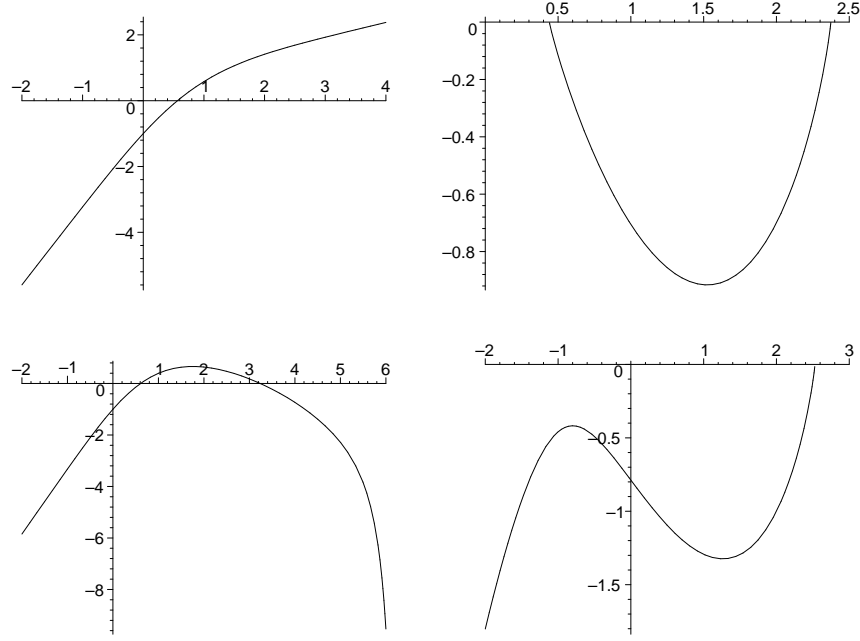


FIGURE 28. Concave solutions of Blasius equation (1.1) on the left, the corresponding Crocco solutions on the right, for $a = 1$, $b = 2$ and respectively $c = -0.7$, $c = -1$.

Therefore, as long as w is positive, w is convex, and so decreasing in such a way that w remains greater than $c_2 - c_1$ on $(\Lambda(a, b, c_1), b]$. However,

$$\lim_{\substack{s \rightarrow \Lambda(a, b, c_1) \\ s > \Lambda(a, b, c_1)}} w(s) = \lim_{\substack{s \rightarrow \Lambda(a, b, c_1) \\ s > \Lambda(a, b, c_1)}} u_2(s) \leq 0.$$

This is a contradiction. □

Let us set

$$c_*(a, b) = \sup\{c < 0 ; \Lambda(a, b, c) = -\infty\}, \quad (8.1)$$

with the convention $\sup \emptyset = -\infty$. As a consequence of Lemma 8.2, if $c < c_*(a, b)$, then $\Lambda(a, b, c) = -\infty$ and if $c \in (c_*(a, b), 0)$, then $\Lambda(a, b, c)$ is finite.

Lemma 8.3. *Let $a, b \in \mathbb{R}$. If $b \leq 0$, then $c_*(a, b) = 0$, and if $b > 0$, then $-\infty < c_*(a, b) < 0$.*

Proof. If $b \leq 0$, then for all $c < 0$, we have $\Lambda(a, b, c) < 0$ and using Lemma 8.1 we obtain $\Lambda(a, b, c) = -\infty$. Since, moreover $\Lambda(a, b, 0) = b$, we deduce $c_*(a, b) = 0$.

Suppose now $b > 0$. For $c < 0$, set $u = u(\cdot ; a, b, c)$. From (3.5) with $s_0 = b$ we obtain for $s \in (\Lambda(a, b, c), b] \cap [0, +\infty)$

$$u(s)^2 = c^2 + 2ac(s - b) + 2 \int_s^b (\eta - s)u'(\eta)^2 d\eta - \frac{1}{3}(b - s)^2(2b + s) \geq c^2 - 2|ac|b - b^3.$$

Thus, for $-c$ large enough, we obtain that u cannot vanish on $[0, b]$, which implies that $\Lambda(a, b, c) < 0$. From Lemma 8.1 we obtain $\Lambda(a, b, c) = -\infty$ for $-c$ large enough, hence $c_*(a, b) > -\infty$.

To obtain the inequality $c_*(a, b) < 0$, we have to distinguish between the cases $a \leq 0$ and $a > 0$. First of all, let us remark that using the convexity of u on $(\Lambda(a, b, c), b] \cap [0, +\infty)$, we

have

$$\forall \eta \in (\Lambda(a, b, c), b] \cap [0, +\infty), \quad c + a(\eta - b) \leq u(\eta) < 0. \quad (8.2)$$

• If $a \leq 0$, then (8.2) gives $c \leq u(\eta) < 0$ for $\eta \in (\Lambda(a, b, c), b] \cap [0, +\infty)$, hence using (3.4) for $s \in (\Lambda(a, b, c), b] \cap [0, +\infty)$

$$\begin{aligned} u(s) &= c + a(s - b) + \int_s^b \frac{\eta(s - \eta)}{u(\eta)} d\eta \\ &\geq c + \frac{1}{c} \int_s^b \eta(s - \eta) d\eta = c - \frac{1}{6c} (b - s)^2 (2b + s). \end{aligned} \quad (8.3)$$

Since the right hand side of (8.3) is positive for $s = 0$ and $-b^{3/2} < c < 0$, we have, for such a c , that $\Lambda(a, b, c) > 0$. Hence, $c_*(a, b) < 0$.

• If $a > 0$, then using (8.2) and (3.3), for $s \in (\Lambda(a, b, c), b] \cap [0, +\infty)$, we get

$$a - u'(s) = - \int_s^b \frac{\eta}{u(\eta)} d\eta \geq - \int_s^b \frac{\eta}{c + a(\eta - b)} d\eta.$$

Hence, for $s \in (\Lambda(a, b, c), b] \cap [0, +\infty)$, we deduce

$$u'(s) \leq a + \frac{b}{a} \int_s^b \frac{d\eta}{c + a(\eta - b)} = a - \frac{b}{a} \ln \left(1 + \frac{a}{c} (s - b) \right).$$

Integrating, we obtain for $s \in (\Lambda(a, b, c), b] \cap [0, +\infty)$

$$u(s) \geq c - \left(a + \frac{b}{a} \right) (b - s) - \frac{b}{a} \left(\frac{c}{a} + s - b \right) \ln \left(1 + \frac{a}{c} (s - b) \right). \quad (8.4)$$

Since the right hand side of (8.4) is positive for $s = 0$ and $-c$ sufficiently small, we obtain, for such a c , that $\Lambda(a, b, c) > 0$. Hence, $c_*(a, b) < 0$ in this case too. \square

Proposition 8.4. *If $(a, b, c) \in \mathbb{R} \times \mathbb{R} \times (-\infty, 0)$ is such that $\Lambda(a, b, c) \in (0, b)$, then Λ is continuous at (a, b, c) .*

Proof. We have $\Lambda(a, b, c) = \inf \{ f'(t; a, b, c) ; 0 \leq t < T_{a,b,c} \}$, hence Λ is upper semicontinuous on $\mathbb{R} \times \mathbb{R} \times (-\infty, 0)$.

Now, to prove that Λ is lower semicontinuous at $(a, b, c) \in \mathbb{R} \times \mathbb{R} \times (-\infty, 0)$, let us set $\lambda = \Lambda(a, b, c)$, consider $\varepsilon \in (0, \lambda)$ and a sequence (a_n, b_n, c_n) which converges to (a, b, c) , and set $\lambda_n = \Lambda(a_n, b_n, c_n)$. Since $\lambda \in (0, b)$, we have

$$\lim_{\substack{s \rightarrow \lambda \\ s > \lambda}} \left(s - \frac{u(s)}{u'(s)} \right) = \lambda,$$

hence there exists $s_0 \in (\lambda, b)$ such that $s_0 - \frac{u(s_0)}{u'(s_0)} > \lambda - \frac{\varepsilon}{2}$. The upper semicontinuity of Λ shows that there exists $n_0 \in \mathbb{N}$ such that $\lambda_n < s_0$ for $n \geq n_0$. Moreover, since $u_n(s_0) \rightarrow u(s_0)$ and $u'_n(s_0) \rightarrow u'(s_0)$ as $n \rightarrow +\infty$, there exists $n_1 \geq n_0$ such that

$$\forall n \geq n_1, \quad s_0 - \frac{u_n(s_0)}{u'_n(s_0)} > \lambda - \varepsilon > 0.$$

Since u_n is negative and convex on $(\lambda_n, b] \cap [0, +\infty)$, we have $u_n(s) \geq u_n(s_0) + u'_n(s_0)(s - s_0)$. Because the right hand side of this inequality vanishes for $s = s_0 - \frac{u_n(s_0)}{u'_n(s_0)}$, we necessarily have $\lambda_n \geq \lambda - \varepsilon$ for $n \geq n_1$. This completes the proof. \square

Proposition 8.5. *Let $a \in \mathbb{R}$ and $b > 0$. The function $c \mapsto \Lambda(a, b, c)$ is an increasing one-to-one mapping from $[c_*(a, b), 0)$ onto $[0, b)$.*

Proof. Set $c_* = c_*(a, b)$. Taking into account Lemma 8.2 and Proposition 8.4, it is sufficient to prove that

$$\lim_{\substack{c \rightarrow c_* \\ c > c_*}} \Lambda(a, b, c) = 0 \quad \text{and} \quad \lim_{c \rightarrow 0^-} \Lambda(a, b, c) = b. \quad (8.5)$$

Let us set $\lambda_c = \Lambda(a, b, c)$. For the first equality, since the map $c \mapsto \lambda_c$ is upper semicontinuous and increasing on $(c_*, 0)$, then $\lambda_c \rightarrow \lambda_{c_*}$ as $c \rightarrow c_*$, $c > c_*$ and $\lambda_{c_*} \geq 0$. On the other hand, for $c < c_*$, we have $\lambda_c = -\infty$. Hence, the map $c \mapsto \lambda_c$ is not continuous at c_* , and thus, from Proposition 8.4, we obtain $\lambda_{c_*} = 0$.

To obtain the second equality of (8.5) for $a \leq 0$, let us take $-c$ sufficiently small to have $\lambda_c > 0$. Letting $s \rightarrow \lambda_c$ in (8.3), we obtain

$$0 \geq c - \frac{1}{6c}(b - \lambda_c)^2(2b + \lambda_c).$$

Hence, $(b - \lambda_c)^2(2b + \lambda_c) \leq 6c^2$ and $\lambda_c \rightarrow b$ as $c \rightarrow 0^-$. For $a > 0$, using (8.4) we obtain in a similar way

$$0 \geq c - \left(a + \frac{b}{a}\right)(b - \lambda_c) - \frac{b}{a} \left(\frac{c}{a} + \lambda_c - b\right) \ln \left(1 + \frac{a}{c}(\lambda_c - b)\right),$$

and hence we obtain a contradiction if $\lambda_c \not\rightarrow b$ as $c \rightarrow 0^-$. \square

Corollary 8.6. *Let $a \in \mathbb{R}$, $b \in \mathbb{R}$ and $\lambda \in (-\infty, b)$. The Blasius boundary problem (1.1-1.2) has exactly one (concave) solution when $0 \leq \lambda < b$, and no solution for $\lambda < 0$, whatever $b > 0$ or $b \leq 0$.*

Proof. This follows immediately from Proposition 8.5 and Lemma 8.1. \square

To finish this section, we give in a very quick way the asymptotic behavior of $f(t; a, b, c)$ as $t \rightarrow T_{a,b,c}$ when $c < c_*(a, b)$. This behavior was already obtained by W.A. Coppel [14] and N. Ishimura and S. Matsui [35].

Proposition 8.7. *Let $a, b \in \mathbb{R}$. If $c \in (-\infty, c_*(a, b))$, then $T := T_{a,b,c}$ is finite and we have*

$$f'(t; a, b, c) \sim \frac{-3}{(T-t)^2} \quad \text{and} \quad f(t; a, b, c) \sim \frac{-3}{T-t} \quad \text{as } t \rightarrow T. \quad (8.6)$$

Proof. Let $f = f(\cdot; a, b, c)$ and $u = u(\cdot; a, b, c)$. We have $\Lambda(a, b, c) = -\infty$ and $-u$ is a positive solution of (2.2) on $(-\infty, b]$. Thanks to item 1 of Theorem 2.2, we have $-u(s) \sim u_*(s)$ as $s \rightarrow -\infty$. In other words,

$$-f''(t) \sim \frac{2}{\sqrt{3}}(-f'(t))^{3/2} \quad \text{as } t \rightarrow T,$$

hence,

$$-\frac{1}{2} \frac{-f''(t)}{(-f'(t))^{3/2}} \rightarrow \frac{1}{\sqrt{3}} \quad \text{as } t \rightarrow T.$$

Then for all $\varepsilon > 0$, there exists $t_\varepsilon > 0$ such that for $t \in (t_\varepsilon, T)$ and all $\tau \in (t, T)$, we have

$$\frac{1}{\sqrt{3}}(\tau - t)(1 - \varepsilon) \leq \frac{1}{\sqrt{-f'(\tau)}} - \frac{1}{\sqrt{-f'(t)}} \leq \frac{1}{\sqrt{3}}(\tau - t)(1 + \varepsilon).$$

Letting $\tau \rightarrow T$, this yields $T < +\infty$ and

$$\frac{3}{(1 + \varepsilon)(T - t)^2} \leq -f'(t) \leq \frac{3}{(1 - \varepsilon)(T - t)^2}.$$

Now, (8.6) follows. \square

9. FINAL REMARKS, ALTERNATIVE PROOFS AND HISTORICAL COMMENTS

9.1. **Vanishing Crocco solutions.** We consider here the Crocco equation in its non-resolved form $uw'' + s = 0$ and we describe solutions that vanish somewhere.

Proposition 9.1. 1. *If $s \mapsto u(s)$ is a C^2 function that has a zero at some point s_0 and that satisfies $u(s)u''(s) + s = 0$ in a neighborhood of s_0 , then we must have $s_0 = 0$.*

2. *There is a unique function w that is analytic in a neighborhood of 0, and such that $w(0) = 0$, $w'(0) = 1$ and $w(s)w''(s) + s = 0$. This function w is defined on $(-\infty, \lambda_+)$ by*

$$w(s) = \begin{cases} -u_-(s) & \text{if } s < 0 \\ 0 & \text{if } s = 0 \\ u_+(s) & \text{if } s \in (0, \lambda_+). \end{cases}$$

3. *The Taylor series $\sum_{n \geq 1} a_n s^n$ of w is given recursively by*

$$a_1 = 1, \quad a_2 = -\frac{1}{2}, \quad a_{n+1} = -\frac{1}{n(n+1)} \sum_{k=1}^{n-1} k(k+1)a_{k+1}a_{n-k+1}. \quad (9.1)$$

The radius of convergence of this series is equal to λ_+ . The first terms are given by

$$w(s) = s - \frac{1}{2}s^2 - \frac{1}{12}s^3 - \frac{1}{36}s^4 - \frac{17}{1440}s^5 - \frac{247}{43200}s^6 - \frac{1819}{604800}s^7 + O(s^9),$$

4. *All other functions u that are analytic in a neighborhood of 0 and such that $u(0) = 0$, and $u(s)u''(s) + s = 0$, are given by $u(s) = \sigma^3 w(s/\sigma^2)$ for $\sigma \neq 0$ and $s < \sigma^2 \lambda_+$.*

Proof. Statement 1 is immediate because otherwise $u''(s_0)$ would be infinite.

Concerning statement 3, if we look for a formal series $\sum a_n s^n$ solution of the Crocco equation with $a_0 = 0$, $a_1 = 1$, then we obtain the recursion formula (9.1).

We now prove by the majorant method that this series has a non-zero radius of convergence. Let (c_n) be the majorizing sequence defined recursively by

$$c_0 = 1, \quad c_1 = \frac{1}{2}, \quad c_n = \sum_{k=1}^{n-1} c_k c_{n-k} \quad \text{for } n \geq 2.$$

This is a majorizing sequence in the sense that we have $|a_{n+1}| \leq c_n$. This is easily obtained by recursion. Now, let us set

$$\widehat{g}(s) = \sum_{n=0}^{+\infty} c_n s^n.$$

We have

$$\widehat{g}(s)^2 = \sum_{n=0}^{+\infty} \left(\sum_{k=0}^n c_k c_{n-k} \right) s^n = 3\widehat{g}(s) - 2 - \frac{s}{2},$$

hence \widehat{g} is the formal expansion of the function $g : s \mapsto \frac{1}{2} (3 - \sqrt{1 - 2s})$. This shows that the series \widehat{g} has a radius of convergence equal to $\frac{1}{2}$, hence the radius R of our formal Crocco solution satisfies $R \geq \frac{1}{2}$.

Moreover, Formula (9.1) shows that the a_n are of constant sign for $n > 1$; therefore the sum of the series must have its first singularity on $[0, +\infty)$ at $s = R$, hence $R = \lambda_+$.

In this manner, we have constructed an analytic Crocco solution with the required properties. Its uniqueness follows from that of the formal series solution. This proves statement 2.

For statement 4, if u is such an analytic solution, then $\sigma := u'(0)$ must be non-zero (otherwise u would coincide with u_* which is not analytic at $s = 0$). As for the case $\sigma = 1$, looking for a

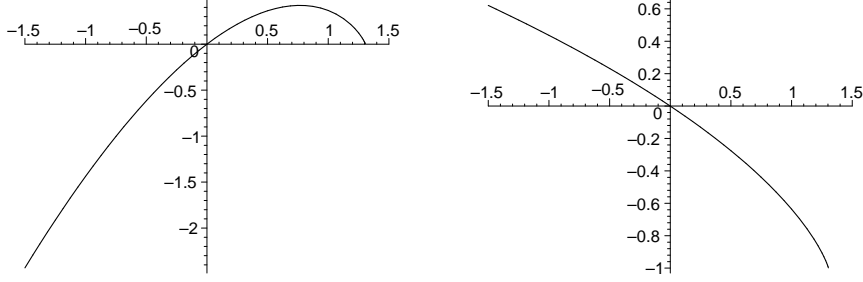


FIGURE 29. The solutions w and y . As $s \rightarrow \lambda_+ \approx 1.303918$, $w(s) \rightarrow 0$ and $y(s) \rightarrow -1$.

formal solution with prescribed derivative at $s = 0$ leads to uniqueness, hence to the function $s \mapsto \sigma^3 w(s/\sigma^2)$. \square

Remark. The fact that Crocco equation $uw'' + s = 0$ has a solution w that is analytic in a neighborhood of $s = 0$ with $w(0) = 0$, $w'(0) = 1$ is not a surprise: the change of variables $w(s) = (1 + y(s))s$ yields the equation $sy'' = -2y' - \frac{1}{1+y}$ with initial conditions $y(0) = 0$, $y'(0) = -\frac{1}{2}$; see Figure 29 for the graphs of w and y . Written as a differential system in $\vec{y} := (y, y')$

$$s \frac{d\vec{y}}{ds} = \vec{f}(s, \vec{y}) \quad \text{with} \quad \vec{f}(s, y_1, y_2) := \left(sy_1, -2y_2 - \frac{1}{1+y_1} \right),$$

it has a singularity of the first kind, hence satisfies the assumptions of Theorem V-2-7 of [29], page 118. This proves that our formal solution is convergent for $|s|$ small enough. However, for completeness we preferred to provide a direct proof based on the majorant method.

9.2. Alternative proofs. The aim of this section is to present three results that can be proved directly from the Crocco equation, *i.e.*, without the use of the vector field (2.12).

9.2.1. Solutions u_- and u_* are unique. (Theorem 2.2, item 2 and Theorem 2.4 item 4, first point). Let $\mu \leq 0$. Suppose that u_1 and u_2 are two positive solutions of the Crocco equation (2.2) on $(-\infty, 0)$ such that $u_1(0^-) = u_2(0^-) = 0$ and $u_1'(0^-) = u_2'(0^-) = \mu$. If $\mu = 0$, we will take $u_1 = u_*$. Let us set $w = u_1 - u_2$. We have $w(0^-) = w'(0^-) = 0$ and

$$\forall s < 0, \quad w''(s) = \frac{sw(s)}{u_1(s)u_2(s)}. \quad (9.2)$$

Integrating twice yields

$$\forall s < 0, \quad w(s) = \int_s^0 \frac{\eta(\eta - s)w(\eta)}{u_1(\eta)u_2(\eta)} d\eta,$$

which shows that for all $s < 0$, w has to vanish between s and 0. In particular, there exists an increasing sequence $s_n < 0$ tending to 0 such that $w(s_n) = 0$. Now, multiplying (9.2) by w' and integrating, we obtain for all $s < 0$ and all $s_n > s$ that

$$\begin{aligned} w'(s_n)^2 - w'(s)^2 &= -\frac{sw(s)^2}{u_1(s)u_2(s)} \\ &\quad - \int_s^{s_n} \left(1 - \frac{\eta u_1'(\eta)}{u_1(\eta)} - \frac{\eta u_2'(\eta)}{u_2(\eta)} \right) \frac{w(\eta)^2}{u_1(\eta)u_2(\eta)} d\eta. \end{aligned} \quad (9.3)$$

If $\mu < 0$, then $u_i(\eta) \sim \mu\eta$ as $\eta \rightarrow 0^-$, for $i = 1, 2$, hence there exists $\delta < 0$ such that

$$\forall \eta \in (\delta, 0), \quad \frac{\eta u'_i(\eta)}{u_i(\eta)} \geq \frac{3}{4},$$

for $i = 1, 2$. If $\mu = 0$, then for all $\eta < 0$ we have

$$1 - \frac{\eta u'_1(\eta)}{u_1(\eta)} = 1 - \frac{\eta \sqrt{3}(-\eta)^{-1/2}}{\frac{2}{\sqrt{3}}(-\eta)^{-3/2}} = -\frac{1}{2}.$$

Therefore, in both cases, equality (9.3) shows that $w'(s_n)^2 - w'(s)^2 \geq 0$ for all $s \in (\delta, 0)$ and all $s_n \in (s, 0)$. Taking the limit as $n \rightarrow +\infty$, we obtain $w'(s)^2 \leq 0$, hence $w' = 0$ on $(\delta, 0)$. This yields $w = 0$ and completes the proof. \square

9.2.2. Any Crocco solution of (2.2-2.3) with $b < 0$ and $c > 0$ is defined at least on $(-\infty, 0)$ and is asymptotic to u_* as $s \rightarrow -\infty$. (Theorem 2.2, item 1). Let u be a solution of (2.2-2.3) with $b < 0 < c$ and (s_-, s_+) be its maximal interval of existence. From Proposition 7.1, s_+ is non-negative. We now prove by contradiction that $s_- = -\infty$. Assume that s_- is finite; then by convexity, $u(s)$ tends to a limit $l \in [0, +\infty]$ as $s \rightarrow s_-$.

If $l \neq 0$ (l finite or not), then $u''(s) = -\frac{s}{u(s)}$ tends to $-\frac{s_-}{l} \in [0, +\infty)$ as $s \rightarrow s_-$, hence u' and u would have a finite limit, contradicting the maximality of (s_-, s_+) .

If $l = 0$, then by convexity u is increasing on $(s_-, 0)$. Identity (3.5) then gives, for an arbitrary $s_0 \in (s_-, 0)$,

$$\begin{aligned} u'(s_0)^2 - u'(s)^2 &= -2s_0 \ln u(s_0) + 2s \ln u(s) + 2 \int_s^{s_0} \ln u(\eta) d\eta \\ &\geq -2s_0 \ln u(s_0) + 2s \ln u(s) + 2(s - s_0) \ln u(s) \\ &= -2s_0 (\ln u(s_0) - \ln u(s)) \rightarrow +\infty \quad \text{as } s \rightarrow s_-, \end{aligned}$$

hence a contradiction. This shows that u is defined at least on $(-\infty, 0)$.

For the asymptotic, we construct by induction sequences $\alpha_n > 0$, $\beta_n > 0$ and $s_n < 0$ such that for all $n \geq 1$, we have

$$\forall s \leq s_n, \quad \alpha_n(-s)^3 \leq u(s)^2 \leq \beta_n(-s)^3, \quad (9.4)$$

and

$$\lim_{n \rightarrow +\infty} \alpha_n = \lim_{n \rightarrow +\infty} \beta_n = \frac{4}{3}. \quad (9.5)$$

First of all, thanks to (3.5) we have

$$\begin{aligned} \forall s \leq 0, \quad u(s)^2 &= u(0)^2 + 2u(0)u'(0)s + 2 \int_s^0 (\eta - s)u'(\eta)^2 d\eta - \frac{1}{3}s^3 \\ &\geq u(0)^2 + 2u(0)u'(0)s - \frac{1}{3}s^3. \end{aligned}$$

Therefore, there exists $\sigma_0 < 0$ such that

$$\forall s \leq \sigma_0, \quad u(s)^2 \geq \frac{1}{4}(-s)^3.$$

Coming back to (2.2) we obtain $u''(s) \leq 2(-s)^{-1/2}$ for $s \leq \sigma_0$ and integrating twice we easily see that there exists $s_1 \leq \sigma_0$ such that

$$\forall s \leq s_1, \quad u(s) \leq 3(-s)^{3/2}.$$

So (9.4) holds for $n = 1$ with $\alpha_1 = \frac{1}{4}$ and $\beta_1 = 9$. Suppose now that for a given integer n , we have α_n , β_n and s_n such that (9.4) holds. Starting from (2.2) and using the first inequality of (9.4), we obtain $u''(s) \leq \frac{1}{\sqrt{\alpha_n}}(-s)^{-1/2}$. Integration gives

$$\exists A_n, B_n > 0, \quad \forall s < s_n, \quad u'(s)^2 \leq \frac{4}{\alpha_n}(-s) + A_n(-s)^{1/2} + B_n.$$

Thus there exists $\sigma_n \leq s_n$ such that

$$\forall s \leq \sigma_n, \quad u'(s)^2 \leq \left(\frac{4}{\alpha_n} + \frac{1}{3n} \right) (-s).$$

From (3.5) we then obtain, for all $s \leq \sigma_n$,

$$\begin{aligned} u(s)^2 &= u(\sigma_n)^2 + 2u(\sigma_n)u'(\sigma_n)s + 2 \int_s^{\sigma_n} (\eta - s)u'(\eta)^2 d\eta \\ &\quad - \frac{1}{3}(s - \sigma_n)^2(s + 2\sigma_n) \leq \left(1 + \frac{4}{\alpha_n} + \frac{1}{3n} \right) \frac{(-s)^3}{3} + s^2\varepsilon_n(s), \end{aligned}$$

where ε_n is bounded on $(-\infty, 0)$. Hence, there exists $\tau_n \leq \sigma_n$ such that

$$\forall s \leq \tau_n, \quad u(s)^2 \leq \frac{1}{3} \left(1 + \frac{4}{\alpha_n} + \frac{1}{2n} \right) (-s)^3.$$

Similarly, using the right inequality of (9.4) and the same method, we obtain

$$\exists \tilde{\tau}_n \leq \sigma_n, \quad \forall s \leq \tilde{\tau}_n, \quad u(s)^2 \geq \frac{1}{3} \left(1 + \frac{4}{\beta_n} - \frac{1}{2n} \right) (-s)^3.$$

Choosing $s_{n+1} := \min(\tau_n, \tilde{\tau}_n)$, we obtain (9.4) at order $n + 1$ with

$$\alpha_{n+1} := \frac{1}{3} \left(1 + \frac{4}{\beta_n} - \frac{1}{2n} \right) \quad \text{and} \quad \beta_{n+1} := \frac{1}{3} \left(1 + \frac{4}{\alpha_n} + \frac{1}{2n} \right).$$

Since $\alpha_1 = \frac{1}{4}$ and $\beta_1 = 9$, we obtain $\alpha_2 = \frac{1}{3} \cdot \frac{17}{18} > \alpha_1$ and $\beta_2 = \frac{1}{3} \cdot \frac{35}{2} < \beta_1$. Then by recursion, we obtain that the sequence (α_n) increases and that (β_n) decreases. Because $\alpha_n \leq \beta_n$ these sequences converge, respectively to α and β , which satisfy

$$\alpha = \frac{1}{3} \left(1 + \frac{4}{\beta} \right) \quad \text{and} \quad \beta = \frac{1}{3} \left(1 + \frac{4}{\alpha} \right).$$

Thus $\alpha = \beta = \frac{4}{3}$. Finally, (9.5) holds and (9.4) gives $u(s) \sim \frac{2}{\sqrt{3}}(-s)^{3/2}$ as $s \rightarrow -\infty$. \square

9.2.3. *The function $\tilde{\Lambda}$ is continuous at $(a_*, c_*) = (-\sqrt{3}, 2/\sqrt{3})$.* (First assertion of Theorem 2.5). Consider a sequence (a_n, c_n) tending to (a_*, c_*) , and set $\lambda_n = \tilde{\Lambda}(a_n, c_n)$ and $u_n = u(\cdot; a_n, -1, c_n)$. Writing (3.6) for $s_0 = -1$ and $s \in [-1, 0]$ we obtain

$$u'_n(s)^2 = a_n^2 - 2s \ln u_n(s) - 2 \ln c_n + 2 \int_{-1}^s \ln u_n(\tau) d\tau. \quad (9.6)$$

Then we obtain

$$2 \int_{-1}^0 s u'_n(s)^2 ds = -a_n^2 + 2 \ln c_n - 6 \int_{-1}^0 s^2 \ln u_n(s) ds,$$

and thus (3.5) written for $s_0 = -1$ and $s = 0$ gives

$$u_n(0)^2 - c_n^2 - 2c_n a_n = -2 \int_{-1}^0 s u'_n(s)^2 ds + \frac{2}{3}, \quad (9.7)$$

$$= a_n^2 - 2 \ln c_n + 6 \int_{-1}^0 s^2 \ln u_n(s) ds + \frac{2}{3}. \quad (9.8)$$

We claim that there is a constant $C_1 > 0$ such that

$$\forall n \in \mathbb{N}, \quad \forall s \in [-1, 0], \quad 0 < u_n(s) \leq C_1. \quad (9.9)$$

On the contrary, it would exist a subsequence $u_{n_k}(0)$ that tends to $+\infty$ as $k \rightarrow +\infty$ and thus for k large enough we should have $u_{n_k}(\tau) < u_{n_k}(0)$ for $y \in [b, 0]$ and thanks to (9.8),

$$u_{n_k}(0)^2 - c_{n_k}^2 - c_{n_k} a_{n_k} \leq \frac{2}{3} - \ln c_{n_k} + \ln u_{n_k}(0),$$

which gives a contradiction as $k \rightarrow +\infty$. Hence (9.9) holds. Coming back to (9.6) we derive that there exists a constant C_2 such that

$$\forall n \in \mathbb{N}, \quad \forall s \in [-1, 0], \quad |u'_n(s)| \leq C_2. \quad (9.10)$$

Using (9.7), (9.10), the fact that for all $s \in [-1, 0)$, $u'_n(s) \rightarrow u'_*(s)$ as $n \rightarrow +\infty$, and the Lebesgue dominated convergence theorem, we obtain that

$$u_n(0)^2 \rightarrow c_*^2 + 2c_*a_* - 2 \int_{-1}^0 s u'_*(s)^2 ds + \frac{2}{3} = u_*(0)^2 = 0 \quad \text{as } n \rightarrow +\infty,$$

and $u_n(0) \rightarrow 0$ as $n \rightarrow +\infty$.

Moreover, taking into account (9.9) and applying the Fatou's Lemma to the nonnegative functions $g_n = \ln C_1 - \ln u_n$, we easily derive that

$$\limsup_{n \rightarrow +\infty} \int_{-1}^0 \ln u_n(s) ds \leq \int_{-1}^0 \ln u_*(s) ds.$$

Then using (9.6) for $s = 0$ leads to

$$0 \leq \limsup_{n \rightarrow +\infty} u'_n(0)^2 \leq a_*^2 - 2 \ln c_* + 2 \int_{-1}^0 \ln u_*(s) ds = u'_*(0)^2 = 0,$$

and $u'_n(0) \rightarrow 0$ as $n \rightarrow +\infty$.

To conclude, let us set $\alpha_n = \max(u'_n(0), u_n(0)^{1/3})$. By Proposition 2.6, item 1 and similarity (2.15), we have

$$0 \leq \lambda_n = \Lambda(u'_n(0), 0, u_n(0)) \leq \Lambda(\alpha_n, 0, \alpha_n^3) = \alpha_n^2 \Lambda(1, 0, 1) \rightarrow 0 \quad \text{as } n \rightarrow +\infty,$$

since $\alpha_n \rightarrow 0$ as $n \rightarrow +\infty$. This completes the proof. \square

9.3. Additional results on concave Blasius solutions. We give here some precisions about concave Blasius solutions, in particular, on the function $c_* : (a, b) \mapsto \sup\{c < 0 ; \Lambda(a, b, c) = -\infty\}$ introduced in (8.1).

Proposition 9.2. 1. *Consider the first order differential equation*

$$\frac{dy}{dx} = \frac{3y - 2x}{x + \frac{2}{y}},$$

in the region $y < 0$ and $xy > -2$. This equation has a unique solution $x \mapsto y_(x)$ which is defined for $-\infty < x < +\infty$ and satisfies $y_*(x) \sim x$ as $x \rightarrow -\infty$. Moreover, this function y_* is concave, increasing and $y_*(x) \rightarrow 0$ as $x \rightarrow +\infty$.*

2. *For $b > 0$, we have*

$$c_*(a, b) = b^{3/2} y_*(b^{-1/2} a). \quad (9.11)$$

Proof. We start from the Crocco equation (2.2). The change of variables

$$x(t) = e^{-t/2} u'(e^t), \quad y(t) = e^{-3t/2} u(e^t),$$

leads to the system

$$\dot{x} = -\frac{1}{2}x - \frac{1}{y}, \quad \dot{y} = x - \frac{3}{2}y. \quad (9.12)$$

Since concave Blasius solutions correspond to negative Crocco solutions, we consider this system only for $y < 0$. The initial conditions $u(b) = c$, $u'(b) = a$, with $b > 0$, correspond to

$$x(\ln b) = b^{-1/2}a, \quad y(\ln b) = b^{-3/2}c. \quad (9.13)$$

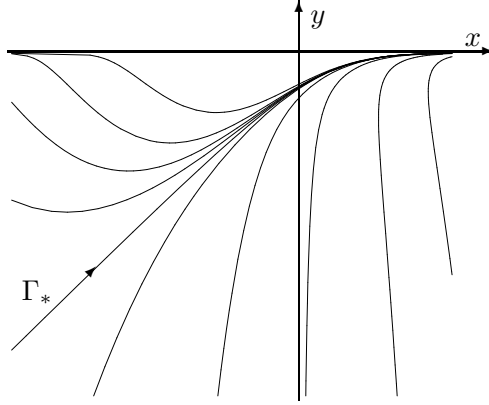


FIGURE 30. The phase portrait of (9.12).

Notice that the vector field (9.12) describes Crocco equation only for $s > 0$. This vector field has no stationary point. A phase plane analysis similar to that of Section 4 shows that there is one and only one orbit, denoted by Γ_* (see Figure 30) with the property that any solution (x, y) that parametrizes Γ_* is such that $\frac{x(t)}{y(t)}$ tends to 1 as $t \rightarrow -\infty$. The orbit Γ_* is the graph of the function $y = y_*(x)$ defined in the proposition. Let (x, y) denote the solution with initial condition (9.13) and (t_-, t_+) denote its maximal interval of definition. Depending on (a, b, c) , the following cases occur.

- If $c < b^{3/2}y_*(b^{-1/2}a)$ (*i.e.*, the trajectory is below Γ_*), then $t_- = -\infty$, t_+ is finite and we have

$$\lim_{t \rightarrow -\infty} y(t) = -\infty, \quad \lim_{t \rightarrow t_+} y(t) = 0.$$

Consequently, the maximal interval of definition of $u(\cdot; a, b, c)$ is $(-\infty, e^{t_+})$. It follows that $f(\cdot; a, b, c)$ is defined on $(-\infty, T_{a,b,c})$ with $0 < T_{a,b,c} < +\infty$ and we obtain $\Lambda(a, b, c) = -\infty$. For an illustration of this case, see the bottom of Figure 28.

- If $c > b^{3/2}y_*(b^{-1/2}a)$, then t_- and t_+ are finite and

$$\lim_{t \rightarrow t_-} y(t) = 0, \quad \lim_{t \rightarrow t_+} y(t) = 0.$$

It follows that the maximal interval of definition of $u(\cdot; a, b, c)$ is (e^{t_-}, e^{t_+}) , hence $f(\cdot; a, b, c)$ is defined on $(-\infty, +\infty)$ and we obtain $\Lambda(a, b, c) = e^{t_-}$. See the top of Figure 28.

- If $c = b^{3/2}y_*(b^{-1/2}a)$, then $t_- = -\infty$, t_+ is finite and we have

$$\lim_{t \rightarrow -\infty} y(t) = -\infty, \quad \lim_{t \rightarrow t_+} y(t) = 0.$$

Therefore, the maximal interval of definition of $u(\cdot; a, b, c)$ is $(0, e^{t+})$, hence $f(\cdot; a, b, c)$ is defined on $(-\infty, +\infty)$ and $\Lambda(a, b, c) = 0$. This proves (9.11). \square

9.4. Reduction of the Blasius equation to a planar vector field. The Blasius differential equation $f''' + ff'' = 0$, of order three, can be reduced to a planar autonomous vector field because it is invariant by the group of transformations $f(t) \mapsto \kappa f(\kappa t)$. Setting

$$g(\xi) = \frac{f'(t)}{f(t)^2}, \quad h(\xi) = \frac{f''(t)}{f(t)^3}, \quad \frac{d\xi}{dt} = f, \quad (9.14)$$

we obtain the equations

$$\frac{dg}{d\xi} = h - 2g^2, \quad \frac{dh}{d\xi} = -h(1 + 3g). \quad (9.15)$$

The change of variables (9.14) was considered in [12] in the more general case of equation $f''' + (m+1)ff'' - 2mf'^2 = 0$ where m is a real parameter. See also [25]. Formulae (9.14) define a change of variables only when the Blasius solution f does not vanish. To a vanishing Blasius solution correspond up to three orbits of the vector field (9.15). These orbits are oriented in the sense of increasing time ξ when f is positive and in the sense of decreasing time ξ when f is negative. It is difficult to follow the function f in the plane (g, h) because the values of t for which $f(t) = 0$ correspond to points at infinity of the vector field (9.15) or to the non-elementary singular point $(0, 0)$ of this vector field.

The reduction of the Blasius equation to a first order equation appeared first in [55], p. 389, and is attributed by H. Weyl to J. von Neumann. Setting

$$f = e^{-s}, \quad f' = e^{-2s}\theta, \quad 2\theta - \frac{d\theta}{ds} = \tau, \quad (9.16)$$

von Neumann obtains the equation

$$\frac{d\tau}{d\theta} = \frac{\tau}{\theta} \frac{1 + \tau + \theta}{2\theta - \tau}. \quad (9.17)$$

Notice that from (9.16) we deduce that

$$\theta = \frac{f'}{f^2} = g, \quad \frac{d\theta}{ds} = \frac{2f'}{f^2} - \frac{f''}{ff'}, \quad \tau = \frac{f''}{ff'} = \frac{h}{g}.$$

Hence, equations (9.17) and (9.15) are equivalent through the change of variables $\theta = g$, $\tau = h/g$. The interest of equation (9.17) and formulae (9.16) is that solving Blasius equation amounts to solving a first order equation, followed by two quadratures. After determining $\tau(\theta)$ from equation (9.17) one find $s(\theta)$, and then $t(\theta)$ from

$$\frac{ds}{d\theta} = \frac{1}{2\theta - \tau(\theta)}, \quad \frac{dt}{d\theta} = \frac{-e^{s(\theta)}}{\theta(2\theta - \tau(\theta))}.$$

Hence $t = t(\theta)$, $f = e^{-s(\theta)}$ parametrizes the solution f to the Blasius equation.

Following von Neumann and Weyl, many authors reduced also the Blasius equation to a first order equation or a planar vector field using various change of variables. Among these authors, we can cite B. Punnis, W.A. Coppel and Y.M. Treve. Setting

$$x = f, \quad y = f', \quad U = \frac{y}{x^2}, \quad V = \frac{1}{x} \frac{dy}{dx}, \quad (9.18)$$

Punnis [42], p. 168, obtains the equation

$$\frac{dV}{dU} = \frac{V}{U} \frac{1 + U + V}{2U - V}. \quad (9.19)$$

From (9.18) we deduce that $U = g$, $V = \frac{h}{g}$. Hence the variables U and V of Punnis are the same than the variables θ and τ of von Neumann. This is not surprising since the first order equation (9.19) obtained by Punnis is the same as the first order equation (9.17) of von Neumann. Setting

$$f = e^s, \quad f' = e^{2s}x, \quad y = \frac{dx}{ds}, \quad \frac{d\sigma}{ds} = -\frac{1}{x},$$

Coppel [14], p. 124, obtains the planar vector field

$$\frac{dx}{d\sigma} = -xy, \quad \frac{dy}{d\sigma} = 2x + y + 6x^2 + 7xy + y^2. \quad (9.20)$$

Notice that equations (9.20) and (9.15) are equivalent through the change of variables $\sigma = -\xi + \text{Const}$, $x = g$ and $y = \frac{h}{g} - 2g$. Setting

$$u_2 = \frac{f'}{f^2}, \quad u_1 u_2 = -\frac{f''}{f^3}, \quad \frac{d\xi}{dt} = f,$$

Treve [47], p. 1220, obtains the planar vector field

$$\frac{du_1}{d\xi} = -u_1(1 - u_1 + u_2), \quad \frac{du_2}{d\xi} = -u_2(u_1 + 2u_2). \quad (9.21)$$

Notice that equations (9.21) and (9.15) are equivalent through the change of variables $u_1 = -\frac{h}{g}$ and $u_2 = g$. See Section 9.5 for more historical informations and a review of the main results obtained with the help of the vector fields (9.17), (9.19), (9.20) and (9.21).

We already noticed that the solutions of the vector (9.15) going to infinity are of particular interest, because they correspond to vanishing Blasius functions. Hence, it should be interesting to study these solutions using the methods of Poincaré. This has been done by some authors, in particular B. Punnis [42] and Y.M. Treve [47]. Setting $X = -\frac{1}{V}$ and $Y = -\frac{U}{V}$ in equation (9.19), Punnis obtains equation

$$\frac{dY}{dX} = \frac{Y}{X} \frac{X - Y - 2}{X + Y - 1}. \quad (9.22)$$

This last equation was previously obtained by C.W. Jones [36]. This author transformed first the Blasius equation into the Crocco equation (without any reference to the work of Crocco [15]) and then, setting $X = \frac{su'}{u}$ and $Y = \frac{s^2}{uu'}$, he obtains equation (9.22). The reason why the Blasius equation reduces to the first order equation (9.22) using two apparently different ways becomes clear when we express all the new variables in terms of f and its derivatives. Actually, Punnis used $U = \frac{f'}{f^2}$, $V = \frac{f''}{ff'}$, and then

$$X = -\frac{1}{V} = -\frac{ff'}{f''}, \quad Y = -\frac{U}{V} = -\frac{f'^2}{ff''}.$$

On the other hand, Jones used $s = f'$, $u = f''$, $u' = -f$, and then

$$X = \frac{su'}{u} = -\frac{ff'}{f''}, \quad Y = \frac{s^2}{uu'} = -\frac{f'^2}{ff''}.$$

Thus, the variables X and Y of Punnis and Jones are identical. Compared to our present work and our variables x, y given by (2.11), these variables are in fact $X = -\frac{x}{y}$, $Y = \frac{1}{xy}$.

9.5. Historical comments.

9.5.1. *The original question.* As we said in the introduction, the Blasius problem (rewritten here for convenience)

$$\begin{aligned} f''' + ff'' &= 0 \quad \text{on } [0, +\infty), \\ f(0) = -a, \quad f'(0) = b, \quad \lim_{t \rightarrow +\infty} f'(t) &= \lambda, \end{aligned} \quad (\mathcal{P}_{a,b,\lambda})$$

first appears, with $a = b = 0$ and $\lambda = 2$, in [7], see also [8]. Without worrying about existence or uniqueness of solution, Blasius is mainly interested in the computation of the value $\alpha := f''(0)$. In the framework of our article, α is such that $\Lambda(0, 0, \alpha) = 2$. By the similarity (2.15), α and $\lambda_1 := \Lambda(0, 0, 1)$ are thus linked by $\alpha^{2/3}\lambda_1 = 2$.

To compute α , Blasius makes use of the formal solution

$$f(t) = \sum_{n=0}^{+\infty} (-1)^n \frac{c_n}{(3n+2)!} t^{3n+2}, \quad (9.23)$$

where the coefficients c_n are given by

$$c_0 = \alpha \quad \text{and} \quad c_{n+1} = \sum_{j=1}^n \binom{3n+2}{3j} c_j c_{n-j}.$$

Thus, the few first terms are $c_1 = \alpha^2$, $c_2 = 11\alpha^3$, $c_3 = 375\alpha^4$, $c_4 = 27897\alpha^5$. The presence of the term $(3n+2)!$ at the denominator in the sum (9.23) leads Blasius to believe that the power solution converges for all $t \in \mathbb{R}$. He then makes use of this power series around $t = 0$ and of certain asymptotic expression for large values of t , adjusting the constant α so as to connect both expressions in a middle region. In this way, Blasius obtains the (erroneous) bounds $1.326 < \alpha < 1.327$.

In 1912, in a short note, C. Töpfer [46] comes back to the paper of Blasius [8] and solves numerically the Blasius equation with initial conditions $f(0) = f'(0) = 0$, $f''(0) = 1$, by using the so-called Runge-Kutta method. He then arrives, without detailing his computations, at the value $\alpha \approx 1.32824$, contradicting the bounds obtained by Blasius. We must notice that neither Blasius, nor Töpfer justify thoroughly the accuracy of their computations.

Thereafter, L. Bairstow [3], with the power series, obtains $\alpha \approx 1.340$, S. Goldstein [23] obtains $\alpha \approx 1.328$, V.M. Falkner [19], by a finite difference method, yields the value $\alpha \approx 1.3282306$, L. Howarth [28] gives $\alpha \approx 1.328228$, and up today, much efforts have been made to get approximated value of α or λ_1 .

From 1968, the Crocco formulation is also used to compute α . For example, A.J. Callegari and M.B. Friedman [13] formulate the Blasius problem in terms of the Crocco variables, show that this problem has an analytical solution, and give the following inequalities: $1.32822 < \alpha < 1.32828$. These bounds for α correspond to $1.65515 < \lambda_1 < 1.65520$. In 1991, K. Vajravelu, E. Soewono and R.N. Mohapatra [49] use the method of Runge-Kutta and a shooting technique to solve numerically the Crocco formulation of the Blasius problem ($\mathcal{P}_{0,0,1}$) and obtain the erroneous value $\alpha \approx 1.32880$ which corresponds to $\lambda_1 \approx 1.65473$. In 1999, J.P. Boyd [9] considers the Blasius equation in the complex plane and gives $\alpha \approx 1,32822934486$.

9.5.2. *About the radius of convergence of the Blasius series.* In 1941, H. Weyl [54] proves that the radius of convergence of the power series (9.23) with $c_0 = 1$ is between 2.620 and 3.915 and chooses to make use of a process of successive and alternating approximations defined by $g_0 = 0$ and $g_{n+1} = \Phi(g_n)$ where

$$\Phi(g)(t) = \exp \left\{ -\frac{1}{2} \int_0^t (t-\zeta)^2 g(\zeta) d\zeta \right\}. \quad (9.24)$$

In this way, he proves that $\alpha < 1.368$ and says that g_3 is a pretty good approximation of f'' .

In 1947, A. Oudart [41] p.123, who seems to be unaware of the paper of Weyl [54], asks the question of knowing if the Blasius formal solution converges on the whole line \mathbb{R} , or not. J. Kuntzmann [37] gives the answer and proves that the radius of convergence R of the power series expansion (9.23) with $c_0 = 1$ is between 2.884 and 3.203.

In 1948, A. Ostrowski [40] improves these bounds. Using two methods, one based on the elementary proof of Borel of the Picard theorem and the second based on majorant series, he provides the bounds $3.1 < R < 3.18$, and announces also that the upper bound can be brought down to 3.14.

If f is the solution of the Blasius equation with initial conditions $f(0) = f'(0) = 0$ and $f''(0) = 1$, then the function $g : t \mapsto -f(-t)$ also is a solution of the Blasius equation, and satisfies $g(0) = g'(0) = 0, g''(0) < 0$. Hence, g does not exist up to $+\infty$, see Section 8. In term of f , this means that f cannot be extended on the whole interval $(-\infty, 0]$. This point of view allows to recover the fact that the radius R is necessarily finite. In [50], W. Walter, introducing appropriate super- and subsolutions, shows that g stops to exist somewhere between 3.098 and 3.151.

Recently, in [9], J.P. Boyd announces $R \approx 3.1273479$, confirmed by calculations of P. Fuchs [21]. Surprisingly, we did not find in the literature any formula relating the asymptotic behavior of the c_n to the radius R . Actually, a study of the Blasius series (9.23) in the complex plane, however out of the scope of the present article, shows that $\frac{c_n}{(3n+2)!} \sim \frac{9}{R^{3n+3}}$. Using (2.14), P. Fuchs [21] obtained that there exist A and φ in \mathbb{R} such that

$$\frac{c_n}{(3n+2)!} = \frac{9}{R^{3n+3}} \left(1 + \frac{A}{n^2} \cos(\sqrt{2} \ln n + \varphi) + O(n^{-3}) \right) \text{ as } n \rightarrow +\infty.$$

A more complete analysis of the singularities would even give a whole asymptotic expansion of c_n . This approach would give very quickly an estimate for R .

9.5.3. *Existence and uniqueness of the solution of the restricted problem $(\mathcal{P}_{0,0,\lambda})$.* The questions of existence and uniqueness of the Blasius problem with $a = b = 0$ is evoked for the first time in 1942, by H. Weyl [55] who proves that the integral operator Φ defined by (9.24) has a fix point g and that $f'' = g$ with initial conditions $f(0) = f'(0) = 0$ yields a solution of the Blasius equation defined on the whole interval $[0, +\infty)$. This solution satisfies $f''(0) = 1$ and $f'(t)$ tends to a positive limit β as $t \rightarrow +\infty$. The function $t \mapsto \kappa f(\kappa t)$, where $\kappa = \sqrt{2/\beta}$, is then the unique solution of the problem $(\mathcal{P}_{0,0,2})$.

In 1960, W.A. Coppel [14], in a long paper essentially concerned with the Falkner-Skan equation

$$f''' + f f'' + \delta(1 - f'^2) = 0, \tag{9.25}$$

with $\delta > 0$, devotes a section to the Blasius equation, and shows by differential inequalities that any convex Blasius solution f does exist up to $+\infty$, that $f'(t)$ has a nonnegative limit as $t \rightarrow +\infty$ and that $f(t)$ is positive for all large t . Then he deduces that there are only three possibilities as $t \rightarrow +\infty$:

$$\begin{array}{lll} \text{either} & f(t) \rightarrow 0, & f'(t) \rightarrow 0, & f''(t) \rightarrow 0, \\ \text{or} & f(t) \rightarrow \mu, & f'(t) \rightarrow 0, & f''(t) \rightarrow 0, & (\mu > 0), \\ \text{or} & f(t) \sim \beta t, & f'(t) \rightarrow \beta, & f''(t) \rightarrow 0, & (\beta > 0). \end{array}$$

Using the vector field (9.20), Coppel obtains that the only solutions of the first type are $f(t) = \frac{3}{t+t_0}$ with $t_0 \in \mathbb{R}$. He finally notices that f is of the last type if $f'(0) \geq 0$, or if $f'(0) < 0$ and $f(0) \leq 0$, and that the Blasius problem $(\mathcal{P}_{0,0,1})$ has one and only one solution.

Coppel also studies the concave solutions of the Blasius equation, and obtains the following possibilities:

$$\begin{aligned} & f(t) \rightarrow -\infty, \quad f'(t) \rightarrow -\infty, \quad f''(t) \rightarrow -\infty, \quad \text{as } t \rightarrow T, \\ \text{or } & f(t) \rightarrow \mu, \quad f'(t) \rightarrow 0, \quad f''(t) \rightarrow 0, \quad \text{as } t \rightarrow +\infty, \quad (\mu > 0), \\ \text{or } & f(t) \sim \beta t, \quad f'(t) \rightarrow \beta, \quad f''(t) \rightarrow 0, \quad \text{as } t \rightarrow +\infty, \quad (\beta > 0). \end{aligned}$$

Using the same vector field (9.20), he then proves that the solution of the first type satisfies

$$f(t) \sim -\frac{3}{T-t} \quad \text{as } t \rightarrow T, \quad t < T. \quad (9.26)$$

Moreover, he proves that for any $\mu > 0$ and $\gamma \neq 0$, the equation of Blasius has one and only one solution defined for all sufficiently large t such that $f(t) \rightarrow \mu$ and $f''(t) \sim \gamma e^{-\mu x}$ as $t \rightarrow +\infty$. His method is quite tedious and makes use of the following integro-differential equation

$$F''(t) = \gamma e^{-\mu x} + \int_x^{+\infty} e^{\mu(\zeta-x)} F(\zeta) F''(\zeta) d\zeta,$$

(with $F = f - \mu$) solving it by the usual fix point argument.

B. Punnis [42] uses (9.22) to show that the solution of the Blasius problem $(\mathcal{P}_{0,0,2})$ has the asymptotic form $f(t) \sim \frac{3}{T-t}$ as $t \rightarrow T$, $t > T$ for some $T < 0$. Coppel (see [14] p. 135) expresses doubts about the arguments of Punnis for proving this behavior, and precisely says that the reasons for asserting that the path in the phase-space necessarily tends to the critical point $(\frac{3}{2}, -\frac{1}{2})$ are not clear to him.

9.5.4. *Existence and uniqueness for the general problem.* In the general case $a, b \in \mathbb{R}$, the questions of existence and uniqueness depend on $\lambda < b$ (concave case) or $\lambda > b$ (convex case).

The concave case is rarely considered, essentially because the physical situations corresponding to it appears later. In particular, in the seventies, the Blasius problem $(\mathcal{P}_{0,1,0})$ arises in the framework of free convection in a porous medium. In 2000, Z. Belhachmi, B. Brighi and K. Taous [4] prove that the Blasius problem $(\mathcal{P}_{a,b,\lambda})$ with $a \in \mathbb{R}$, $b \in \mathbb{R}$ and $\lambda < b$ has one and only one solution if $b > 0$ and $\lambda \in [0, b]$, and no solutions if $\lambda < 0$. The authors prove directly on the Blasius equation that if $b \leq 0$, then any solution f of the Blasius equation verifying $f''(0) = c < 0$ does not exist up to $+\infty$, and that if $b > 0$, then there exists a negative real number $c_* = c_*(a, b)$ such that the function $c \mapsto \Lambda(a, b, c)$ is one-to-one and increasing from $[c_*, 0]$ onto $[0, b]$. In the present paper, for completeness, we gave in Section 8 a proof of the same result with the use of Crocco equation.

In the concave case, the question of uniqueness for the Blasius problem, is easy to solve, because, if there is a pair of distinct solutions f_1, f_2 and if $f_1''(0) > f_2''(0)$, then the function $g = f_1' - f_2'$ satisfies $g(0) = g(+\infty) = 0$ and $g'(0) > 0$. It follows that g has a positive maximum at some point $t_0 > 0$ such that $g(t) > 0$ for $0 < t < t_0$, but then we obtain

$$g''(t_0) = f_1'''(t_0) - f_2'''(t_0) = -f_1''(t_0) \int_0^{t_0} g(t) dt > 0,$$

and a contradiction. We see also that this argument does not work for convex solutions, and we will see below that the situation in this latter case is indeed very different. Let us finally notice that, in 1967, Y.M. Treve [47] studies the Blasius equation with initial conditions given by $f(0) = 1$, $f'(0) = b > 0$ and $f''(0) = c < 0$. Using the vector field (9.21) corresponding to the Blasius equation, Treve shows that there are values of c for which the solutions behave at infinity as mentioned by Coppel [14].

In the convex case, the results of existence and uniqueness for $b \geq 0$ are scattered in many papers concerned with the Falkner-Skan equation (9.25). Let us give some key steps. In 1945,

R. Iglisch and D. Grohne [34] obtain existence for $a \leq a_* \approx 1.2385$, $b = 0$, $\lambda = 1$. In 1964, P. Hartman [27] completes this result and proves existence for $a \in \mathbb{R}$, $0 < b < 1$ and $\lambda = 1$. In 1971, K.K. Tam [44] considers the Blasius problem $(\mathcal{P}_{a,0,1})$ with $a \leq 0$, but his work contains some mistakes. Moreover, in spite of the title, the approach is not so elementary.

The proofs of uniqueness of the solution of the problem $(\mathcal{P}_{a,b,\lambda})$ depend on the introduction of some changes of variables. In the case $a \in \mathbb{R}$, $b \geq 0$ and $\lambda > b$, the first change of variables for a uniqueness result consists of setting $z = f$ and $v = f'$. Blasius equation is then transformed into

$$v'' = -\frac{v'}{v}(v' + z),$$

where $'$ denotes the derivation with respect to z . To our knowledge, the first author to consider this transformation is S. Furuya [22] in 1953, who obtains uniqueness for the Falkner-Skan equation (9.25) (with $0 \leq \delta \leq 1$) subjected to the boundary conditions $f(0) = f'(0) = 0$ and $f'(t) \rightarrow \lambda > 0$ as $t \rightarrow +\infty$. It is easy to see that his proof can be extended to the case $f(0) = -a \geq 0$. This is done in 1954 by R. Iglisch [33] for all $\delta \geq 0$. Moreover, Iglisch proves, by an additional argument, that uniqueness also holds for $a \in (0, a_*)$. In [14], the uniqueness proof given by Coppel for the Falkner-Skan equation, contains the Blasius problem $(\mathcal{P}_{a,b,\lambda})$ for $a \leq 0$, $b \geq 0$ and $\lambda > b$. See also W.R. Utz [48].

To overcome the difficulty appearing for $a > 0$, P. Hartman [27] sets $z = f$ and $v = f'^2$, arrives to the equation

$$v'' = -\frac{zv'}{\sqrt{v}}, \quad (9.27)$$

and obtains uniqueness for all $a \in \mathbb{R}$ and $b \geq 0$. His proof, given for the Falkner-Skan equation, is quite complicated and depends on the introduction of suitable further transformations, but looking carefully, in the case of Blasius equation, the proof of P. Hartman can be simplified, and the successive transformations reduce to the Crocco transformation. Notice also that the equation (9.27) is used by N. Ishimura and S. Matsui [35] to prove the asymptotic (9.26).

For $b \geq 0$, the Crocco equation is the most elementary way to obtain uniqueness for all a . Indeed, let us assume that f_1, f_2 are two distinct solutions of the Blasius problem, set $c_i = f_i''(0)$ ($i = 1, 2$) and suppose that $c_1 > c_2$. We obtain $u_1, u_2 : [b, \lambda) \rightarrow \mathbb{R}$ solutions of the Crocco equation, and if $w = u_1 - u_2$ we have

$$\forall s \in (b, \lambda), \quad w''(s) = u_1''(s) - u_2''(s) = \frac{-s}{u_1(s)} + \frac{s}{u_2(s)} = \frac{sw(s)}{u_1(s)u_2(s)}.$$

Because u_1 and u_2 are positive, we obtain $w'' > 0$ as long as $w > 0$. Since $w(0) = c_1 - c_2 > 0$ and $w'(0) = a - a = 0$ we obtain that w increases, hence a contradiction with the fact that $w(s) \rightarrow 0$ as $s \rightarrow \lambda$. These arguments are more or less used by A.J. Callegari, M.B. Friedman [13] and K. Vajravelu, E. Soewono and R.N. Mohapatra [49].

REFERENCES

- [1] E. H. Aly, L. Elliott, and D. B. Ingham, *Mixed convection boundary-layer flow over a vertical surface embedded in a porous medium*, Eur. J. Mech. B Fluids, 22 (2003), 529-543.
- [2] V. I. Arnold, "Dynamical Systems V, Bifurcation Theory and Catastrophe Theory," Encyclopedia Math. Sci., Vol.5, Springer-Verlag, Berlin/New York, 1993.
- [3] L. Bairstow, *Skin friction*, The Journal of the Royal Aeronautical Society, 29 (1925), 3-23.
- [4] Z. Belhachmi, B. Brighi, and K. Taous, *On the concave solutions of the Blasius equation*, Acta Math. Univ. Comenianae, 69 (2000), 199-214.
- [5] E. Benoît Ed., "Dynamic Bifurcations," Proceedings Luminy 1990, Lect. Notes Math. Vol.1493, Springer-Verlag, 1991.
- [6] E. Benoît, J.-L. Callot, F. Diener, and M. Diener, *Chasse au canard*, Collect. Math., 32 (1981), 37-119.

- [7] H. Blasius, "Thesis," Göttingen (1907).
- [8] H. Blasius, *Grenzschichten in Flüssigkeiten mit kleiner Reibung*, Zeitschr. Math. Phys., 56 (1908), 1-37.
- [9] J. P. Boyd, *The Blasius function in the complex plane*, Experiment. Math., 8 (1999), 381-394.
- [10] B. Brighi, *Deux problèmes aux limites pour l'équation de Blasius*, Revue Math. Ens. Sup., 8 (2001), 833-842.
- [11] B. Brighi and J.-D. Hoernel, *Recent advances on similarity solutions arising during free convection*, "Progress in Nonlinear Differential Equations and Their Applications," 63, Birkhäuser Verlag Basel/Switzerland (2005), 83-92.
- [12] B. Brighi and T. Sari, *Blowing up coordinates for a similarity boundary layer equation*, Discrete Cont. Dyn. Systems, 12 (2005), 929-948.
- [13] A. J. Callegari and M. B. Friedman, *An analytical solution of a nonlinear singular boundary value problem in the theory of viscous fluids*, J. Math. Anal. Appl., 21 (1968), 510-529.
- [14] W. A. Coppel, *On a differential equation of boundary layer theory*, Phil. Trans. Roy. Soc. London, Ser. A, 253 (1960), 101-136.
- [15] L. Crocco, *Sull strato limite laminare nei gas lungo una lamina piana*, Rend. Math. Appl. Ser. 5, 21 (1941), 138-152.
- [16] F. Diener and M. Diener, *Ducks and rivers: three existence results*, in "Nonstandard Analysis in Practice," F. Diener and M. Diener (Eds.), Universitext, Springer-Verlag, Berlin/New York (1995), 205-224.
- [17] F. Dumortier and R. Roussarie, "Canard Cycles and center manifolds," Mem. Amer. Math. Soc. 577 (1996).
- [18] W. Eckhaus, *Relaxation oscillations including a standard chase on French ducks*, in Asymptotic Analysis II, surveys and new trends, Lecture Note Math. 985, Springer-Verlag, Berlin/New York (1984), 449-494.
- [19] V. M. Falkner, *A method of numerical solution of differential equations*, Phil. Mag. Jo. of Sc., 21 (1936), 624-640.
- [20] A. Fruchard and R. Schäfke, *Exceptional complex solutions of the forced van der Pol equation*, Funkcialaj Ekvacioj, 42 (1999), 201-223.
- [21] P. Fuchs, *Etude de la série de Blasius*, Rapport de stage de Magistère, U.H.A., Mulhouse (2007).
- [22] S. Furuya, *Note on a boundary value problem*, Comment. Math. Univ. St Paul, 1 (1953), 81-83.
- [23] S. Goldstein, *Concerning some solutions of the boundary layer equations in hydro-dynamics*, Proc. Camb. Philos. Soc., 26 (1930), 1-30.
- [24] M. Guedda, *Similarity and pseudosimilarity solutions of degenerate boundary layer equations*, in "Handbook of Differential Equations, Stationary Partial Differential Equations," 4, M. Chipot (Ed.) (2007), 117-198.
- [25] J.-S. Guo and J.-C. Tsai, *The structure of the solutions for a third order differential equation in boundary layer theory*, Japan J. Indust. Appl. Math., 22 (2005), 311-351.
- [26] P. Hartman, *On the asymptotic behavior of a differential equation in boundary layer theory*, ZAMM, 44 (1964), 123-128.
- [27] P. Hartman, "Ordinary Differential Equations," Wiley, 1964.
- [28] L. Howarth, *On the solution of the laminar boundary layer equation*, Proc. Royal Soc. London A, 164 (1938), 547-579.
- [29] P.-F. Hsieh and Y. Sibuya, "Basic Theory of Ordinary Differential Equations," Universitext, Springer-Verlag, 1999.
- [30] J. H. Hubbard and B. H. West, "Differential Equations: A Dynamical System Approach," Springer-Verlag, 1991.
- [31] M. Y. Hussaini and W. D. Laikin, *Existence and non-uniqueness of similarity solutions of a boundary layer problem*, Quart. J. Mech. Appl. Math., 39 (1986), 15-24.
- [32] M. Y. Hussaini, W. D. Laikin, and A. Nachman, *On similarity solutions of a boundary layer problem with an upstream moving wall*, SIAM J. Appl. Math., 47 (1987), 699-709.
- [33] R. Iglisch, *Elementarer Beweis für die Eindeutigkeit der Strömung in der laminaren Grenzschicht zur Potentialströmung $U = u_1 x^m$ mit $m \geq 0$ bei Absaugen und Ausblasen*, ZAMM, 34 (1954), 441-443.
- [34] R. Iglisch and D. Grohne, *Die laminare Grenzschicht an der längs angeströmten ebenen Platte mit schrägem Absaugen und Ausblasen*, Veröffentlichung des Mathematischen Instituts der Technischen Hochschule, Braunschweig. Bericht 1/45, (1945).
- [35] N. Ishimura and S. Matsui, *On blowing up solutions of the Blasius equation*, Discrete Cont. Dyn. Systems, 9 (2003), 985-992.
- [36] C.W. Jones, *On reducible non-linear differential equations occuring in mechanics*, Proc. Royal Soc. London A, 217 (1953), 327-343.
- [37] J. Kuntzmann, Interm. Rech. Math., (1947).
- [38] C. Lobry, T. Sari, and S. Touhami, *On Tykhonov's theorem for convergence of solutions of slow and fast systems*, Electron. J. Differential Equations 19 (1998), 1-22.

- [39] O.A. Oleinik and V.N. Samokhin, “Mathematical Models in Boundary Layer Theory,” Chapman and Hall/CRC, Boca Raton, 1999.
- [40] A. Ostrowski, *Sur le rayon de convergence de la série de Blasius*, C. R. Acad. Sci. Paris, 227 (1948), 580-582.
- [41] A. Oudart, *Les méthodes scientifiques de la couche limite laminaire*, Publ. Sci. Tech. Ministère de l’Air, 213 (1948), 1-127.
- [42] B. Punnis, *Zur Differentialgleichung der Plattengrenzschicht von Blasius*, Arch. Math. (Basel), 7 (1956), 165-171.
- [43] E. Soewono, K. Vajravelu, and R.N. Mohapatra, *Existence and nonuniqueness of solutions of a singular non linear boundary layer problem*, J. Math. Anal. Appl., 159 (1991), 251-270.
- [44] K. K. Tam, *An elementary existence proof for the Blasius equation*, ZAMM, 51 (1971), 318-319.
- [45] A. N. Tikhonov, *Systems of differential equations containing small parameters multiplying the derivatives*, Mat. Sb., 31 (1952), 575-586.
- [46] C. Töpfer, *Bemerkungen zu dem Aufsatz von H. Blasius “Grenzschichten in Flüssigkeiten mit kleiner Reibung”*, Zeitschr. Math. Phys., 60 (1912), 397-398.
- [47] Y. M. Treve, *On a class of solutions of the Blasius equation with applications*, SIAM J. Appl. Math., 15:5 (1967), 1209-1227.
- [48] W. R. Utz, *Existence of solutions of a generalized Blasius equation*, J. Math. Anal. Appl., 66 (1978), 55-59.
- [49] K. Vajravelu, E. Soewono, and R. N. Mohapatra, *On solutions of some singular nonlinear differential equations in boundary layer theory*, J. Math. Anal. Appl., 155 (1991), 499-512.
- [50] W. Walter, “Differential and Integral Inequalities,” Springer-Verlag, 1970.
- [51] J. Wang, W. Gao, and Z. Zhang, *Singular nonlinear boundary value problems arising in boundary layer theory*, J. Math. Anal. Appl., 233 (1999), 246-256.
- [52] W. Wasow, “Asymptotic Expansions for Ordinary Differential Equations,” Krieger, New York, 1976.
- [53] P.D. Weidman, *New solutions for laminar boundary layers with cross flow*, Z. Angew. Math. Phys., 48:2 (1997), 341-356.
- [54] H. Weyl, *Concerning the differential equations of some boundary-layer problems*, Proc. Nat. Acad. Sci., 27 (1941), 578-583.
- [55] H. Weyl, *On the differential equations of the simplest boundary-layer problems*, Ann. Math., 43 (1942), 381-407.
- [56] A. K. Zvonkin and M. A. Shubin, *Nonstandard Analysis and Singular Perturbations of Ordinary Differential Equations*, Uspekhi Mat. Nauk., 39 (1984), 77-127. English transl.: Russian Math. Surv., 39 (1984), 69-131.

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