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# Factorization formulas for Macdonald polynomials

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## Abstract

The aim of this note is to give some factorisation formulas for different versions of the Macdonald polynomials when the parameter  $t$  is specialized at roots of unity, generalizing those given in [9] for Hall-Littlewood functions.

## 1 Introduction

In [9], Lascoux, Leclerc and Thibon give some factorisation formulas for Hall-Littlewood functions when the parameter  $q$  is specialized at roots of unity. They also give formulas in terms of cyclic characters of the symmetric group. In this article, we give a generalization of these specializations for different versions of the Macdonald polynomials and we obtain similar formulas in terms of plethysms and cyclic characters. In the last section, we give congruence formulas for  $(q, t)$ -Kostka polynomials using Schur functions in the alphabet of the powers of the parameter  $t$ . We will mainly follow the notations of [11].

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## 2 Preliminaries

For a partition  $\lambda = (\lambda_1, \dots, \lambda_n)$ ,  $\lambda_1 \geq \dots \geq \lambda_n$ , we write  $l(\lambda)$  its length,  $|\lambda|$  its weight,  $m_i(\lambda)$  the multiplicity of the part of length  $i$  and  $\lambda'$  its conjugate partition. Let  $q$  and  $t$  be two indeterminates and  $F = \mathbb{Q}(q, t)$ . Let  $\Lambda_F$  be the ring of symmetric functions over the field  $F$ . Let denote by  $\langle \cdot, \cdot \rangle_{q,t}$  the inner product on  $\Lambda_F$  defined on the powersums by

$$\langle p_\lambda, p_\mu \rangle_{q,t} = \delta_{\lambda\mu} z_\lambda(q, t),$$

where

$$z_\lambda(q, t) = \prod_{i \geq 1} (m_i)! i^{m_i(\lambda)} \prod_{i=1}^{l(\lambda)} \frac{1 - q^{\lambda_i}}{1 - t^{\lambda_i}}.$$

The special case  $\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_{q=0}$  is the usual inner product.

Let  $\{P_\lambda(x; q, t)\}_\lambda$  be the family of Macdonald polynomials obtained by orthogonalization of the Schur functions basis with respect to the inner product  $\langle \cdot, \cdot \rangle_{q,t}$ . Let us define a normalization of these functions by

$$Q_\lambda(x; q, t) = \frac{1}{\langle P_\lambda(x; q, t), P_\lambda(x; q, t) \rangle_{q,t}} P_\lambda(x; q, t).$$

It is clear from the previous definitions that the families  $\{P_\lambda(x; q, t)\}_\lambda$  and  $\{Q_\mu(x; q, t)\}_\mu$  are dual to each other with respect to the inner product  $\langle \cdot, \cdot \rangle_{q,t}$  (c.f. [11], Chap. I, section 4 and Chap. VI, formula (2.7)).

**Proposition 2.1** [[11], VI, (4.13)] *Let  $x = (x_1, x_2, \dots)$  and  $y = (y_1, y_2, \dots)$  be two alphabets. The Macdonald polynomials  $\{P_\lambda(x; q, t)\}_\lambda$  and  $\{Q_\lambda(x; q, t)\}_\lambda$  satisfy the Cauchy formula*

$$\sum_\lambda P_\lambda(x; q, t) Q_\lambda(y; q, t) = \prod_{i,j} \frac{(tx_i y_j; q)_\infty}{(x_i y_j; q)_\infty}, \quad (1)$$

where  $(a; q)_\infty$  is the infinite product  $\prod_{r \geq 0} (1 - aq^r)$ .

We consider the following algebra homomorphism

$$\begin{aligned} ' : \Lambda_F &\longrightarrow \Lambda_F \\ f(x) &\longmapsto f'(x) = f\left(\frac{1-q}{1-t}x\right). \end{aligned}$$

The images of the powersums  $(p_k)_{k \geq 1}$  by these morphisms are

$$p'_k(x) = \frac{1 - q^k}{1 - t^k} p_k(x).$$

Let us consider the following modified version of the Macdonald polynomial

$$Q'_\mu(x; q, t) = Q_\mu\left(\frac{1-q}{1-t}x; q, t\right).$$

We can see that the families  $\{Q'_\mu(x; q, t)\}_\mu$  and  $\{P_\lambda(x; q, t)\}_\lambda$  are dual to each other with respect to the usual inner product.

**Proposition 2.2** *Let  $x = (x_1, x_2, \dots)$  and  $y = (y_1, y_2, \dots)$  be two alphabets. The Macdonald polynomials  $\{P_\lambda(x; q, t)\}_\lambda$  and  $\{Q'_\lambda(x; q, t)\}_\lambda$  satisfy the following Cauchy formula*

$$\sum_\lambda P_\lambda(x; q, t) Q'_\lambda(y; q, t) = \prod_{i,j} \frac{1}{1 - x_i y_j}. \quad (2)$$

*Proof.* By Proposition 2.1, we have

$$\sum_{\lambda} P_{\lambda}(x; q, t) Q_{\lambda}(y; q, t) = \prod_{i,j} \prod_{r \geq 0} \frac{1 - tx_i y_j q^r}{1 - x_i y_j q^r}.$$

Since the map  $x \mapsto x/(1-t)$  corresponds to the transformation of the alphabet  $\{x_1, x_2, \dots\}$  into the alphabet  $\{x_i t^j, i \geq 1, j \geq 0\}$ , a straightforward computation shows that

$$\sum_{\lambda} P_{\lambda}(x; q, t) Q_{\lambda} \left( \frac{y}{1-t}; q, t \right) = \prod_{r \geq 0} \prod_{i,j} \frac{1}{1 - x_i y_j q^r}.$$

This means that the families  $\{P_{\lambda}(x; q, t)\}_{\lambda}$  on the alphabet  $x$  and  $\left\{Q_{\mu} \left( \frac{y}{1-q}; q, t \right)\right\}_{\mu}$  on the alphabet  $y/(1-q)$  are dual to each other with respect to the usual inner product. Since the transformation of alphabets  $y \mapsto y/(1-q)$  is invertible and the inverse map is given by  $y \mapsto (1-q)y$ , it follows that

$$\sum_{\lambda} P_{\lambda}(x; q, t) Q_{\lambda} \left( \frac{1-q}{1-t} y; q, t \right) = P_{\lambda}(x; q, t) Q'_{\lambda}(y; q, t) = \prod_{i,j} \frac{1}{1 - x_i y_j}.$$

□

We recall some definitions of combinatorial quantities associated to a cell  $s = (i, j)$  of a given partition. The arm length  $a(s)$ , arm-colength  $a'(s)$ , leg length  $l(s)$  and leg-colength  $l'(s)$  are respectively the number of cells at the east, at the west, at the south and at the north of the cell  $s$  (c.f. [11], Chap. VI, formula (6.14)), i.e

$$\begin{aligned} a(s) &= \lambda_i - j & , & & a'(s) &= j - 1, \\ l(s) &= \lambda'_j - i & , & & l'(s) &= i - 1. \end{aligned}$$

We also define the quantity

$$n(\lambda) = \sum_i (i-1) \lambda_i.$$

Let  $J_{\mu}(x; q, t)$  be the symmetric function with two parameters defined by

$$J_{\mu}(x; q, t) = c_{\mu}(q, t) P_{\mu}(x; q, t) = c'_{\mu}(q, t) Q_{\mu}(x; q, t), \quad (3)$$

with

$$c_{\mu}(q, t) = \prod_{s \in \mu} (1 - q^{a(s)} t^{l(s)+1}) \quad \text{and} \quad c'_{\mu}(q, t) = \prod_{s \in \mu} (1 - q^{a(s)+1} t^{l(s)}).$$

The symmetric function  $J_{\mu}(x; q, t)$  is called the integral form of  $P_{\mu}(x; q, t)$  or  $Q_{\mu}(x; q, t)$  (c.f. [11], Chap. VI, section 8). Using this integral form, we can define an other modified version of the Macdonald polynomial and the  $(q, t)$ -Kostka polynomials  $K_{\lambda, \mu}(q, t)$  by

$$J_{\mu} \left( \frac{x}{1-t}; q, t \right) = \sum_{\lambda} K_{\lambda, \mu}(q, t) s_{\lambda}. \quad (4)$$

In [4], Haglund, Haiman and Loehr consider a modified version of  $J_\mu\left(\frac{x}{1-t}; q, t\right)$  and introduce other  $(q, t)$ -Kostka polynomials  $\tilde{K}_{\lambda, \mu}(q, t)$  by defining the functions

$$\tilde{H}_\mu(x; q, t) = t^{n(\mu)} J_\mu\left(\frac{x}{1-t^{-1}}; q, t^{-1}\right) = \sum_{\lambda} \tilde{K}_{\lambda, \mu}(q, t) s_\lambda. \quad (5)$$

They give a combinatorial interpretation of this modified version expanded on the monomials basis by defining two statistics (major index and inversions) on arbitrary fillings by integers of  $\mu$ .

**Remark 2.1** Let  $\mu$  and  $\rho$  be two partitions of the same weight. We have

$$X_\rho^\mu(q, t) = \langle \tilde{H}_\mu(q, t), p_\rho(x) \rangle,$$

where  $X_\rho^\mu(q, t)$  is the Green polynomial with two variables, defined by

$$X_\rho^\mu(q, t) = \sum_{\lambda} \chi_\rho^\lambda \tilde{K}_{\lambda, \mu}(q, t).$$

Here  $\chi_\rho^\lambda$  is the value of the irreducible character of the symmetric group corresponding to the partition  $\lambda$  on the conjugacy class indexed by  $\rho$ . For related topics, see [12, 13].

### 3 Plethystic formula

In this section, we prove a plethystic formula for Macdonald polynomials indexed by rectangular partitions when the second parameter  $t$  is specialized at primitive roots of unity.

**Proposition 3.1** [[11], VI, (6.11')] *Let  $l$  be a positive integer and  $\lambda$  be a partition such that  $l(\lambda) \leq l$ . The Macdonald polynomials  $P_\lambda(x; t, q)$  on the alphabet  $\{x_i = t^i, 0 \leq i \leq l-1\}$ , and  $x_i = 0, \forall i \geq l$ , can be written*

$$P_\lambda(1, t, \dots, t^{l-1}; q, t) = t^{n(\lambda)} \prod_{s \in \lambda} \frac{1 - q^{a'(s)} t^{l-l'(s)}}{1 - q^{a(s)} t^{l(s)+1}}. \quad (6)$$

**Corollary 3.1** *Let  $l$  be a positive integer and  $\lambda$  a partition such that  $l(\lambda) \leq l$ . For  $\zeta$  a primitive  $l$ -th root of unity, the Macdonald polynomials  $P_\lambda(x; q, t)$  satisfy the specialization*

$$P_\lambda(1, \zeta, \zeta^2, \dots, \zeta^{l-1}; q, \zeta) = \begin{cases} (-1)^{(l-1)r} & \text{if } \lambda = (r^l) \text{ for some } r \geq 0, \\ 0 & \text{otherwise.} \end{cases} \quad (7)$$

*Proof.* Supplying zeros at the end of  $\lambda$ , we consider the partition  $\lambda$  as a sequence of length exactly equal to  $l$ . The multiplicity of 0 in  $\lambda$  is  $m_0 = l - l(\lambda)$ . We will denote by  $\varphi_r(t)$  the polynomial

$$\varphi_r(t) = (1-t)(1-t^2)\dots(1-t^r).$$

Let

$$f(t) = \frac{(1-t^l)(1-t^{l-1})\dots(1-t^{l-l(\lambda)})(1-t^{l-l(\lambda)-1})\dots(1-t^2)(1-t)}{\varphi_{m_0}(t)\varphi_{m_1}(t)\varphi_{m_2}(t)\dots\dots}$$

be the product of factors of the form  $1 - q^0 t^\alpha$  for some  $\alpha > 0$  in the formula (6). If we suppose that  $f(\zeta) \neq 0$ , the factor  $1 - t^l$  should be contained in one of  $\varphi_{m_i}(t)$ . This means that there exists  $i \geq 0$  such that  $m_i \geq l$ . Since we consider  $\lambda$  as a sequence of length exactly  $l$ , this implies the condition  $m_r = l$  for some  $r \geq 0$ . Thus, if  $P_\lambda(1, \zeta, \zeta^2, \dots, \zeta^{l-1}; q, \zeta) \neq 0$ , the shape of  $\lambda$  should be  $(r^l)$ .

Suppose now that  $\lambda = (r^l)$ . By Proposition 3.1, it follows that

$$\begin{aligned} P_\lambda(1, \zeta, \zeta^2, \dots, \zeta^{l-1}; q, \zeta) &= \zeta^{n(\lambda)} \prod_{s \in \lambda} \frac{1 - q^{a'(s)} \zeta^{l-l'(s)}}{1 - q^{a(s)} \zeta^{1+l(s)}} \\ &= \zeta^{n(\lambda)} \prod_{(i,j) \in \lambda} \frac{1 - q^{j-1} \zeta^{l-(i-1)}}{1 - q^{r-j} t^{l-i+1}} \\ &= \zeta^{n(\lambda)} \prod_{i=1}^l \prod_{j=1}^r \frac{1 - q^{j-1} \zeta^{l-i+1}}{1 - q^{r-j} t^{l-i+1}}. \end{aligned}$$

For each  $i$ , it is easy to see that

$$\prod_{j=1}^r \frac{1 - q^{j-1} \zeta^{l-i+1}}{1 - q^{r-j} t^{l-i+1}} = 1.$$

Hence, we obtain

$$P_\lambda(1, \zeta, \zeta^2, \dots, \zeta^{l-1}; q, \zeta) = \zeta^{n(\lambda)},$$

and it follows immediately from the definition of  $n(\lambda)$  that

$$\zeta^{n(\lambda)} = \zeta^{l(l-1)r/2} = (-1)^{(l-1)r}.$$

□

**Theorem 3.1** *Let  $l$  and  $r$  be two positive integers and  $\zeta$  a primitive  $l$ -th root of unity. The Macdonald polynomials  $Q'_{r^l}(x; q, t)$  satisfy the following specialization formula at  $t = \zeta$*

$$Q'_{(r^l)}(x; q, \zeta) = (-1)^{(l-1)r} (p_l \circ h_r)(x). \quad (8)$$

*Proof.* Recall that

$$\sum_{\lambda} P_{\lambda}(x; q, t) Q'_{\lambda}(y; q, t) = \prod_{i,j} \frac{1}{1 - x_i y_j}.$$

If we let  $x_i = \zeta^{i-1}$ , for  $i = 1, 2, \dots, l$ , and  $x_i = 0$ , for  $i > l$ , and  $t = \zeta$ , we obtain

$$\sum_{\lambda} P_{\lambda}(1, \zeta, \zeta^2, \dots, \zeta^{l-1}; q, \zeta) Q'_{\lambda}(y; q, \zeta) = \prod_{j \geq 1} \prod_{i=1}^l \frac{1}{1 - \zeta^{i-1} y_j}. \quad (9)$$

By Corollary 3.1, the left hand side of (9) is equal to

$$\sum_{r \geq 0} (-1)^{(r-1)l} Q'_{(r^l)}(y; q, \zeta).$$

Since  $\prod_{i=1}^l (1 - \zeta^{i-1} t) = 1 - t^l$ , the right hand side of (9) coincides with  $\sum_{r \geq 0} h_r(y^l)$ , where  $y^l$  denotes the alphabet  $(y_1^l, y_2^l, \dots)$ . Comparing the degrees, we can conclude that

$$Q'_{(r^l)}(y; q, \zeta) = (-1)^{(l-1)r} h_r(y^l) = (-1)^{(l-1)r} (p_l \circ h_r)(y).$$

□

**Example 3.2** For  $\lambda = (222)$  and  $l = 3$ , we can compute the specialization

$$\begin{aligned} Q'_{(222)}(x; q, e^{\frac{2i\pi}{3}}) &= -s_{321} + s_{33} + s_{411} - s_{51} + s_6 + s_{222} \\ &= p_3 \circ h_2(x). \end{aligned}$$

In order to give a similar formula for the modified versions of the integral form of the Macdonald polynomials, we give a formula for the specialization of the constant  $c'_{r^l}(t, q)$  at  $t$  a primitive  $l$ -th root of unity.

**Lemma 3.3** *Let  $l$  and  $r$  be two positive integers and  $\zeta$  a  $l$ -th primitive root of unity. The normalization constant  $c'_{\lambda}(q, t)$  satisfies the following specialization at  $t = \zeta$*

$$c'_{r^l}(q, \zeta) = \prod_{i=1}^r (q^{il} - 1). \quad (10)$$

*Proof.* Recall the definition of the normalization constant

$$c'_{r^l}(q, t) = \prod_{s \in \mu} (1 - q^{a(s)+1} t^{l(s)}) = \prod_{i=1}^r \prod_{j=1}^l (1 - q^{r-i+1} t^j) = \prod_{i=1}^r \prod_{j=1}^l (1 - q^i t^j).$$

Specializing  $t$  at  $\zeta$  a  $l$ -th primitive root of unity, we obtain

$$c'_{r^l}(q, \zeta) = \prod_{i=1}^r \prod_{j=1}^l (1 - q^{r-i+1} \zeta^j) = \prod_{i=1}^r (q^{il} - 1).$$

□

**Corollary 3.2** *With the same notations as in Theorem 3.1, the modified integral form of the Macdonald polynomials  $\tilde{H}_\mu(x; q, t)$  satisfy a similar formula at  $t = \zeta$ ,*

$$\tilde{H}_{(r^l)}(x; q, \zeta) = \prod_{i=1}^r (q^{il} - 1) p_l \circ h_r \left( \frac{x}{1-q} \right). \quad (11)$$

*Proof.* Using the definition 3 of the integral form of the Macdonald polynomials

$$\begin{aligned} J_{(r^l)} \left( \frac{x}{1-t}; q, t \right) &= c'_{r^l}(q, t) Q_{(r^l)} \left( \frac{x}{1-t}; q, t \right) \\ &= c'_{r^l}(q, t) Q'_{(r^l)} \left( \frac{x}{1-q}; q, t \right). \end{aligned}$$

This expression can be rewritten in terms of plethysms by the powersum  $p_1$ , consequently

$$J_{(r^l)} \left( \frac{x}{1-t}; q, t \right) = c'_{r^l}(q, t) \left( Q'_{(r^l)}(\cdot; q, t) \circ \frac{1}{1-q} p_1 \right) (x).$$

By specializing in this equality,  $t$  at a primitive  $l$ -th root of unity  $\zeta$ , using Theorem 3.1, we obtain

$$\begin{aligned} J_{(r^l)} \left( \frac{x}{1-\zeta}; q, \zeta \right) &= c'_{r^l}(q, \zeta) \left( Q'_{(r^l)}(\cdot; q, \zeta) \circ \frac{1}{1-q} p_1 \right) (x) \\ &= c'_{r^l}(q, \zeta) (-1)^{r(l-1)} \left( (p_l \circ h_r) \circ \frac{1}{1-q} p_1 \right) (x). \end{aligned}$$

As plethysm is associative, we can write

$$\begin{aligned} J_{(r^l)} \left( \frac{x}{1-\zeta}; q, \zeta \right) &= c'_{r^l}(q, \zeta) (-1)^{r(l-1)} p_l \circ \left( h_r \circ \frac{1}{1-q} p_1 \right) (x) \\ &= c'_{r^l}(q, \zeta) (-1)^{r(l-1)} p_l \circ h_r \left( \frac{x}{1-q} \right). \end{aligned}$$

Using the formula of Lemma 3.3 and  $\zeta^{n(\lambda)} = (-1)^{(l-1)r}$ . we obtain the formula. □

**Example 3.4** For  $\lambda = (222)$  and  $l = 3$ , we can compute

$$\begin{aligned} \tilde{H}_{(222)}(x; q, i) &= s_{611} + s_8 - s_{5111} - s_{71} + q^4(s_{11111111} + s_{311111} - s_{2111111} - s_{41111}) + \\ &\quad (q^4 + 1)(s_{2222} + s_{332} + s_{4211} + s_{44} - s_{431} - s_{3221}) \\ &= (1 - q^4)(1 - q^8) p_4 \circ h_2 \left( \frac{x}{1-q} \right). \end{aligned}$$

**Remark 3.5** At  $t = \zeta$ , a primitive  $l$ -th root of unity, the inverse of the norm of the Macdonald polynomial  $P_{(r^l)}(x; q, t)$  satisfies

$$\frac{1}{\langle P_{(r^l)}(x; q, t), P_{(r^l)}(x; q, t) \rangle_{q,t}} \Big|_{t=\zeta} = 0.$$

Consequently, we obtain the following specializations

$$Q_{(r^l)}(x; q, \zeta) = 0 \quad \text{and} \quad J_{(r^l)}(x; q, \zeta) = 0.$$

## 4 Pieri formula at roots of unity

In order to prove the factorization formulas, we give an auxiliary result, in Proposition 4.1, on the coefficients of Pieri formula at root of unity (c.f. [11], Chap. VI, formula (6.24 ii))

$$Q'_\mu(x; q, t)g'_r(x; q, t) = \sum_{\lambda} \psi_{\lambda/\mu}(q, t) Q'_\lambda(x; q, t), \quad (12)$$

with

$$\forall r \geq 0, g'_r \left( \frac{1-q}{1-t}x; q, t \right) = \sum_{|\lambda|=n} z_\lambda(q, t)^{-1} p_\lambda \left( \frac{1-q}{1-t}x \right).$$

Let  $\lambda$  and  $\mu$  be partitions such that  $\lambda/\mu$  is an horizontal ( $r$ -)strip  $\theta$ . Let  $C_{\lambda/\mu}$  (resp.  $R_{\lambda/\mu}$ ) be the union of columns (resp. rows) of  $\lambda$  that intersects with  $\theta$ , and  $D_{\lambda/\mu} = C_{\lambda/\mu} - R_{\lambda/\mu}$  the set theoretical difference. Then it can be seen from the definition that for each cell  $s$  of  $D_{\lambda/\mu}$  (resp.  $D_{\tilde{\lambda}/\tilde{\mu}}$ ) there exists a unique connected component of  $\theta$  (resp.  $\tilde{\theta}$ ), which lies in the same row as  $s$ . We denote the corresponding component by  $\theta_s$  (resp.  $\tilde{\theta}_s$ ).

Suppose that  $l$  and  $r$  are positive integers. Set  $\tilde{\lambda} = \lambda \cup (r^l)$  and  $\tilde{\mu} = \mu \cup (r^l)$ . We shall consider the difference between  $D_{\tilde{\lambda}/\tilde{\mu}}$  and  $D_{\lambda/\mu}$ . It can be seen that there exists a projection

$$p = p_{\lambda/\mu} : D_{\tilde{\lambda}/\tilde{\mu}} \longrightarrow D_{\lambda/\mu}.$$

The cardinality of the fiber of each cell  $s = (i, j) \in D_{\lambda/\mu}$  is exactly one or two. Let  $J_s$  denote the set of second coordinates of the cells in  $\theta_s$ . If all elements of  $J_s$  are all strictly larger than  $r$ , the fiber  $p^{-1}(s)$  consists of a single element  $s = (i, j)$ . If all elements of  $J_s$  are strictly smaller than  $r$ , then the fiber  $p^{-1}(s)$  consists of a single element  $\tilde{s} := (i, j + l)$ . In the case where  $J_s$  contains  $r$ , then the fiber  $p^{-1}(s)$  consists of exactly two elements  $s = (i, j)$  and  $\tilde{s} = (i, j + l)$ . For the case where  $r \in J_s$ , we have the following lemma, which follows immediately from the definition of the projection  $p = p_{\lambda/\mu}$ .

**Lemma 4.1** *Let  $s = (i, j)$  be a cell of  $D_{\lambda/\mu}$  and  $\tilde{s} = (i, j + l)$  be a cell of  $D_{\tilde{\lambda}/\tilde{\mu}}$  such that  $r \in J_s$ . The arm length, the arm-colength, the leg length and the leg-colength satisfy the following properties*

$$a_{\tilde{\mu}}(s) = a_{\tilde{\lambda}}(\tilde{s}), \quad l_{\tilde{\mu}}(s) - l_{\tilde{\lambda}}(\tilde{s}) = l, \quad (13)$$

$$a_{\tilde{\mu}}(\tilde{s}) = a_{\mu}(s), \quad l_{\tilde{\mu}}(\tilde{s}) = l_{\mu}(s), \quad (14)$$

$$a_{\tilde{\lambda}}(s) = a_{\lambda}(s), \quad l_{\tilde{\lambda}}(s) - l_{\lambda}(s) = l. \quad (15)$$

**Proposition 4.1** *Let  $\lambda$  and  $\mu$  be two partitions such that  $\mu \subset \lambda$  and  $\theta = \lambda - \mu$  an horizontal strip. Let  $r$  and  $l$  be positive integers and  $\zeta$  a primitive root of unity. It follows that*

$$\psi_{\lambda \cup (r^l)/\mu \cup (r^l)}(q, \zeta) = \psi_{\lambda/\mu}(q, \zeta).$$

*Proof.* Recall that for a cell  $s$  of the partition  $\nu$ ,

$$\psi_{\lambda/\mu}(q, t) = \prod_{s \in D_{\lambda/\mu}} \frac{b_{\mu}(s)}{b_{\lambda}(s)},$$

where

$$b_{\nu}(s) = \frac{1 - q^{a_{\nu}(s)} t^{l_{\nu}(s)+1}}{1 - q^{a_{\nu}(s)+1} t^{l_{\nu}(s)}}.$$

If  $s = (i, j) \in \lambda$  satisfies the condition  $r < j$  for all  $j \in J_s$ , then the fiber  $p^{-1}(s)$  of the projection  $p$  is  $\{s = (i, j)\}$ , and we have  $a_{\mu}(s) = a_{\bar{\mu}}(s)$ ,  $a_{\lambda}(s) = a_{\bar{\lambda}}(s)$  and  $l_{\mu}(s) + l = l_{\bar{\mu}}(s)$ ,  $l_{\lambda}(s) + l = l_{\bar{\lambda}}(s)$ . It is clear from these identities that  $b_{\mu}(s)/b_{\lambda}(s) = b_{\bar{\mu}}(s)/b_{\bar{\lambda}}(s)$  at  $t = \zeta$  in this case. Suppose that  $s$  satisfies  $j < r$  for all  $j \in J_s$ . In this case, the fiber  $p^{-1}(s)$  consists of a single element  $\{\tilde{s} = (i, j + l)\}$ , and we have  $a_{\mu}(s) = a_{\bar{\mu}}(s)$  and  $a_{\lambda}(s) = a_{\bar{\lambda}}(s)$  and  $l_{\mu}(s) = l_{\bar{\mu}}(s)$  and  $l_{\lambda}(s) = l_{\bar{\lambda}}(s)$ . Hence we have  $b_{\mu}(s)/b_{\lambda}(s) = b_{\bar{\mu}}(s)/b_{\bar{\lambda}}(s)$ . Consider the case where  $r \in J_s$ . In this case, the fiber  $p^{-1}(s)$  consists of two elements  $\{s, \tilde{s}\}$ . Let us consider

$$\prod_{u \in p^{-1}(s)} \frac{b_{\bar{\mu}}(u)}{b_{\bar{\lambda}}(u)} = \frac{1 - q^{a_{\bar{\mu}}(s)} t^{l_{\bar{\mu}}(s)+1}}{1 - q^{a_{\bar{\mu}}(s)+1} t^{l_{\bar{\mu}}(s)}} \frac{1 - q^{a_{\bar{\lambda}}(s)+1} t^{l_{\bar{\lambda}}(s)}}{1 - q^{a_{\bar{\lambda}}(s)} t^{l_{\bar{\lambda}}(s)+1}} \frac{1 - q^{a_{\bar{\mu}}(\tilde{s})} t^{l_{\bar{\mu}}(\tilde{s})+1}}{1 - q^{a_{\bar{\mu}}(\tilde{s})+1} t^{l_{\bar{\mu}}(\tilde{s})}} \frac{1 - q^{a_{\bar{\lambda}}(\tilde{s})+1} t^{l_{\bar{\lambda}}(\tilde{s})}}{1 - q^{a_{\bar{\lambda}}(\tilde{s})} t^{l_{\bar{\lambda}}(\tilde{s})+1}}.$$

By (13) it follows that

$$\left\{ \frac{1 - q^{a_{\bar{\mu}}(s)} t^{l_{\bar{\mu}}(s)+1}}{1 - q^{a_{\bar{\mu}}(s)+1} t^{l_{\bar{\mu}}(s)}} \right\}^{-1} \Big|_{t=\zeta} = \frac{1 - q^{a_{\bar{\lambda}}(\tilde{s})+1} t^{l_{\bar{\lambda}}(\tilde{s})}}{1 - q^{a_{\bar{\lambda}}(\tilde{s})} t^{l_{\bar{\lambda}}(\tilde{s})+1}} \Big|_{t=\zeta}.$$

It also follows by (14)

$$\frac{1 - q^{a_{\bar{\mu}}(\tilde{s})} t^{l_{\bar{\mu}}(\tilde{s})+1}}{1 - q^{a_{\bar{\mu}}(\tilde{s})+1} t^{l_{\bar{\mu}}(\tilde{s})}} \Big|_{t=\zeta} = \frac{1 - q^{a_{\mu}(s)} t^{l_{\mu}(s)+1}}{1 - q^{a_{\mu}(s)+1} t^{l_{\mu}(s)}} \Big|_{t=\zeta},$$

and from (15)

$$\frac{1 - q^{a_{\bar{\lambda}}(s)+1} t^{l_{\bar{\lambda}}(s)}}{1 - q^{a_{\bar{\lambda}}(s)} t^{l_{\bar{\lambda}}(s)+1}} \Big|_{t=\zeta} = \frac{1 - q^{a_{\lambda}(s)+1} t^{l_{\lambda}(s)}}{1 - q^{a_{\lambda}(s)} t^{l_{\lambda}(s)+1}} \Big|_{t=\zeta}.$$

Therefore, it follows that

$$\prod_{u \in p^{-1}(s)} \frac{b_{\bar{\mu}}(u)}{b_{\bar{\lambda}}(u)} = \frac{b_{\mu}(s)}{b_{\lambda}(s)}.$$

Combining these, the assertion follows.  $\square$

## 5 Factorization formulas

In this section, we shall show factorization formulas for different kinds of Macdonald polynomials at roots of unity.

**Theorem 5.1** *Let  $l$  be a positive integer and  $\zeta$  a primitive  $l$ -th root of unity. Let  $\mu = (1^{m_1}, 2^{m_2}, \dots, n^{m_n})$  be a partition of a positive integer  $n$ . For each  $i$ , let  $m_i = lq_i + r_i$  with  $0 \leq r_i \leq l-1$  and let  $\bar{\mu} = (1^{r_1} 2^{r_2} \dots m^{r_n})$ . The function  $Q'_\mu$  satisfy the following factorisation formula at  $t = \zeta$*

$$Q'_\mu(x; q, \zeta) = \left( Q'_{(1^l)}(x; q, \zeta) \right)^{q_1} \left( Q'_{(2^l)}(x; q, \zeta) \right)^{q_2} \cdots \left( Q'_{(n^l)}(x; q, \zeta) \right)^{q_n} Q'_{\bar{\mu}}(x; q, \zeta). \quad (16)$$

*Proof.* We shall show that the  $\mathbb{C}$ -linear map defined by

$$\begin{aligned} f_r : \Lambda_F &\longrightarrow \Lambda_F \\ Q'_\mu(x; q, \zeta) &\longmapsto Q'_{\mu \cup (r^l)}(x; q, \zeta), \end{aligned}$$

is an  $\Lambda_{\mathbb{C}(q)}$ -linear map. Let  $\zeta$  be a primitive  $l$ -th root of unity. From (12), we have

$$Q'_\mu(x; q, \zeta) g'_k(x; q, \zeta) = \sum_{\lambda} \psi_{\lambda/\mu}(q, \zeta) Q'_\lambda(x; q, \zeta),$$

where the sum is taken over the partitions  $\lambda$  such that  $\lambda - \mu$  is an horizontal  $k$ -strip. Using the result of Proposition 4.1, it follows that

$$\begin{aligned} Q'_{\mu \cup (r^l)}(x; q, \zeta) g'_k(x; q, \zeta) &= \sum_{\lambda} \psi_{\lambda \cup (r^l)/\mu \cup (r^l)}(q, \zeta) Q'_{\lambda \cup (r^l)}(x; q, \zeta) \\ &= \sum_{\lambda} \psi_{\lambda/\mu}(q, \zeta) Q'_{\lambda \cup (r^l)}(x; q, \zeta). \end{aligned}$$

Consequently, for each  $r \geq 1$ , the multiplication by  $g_k$  commutes with the morphism  $f_r$ . Since the family  $\{g'_k(x; q, \zeta_l)\}_{k \geq 1}$  generates the algebra  $\Lambda_{\mathbb{C}(q)}$  (see [11], Chap. VI, formula (2.12)), the map  $f_r$  is  $\Lambda_{\mathbb{C}(q)}$ -linear. This implies that

$$\begin{aligned} \forall F \in \Lambda_{\mathbb{C}(q)}, f_r(F(x)) &= F(x) f_r(1) \\ &= F(x) Q'_{r^l}(x; q, \zeta). \end{aligned}$$

□

**Corollary 5.1** *With the same notation as in Theorem 3.1, we have the following factorisation formula for the Macdonald polynomials  $\tilde{H}_\mu(x; q, t)$*

$$\tilde{H}_\mu(x; q, \zeta) = \left( \tilde{H}_{(1^l)}(x; q, \zeta) \right)^{q_1} \left( \tilde{H}_{(2^l)}(x; q, \zeta) \right)^{q_2} \cdots \left( \tilde{H}_{(n^l)}(x; q, \zeta) \right)^{q_n} \tilde{H}_{\bar{\mu}}(x; q, \zeta). \quad (17)$$

*Proof.* If we define

$$\Psi_{\lambda/\mu}(q, t) := \psi_{\lambda/\mu}(q, t) \frac{c'_\mu(q, t)}{c'_\lambda(q, t)},$$

then the Pieri formula for the modified integral form  $J_\mu\left(\frac{x}{1-t}; q, t\right)$  is written as follows

$$J_\mu\left(\frac{x}{1-t}; q, t\right) g'_r\left(\frac{x}{1-q}\right) = \sum_\lambda \Psi_{\lambda/\mu}(q, t) J_\lambda\left(\frac{x}{1-t}; q, t\right),$$

where the sum is over the partitions  $\lambda$  such that  $\lambda - \mu$  is a horizontal  $k$ -strip.

Let a positive integer  $r$  be arbitrarily fixed, and  $\tilde{\nu}$  denote the partition  $\nu \cup (r^l)$ . Since we have already shown that  $\psi_{\tilde{\lambda}/\tilde{\mu}}(q, \zeta) = \psi_{\lambda/\mu}(q, \zeta)$ , it suffices to show that

$$\frac{c'_{\tilde{\mu}}(q, \zeta)}{c'_{\tilde{\lambda}}(q, \zeta)} = \frac{c'_\mu(q, \zeta)}{c'_\lambda(q, \zeta)}.$$

We shall actually show that

$$\frac{c'_{\tilde{\mu}}(q, \zeta)}{c'_\mu(q, \zeta)} = \frac{c'_{\tilde{\lambda}}(q, \zeta)}{c'_\lambda(q, \zeta)}.$$

It follows from the definition that

$$\begin{aligned} \frac{c'_{\tilde{\mu}}(q, t)}{c'_\mu(q, t)} &= \frac{\prod_{s \in \tilde{\mu}} (1 - q^{a_{\tilde{\mu}}(s)+1} t^{l_{\tilde{\mu}}(s)})}{\prod_{s \in \mu} (1 - q^{a_{\tilde{\mu}}(s)+1} t^{l_{\tilde{\mu}}(s)})} \\ &= \frac{\prod_{\substack{s \in \tilde{\mu} \\ s \notin (r^l)}} (1 - q^{a_{\tilde{\mu}}(s)+1} t^{l_{\tilde{\mu}}(s)})}{\prod_{s \in \mu} (1 - q^{a_{\tilde{\mu}}(s)+1} t^{l_{\tilde{\mu}}(s)})} \prod_{s \in (r^l) \subset \tilde{\mu}} (1 - q^{a_{\tilde{\mu}}(s)+1} t^{l_{\tilde{\mu}}(s)}). \end{aligned}$$

The Young diagram of the partition  $\tilde{\mu}$  is the disjoint union of the cells  $\{\tilde{s} \in \tilde{\mu} | s \in \mu\}$  and  $(r^l)$ . For each  $s \in \mu$ , we have as seen in previous theorem that  $a_{\tilde{\mu}}(\tilde{s}) = a_\mu(s)$ , and  $l_{\tilde{\mu}}(\tilde{s}) = l_\mu(s)$  or  $l_\mu(s) + l$ . Hence at  $t = \zeta$ , we have

$$\frac{c'_\mu(q, \zeta)}{c'_{\tilde{\mu}}(q, \zeta)} = \prod_{s \in (r^l) \subset \tilde{\mu}} (1 - q^{a_{\tilde{\mu}}(s)+1} \zeta^{l_{\tilde{\mu}}(s)}) \quad \text{and} \quad (18)$$

$$\frac{c'_\lambda(q, \zeta)}{c'_{\tilde{\lambda}}(q, \zeta)} = \prod_{s \in (r^l) \subset \tilde{\lambda}} (1 - q^{a_{\tilde{\lambda}}(s)+1} \zeta^{l_{\tilde{\lambda}}(s)}). \quad (19)$$

Although there is a difference between the positions where the block  $(r^l)$  is inserted in the Young diagram of  $\mu$  and  $\lambda$ , (3.1) and (3.2) coincide at  $t = \zeta$ , since  $a_{\tilde{\mu}}(s) = a_{\tilde{\lambda}}(s)$  for each  $s \in (r^l)$ . Thus we have

$$\frac{c'_\mu(q, \zeta)}{c'_{\tilde{\mu}}(q, \zeta)} = \frac{c'_\lambda(q, \zeta)}{c'_{\tilde{\lambda}}(q, \zeta)}.$$

□

Let  $\nu = (\nu_1, \dots, \nu_p)$  be a partition. For some  $l \geq 0$ , we denote by  $\nu^l$  the partition where each part of  $\nu$  is repeated  $l$  times. We can give a more explicit expression for the factorisation formula in the special case where  $\mu = \nu^l$ .

**Corollary 5.2** *Let  $\nu$  be a partition and  $l$  a positive integer. We have the following special cases for the factorisation formulas*

$$Q'_{\nu^l}(X; q, \zeta) = (-1)^{(l-1)|\nu|} p_l \circ h_\nu(x) , \quad (20)$$

$$\tilde{H}_{\nu^l}(X; q, \zeta) = \prod_{j=1}^{l(\nu)} \prod_{i=1}^{\nu_j} (q^{il} - 1) p_l \circ h_\nu \left( \frac{x}{1-q} \right) . \quad (21)$$

**Example 5.2** For  $\lambda = (222111)$  and  $k = 3$ , we can compute the specialization

$$\begin{aligned} Q'_{222111}(x; q, e^{2i\pi/3}) &= -s_{22221} - s_{321111} + s_{3222} + s_{33111} - s_{3321} + 3s_{333} + s_{411111} \\ &\quad - 2s_{432} + 2s_{441} - s_{51111} + 2s_{522} - 2s_{54} + s_{6111} - 2s_{621} + 2s_{63} \\ &\quad + s_{711} - s_{81} + s_9 + s_{222111} \\ &= p_3 \circ h_{21}(x). \end{aligned}$$

## 6 A generalization of the plethystic formula

In this section, using the factorisation formula given in Theorem 5.1, we shall give a generalization of the plethystic formula obtained by specializing Macdonald polynomials at roots of unity in Theorem 3.1. For  $\lambda$  a partition, let consider the following map which is the plethystic substitution by the powersum  $p_\lambda$

$$\begin{aligned} \Psi_\lambda : \Lambda_F &\longrightarrow \Lambda_F \\ f &\longmapsto p_\lambda \circ f . \end{aligned}$$

**Lemma 6.1** *Let  $\lambda$  and  $\mu$  be two partitions, the maps  $\Psi$  satisfy the multiplicative rule*

$$\Psi_\lambda(f) \Psi_\mu(f) = \Psi_{\lambda \cup \mu}(f) .$$

**Proposition 6.1** *Let  $d$  be an integer such that  $d|l$  and  $\zeta_d$  be a primitive  $d$ -th root of unity,*

$$Q'_{(r^l)}(x; q, \zeta_d) = (-1)^{\frac{rl(d-1)}{d}} p_d^{l/d} \circ h_r(x) . \quad (22)$$

*Proof.* Let  $d$  and  $l$  be two integers such that  $d$  divide  $l$ . Let  $\mu = (r^l)$  the rectangle partition with parts of length  $r$ . Using the factorisation formula described in Theorem 5.1, we can write

$$Q'_{(r^l)}(x; q, \zeta_d) = \left( Q'_{(r^d)}(x; q, \zeta_d) \right)^{l/d} . \quad (23)$$

With the specialization formula at root of unity written in Theorem 3.1, we have

$$\begin{aligned}
\left(Q'_{(r^d)}(x; q, \zeta_d)\right)^{l/d} &= \left((-1)^{(d-1)r} p_d \circ h_r(x)\right)^{l/d} \\
&= (-1)^{\frac{lr(d-1)}{d}} (p_d \circ h_r(x))^{l/d}
\end{aligned}$$

Using the Lemma 6.1, we obtain

$$\left(Q'_{(r^d)}(x; q, \zeta_d)\right)^{l/d} = (-1)^{\frac{lr(d-1)}{d}} p_d^{l/d} \circ h_r(x) .$$

Finally, we obtain by the factorization formula of Theorem 5.1

$$Q'_{(r^l)}(x; q, \zeta_d) = (-1)^{\frac{lr(d-1)}{d}} p_d^{l/d} \circ h_r(x) .$$

□

Using the same proof, we can write a similar specialization for integral forms of the Macdonald Polynomials.

**Corollary 6.1** *With the same notations as in Proposition 6.1, the modified Macdonald polynomials  $\tilde{H}_\lambda(x; q, t)$  satisfy the same specialization*

$$\tilde{H}_{(r^l)}(x; q, \zeta_d) = \prod_{i=1}^r (q^{il} - 1) p_d^{l/d} \circ h_r \left( \frac{x}{1-q} \right) . \quad (24)$$

**Example 6.2** For  $\lambda = (222222)$ , i.e  $r = 2$  and  $l = 6$  and  $d = 3$ , we can compute

$$\begin{aligned}
Q'_{(222222)}(x; q, e^{2i\pi/3}) &= -s_{322221} + s_{33222} + 2s_{333111} - 2s_{33321} + 2s_{3333} + s_{422211} - 2s_{432111} \\
&\quad + s_{43221} + 2s_{441111} - s_{4422} + 4s_{444} + s_{522111} - 2s_{52221} + s_{53211} - 2s_{54111} \\
&\quad + s_{5421} - 4s_{543} + 3s_{552} - s_{621111} + 2s_{6222} + s_{63111} - 2s_{6321} + 4s_{633} \\
&\quad + s_{6411} - 3s_{651} + 3s_{66} + s_{711111} - 2s_{732} + 2s_{741} - s_{81111} + 2s_{822} - 2s_{84} \\
&\quad + s_{9111} - 2s_{921} + 2s_{93} + s_{1011} - s_{111} + s_{12} + s_{222222} \\
&= p_3^2 \circ h_2(x) = p_{(33)} \circ h_2(x).
\end{aligned}$$

## 7 Macdonald polynomials at roots of unity and cyclic characters of the symmetric group

In the following, we will denote the symmetric group of order  $k$  by  $\mathfrak{S}_k$ . Let  $\Gamma \subset \mathfrak{S}_k$  be a cyclic subgroup generated by an element of order  $r$ . As  $\Gamma$  is a commutative subgroup its irreducible representations are one-dimensional vector spaces. The corresponding maps  $(\gamma_j)_{j=0 \dots r-1}$  can be defined by

$$\begin{aligned}
\gamma_j : \Gamma &\longrightarrow GL(\mathbb{C}) \simeq \mathbb{C}^* \\
\tau &\longmapsto \zeta_r^j ,
\end{aligned}$$

where  $\zeta_r$  is a  $r$ -th primitive root of unity (See [14] for more details). In [3], Foulkes considered the Frobenius characteristic of the representations of  $\mathfrak{S}_k$  induced by these irreducible representations and obtained an explicit formula that we will give in the next proposition. Let  $k$  and  $n$  be two positive integers such that  $u = (k, d)$  (the greater common divisor between  $k$  and  $n$ ) and  $d = u \cdot m$ . Let us define the Ramanujan (or Von Sterneck) sum  $c(k, d)$  by

$$c(k, d) = \frac{\mu(m)\phi(d)}{\phi(m)}$$

where  $\mu$  is the Moebius function and  $\phi$  the Euler totient. The quantity  $c(k, d)$  corresponds to the sum of the  $k$ -th powers of the primitive  $d$ -th roots of unity (the previous expression was given first by Hölder in [6]).

**Proposition 7.1** *Let  $\tau$  be a cyclic permutation of length  $k$  and  $\Gamma$  the maximal cyclic subgroup of  $\mathfrak{S}_k$  generated by  $\tau$ . Let  $j$  be a positive integer less than  $k$ . The Frobenius characteristic of the representation of  $\mathfrak{S}_k$  induced by the irreducible representation of  $\Gamma$ ,  $\gamma_j : \tau \mapsto \zeta_r^j$ , is given by*

$$l_k^{(j)}(x) = \frac{1}{k} \sum_{d|k} c(j, d) p_d^{k/d}(x). \quad (25)$$

**Example 7.1** *For  $\mathfrak{S}_6$  and  $k = 2$ , the cyclic character  $l_6^{(2)}$  expanded on powersums and Schur basis is*

$$\begin{aligned} l_6^{(2)} &= \frac{1}{6} (p_{111111} + p_{222} - p_{33} - p_6) \\ &= s_{51} + 2s_{42} + s_{411} + 3s_{321} + 2s_{3111} + s_{222} + s_{2211} + s_{21111}. \end{aligned}$$

**Theorem 7.2** *Let  $r$  and  $l$  be two positive integers. The specialization of the Macdonald polynomials indexed by the rectangle partition  $(r^l)$  at a primitive  $l$ -th root of unity is equivalent to*

$$Q'_{(r^l)}(x; q, t) \pmod{\Phi_l(t)} = \sum_{j=0}^{l-1} t^j (l_l^{(j)} \circ h_r)(x). \quad (26)$$

*Proof.* We will first give a generalization of the Moebius inversion formula due to E. Cohen (see [1] for the original work and [2] for a simpler proof). Let

$$P(q) = \sum_{k=0}^{n-1} a_k q^k,$$

be a polynomial of degree less than  $n - 1$  with coefficients  $a_k$  in  $\mathbb{Z}$ .  $P$  is said to be even modulo  $n$  if

$$(i, n) = (j, n) \implies a_i = a_j.$$

**Lemma 7.3** *The polynomial  $P$  is even modulo  $n$  if and only if for every divisor  $d$  of  $n$ , the residue of  $P$  modulo the  $d$ -th cyclotomic polynomial  $\Phi_d$  is a constant  $r_d$  in  $\mathbb{Z}$ . In this case, one has*

$$a_k = \frac{1}{n} \sum_{d|n} c(k, d) r_d \quad \text{and} \quad r_d = \sum_{t|n} c(n/d, t) a_{n/t}.$$

Let  $d$  be an integer such that  $d|l$ . By expanding  $Q'_{(r^l)}(x; q, t)$  (and more generally  $Q'_\lambda(x; q, t)$ ) on the Schur basis, we can define a kind of  $(q, t)$ -Kostka polynomials  $K'_{\mu, (r^l)}(q, t)$  by

$$Q'_{(r^l)}(x; q, t) = \sum_{\mu} K'_{\mu, (r^l)}(q, t) s_{\mu}(x).$$

Let  $\mu$  be a partition and  $d$  an integer such that  $d|l$ . The polynomial  $P_{\mu}^q(t) = \sum_{j=0}^{l-1} a_j(q) t^j$  is the residue modulo  $1 - t^l$  of the  $(q, t)$ -Kostka polynomial  $K'_{\mu, (r^l)}(q, t)$ , if and only if, for all  $\zeta_d$  primitive  $d$ -th root of unity,

$$P_{\mu}^q(\zeta_d) = K'_{\mu, (r^l)}(q, \zeta_d).$$

Using Theorem 5.1, one has

$$P_{\mu}^q(\zeta_d) = (-1)^{(d-1)rl/d} \langle p_d^{l/d} \circ h_r(x), s_{\mu}(x) \rangle.$$

Consequently  $P(\zeta_d)$  is an integer since the entries of the transition matrix between the powersums and the Schur functions are all integers. Using the Lemma 7.3, we obtain

$$\begin{aligned} a_j(q) &= \frac{1}{l} \sum_{d|l} c(j, d) \langle p_d^{l/d} \circ h_r(x), s_{\mu}(x) \rangle \\ &= \langle l_l^{(j)} \circ h_r(x), s_{\mu}(x) \rangle. \end{aligned}$$

□

**Example 7.4** *Let define  $g$  as the right hand side of 26 for  $r = 2$  and  $l = 3$*

$$\begin{aligned} g(t) &= l_3^{(0)} \circ h_2 + t l_3^{(1)} \circ h_2 + t^2 l_3^{(2)} \circ h_2 \\ &= s_{411} + (t^2 + t + 1)s_{42} + t(t + 1)s_{51} + t(t + 1)s_{321} + s_{222} + s_{33} + s_6. \end{aligned}$$

*The specialization of  $g(t)$  and  $Q'_{222}(X; q, t)$  at  $t$  the 3-th primitive roots of unity satisfy*

$$\begin{aligned} g(j) = Q'_{222}(X; q, j) &= s_{411} - s_{51} - s_{321} + s_{222} + s_{33} + s_6, \\ g(j^2) = Q'_{222}(X; q, j^2) &= s_{411} - s_{51} - s_{321} + s_{222} + s_{33} + s_6. \end{aligned}$$

**Corollary 7.1** *For two positive integers  $r$  and  $l$ , the same residue formulas occurs for the modified Macdonald polynomials  $\tilde{H}_{(r^l)}(x; q, t)$*

$$\tilde{H}_{(r^l)}(x; q, t) \pmod{\Phi_l(t)} \equiv \prod_{i=1}^r (q^{il} - 1) \sum_{j=0}^{l-1} t^j (l_l^{(j)} \circ h_r) \left( \frac{x}{1 - q} \right). \quad (27)$$

## 8 Congruences for $(q, t)$ -Kostka polynomials

For a given partition  $\lambda$ , let denote by  $\tilde{s}_\lambda^{(q)}$  the symmetric function defined by

$$\tilde{s}_\lambda^{(q)}(x) = s_\lambda \left( \frac{x}{1-q} \right).$$

Let  $\star$  be the internal product on  $\Lambda_F$  defined by (see [11], Chap. I, formula (7.12))

$$p_\lambda \star p_\mu = \delta_{\lambda, \mu} z_\lambda p_\lambda.$$

**Proposition 8.1** *Let  $r$  and  $l$  be two positive integers and  $\mu$  a partition of weight  $nl$ . Let denote by  $\Phi_l(t)$  the cyclotomic polynomial of order  $l$ . The  $(q, t)$ -Kostka polynomial  $\tilde{K}_{(r^l), \mu}(q, t)$  satisfy the following congruence modulo  $\Phi_l(t)$*

$$\tilde{K}_{\mu, (r^l)}(q, t) \equiv \prod_{i=1}^r (q^{il} - 1) \tilde{s}_\mu^{(q)}(1, t, t^2, \dots, t^{l-1}) \pmod{\Phi_l(t)}. \quad (28)$$

More generally, for all partitions  $\nu = (\nu_1, \dots, \nu_p)$  of weight  $r$ ,

$$\tilde{K}_{\mu, \nu^l}(q, t) \equiv \prod_{j=1}^{l(\nu)} \prod_{i=1}^{\nu_j} (q^{il} - 1) \widetilde{h_{l\nu} \star s_\mu}^{(q)}(1, t, t^2, \dots, t^{l-1}) \pmod{\Phi_l(t)}, \quad (29)$$

where  $l\nu = (l\nu_1, \dots, l\nu_p)$ .

*Proof.* Let  $\zeta$  be a primitive root of unity and  $Z_l = \{1, \zeta, \dots, \zeta^{l-1}\}$  be the alphabet of the  $l$ -roots of unity. Using  $\lambda$ -ring notations (see [7] for more details) and Theorem 3.1, we have for all positive integer  $r$

$$\tilde{H}_{r^l}(X; q, \zeta) = \prod_{i=1}^r (q^{il} - 1) (p_l \circ h_r) \left( \frac{x}{1-q} \right) = h_{lr} \left( \frac{Z_l x}{1-q} \right).$$

Consequently, for all partitions  $\mu$  of size  $rl$ , we can write

$$\tilde{K}_{\mu, r^l}(q, \zeta) = \prod_{i=1}^r (q^{il} - 1) \tilde{s}_\mu^{(q)}(Z_l),$$

which is equivalent to the first statement of the theorem. The second statement follows from the following identity

$$(p_l \circ h_\nu) \left( \frac{x}{1-q} \right) = (h_{l\nu} \star h_{lr}) \left( \frac{Z_l x}{1-q} \right).$$

□

**Example 8.1** *Let consider  $r = 2$  and  $l = 3$ . For  $\mu = (222)$ , we have*

$$\tilde{K}_{222, 222}(q, j) = \tilde{K}_{222, 222}(q, j^2) = 1 + q^3 \quad \text{and} \quad (1 - q^6)(1 - q^3) \tilde{s}_{222}^{(q)}(1, j, j^2) = 1 + q^3.$$

## References

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