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APPLICATION OF A REGULARIZED SGBEM FORMULATION TO FEM-BEM COUPLING IN ELASTOSTATICS

Saida Mouhoubi^(a), Laurent Ulmet^(a), Marc Bonnet^(b)

(a) Laboratoire de Génie Civil d'Egletons, Université de Limoges, France

(b) Laboratoire de Mécanique des Solides (CNRS UMR 7649), Ecole Polytechnique, France

mouhoubi@lgce0.unilim.fr, ulmet@unilim.fr, bonnet@lms.polytechnique.fr

Abstract This paper concerns a procedure, based on the symmetric Galerkin boundary element equations, for computing the stiffness matrix of an elastic region without body forces in terms of boundary nodal displacements. This domain can thus be treated as a macro-element in a FEM calculation. In the present case, the implementation has been done in the CASTEM 2000 FEM environment. Sample numerical examples are presented; they yield numerical results in good agreement with the corresponding exact solutions.

Keywords: Symmetric Galerkin BEM, FEM/BEM coupling, elastostatics

1. INTRODUCTION

The boundary element methods (BEMs) are often disadvantaged with respect to finite element methods (FEMs) due to the fully populated and unsymmetric character of the matrix resulting from the discretization of the integral equations. This fact has in particular hindered the development of coupled BEM-FEM solution procedures. Although the BEM-modelled region will in any case lead to fully populated matrices, it is possible to reduce both computer time and storage by devising symmetric coupling procedures.

In this paper, this goal is achieved by exploiting the fact that the Symmetric Galerkin BEM (SGBEM) equations, adequately combined, yield the strain energy corresponding to any given boundary displacement field, i.e. provide the (symmetric) stiffness matrix of the elastic region modelled by BEM. This technique thus treats the latter as a macro-element and is therefore very well suited for implementation into

a FEM code. It is currently being implemented by the present authors into the CASTEM 2000 [4] FEM environment. Some preliminary numerical results are presented in section 5

The choice of this approach, in which the SGBEM is introduced into FEM, can be justified by the historic advance of FEM, for which many powerful and popular packages are available, and whose environments offer consistent backgrounds for further developments (the CASTEM 2000 environment is, by design, very flexible in this respect).

2. SYMMETRIC GALERKIN BEM FORMULATION

Consider a three-dimensional elastic body Ω , whose boundary Γ is a piecewise closed surface: $\Gamma = \cup_{\alpha} \Gamma^{\alpha}$, $\alpha = 1, \dots$ where each Γ^{α} is a closed surface that may be split into a displacement-prescribed part Γ_u^{α} and a traction-prescribed part Γ_t^{α} such that $\Gamma^{\alpha} = \Gamma_u^{\alpha} \cup \Gamma_t^{\alpha}$ and $\Gamma_u^{\alpha} \cap \Gamma_t^{\alpha} = \emptyset$. Let (\mathbf{u}, \mathbf{t}) denote any compatible pair of displacements and tractions on the boundary Γ , i.e. \mathbf{u}, \mathbf{t} are the trace on the boundary of an elastostatic state $\mathbf{u}, \boldsymbol{\sigma}(\mathbf{u})$ in Ω (where $\boldsymbol{\sigma}(\mathbf{u}) \equiv \mathbf{C} : \nabla \mathbf{u}$ is the tensor of elastic stresses, \mathbf{C} denoting the fourth-order tensor of elastic moduli). In the absence of body forces, the elastic strain energy associated with (\mathbf{u}, \mathbf{t}) can be expressed in terms of boundary fields (Clapeyron formula):

$$W = \frac{1}{2} \int_{\Omega} \boldsymbol{\sigma}(\mathbf{x}) : \boldsymbol{\varepsilon}(\mathbf{x}) \, d\Omega = \frac{1}{2} \int_{\Gamma} \mathbf{t}(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x}) \, d\Gamma_x \quad (1)$$

Following Hsiao [5], we will refer to the second term in the above equation as *boundary energy*.

Since (\mathbf{u}, \mathbf{t}) are assumed to be compatible, the boundary energy is in fact a function of either \mathbf{u} or \mathbf{t} alone. In order to build a stiffness operator in the usual sense, the displacement \mathbf{u} is taken as the primary variable and $W \equiv W(\mathbf{u})$ in Eq. (1).

The explicit form of the mathematical link between \mathbf{u} and \mathbf{t} on Γ is either the displacement boundary integral equation (Somigliana identity) or, equivalently, the traction boundary integral equation. Here, both integral equations are invoked in Galerkin, i.e. weighted residual, form (see the survey paper [3] for more details and many references on the symmetric Galerkin boundary element methods).

First consider the traction \mathbf{t} as being induced by given displacements \mathbf{u} (i.e., $\Gamma_u^{\alpha} = \Gamma^{\alpha}$ and $\Gamma_t^{\alpha} = \emptyset$). The following governing Galerkin integral equation for \mathbf{t} is obtained by weighting the displacement BIE with a traction-like test function $\tilde{\mathbf{t}}$ and following a regularization procedure:

$$\text{Find } \mathbf{t} \in \mathcal{V}_T \quad \mathcal{B}_{tt}(\mathbf{t}, \tilde{\mathbf{t}}) = \mathcal{B}_{ut}(\mathbf{u}, \tilde{\mathbf{t}}) + \mathcal{D}_{ut}(\mathbf{u}, \tilde{\mathbf{t}}) \quad (\forall \tilde{\mathbf{t}} \in \mathcal{V}_T) \quad (2)$$

where

$$\mathcal{B}_{tt}(\mathbf{t}, \tilde{\mathbf{t}}) = \int_{\Gamma} \int_{\Gamma} t_k(\mathbf{x}) \tilde{t}_i(\tilde{\mathbf{x}}) U_i^k(\mathbf{x}, \tilde{\mathbf{x}}) d\Gamma_{\tilde{\mathbf{x}}} d\Gamma_x \quad (3)$$

$$\begin{aligned} \mathcal{B}_{ut}(\mathbf{u}, \tilde{\mathbf{t}}) &= \sum_{\alpha} \int_{\Gamma^{\alpha}} \int_{\Gamma^{\alpha}} \tilde{t}_k(\tilde{\mathbf{x}}) [u_i(\mathbf{x}) - u_i(\tilde{\mathbf{x}})] T_i^k(\tilde{\mathbf{x}}, \mathbf{x}) d\Gamma_{\tilde{\mathbf{x}}} d\Gamma_x \\ &\quad + \sum_{\tilde{\alpha} \neq \alpha} \int_{\Gamma^{\alpha}} \int_{\Gamma^{\tilde{\alpha}}} \tilde{t}_k(\tilde{\mathbf{x}}) u_i(\mathbf{x}) T_i^k(\tilde{\mathbf{x}}, \mathbf{x}) d\Gamma_{\tilde{\mathbf{x}}} d\Gamma_x \end{aligned} \quad (4)$$

$$\mathcal{D}_{ut}(\mathbf{u}, \tilde{\mathbf{t}}) = \sum_{\alpha} \kappa_{\alpha} \int_{\Gamma^{\alpha}} u_k(\mathbf{x}) \tilde{t}_k(\mathbf{x}) d\Gamma_x \quad (5)$$

Then consider the displacement \mathbf{u} as induced by given tractions \mathbf{t} (i.e., $\Gamma_t^{\alpha} = \Gamma^{\alpha}$ and $\Gamma_u^{\alpha} = \emptyset$). The following governing Galerkin integral equation for \mathbf{u} is obtained by weighing the traction BIE with a displacement-like test function $\tilde{\mathbf{u}}$ and following a regularization procedure [1] which is not shown here:

$$\text{Find } \mathbf{u} \in \mathcal{V}_u \quad \mathcal{B}_{uu}(\mathbf{u}, \tilde{\mathbf{u}}) = \mathcal{B}_{tu}(\mathbf{t}, \tilde{\mathbf{u}}) + \mathcal{D}_{tu}(\mathbf{t}, \tilde{\mathbf{u}}) \quad (\forall \tilde{\mathbf{u}} \in \mathcal{V}_u) \quad (6)$$

where

$$\mathcal{B}_{uu}(\mathbf{u}, \tilde{\mathbf{u}}) = \int_{\Gamma} \int_{\Gamma} (Ru)_{iq}(\mathbf{x}) (R\tilde{u})_{ks}(\tilde{\mathbf{x}}) B_{ikqs}(\mathbf{x}, \tilde{\mathbf{x}}) d\Gamma_{\tilde{\mathbf{x}}} d\Gamma_x \quad (7)$$

$$\begin{aligned} \mathcal{B}_{tu}(\mathbf{t}, \tilde{\mathbf{u}}) &= \sum_{\alpha} \int_{\Gamma^{\alpha}} \int_{\Gamma^{\alpha}} t_k(\mathbf{x}) [\tilde{u}_i(\tilde{\mathbf{x}}) - \tilde{u}_i(\mathbf{x})] T_i^k(\mathbf{x}, \tilde{\mathbf{x}}) d\Gamma_{\tilde{\mathbf{x}}} d\Gamma_x \\ &\quad + \sum_{\tilde{\alpha} \neq \alpha} \int_{\Gamma^{\alpha}} \int_{\Gamma^{\tilde{\alpha}}} t_k(\mathbf{x}) \tilde{u}_i(\tilde{\mathbf{x}}) T_i^k(\mathbf{x}, \tilde{\mathbf{x}}) d\Gamma_{\tilde{\mathbf{x}}} d\Gamma_x \end{aligned} \quad (8)$$

$$\mathcal{D}_{tu}(\mathbf{t}, \tilde{\mathbf{u}}) = \sum_{\alpha} (\kappa_{\alpha} - 1) \int_{\Gamma^{\alpha}} t_k(\mathbf{x}) \tilde{u}_k(\mathbf{x}) d\Gamma_x \quad (9)$$

In Eqs. (2) and (6), \mathcal{V}_u and \mathcal{V}_T denote the spaces of admissible displacements and tractions on the boundary, and $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{t}}$ are test functions. Also, $U_i^k(\mathbf{x}, \tilde{\mathbf{x}})$ and $T_i^k(\mathbf{x}, \tilde{\mathbf{x}})$ are the i -components of the Kelvin fundamental solution (displacement and traction generated at $\tilde{\mathbf{x}}$ by a unit point force applied at \mathbf{x} along the k -direction). The unit normal \mathbf{n} is exterior to Ω . In Eqs. (5) and (9), one has either $\kappa_{\alpha} = 0$ (if \mathbf{n} is exterior to Γ_{α} , e.g. when Γ_{α} is the external boundary of a bounded body) or $\kappa_{\alpha} = 1$ (if \mathbf{n} is interior to Γ_{α} , e.g. when Γ_{α} is the boundary of an unbounded medium or of a cavity).

The SGBIE formulations (2)–(5) and (6)–(9) are in regularized form: the various double surface integrations consist of a weakly singular inner

integral followed by a regular outer integral. This is achieved by means of a combined use of indirect regularization and Stokes theorem (see e.g. [1, 2]). In particular, the latter operation uses the surface curl operator, first introduced in [6] and defined (with e_{jfq} : components of the permutation tensor) by:

$$(Ru)_{iq} = e_{jfq}n_j u_{i,f} \quad (10)$$

and the fourth-order tensor B_{ikqs} , given by:

$$\frac{1}{\mu^2} B_{ikqs} = -4\delta_{qs}F_{,ik} + [4\delta_{ik}\delta_{qs} - 4\nu\delta_{is}\delta_{kq} - 2(1-\nu)\delta_{iq}\delta_{ks}]F_{,pp}$$

with

$$\begin{aligned} F(\tilde{\mathbf{x}} - \mathbf{x}) &= \frac{r}{16\pi\mu(1-\nu)} \\ F_{,pp}(\tilde{\mathbf{x}} - \mathbf{x}) &= \frac{1}{8\pi\mu(1-\nu)r} \\ F_{,ik}(\tilde{\mathbf{x}} - \mathbf{x}) &= \frac{1}{16\pi\mu(1-\nu)r}(\delta_{ik} - r_{,i^T,k}) \end{aligned}$$

3. CONSTRUCTION OF THE STIFFNESS MATRIX

Assume now that the SGBEM equations (2)–(5) and (6)–(9) have been discretized using boundary element interpolations of the surfaces Γ^α and the boundary fields (\mathbf{u}, \mathbf{t}) and associated test functions $(\tilde{\mathbf{u}}, \tilde{\mathbf{t}})$. Further, let the latter be chosen specifically as $\tilde{\mathbf{u}} = \mathbf{u}$ and $\tilde{\mathbf{t}} = \mathbf{t}$.

In that case, the discretized SGBEM equations (2) and (6) describing respectively the Dirichlet and Neumann problems yield the identities:

$$\{\mathbf{t}\}^T [\mathbf{B}_{tt}] \{\mathbf{t}\} = \{\mathbf{u}\}^T ([\mathbf{B}_{ut}] + [\mathbf{D}_{ut}]) \{\mathbf{t}\} \quad (11)$$

$$\{\mathbf{u}\}^T [\mathbf{B}_{uu}] \{\mathbf{u}\} = \{\mathbf{t}\}^T ([\mathbf{B}_{tu}] + [\mathbf{D}_{tu}]) \{\mathbf{u}\} \quad (12)$$

Then, subtracting Eq. (11) from Eq. (12), we obtain:

$$\{\mathbf{u}\}^T [\mathbf{B}_{uu}] \{\mathbf{u}\} - \{\mathbf{t}\}^T [\mathbf{B}_{tt}] \{\mathbf{t}\} = \{\mathbf{t}\}^T [\mathbf{D}_{tu}] \{\mathbf{u}\} - \{\mathbf{u}\}^T [\mathbf{D}_{ut}] \{\mathbf{t}\} \quad (13)$$

having used the symmetry property $[\mathbf{B}_{ut}]^T = [\mathbf{B}_{tu}]$. Besides, one can observe from Eqs. (5) and (9) that:

$$\begin{aligned} \mathcal{D}_{tu}(\mathbf{t}, \mathbf{u}) - \mathcal{D}_{tu}(\mathbf{u}, \mathbf{t}) &= [(\kappa - 1) - \kappa] \int_{\Gamma} \mathbf{t}(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x}) \, d\Gamma_x \\ &= - \int_{\Gamma} \mathbf{t}(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x}) \, d\Gamma_x \\ &= -2W(\mathbf{u}) \end{aligned}$$

Going back to the discretized problem, one then has in an approximate sense (see [8] for a detailed discussion of this issue):

$$\{\mathbf{t}\}^T [\mathbf{D}_{tu}] \{\mathbf{u}\} - \{\mathbf{u}\}^T [\mathbf{D}_{ut}] \{\mathbf{t}\} = -2W(\{\mathbf{u}\})$$

or, returning to Eq. (13):

$$2W(\{\mathbf{u}\}) = \{\mathbf{t}\}^T [\mathbf{B}_{tt}] \{\mathbf{t}\} - \{\mathbf{u}\}^T [\mathbf{B}_{uu}] \{\mathbf{u}\} \quad (14)$$

Finally, solving Eq. (11) for $\{\mathbf{t}\}$ (note that this system is always uniquely solvable) and substituting the result into Eq. (14) yields the result:

$$W = \frac{1}{2} \{\mathbf{u}\}^T [\mathbf{K}] \{\mathbf{u}\} \quad (15)$$

where the stiffness matrix $[\mathbf{K}]$ is given by:

$$[\mathbf{K}] = ([\mathbf{B}_{ut}] + [\mathbf{D}_{ut}])([\mathbf{B}_{tt}]^{-1})([\mathbf{B}_{tu}] + [\mathbf{D}_{ut}]^T) - [\mathbf{B}_{uu}] \quad (16)$$

Thus, the above equation provides the symmetric stiffness matrix for the subdomain Ω , which can be added to the stiffness matrix for the complementary subdomain, obtained e.g. by the FEM approach. It is however important to point out that Eqs. (11) and (12) are not exactly satisfied by the same pair $\{\mathbf{u}\}, \{\mathbf{t}\}$ due to discretization error (in other words Eqs. (13) is true only in an approximate sense), although their continuous versions are exactly satisfied.

4. NUMERICAL IMPLEMENTATION

The evaluation of all integrals calls for the integration of kernels having at most a $1/r$ singularity, i.e. leading to simple surface integrations that are convergent in the usual sense; this is the main advantage of the analytical regularization. The present Galerkin procedure employs standard BEM modeling: the boundary Γ of the subdomain Ω is divided into quadrilateral, eight-noded isoparametric elements. In order to evaluate numerically the entries of the matrices $[\mathbf{B}_{uu}]$ etc., the double surface integrals over all pairs of quadrilateral elements must be computed and three situations arise as follows:

(i) *Disjoint elements* : the two elements do not have any common edge or vertex, so the double integration is nonsingular and is evaluated by using the standard Gaussian quadrature product rule with a variable order. The severity index, as defined in [7], is computed, in order to adjust the number of Gauss points to the distance between both elements. When this distance is sufficiently large, the evaluation of the double reg-

ular integrals can be accurately achieved with (2x2) Gauss points on each element.

(ii) *Coincident elements* : the inner integral being weakly singular, its evaluation is based on a subdivision of the element into triangles for which the common shared vertex is a Gauss point for the (nonsingular) outer integration. This transformation introduces a Jacobian which cancels the weak singularity of the inner integral. The evaluation of the outer one is straightforward using Gaussian quadrature. One notice that in practice this integration procedure slightly alters the overall symmetry due to the use of different quadrature schemes for inner and outer integrations. Another integration scheme, which preserves exact symmetry by treating the double integration as a whole and using relative intrinsic coordinates, is under implementation.

(iii) *Adjacent elements*: this case is specific to SGBEMs, and is like (ii), a case of singular integration, but the singularity is weaker and occurs when the outer point and the inner one are simultaneously located on the side or on the vertex shared by the two elements. This integration is, as of now, performed for simplicity using a nonsingular (Gaussian) integration scheme, and in particular the numerical examples presented in the next section rely on this imperfect treatment. However, another integration scheme, in which this weak singularity is correctly treated, is currently under implementation.

Once the stiffness matrix $[\mathbf{K}]$ is computed, coupling can directly be performed thanks to the standard high-level operators available in the CASTEM 2000 environment, which confers a great versatility to the method.

5. NUMERICAL EXAMPLES

Several example elastostatic problems for which analytical solutions are available are now presented in order to illustrate the accuracy and versatility of the three dimensional SGBEM-FEM coupling.

Rectangular prism submitted to uniaxial tension. An elastic bar (length: 3 units, $E = 1$, $\nu = 0.3$) with square section (Fig. 5) is divided into 3 subregions: the surface $\partial\Omega_2$ is discretized by quadrilateral boundary elements, while Ω_1 and Ω_3 are filled with standard parallelepipedic finite elements. Symmetry conditions (i.e. zero normal displacement and tangential tractions) are prescribed on the coordinate planes $x_i = 0$, $\forall i \in [1, 2, 3]$ (notice that displacements can be prescribed without difficulty along boundary elements thanks to standard CASTEM

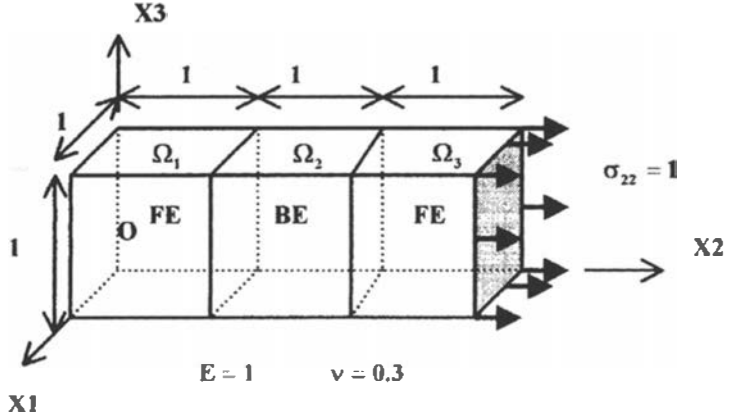


Figure 1 Rectangular prism submitted to uniaxial tension : discretization of the 3 subregions

2000 operators). A uniform traction $\sigma_{22} = 1$ is imposed on the bar end defined by $x_2 = 3$.

Displacements in the BEM subregion are found to be in good agreement with the analytical solution. Table 1 contains the results for the displacements components when 24 quadrilateral boundary elements are used for $\partial\Omega_2$.

Table 1 Nodal displacements along the edge $x_1 = x_3$ in the B.E. subregion.

direction	x_2	Numerical	Analytical	Relative error (%)
Longitudinal	1.00	0.99722	1.00000	-0.278
	1.25	1.24620	1.25000	-0.380
	1.50	1.50000	1.50000	+0.000
	1.75	1.75000	1.75000	+0.000
	2.00	2.00107	2.00000	+0.107
Transversal	1.00	0.29904	0.3	-0.095
	1.25	0.30180	0.3	+0.180
	1.50	0.29550	0.3	-0.449
	1.75	0.30246	0.3	+0.246
	2.00	0.30069	0.3	+0.069

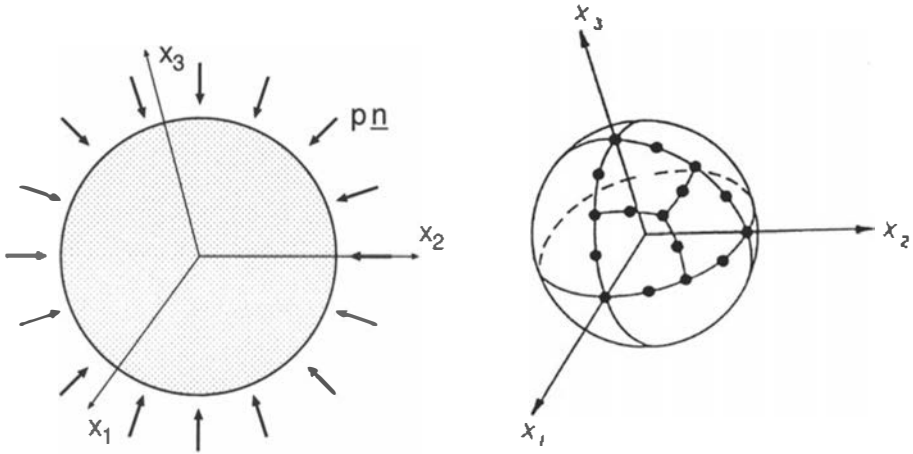


Figure 2 Sphere submitted to external uniform pressure(a), three quadrilateral boundary elements (b).

One note that this problem is characterized by discontinuous tractions along the edges of the BE subdomain. This point, which is somewhat cumbersome to deal with in regular collocation methods (especially if one wants to equate the number of equations and unknowns), is taken into account without any special consideration, excepted the mention of these discontinuities in the data field. This point constitutes an additional useful feature of SGBIE.

Sphere submitted to external uniform pressure. The problem of a sphere of radius $r = 1$ submitted to external uniform pressure (Fig. 2a) is considered, again using $E = 1$ and $\nu = 0.3$. The FE-BE interface is made of 24 quadratic quadrilateral isoparametric elements (Fig. 3a) and 74 nodes. The load is applied on the external boundary of the FE subregion.

The relative error produced by the coupled FEM-SGBEM analysis, for the radial displacement u_r in the BE subregion is found not to exceed 0.0325%, which is a quite good accuracy given the relatively coarse mesh.

Spherical cavity in infinite body under uniform pressure.

Finally, the hollow spherical cavity in an infinite elastic body (with again $E = 1$ and $\nu = 0.3$), submitted to a uniform internal pressure, is studied. Figure 3b illustrates the discretization. The mesh of the BE-FE interface is again made of 24 elements. The spherical symmetry of the problem is, like in the previous example, not exploited. Here, the stiffness matrix

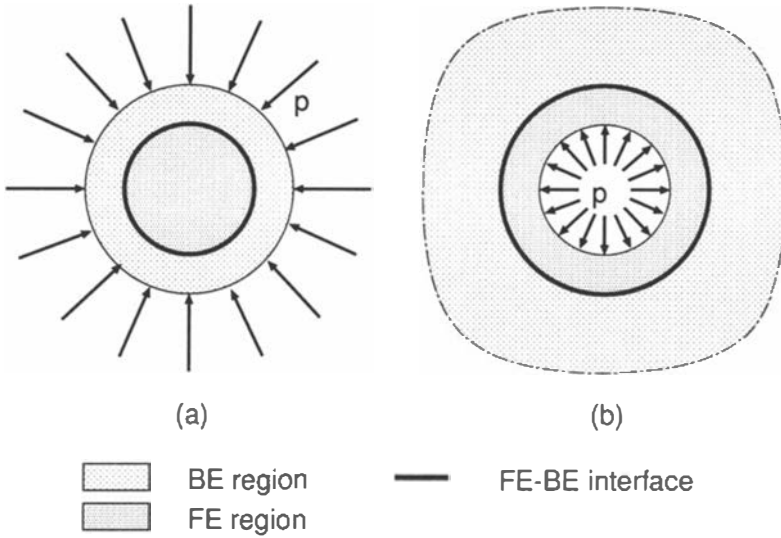


Figure 3 BE and FE domains for (a) the sphere under uniform external pressure. (b) the spherical cavity in an infinite body under to uniform pressure

$[K]$ evaluated with Eq. (16) accounts for the infinite elastic region, with implicitly enforced decay conditions at infinity.

The relative error produced by the coupled FEM-SGBEM analysis, for the radial displacement u_r in the BE-FE interface is found not to exceed 0.23%, which is again good given the relatively coarse mesh. The higher relative error (the FE-BE mesh being the same as for the finite sphere problem) is probably explained by the fact that the BE-modelled region is much larger (recall that Eq. (16) relies on an equality which is only approximate for discretized problems).

6. CONCLUSION

A procedure for coupling the symmetric Galerkin boundary element method and the finite element method has been developed for three-dimensional problems. The procedure creates the stiffness matrix for the BEM-modelled subdomain, for which elastic equilibrium without body forces is assumed. This subdomain is thus treated as a macroelement. This procedure has been implemented into the CASTEM 2000 FEM environment.

Sample numerical examples (rectangular prism submitted to uniform tension, characterized by discontinuous tractions; sphere subjected to external pressure; spherical cavity in infinite body under uniform pres-

sure) yield numerical results in good agreement with the corresponding exact solutions.

References

- [1] BONNET, M. Regularized direct and indirect symmetric variational BIE formulations for three-dimensional elasticity. *Engng. Anal. with Bound. Elem.*, **15**, 93–102 (1995).
- [2] BONNET, M. *Boundary integral equations methods for Solids and Fluids*. John Wiley and sons (1999).
- [3] BONNET, M., MAIER, G., POLIZZOTTO, C. On symmetric galerkin boundary element method. *Appl. Mech. Rev.*, **51**, 669–704 (1998).
- [4] CASTEM 2000. see website <http://www.castem.org:8001> (1999). CASTEM 2000 is a research FEM environment; its development is sponsored by the French Atomic Energy Commission (*Commissariat à l’Energie Atomique – CEA*).
- [5] HSIAO, G C. The coupling of boundary element and finite element methods. *Z. Angew. Math. Mech.*, **70**, 493–503 (1990).
- [6] NEDELEC, J-C. Approximation des équations intégrales par éléments finis. Etude d’erreur. In J. L. Lions R. Dautray (ed.), *Analyse mathématique et calcul scientifique pour les sciences et les techniques (chap. 13)*, pp. 953–968. Masson, Paris (1985).
- [7] REZAYAT, M., SHIPPY, D. J., RIZZO, F. J. On time-harmonic elastic wave analysis by the boundary element method for moderate to high frequencies. *Comp. Meth. in Appl. Mech. Engng.*, **55**, 349–367 (1986).
- [8] SIRTORI, S, MICCOLI, S, KORACH, E. Symmetric coupling of finite elements and boundary elements. In J. H. Kane, G. Maier, N. Tosaka, S. N. Atluri (eds.), *Advances in Boundary Element Techniques*, pp. 407–427. Springer-Verlag (1993).