Applications of Boundary Integral Equations for Solving Some Identification Problems in Elasticity

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APPLICATIONS OF BOUNDARY INTEGRAL EQUATIONS FOR SOLVING SOME IDENTIFICATION PROBLEMS IN ELASTICITY

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Abstract The identification of distributed elastic moduli of damaged materials or crack like defects in elastic media are inverse problems known as generalized elastic tomography. It consists of recovering the damaged zone or the crack in a 3D body using mechanical overdetermined boundary data. For 21 distributed unknowns which are small perturbations of elastic isotropy, a linear system of rank 5 may be derived directly from the observation equations which involves both domains and boundary integrals, with actual mechanical fields and the proposed adjoint fields. It is found that the generalization of Calderon's method in elasto-statics provided a linear system of rank 5, hence identification problems for small symmetry up to 5 elastic moduli fields could be solved. Finally, the problem of identification of a plane crack in 3D elastic body illustrates the ability of the observation equation method to provide closed form solution for the identification of the crack plane and the crack geometry.

Introduction

The identification of spatially distributed elastic moduli is an inverse problem generally denoted as generalized elastic tomography. In a certain number of cases the origin of distributed elastic moduli is damage or microcracking. This also implies that the interior distribution is a priori unknown and generally not measurable by direct methods. One can only imagine to know certain quantities like applied forces or displacements on the boundary of the body and to identify the distribution of interior of elastic moduli from these measurements.

This problem has been discussed in the last years in a series of papers generally devoted to the linearized identification problem and a series of results have been obtained. Thus, the linearized identification problem in elasticity has been solved for the isotropic case (2 unknowns : Young and Poisson moduli fields) by Ikehata (1990), Nakamura and Uhlmann (1991). The case of general anisotropy has been discussed by Constantinescu (1994) under the assumption that the orientation angles of the material symmetries and the eigentensors are known a priori. Theses works generalize to elasticity previous results obtained for inverse electricty problems, Calderon (1980), Kohn and Vogelius (1984).

The case of general anisotropie present a series of difficulties in both electricity and elasticity. Kohn and Vogelius (1987) discussed an example of non uniqueness in anisotropic conductivity problem and Constantinescu (1994) made similar analyses for anisotropic elasticity.

The present paper is first devoted to such problems by presenting a series of integrals equations which can be used for solving elastic identification problems. The method considered here is based on the observation equation. As a starting point, we shall consider the observation equation established initially for the electrostatic linear identification problem by Calderon (1980). This equation conducts to different formulations of the identification problem and to uniqueness results in the isotropic and anisotropic cases. Finally the paper is devoted to plane crack identification in 3D elastic body. Most works in the literature are concerned with the control of the crack and the use of classical boundary integral equations with singular Green function kernels. These methods are used for numerical purposes. Here we shall rather discuss the observable equation method using appropriate adjoint fields which conducts to a closed form solution of the inverse problem, recently given by Andrieux et al (1997).

1. THE DIRECT AND THE INVERSE PROBLEM

Let us consider an elastic body Ω under the hypothesis of small strains and rotations. The displacement, strain and respectively stress fields, denoted by $\boldsymbol{u}, \boldsymbol{\epsilon}, \boldsymbol{\sigma}$ are subject, considering the absence of a body force, to the following set of equations:

$$\begin{aligned} \boldsymbol{\varepsilon}(\boldsymbol{u}) &= \frac{1}{2}(\nabla + \nabla^T)\boldsymbol{u} \\ \boldsymbol{\sigma} &= \boldsymbol{L} : \boldsymbol{\varepsilon} \\ \text{div } \boldsymbol{\sigma} &= \boldsymbol{0} \end{aligned} \tag{1}$$

where $\boldsymbol{L}$ represents the forth order tensor of elastic moduli.

The *direct elasticity problem* consists of computing a solution of the system of partial differential equations (1) with known elastic moduli $\boldsymbol{L}$ and given one of the following boundary conditions on $\partial\Omega$: imposed displacements $\boldsymbol{u}|_{\partial\Omega}$ or imposed tractions $\boldsymbol{\sigma n}|_{\partial\Omega} = \boldsymbol{T}$. The pairs of corresponding boundary conditions $(\boldsymbol{u}, \boldsymbol{T})$ can be more generally described in terms of the Dirichlet-to-Neumann data map:

$$\Lambda_{\boldsymbol{L}} : \boldsymbol{u} \longrightarrow \Lambda_{\boldsymbol{L}}(\boldsymbol{u}) = \boldsymbol{T} \tag{2}$$

which maps a given boundary displacement in the corresponding boundary traction.

The *inverse elasticity problem* seeks to determine the *unknown* elastic moduli $\boldsymbol{L}$ from the partial knowledge of the Dirichlet-to-Neumann data map $\Lambda_{\boldsymbol{L}}$. This means from a series of overdetermined boundary conditions, i.e. simultaneously known displacements $\boldsymbol{u}$ and tractions $\boldsymbol{T}$.

Let us now suppose that the elastic moduli $\boldsymbol{L}$ can be expressed as:

$$\boldsymbol{L}(\boldsymbol{x}) = \boldsymbol{L}_0 + \delta\boldsymbol{L}(\boldsymbol{x}) \qquad\qquad \forall \boldsymbol{x} \in \Omega$$

where $\boldsymbol{L}_0$ is a homogenous distribution of isotropic elastic moduli and $\delta\boldsymbol{L}(\boldsymbol{x})$ is a small perturbation of $\boldsymbol{L}_0$ representing a heterogeneous distribution of anisotropic elastic moduli.

The *linearized identification problem* seeks to determine only the anisotropic perturbation $\delta\boldsymbol{L}(\boldsymbol{x})$ from the partial knowledge of the Dirichlet-to-Neumann data map.

2. THE OBSERVATION EQUATION

The derivation of an explicit observation equation using the boundary data $(\boldsymbol{u}, \boldsymbol{T})$ is a classical result (Ikehata (1990), Nakamura and Uhlmann (1991), Bui (1993)) and a straightforward generalization of the technique

used previously in electricity (Calderon (1980), Friedmann and Vogelius (1989)).

The observation equation is based on the variational formulation of the direct problem (1) and the use of special choosen adjoint test functions.

Let us consider the adjoint elastic fields $\boldsymbol{u}_0^*$, elastic solution on Ω with elastic moduli $\boldsymbol{L}_0$ and boundary conditions given by the traction vector $\boldsymbol{T}^*$.

For the same given traction vector $\boldsymbol{T}$ on $\partial\Omega$, let us denote two actual displacement fields:

- the *perturbed* solution $\boldsymbol{u}$, computed with elastic moduli $\boldsymbol{L} = \boldsymbol{L}_0 + \delta\boldsymbol{L}$, and
- the *unperturbed* solution $\boldsymbol{u}_0$ computed with unperturbed elastic moduli $\boldsymbol{L}_0$

For the linearized inverse problem, where the unknown is the perturbation $\delta\boldsymbol{L}$, one can write two observation equations corresponding to the actual displacements fields. Thus, the nonlinear observation equation is given by:

$$
\begin{aligned}
-\int_\Omega \nabla \boldsymbol{u}_0^* : \delta\boldsymbol{L} : \nabla \boldsymbol{u}\, d\Omega &= \int_{\partial\Omega} (\boldsymbol{u}^* \cdot \boldsymbol{T} - \boldsymbol{u} \cdot \boldsymbol{T}^*)\, dS \qquad (3)\\
&= \int_{\partial\Omega} (\boldsymbol{u}^* \cdot \Lambda_{\boldsymbol{L}}(\boldsymbol{u}) - \boldsymbol{u} \cdot \Lambda_{\boldsymbol{L}}(\boldsymbol{u}^*))\, dS := \mathcal{D}
\end{aligned}
$$

while the linearized observation equation is similarly expressed as:

$$
\begin{aligned}
-\int_\Omega \nabla \boldsymbol{u}_0^* : \delta\boldsymbol{L} : \nabla \boldsymbol{u}_0\, d\Omega &= \int_{\partial\Omega} (\boldsymbol{u}^* \cdot \boldsymbol{T} - \boldsymbol{u} \cdot \boldsymbol{T}^*)\, dS \qquad (4)\\
&= \int_{\partial\Omega} (\boldsymbol{u}^* \cdot \Lambda_{\boldsymbol{L}}(\boldsymbol{u}) - \boldsymbol{u} \cdot \Lambda_{\boldsymbol{L}}(\boldsymbol{u}^*))\, dS := \mathcal{D}
\end{aligned}
$$

The right hand side $\mathcal{D}$ represents the boundary data and is in both cases the same. Its value is known from the true boundary data on the actual solid $(\boldsymbol{u}, \boldsymbol{T})$ and from the computed boundary values of the choosen adjoint fields on the unperturbed solid $(\boldsymbol{u}^*, \boldsymbol{T}^*)$.

From the general form of the observation equation it is obvious that a convenient method for computing field distributions is provided by the *boundary integrals equations.*

3. THE LOADING TYPES AND THE PARAMETERS

In the sequel, we shall adress the identification problem as expressed by the linearized observation equation (4).

One can remark that, the equation (4) alone is not sufficient for the identification of the anisotropic moduli tensor $\delta \boldsymbol{L}(x)$ with the 21 components $\delta L_{ijkl}(\boldsymbol{x})$. The identification depends on the different "types" of mechanical loadings provided during the experiments on the body and creating the data set: $\boldsymbol{Y} = \{\mathcal{D}_m\}_{m \in M}$.

In order to obtain identifiability, one important task is to provide data, rich enough to span the whole space of the 21 unknown functions $\boldsymbol{X} = \{\delta L_{ijkl}(\boldsymbol{x})\}$, or in other terms to construct a one-to-one mapping $\boldsymbol{X} \longrightarrow \boldsymbol{Y}$.

As it will be shown in the next sections on different examples, this task is not a straightforward one.

4. AN EXAMPLE OF NON-UNIQUENESS

The identifiability of the perturbation of elastic moduli $\delta \boldsymbol{L}$ from the complete knowledge of the Dirichlet-to-Neumann data map, has been proven in the case of an isotropic perturbation in a series of papers (Ikehata (1990), Nakamura and Uhlmann (1991)). Constantinescu (1994, 1995) studied the case where 3 Euler angles associated with the material symmetries and 6 second order eigentensors $\boldsymbol{N}_k$, $k = 1, 6$ are known (with unit norm $tr \boldsymbol{N}_k \boldsymbol{N} = 1$) and conjectured the identifiability of 6 Kelvin's eigenelastic moduli c_k, $\boldsymbol{L} = c_k \boldsymbol{N}_k \otimes \boldsymbol{N}_k$, from the Dirichlet-to-Neumann map. The complet anisotropic case, without apriori knowledge of the anisotropy or of the spatial distribution is to our knowledge still an open problem as presented in the sequel.

Let us suppose that a symmetric and positive definite tensor of elastic moduli $\boldsymbol{L}(\boldsymbol{x})$ is given. Is it possible to find another tensor of elastic moduli $\boldsymbol{L}'(\boldsymbol{y})$ over the same domain Ω, such that the Dirichlet-to-Neumann maps coincide $\Lambda_{\boldsymbol{L}} = \Lambda_{\boldsymbol{L}'}$, for both elastic bodies ?

Based on a result on the conductivity problem by Kohn and Vogelius (1987), Constantinescu (1994) gave in certain cases a positive answer to this nonuniqueness question.

Let us choose a one-to-one differentiable map on Ω, $\boldsymbol{\psi} : \boldsymbol{x} \to \boldsymbol{y} = \boldsymbol{\psi}(\boldsymbol{x})$ such that $\boldsymbol{\psi}(\boldsymbol{x}) = \boldsymbol{x}$ on the boundary $\partial\Omega$.

For an actual displacement field $\boldsymbol{u}(\boldsymbol{x})$ corresponding to a solution with elastic moduli $\boldsymbol{L}$, let us define the displacement field $\boldsymbol{u}'(\boldsymbol{y})$ as the "parallel transport" of the actual field:

$$\boldsymbol{u}'(\boldsymbol{y}) = \boldsymbol{u}(\boldsymbol{\psi}^{-1}(\boldsymbol{y})) \tag{5}$$

If we define the elastic moduli $\boldsymbol{L}'$ as:

$$\boldsymbol{L}'_{ijkl}(\boldsymbol{x}) = |\det(\boldsymbol{\psi})|^{-1} \boldsymbol{L}_{inkm}(\boldsymbol{x})\, \boldsymbol{\psi}_{n,j}(\boldsymbol{x})\, \boldsymbol{\psi}_{m,l}(\boldsymbol{x}) \tag{6}$$

we remark that $\boldsymbol{u}'$ is an elastic solution with moduli $\boldsymbol{L}'$. Moreover $\boldsymbol{u}$ and $\boldsymbol{u}'$ have the same boundary displacement, the same traction vector $\boldsymbol{T}$ and will create the same deformation energy.

It can be checked that $\boldsymbol{L}'$ is positive definite, because $\boldsymbol{L}$ is assumed positive definite and $\det(\boldsymbol{\psi}) = 0$. However, the symmetries conditions:

$$L'_{ijkl} = L'_{jikl} = L'_{klij} \tag{7}$$

have yet to be satisfied in order to obtain a complete elasticity tensor.

A closer look at the preceding system of equations (7) shows that, for a given $\boldsymbol{L}$, there are more equations than unknowns. For example, in the linear case, there are 18 linearized equations for determining the vector field $\boldsymbol{\psi}(\boldsymbol{x}) - \boldsymbol{x}$, which vanishes on the boundary. The linearized solution of (7) is generally $\boldsymbol{\psi}(\boldsymbol{x}) - \boldsymbol{x} = 0$, except for some particular $\boldsymbol{L}$. The non-uniqueness in this identification problem corresponds to some particular choice of both $\boldsymbol{L}$ and $\boldsymbol{\psi}$ satisfying (7).

In what follows, we shall restrict ourself to the case of a homogeneous isotropic $\boldsymbol{L}_0$, but we shall accept that the perturbation $\delta\boldsymbol{L}(\boldsymbol{x}) = \boldsymbol{L}(\boldsymbol{x}) - \boldsymbol{L}_0$ may be anisotropic and heterogenous.

5. THE LINEARIZED INVERSE PROBLEM

5.1. GENERALIZATION OF CALDERON'S METHOD

In order to find the unknown perturbation of the elastic moduli $\delta\boldsymbol{L}(\boldsymbol{x})$, we will assume that $\delta\boldsymbol{L}(\boldsymbol{x}) = \boldsymbol{0}$ on $\partial\Omega$, and apply the technique outlined by Calderon (1980) in electricty or Ikehata (1990) in isotropic elasticity.

Let us introduce the complex vectors $\boldsymbol{Z}, \boldsymbol{Z}^* \in \boldsymbol{C}^3$, by:

$$\boldsymbol{Z} = \frac{1}{2}(\boldsymbol{\zeta} + i\boldsymbol{\zeta}^{\perp}) \qquad\qquad \boldsymbol{Z}^* = \frac{1}{2}(\boldsymbol{\zeta} - i\boldsymbol{\zeta}^{\perp}) \tag{8}$$

where $\boldsymbol{\zeta} = (\zeta_1, \zeta_2, \zeta_3) \in \boldsymbol{R}^3$ and $\boldsymbol{\zeta}^{\perp}$ is a vector perpendicular to $\boldsymbol{\zeta}$ and having the same norm.

The elastic field $\boldsymbol{u}_0$ is introduced using the scalar harmonic field $B_0(\boldsymbol{x}) := \exp(-i\boldsymbol{Z} \cdot \boldsymbol{x})$ as follows:

$$\begin{aligned} \boldsymbol{u}_0 &:= \nabla B_0(\boldsymbol{x}), & \boldsymbol{\varepsilon}_0 &= -\boldsymbol{Z} \otimes \boldsymbol{Z} \exp(-i\boldsymbol{Z} \cdot \boldsymbol{x}), \\ \boldsymbol{\sigma}_0 &:= 2\mu_0 \nabla\nabla B_0(\boldsymbol{x}), & \boldsymbol{T} &= -2\mu_0 \boldsymbol{Z}(\boldsymbol{Z} \cdot \boldsymbol{n}) \exp(-i\boldsymbol{Z} \cdot \boldsymbol{x}) \end{aligned} \tag{9}$$

where $\boldsymbol{n}$ is the unit outward normal on the boundary $\partial\Omega$ and μ_0 is the shear moduli corresponding to $\boldsymbol{L}_0$.

Similarly the adjoint elastic field $\boldsymbol{u}_0^*$ is derived from the scalar harmonic field $B_0^*(\boldsymbol{x}) := \exp(-i\boldsymbol{Z}^* \cdot \boldsymbol{x})$;

$$\begin{aligned} \boldsymbol{u}_0^* &:= \nabla B_0^*(\boldsymbol{x}), \qquad \boldsymbol{\varepsilon}_0^* = -\boldsymbol{Z}^* \otimes \boldsymbol{Z}^* \exp(-i\boldsymbol{Z}^* \cdot \boldsymbol{x}), \\ \boldsymbol{\sigma}_0^* &= 2\mu_0 \nabla\nabla B_0^*(\boldsymbol{x}), \quad \boldsymbol{T}^* = -2\mu_0 \boldsymbol{Z}^* (\boldsymbol{Z}^* \cdot \boldsymbol{n}) \exp(-i\boldsymbol{Z}^* \cdot \boldsymbol{x}) \end{aligned} \tag{10}$$

The linear observation equation (4) becomes:

$$\begin{aligned} \int_\Omega (\boldsymbol{Z} \otimes \boldsymbol{Z}) : \delta\boldsymbol{L}(\boldsymbol{x}) \quad &: \quad (\boldsymbol{Z}^* \otimes \boldsymbol{Z}^*) \exp(-i\boldsymbol{\zeta} \cdot \boldsymbol{x}) d\Omega_{\boldsymbol{x}} \\ &= \int_{\partial\Omega} (\boldsymbol{u}^* \cdot \boldsymbol{T} - \boldsymbol{u} \cdot \boldsymbol{T}^*)\, dS := \mathcal{D}(\boldsymbol{\zeta}, \boldsymbol{\zeta}^\perp) \end{aligned} \tag{11}$$

The right hand side of (11) is the data $D(\boldsymbol{\zeta}, \boldsymbol{\zeta}^\perp)$.

Let us extend formally the definition of $\delta\boldsymbol{L}(\boldsymbol{x})$ on the whole space $\boldsymbol{R}^3$ by letting $\delta\boldsymbol{L}(\boldsymbol{x}) = \boldsymbol{0}, \boldsymbol{x} \in \boldsymbol{R}^3 \setminus \Omega$ and introduce the Fourier transform $\mathcal{F}[\delta\boldsymbol{L}](\boldsymbol{\zeta})$ of $\delta\boldsymbol{L}(\boldsymbol{x})$. Then equation (11) becomes:

$$(\boldsymbol{Z} \otimes \boldsymbol{Z}) : \mathcal{F}[\delta\boldsymbol{L}](\boldsymbol{\zeta}) : (\boldsymbol{Z}^* \otimes \boldsymbol{Z}^*) = \int_{\partial\Omega} (\boldsymbol{u}^* \cdot \boldsymbol{T} - \boldsymbol{u} \cdot \boldsymbol{T}^*)\, dS \tag{12}$$

5.2. THE LOADING TYPES AND PARAMETERS

In order to construct a linear system for the Fourier transform $\mathcal{F}[\delta\boldsymbol{L}](\boldsymbol{\zeta})$, let us choose a series of $\boldsymbol{\zeta}^\perp$ in the the plane perpendicular to $\boldsymbol{\zeta}$. Their position is completely determined by an angle in this plan denoted by ϕ. An "experiment" and the corresponding adjoint field are now completly characterized by $\boldsymbol{\zeta}$ and ϕ. Using this notation one can write equation (12) in the following form:

$$a_{ij}(\boldsymbol{\zeta}; \phi) \mathcal{F}[\delta\boldsymbol{L}]_{ijhk}(\boldsymbol{\zeta}) b_{hk}(\boldsymbol{\zeta}; \phi) = \mathcal{D}(\boldsymbol{\zeta}; \phi). \tag{13}$$

with $a_{ij}(\boldsymbol{\zeta}; \phi)$ and $b_{hk}(\boldsymbol{\zeta}; \phi)$ polynomes of degree 4 in $\cos(\phi)$.

Choosing different "types" of experiments with different angles $\phi_\alpha, \alpha = 1, 2, 3, \ldots$ creates the linear system for the Fourier transform:

$$a_{ij}(\boldsymbol{\zeta}; \phi_\alpha) \mathcal{F}[\delta\boldsymbol{L}]_{ijhk}(\boldsymbol{\zeta}) b_{hk}(\boldsymbol{\zeta}; \phi_\alpha) = \mathcal{D}(\boldsymbol{z}; \phi_\alpha) \alpha = 1, 2, 3, \ldots.$$

The general anisotropic case corresponds to 21 elastic coefficients which can be expressed by: 6 eigenelastic moduli c_k corresponding to the eigentensors $\boldsymbol{N}_k$, $k = 1, \ldots, 6$, (subject to the normalization condition $\mathrm{tr}\boldsymbol{N}_k \otimes \boldsymbol{N}_k = 1$), 6 traces $\mathrm{tr}\boldsymbol{N}_k$, 6 traces $\mathrm{tr}\boldsymbol{N}_k \otimes \boldsymbol{N}_k \otimes \boldsymbol{N}_k$ and 3 Euler angles of the material symmetry, Cowin and Mehrabadi (1990). If the

Euler angles of the material symmetry and the 6 eigentensors $\boldsymbol{N}_k$ are known, the variations $\delta c_k(\boldsymbol{z})$ are determined by the 6×6 linear systems

$$\sum_{k=1}^{6} \mathcal{F}\delta c_k(\boldsymbol{\zeta})(\boldsymbol{Z} \cdot \boldsymbol{N}_k \cdot \boldsymbol{Z})(\boldsymbol{Z}^* \cdot \boldsymbol{N}_k \cdot \boldsymbol{Z}^*) = \mathcal{D}(\zeta; \phi_l), l = 1, 6 \qquad (14)$$

Unfortunately, the construction (14) cannot determine more than 5 unknowns,because it can be proved that the rank of the linear system is only 5. This is due to the fact that $\boldsymbol{s}_0^* := 2\mu_0 \nabla\nabla B_0^*(\boldsymbol{x})$, is a deviatoric field and thus it does not span the space of symetric second order tensors. To obtain higher rank of the linear system, more adjoint fields with non zero traces are needed. Even with the incomplete adjoint fields, the equation (14) may ensure the identifiability of some elastic moduli or orientation angles but not all, for hexagonal (5), cubic (3) and isotropic (2) symmetries. For tetragonal (6), and trigonal (6), we need complementary informations (for example one orientation angle). The identification of the orthotropic symmetry (9) is possible if the orientation angles of the material symmetry are known. Finally, the identification of the general triclinic elastic perturbation (21 unknowns), by the present method, is still an open problem, since we need new appropriate adjoint fields to complete those given above.

6. PLANAR CRACK IDENTIFICATION IN ELASTOSTATICS

In this section, we present the analytical solution to the inverse problem of the identification of a planar crack in 3D elasticity, given recently by Andrieux et al (1997) who derived their solution using the observation equation. The crack opening displacement discontinuity $[\![\boldsymbol{u}]\!]$ satisfies the integral equation over the unknown crack $\mathcal{S}$ with the normal $\boldsymbol{N}$ to one crack face:

$$\begin{aligned} \int_{\mathcal{S}} [\![\boldsymbol{u}]\!] \cdot \boldsymbol{L}_0 : \boldsymbol{\varepsilon}(\boldsymbol{v}) \cdot \boldsymbol{N} d\mathcal{S} &= \int_{\partial\Omega} (\boldsymbol{T}(\boldsymbol{w}) \cdot \boldsymbol{v} - \boldsymbol{w} \cdot \boldsymbol{L}_0 : \boldsymbol{\varepsilon}(\boldsymbol{v}) \cdot \boldsymbol{n}) d\mathcal{S} \qquad (15) \\ &= \mathcal{D}(\boldsymbol{v}; \boldsymbol{w}, \boldsymbol{T}(\boldsymbol{w})), \end{aligned}$$

where $\boldsymbol{v}(\boldsymbol{x})$ is the adjoint field corresponding to the *uncracked* body, hence a twice differentiable field. The right hand side $\mathcal{D}$ is a linear functional of $\boldsymbol{v}$, with coefficients depending on the data pair $(\boldsymbol{w}, \boldsymbol{T}(\boldsymbol{w}))$. The observation equation (15) is very similar to (3). By choosing appropriately the adjoint fields, one can determine the normal $\boldsymbol{n}$, the crack plane position and what is more important, the crack geometry. The normal is determined by the adjoint fields $\boldsymbol{v}$, linear functions of the coordinates

such that the corresponding stresses are constant, defined by:

$$v_k^{ij} = (\boldsymbol{L}^{-1})_{klmn}(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm})x_l \tag{16}$$

with all indexes taking values in the set $\{1, 2, 3\}$.

Two data sets $(\boldsymbol{w}_1, \boldsymbol{T}(\boldsymbol{w}_1))$ and $(\boldsymbol{w}_2, \boldsymbol{T}(\boldsymbol{w}_2))$ are required for identifying the normal $\boldsymbol{N}$ on the crack plane. The proof, more technical than difficult, can be found in Andrieux et al (1997). Once the normal is determined, a change of coordinates permits to define the axes Ox_1x_2 parallel to the crack plane. The position of the crack plane defined as $x_3 = c$ is then determined by the adjoint field $\boldsymbol{v}(\boldsymbol{x})$, a polynome of second order of x_1 and x_2 corresponding to the bending solution in elasticity.

Let us sketch the determination of the crack geometry with more details. The adjoint fields are defined by the following equations. One introduces the index $\boldsymbol{\zeta} = (\zeta_1, \zeta_2, 0)$ running over $\boldsymbol{R}^2$ and the family of complex vector fields $\boldsymbol{Z}$ and its conjugate $\boldsymbol{Z}^*$ belonging to the complex vectorial space $\boldsymbol{C}^3$ defined by:

$$\boldsymbol{Z} = (\boldsymbol{\zeta} + i|\boldsymbol{\zeta}|\boldsymbol{N}) \qquad \boldsymbol{Z}^* = (\boldsymbol{\zeta} - i|\boldsymbol{\zeta}|\boldsymbol{N}) \tag{17}$$

Two kinds of adjoint fields are considered :

$$\boldsymbol{v}^+(\boldsymbol{x}, \boldsymbol{\zeta}) = \nabla \exp(-i\boldsymbol{Z} \cdot \boldsymbol{x}) + \nabla \exp(-i\boldsymbol{Z}^* \cdot \boldsymbol{x}), \tag{18}$$

$$\boldsymbol{v}^-(\boldsymbol{x}, \cdot) = \nabla \exp(-i\boldsymbol{Z} \cdot \boldsymbol{x}) - \nabla \exp(-i\boldsymbol{Z}^* \cdot \boldsymbol{x}). \tag{19}$$

Equation (15) can be splitted into 2 equations for the Fourier transforms of the normal and tangential components of the discontinuity $\boldsymbol{D}(\boldsymbol{\zeta}) := [\![\boldsymbol{u}]\!](\boldsymbol{\zeta})$.

$$\mathcal{F}[D_3](\boldsymbol{\zeta}) = \frac{1+\nu}{2E|\boldsymbol{\zeta}|^2}\mathcal{D}(\boldsymbol{v}^+(\boldsymbol{\zeta}); \boldsymbol{w}, \boldsymbol{T}(\boldsymbol{w})), \tag{20}$$

$$\zeta_1\mathcal{F}[D_1](\boldsymbol{\zeta}) + \zeta_2\mathcal{F}[D_2](\boldsymbol{\zeta}) = \frac{1+\nu}{2iE|\boldsymbol{\zeta}|}\mathcal{D}(\boldsymbol{v}^-(\boldsymbol{\zeta}); \boldsymbol{w}, \boldsymbol{T}(\boldsymbol{w})). \tag{21}$$

where E, ν are respectively the Young modulus and the Poisson ratio.

Choosing the indexes $\boldsymbol{\zeta} = (\zeta_1, 0, 0)$, $\boldsymbol{\zeta} = (0, \zeta_2, 0)$, we obtain from equation (21) the partial Fourier transforms of the tangential components of the discontinuity. And finally, the crack discontinuity, hence the crack itself, is explicitly known from the boundary data $(\boldsymbol{w}, \boldsymbol{T}(\boldsymbol{w}))$ as proven in Andrieux et al. (1997).

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