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Abstract. Procedures based on group representation theory, allowing the exploitation of geometrical symmetry in symmetric Galerkin BEM formulations of 3D elastodynamic problems, are developed. They are applicable for both commutative and noncommutative finite symmetry groups and to partial geometrical symmetry, where the boundary has two disconnected components, one of which is symmetric.

1 Introduction

When a linear boundary-value problem (BVP) exhibits geometrical symmetry, taking full advantage of it yields substantial computational benefits. In Bossavit [4], the linear representation theory for finite groups [8,9] is shown to lead to the correct definition of (i) decomposition of function spaces into orthogonal subspaces of symmetric, skew-symmetric,... functions, and (ii) reconstruction of the global solution from these components; the (domain-based, FEM-oriented) weak formulation is thus recast into a block-diagonal form, each 'subproblem' being defined on a 'symmetry cell' (a subdomain of smallest measure that, under the action of the symmetry group, generates the entire initial domain) and associated to the corresponding projection of the boundary data. The procedure, being essentially an elaborate superposition technique, assumes linear constitutive properties. Similar principles are used by Allgower et al. [1] to block-diagonalize the discretized equations.

The adaptation of the former approach to boundary element methods (BEMs) is not straightforward, mostly because the symmetry cell usually involve the definition of new boundaries, a feature which is unimportant in FEMs but clearly undesirable in BEMs, where subproblems should be stated only on symmetry cells of the boundary. In an earlier work [2], this issue was adressed for collocation BEMs and commutative symmetry groups (see also [6]). Using standard methods to set up and solve the matrix equations, the theoretical computational gains (in relative terms, compared to using the same discretization without symmetry) were found to be $1/n$, $1/n$ and $1/n^2$ for the matrix storage requirement, matrix set-up time and solution time, respectively, where n is the number of elements in the symmetry group (e.g. $n = 8$ for the group of symmetries with respect to three orthogonal planes).

This contribution aims at extending the concepts and results of [2] in three directions. Firstly, the exploitation of geometrical symmetry is considered here in the framework of symmetric Galerkin BEM formulations. Secondly, procedures are developed for both commutative or noncommutative symmetry groups. Thirdly, the approach is generalized to partial geometrical symmetry, where the boundary has two (or more) disconnected components, one (or more) of which being invariant under a symmetry group. For instance, in defect identification problems, bodies with external geometrical symmetry but containing internal cracks, voids, inclusions... of arbitrary shape and location might be encountered. The formulations developed herein are expected to bring significant gains in computational efficiency by exploiting symmetries of the external boundary.

2 Governing equations

In this paper, the use of geometrical symmetry is fully developed for the Neumann BVP of linear elastodynamics in the frequency-domain, using double-layer integral representations. These BVPs are chosen as representative model problems, and the developments to follow are expected to be easily adaptable to other scalar or vector linear BVPs (e.g. from electromagnetics).

The displacement vector $\boldsymbol{u}$, strain tensor $\boldsymbol{\varepsilon}$ and stress tensor $\boldsymbol{\sigma}$ in a three-dimensional isotropic elastic medium are governed by the dynamic equilibrium, constitutive and compatibility field equations:

$$\begin{aligned} \operatorname{div}\boldsymbol{\sigma} + \rho\omega^2\boldsymbol{u} + \boldsymbol{f} &= \boldsymbol{0} \\ \boldsymbol{\sigma} = \mu\big[\tfrac{2\nu}{1-2\nu}\operatorname{Tr}(\boldsymbol{\varepsilon})\mathbf{1} + 2\boldsymbol{\varepsilon}\big] \\ \boldsymbol{\varepsilon} = (\boldsymbol{\nabla}\boldsymbol{u} + \boldsymbol{\nabla}^T\boldsymbol{u})/2 \end{aligned} \tag{1}$$

(with μ: shear modulus, ν: Poisson ratio, ρ: mass per unit volume, $\boldsymbol{f}$: body force distribution), which, upon elimination of $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$, yield the well-known Navier equation, an elliptic second-order vector PDE for the primary field $\boldsymbol{u}$.

In particular, a time-harmonic unit point force (i.e. $\boldsymbol{f} = \delta(\boldsymbol{x} - \tilde{\boldsymbol{x}})\boldsymbol{e}_k$) applied in an infinite elastic body at the fixed point $\tilde{\boldsymbol{x}}$ and along the k-direction defines at $\boldsymbol{x} \in \mathbb{R}^3 \setminus \{\tilde{\boldsymbol{x}}\}$ the well-known elastodynamic *fundamental solution*. The fundamental displacement $U_i^k(\tilde{\boldsymbol{x}}, \boldsymbol{x})$, stress tensor $\Sigma_{ij}^k(\tilde{\boldsymbol{x}}, \boldsymbol{x})$ and traction vector $T_i^k(\tilde{\boldsymbol{x}}, \boldsymbol{x})$ are given by:

$$U_i^k(\tilde{\boldsymbol{x}}, \boldsymbol{x}) = 2(1-\nu)[F_{,aa} + k_L^2 F]\delta_{ik} - F_{,ik} \tag{2}$$

$$\Sigma_{ij}^k(\tilde{\boldsymbol{x}}, \boldsymbol{x}) = \mu\Big[\frac{2\nu}{1-2\nu}\delta_{ij}U_{a,a}^k + U_{i,j}^k + U_{j,i}^k\Big] \tag{3}$$

$$T_i^k(\tilde{\boldsymbol{x}}, \boldsymbol{x}) = \Sigma_{ij}^k n_j \tag{4}$$

in terms of the Somigliana potential [5] F:

$$F(\tilde{\boldsymbol{x}}, \boldsymbol{x}) = \frac{1}{4\pi\mu k_T^2}(e^{ik_L r} - e^{ik_T r})\frac{1}{r} \tag{5}$$

($k_T^2 = \rho\omega^2/\mu$ and $k_L^2 = \kappa k_T^2$, with $\kappa = \frac{1-2\nu}{2(1-\nu)}$: transversal and longitudinal wave numbers; $r = |\boldsymbol{x} - \tilde{\boldsymbol{x}}| = [(\boldsymbol{x}-\tilde{\boldsymbol{x}}).(\boldsymbol{x}-\tilde{\boldsymbol{x}})]^{1/2}$: Euclidian distance between $\boldsymbol{x}, \tilde{\boldsymbol{x}}$). F satisfies the equation:

$$F_{,aabb} = \frac{k_T^2}{4\pi\mu}(\kappa^2 e^{ik_L r} - e^{ik_T r})\frac{1}{r} \tag{6}$$

In this paper, solutions to (1) are assumed to be given by a double-layer integral representation formula:

$$u_k(\tilde{\boldsymbol{x}}) = \int_S T_i^k(\tilde{\boldsymbol{x}}, \boldsymbol{x})\phi_i(\boldsymbol{x})\,\mathrm{d}S_x =: [\boldsymbol{A}\boldsymbol{\phi}](\tilde{\boldsymbol{x}}) \tag{7}$$

where S is a bounded surface, either closed or open (or possibly a set of several such surfaces) and the density $\boldsymbol{\phi}$ depends on the boundary conditions; the case of an open surface is usually associated with scattering of elastodynamic waves by cracks. Representations of the form (7) are often used to formulate boundary integral equations (BIEs) for interior or exterior problems on the domain Ω bounded by S with Neumann boundary data $\bar{\boldsymbol{p}}$ over S. In particular, such problems lead to symmetric Galerkin BIE (SGBIE) formulations through a weighted-residual statement of the Neumann boundary condition:

$$\int_S [\boldsymbol{T}^n \boldsymbol{A}\boldsymbol{\phi}](\tilde{\boldsymbol{x}}).\tilde{\boldsymbol{\phi}}^\star(\tilde{\boldsymbol{x}})\,\mathrm{d}S_{\tilde{x}} = \int_S \bar{\boldsymbol{p}}(\tilde{\boldsymbol{x}}).\tilde{\boldsymbol{\phi}}^\star(\tilde{\boldsymbol{x}})\,\mathrm{d}S_{\tilde{x}} \qquad (\forall \tilde{\boldsymbol{\phi}} \in \mathcal{V}) \tag{8}$$

where the traction vector operator $\boldsymbol{T}^n\boldsymbol{u}$ is defined by $\boldsymbol{T}^n\boldsymbol{u} = \boldsymbol{\sigma}(\boldsymbol{u}).\boldsymbol{n}$. The operation $[\boldsymbol{T}^n\boldsymbol{A}\boldsymbol{\phi}](\tilde{\boldsymbol{x}})$ gives rise to hypersingular kernels involving a r^{-3} singularity. After a well-documented regularization process [3,7] involving two integrations by parts over S, the actual SGBIE formulation, which is the basis for the present development, is:

$$\mathcal{A}(\boldsymbol{\phi}, \tilde{\boldsymbol{\phi}}^\star) = \mathcal{L}(\tilde{\boldsymbol{\phi}}^\star) \qquad (\forall \tilde{\boldsymbol{\phi}} \in \mathcal{V} = [H^{1/2}(S)]^3) \tag{9}$$

where the linear form $\mathcal{L}$ and the symmetric bilinear form $\mathcal{A}$ are given by:

$$\mathcal{L}(\tilde{\boldsymbol{\phi}}^\star) = \frac{1}{2}\int_S \bar{\boldsymbol{p}}(\tilde{\boldsymbol{x}}).\tilde{\boldsymbol{\phi}}(\tilde{\boldsymbol{x}})\,\mathrm{d}S_{\tilde{x}} \tag{10}$$

$$\mathcal{A}(\boldsymbol{\phi}, \tilde{\boldsymbol{\phi}}^\star) = \int_S\int_S B(\boldsymbol{x}, \tilde{\boldsymbol{x}}; \boldsymbol{\phi}, \tilde{\boldsymbol{\phi}}^\star)\,\mathrm{d}S_x\,\mathrm{d}S_{\tilde{x}} \tag{11}$$

$$B(\boldsymbol{x}, \tilde{\boldsymbol{x}}; \boldsymbol{\phi}, \tilde{\boldsymbol{\phi}}^\star) = B_{ikqs}(\tilde{\boldsymbol{x}}, \boldsymbol{x})R_s\phi_k(\boldsymbol{x})R_q\tilde{\phi}(\tilde{\boldsymbol{x}}) + k_T^2 A_{ik}(\tilde{\boldsymbol{x}}, \boldsymbol{x})\phi_k(\boldsymbol{x})\tilde{\phi}(\tilde{\boldsymbol{x}})$$

where the two kernel functions B_{ikqs} and A_{ik} are given by:

$$B_{ikqs}(\tilde{\boldsymbol{x}}, \boldsymbol{x}) = -e_{iep}e_{kgr}\mu^2[4\nu\delta_{pq}\delta_{rs} + 2(1-\nu)(\delta_{pr}\delta_{qs} + \delta_{ps}\delta_{qr})]F_{,eg}$$

$$\begin{aligned} A_{ik}(\tilde{\boldsymbol{x}}, \boldsymbol{x}) = \Big[&[2(1-\nu)(\delta_{ik}\delta_{j\ell} + \delta_{jk}\delta_{i\ell}) + \frac{2\nu}{\kappa}\delta_{ij}\delta_{k\ell}]\frac{F_{,aabb}}{k_T^2} \\ &+ (1-2\nu)(\delta_{ik}F_{,j\ell} + \delta_{j\ell}F_{,ik} + \delta_{jk}F_{,i\ell} + \delta_{i\ell}F_{,jk}) \\ &+ \frac{4\nu^2}{1-2\nu}\delta_{ij}\delta_{k\ell}F_{,aa} + 4\nu(\delta_{ij}F_{,k\ell} + \delta_{k\ell}F_{,ij})\Big]n_j(\tilde{\boldsymbol{x}})n_\ell(\boldsymbol{x}) \end{aligned}$$

and where $R_i f$ denotes the i-th component of the *surface curl* of a scalar function f [7] (e_{abc}: permutation tensor):

$$R_i f = e_{ijk} n_j f_{,k} \tag{12}$$

Besides, if the surface S is defined by a mapping $\Delta \subset \mathbb{R}^2 \to S$, $\boldsymbol{\xi} \to \boldsymbol{x}(\boldsymbol{\xi})$, one has

$$[R_i f](\boldsymbol{x}(\boldsymbol{\xi}))\, \mathrm{d}S_x = [\partial_{\xi_1} f \partial_{\xi_2} x_i - \partial_{\xi_2} f \partial_{\xi_1} x_i]\, \mathrm{d}\boldsymbol{\xi} \tag{13}$$

which shows in particular that R_i is a tangential differential operator. Both B_{ikqs} and A_{ik} are weakly singular in view of (5) and (6) and have symmetry properties which ensure the overall symmetry of $\mathcal{A}(\phi, \tilde{\phi}^\star)$ through:

$$B(\boldsymbol{x}, \tilde{\boldsymbol{x}}; \boldsymbol{\phi}, \tilde{\boldsymbol{\phi}}^\star) = B(\tilde{\boldsymbol{x}}, \boldsymbol{x}; \boldsymbol{\phi}, \tilde{\boldsymbol{\phi}}^\star) = B(\boldsymbol{x}, \tilde{\boldsymbol{x}}; \tilde{\boldsymbol{\phi}}^\star, \boldsymbol{\phi}) \tag{14}$$

3 Geometrical symmetry assumptions

The most important assumption for the present purposes is that the boundary S has either full or partial geometrical symmetry. By this, we mean that there exists a finite group $\mathcal{S} = \{s_1, \ldots, s_n\}$ of n isometries of $\mathbb{R}^3$ (n is the *order* of $\mathcal{S}$) and a partition of the boundary S into two disconnected components S^1, S^2 such that S^1 is invariant under $\mathcal{S}$ whereas S^2 is not:

$$(\forall s \in \mathcal{S}) \qquad s(S^1) = S^1$$

One can therefore introduce a *symmetry cell* for S^1, i.e. a subset C of S^1 such that

$$\mathrm{Area}(C) = \mathrm{Area}(S^1)/n \qquad \text{and} \quad S^1 = \bigcup_{s \in \mathcal{S}} s(C)$$

For example, S^1 is the (symmetric) external boundary while S^2 is a (collection of) interior hole(s) or crack(s) of arbitrary shape and location. Full symmetry refers to the case where $S^1 = S$ and $S^2 = \emptyset$, i.e. the whole boundary S (and hence also Ω) is invariant under $\mathcal{S}$. Recall that an isometry of $\mathbb{R}^3$ is a linear application $s : \mathbb{R}^3 \to \mathbb{R}^3$ such that $|s\boldsymbol{x}| = |\boldsymbol{x}|$ ($\forall \boldsymbol{x} \in \mathbb{R}^3$), where $|\boldsymbol{x}| \equiv (\boldsymbol{x}.\boldsymbol{x})^{1/2}$ is the usual Euclidean norm in $\mathbb{R}^3$.

Exploiting (partial) symmetry in the SGBEM formulation (9) essentially consists in transforming integrals over S^1 into integrals over C, so that the matrix operators produced by the discretization process are of smaller size than those corresponding to the original integral equations. Note that no symmetry is assumed regarding the Neumann data $\bar{\boldsymbol{p}}$.

In addition to geometrical symmetry, one must assume that the material properties are also invariant under the symmetry group $\mathcal{S}$. Accordingly, the bilinear form $\mathcal{A}$ is said to have the *equivariance property* if:

$$\mathcal{A}(\boldsymbol{u}_s, \boldsymbol{v}_s) = \mathcal{A}(\boldsymbol{u}, \boldsymbol{v}) \qquad (\forall s \in \mathcal{S},\ \forall \boldsymbol{u}, \boldsymbol{v} \in \mathcal{V}^1) \tag{15}$$

(where $\boldsymbol{u}_s(\boldsymbol{x}) \equiv s\boldsymbol{u}(s^{-1}\boldsymbol{x})$) which is a straightforward adaptation of the definition proposed in [4]. Since S^1 is invariant under $\mathcal{S}$, the changes of variables $(\boldsymbol{x}, \tilde{\boldsymbol{x}}) \to (s\boldsymbol{x}, s\tilde{\boldsymbol{x}})$ imply:

$$B(\boldsymbol{x}, \tilde{\boldsymbol{x}}; \boldsymbol{u}_s, \boldsymbol{v}_s) = B(s\boldsymbol{x}, s\tilde{\boldsymbol{x}}; s\boldsymbol{u}, s\boldsymbol{v}) \qquad (\forall \boldsymbol{x}, \tilde{\boldsymbol{x}} \in S^1)$$

and (15) thus implies:

$$B(s\boldsymbol{x}, s\tilde{\boldsymbol{x}}; s\boldsymbol{u}, s\boldsymbol{v}) = B(\boldsymbol{x}, \tilde{\boldsymbol{x}}; \boldsymbol{u}, \boldsymbol{v}) \qquad (\forall s \in \mathcal{S}, \forall \boldsymbol{x}, \tilde{\boldsymbol{x}} \in S^1) \tag{16}$$

In fact, it is easy to check that:

$$B_{ikqs}(s\tilde{\boldsymbol{x}}, s\boldsymbol{x}) = s_{ij}s_{k\ell}s_{qr}s_{st}B_{j\ell rt}(\tilde{\boldsymbol{x}}, \boldsymbol{x})$$

$$A_{ik}(s\tilde{\boldsymbol{x}}, s\boldsymbol{x}) = s_{ij}s_{k\ell}A_{j\ell}(\tilde{\boldsymbol{x}}, \boldsymbol{x})$$

$$\{R_i[f_s]_k \, \mathrm{d}S\}(s\boldsymbol{x}) = s_{ij}s_{k\ell}\{R_j[f_\ell]_b \, \mathrm{d}S\}(\boldsymbol{x})$$

from (26_2) and the identities:

$$\boldsymbol{r}(s\tilde{\boldsymbol{x}}, s\boldsymbol{x}) = s\boldsymbol{r}(\tilde{\boldsymbol{x}}, \boldsymbol{x}) \qquad r(s\tilde{\boldsymbol{x}}, s\boldsymbol{x}) = r(\tilde{\boldsymbol{x}}, \boldsymbol{x})$$

where $\boldsymbol{r}(\tilde{\boldsymbol{x}}, \boldsymbol{x}) \equiv \boldsymbol{x} - \tilde{\boldsymbol{x}}$ and $r(\tilde{\boldsymbol{x}}, \boldsymbol{x}) = |\tilde{\boldsymbol{x}}, \boldsymbol{x}|$; then (16) follows easily.

An immediate and useful consequence of property (16) is:

$$B(s\boldsymbol{x}, \tilde{s}\tilde{\boldsymbol{x}}; s\boldsymbol{u}, \tilde{s}\boldsymbol{v}) = B(t\boldsymbol{x}, \tilde{\boldsymbol{x}}; t\boldsymbol{u}, \boldsymbol{v}) \qquad (\forall s, \tilde{s} \in \mathcal{S}) \quad \text{with } t = s\tilde{s}^{-1} \tag{17}$$

4 Using geometrical symmetry: the Abelian case

The present approach is based on the exploitation of some basic results from the the theory of linear representations of finite groups. In this respect, Abelian (or commutative, i.e. $\forall s, t \in \mathcal{S}$, $st = ts$) and non-Abelian symmetry groups lead to quite different formulations. In this section, $\mathcal{S}$ is a *commutative* finite group of order n; this includes the common cases of group P_m of symmetries w.r.t. m orthogonal planes (with $n = 2^m$ and $1 \leq m \leq d$) and the group $C_n = \{\mathrm{Id}, r, r^2, \ldots, r^{n-1}\}$ of cyclic symmetry generated by a rotation r of angle $2\pi/n$, with $2 \leq n$). The non-Abelian case is deferred to section 5.

Review of basic definitions [4,8,9]. Any finite Abelian group $\mathcal{S}$ of order n possess n *irreducible linear representations*, i.e. n applications $\rho_\nu \colon \mathcal{S} \to \mathbb{C}$ which satisfy the following relations:

$$|\rho_\nu(s)| = 1 \qquad \rho_\nu(st) = \rho_\nu(s)\rho_\nu(t) \qquad \rho_\nu(s^{-1}) = \rho_\nu^\star(s) \tag{18}$$

for any $s, t \in \mathcal{S}$, $1 \leq \nu \leq n$ ($z^\star$ denotes the complex conjugate of z), as well as the 'orthogonality relation':

$$\frac{1}{n}\sum_{s \in \mathcal{S}} \rho_\nu(s)\rho_\mu^\star(s) = \delta_{\mu\nu} \tag{19}$$

The ρ_ν are known for all usual groups; they are shown in table 1 for the groups P_1, P_2, P_3 while for the group C_n, one has

$$\rho_\nu(r) = \exp(2i\pi\nu/n) \qquad (\nu = 0, \ldots, n-1) \tag{20}$$

The ρ_ν: $\mathcal{S} \to \mathbb{C}$ can be view as a group isomorphism between $\mathcal{S}$ and $\mathrm{GL}(\mathbb{C})$, the multiplicative group of linear endomorphisms of $\mathbb{C}$, and are said to be *of degree one*. In contrast, when $\mathcal{S}$ is not commutative, some of the irreducible linear representations are necessarily of degree ≥ 2.

Any vector function $\boldsymbol{v}$ defined on S^1 then admits the decomposition [4]:

$$\boldsymbol{v} = \sum_{\nu=1}^{n} \boldsymbol{P}_\nu \boldsymbol{v} \tag{21}$$

where the linear operators $\boldsymbol{P}_\nu$ defined by:

$$\boldsymbol{v} \to [\boldsymbol{P}_\nu \boldsymbol{v}](\boldsymbol{x}) = \frac{1}{n} \sum_{t \in \mathcal{S}} \rho_\nu(t) t^{-1} \boldsymbol{v}(t\boldsymbol{x}) \tag{22}$$

are readily shown using (19) to be orthogonal projectors for the L^2 scalar product: if $\mathcal{V}$ is a space of functions defined on S^1, then one has

$$\int_{S^1} (\boldsymbol{P}_\mu \boldsymbol{v}).(\boldsymbol{P}_\nu \boldsymbol{v})^\star \, \mathrm{d}S = 0 \qquad \mu \neq \nu$$

and

$$\mathcal{V} = \bigoplus_{\nu=1}^{n} \boldsymbol{P}_\nu \mathcal{V}$$

Let $\boldsymbol{v}_\nu$ denote the restriction on C of $\boldsymbol{P}_\nu v$. Then, from the properties (18), it is easy to show that, for any $\boldsymbol{x} \in C$ and any $s \in \mathcal{S}$:

$$[\boldsymbol{P}_\nu \boldsymbol{v}](s\boldsymbol{x}) = \rho_\nu(s^{-1}) s \boldsymbol{v}_\nu(\boldsymbol{x}) \tag{23}$$

P_1	Id	s_1
ρ_1	+1	+1
ρ_2	+1	-1

P_2	Id	s_1	s_2	s_1s_2
ρ_1	+1	+1	+1	+1
ρ_2	+1	-1	+1	-1
ρ_3	+1	+1	-1	-1
ρ_4	+1	-1	-1	+1

P_3	Id	s_1	s_2	s_3	s_2s_3	s_1s_3	s_1s_2	$s_1s_2s_3$
ρ_1	+1	+1	+1	+1	+1	+1	+1	+1
ρ_2	+1	+1	+1	-1	-1	-1	+1	-1
ρ_3	+1	-1	+1	+1	+1	-1	-1	-1
ρ_4	+1	-1	+1	-1	-1	+1	-1	+1
ρ_5	+1	+1	-1	+1	-1	+1	-1	-1
ρ_6	+1	+1	-1	-1	+1	-1	-1	+1
ρ_7	+1	-1	-1	+1	-1	-1	+1	+1
ρ_8	+1	-1	-1	-1	+1	+1	+1	-1

Table 1. Irreducible representations for plane symmetries with respect to one, two and three coordinate planes (respective orders $n = 2, 4, 8$).

Moreover, let $I_s = \{\boldsymbol{x} \in C, s\boldsymbol{x} \in C\}$, i.e. I_s is the set of points of C whose images under a given s are in C (in fact, such points necessarily belong to ∂C for C to be actually a symmetry cell). Identity (23) then implies that the $\boldsymbol{v}_\nu$ obtained from a given $\boldsymbol{v} \in \mathcal{V}$ must satisfy the constraints:

$$\boldsymbol{v}_\nu(s\boldsymbol{x}) - \rho_\nu(s^{-1})s\boldsymbol{v}_\nu(\boldsymbol{x}) = 0 \qquad (\forall s \in \mathcal{S},\ \forall \boldsymbol{x} \in I_s) \tag{24}$$

Let $\mathcal{V}_\nu$ denote the set of functions defined on C for which (24) holds.

Finally, for any $\boldsymbol{x} \in C$ and any $s \in \mathcal{S}$, the value $\boldsymbol{v}(s\boldsymbol{x})$ of $\boldsymbol{v}$ at the image $s\boldsymbol{x}$ of $\boldsymbol{x}$ can be expressed in terms of the $\boldsymbol{v}_\nu$:

$$\boldsymbol{v}(s\boldsymbol{x}) = \sum_{\nu=1}^{n} \rho_\nu(s^{-1})s\boldsymbol{v}_\nu(\boldsymbol{x}) \tag{25}$$

Exploiting partial symmetry. Under the assumption of partial geometrical symmetry, one can map each $s(C)$ onto C by $\boldsymbol{x} \in C \to \boldsymbol{z} = s\boldsymbol{x} \in s(C)$ and express integrals over S^1 as sums of integrals over C, with the help of the identities

$$\mathrm{d}S(s\boldsymbol{x}) = \mathrm{d}S(\boldsymbol{x}) \qquad \boldsymbol{n}(s\boldsymbol{x}) = s[\boldsymbol{n}(\boldsymbol{x})] \tag{26}$$

which stem from the fact that s is an isometry. In particular, the bilinear form $\mathcal{A}(\boldsymbol{\phi}, \tilde{\boldsymbol{\phi}}^\star)$ and linear form $\mathcal{L}(\tilde{\boldsymbol{\phi}}^\star)$ defined by (11) and (10) take the form:

$$\mathcal{A}(\boldsymbol{\phi}, \tilde{\boldsymbol{\phi}}^\star) = \mathcal{B}(\boldsymbol{\phi}^1, \tilde{\boldsymbol{\phi}}^{1\star}) + \mathcal{C}(\boldsymbol{\phi}^1, \tilde{\boldsymbol{\phi}}^{2\star}) + \mathcal{C}^T(\boldsymbol{\phi}^2, \tilde{\boldsymbol{\phi}}^{1\star}) + \mathcal{D}(\boldsymbol{\phi}^2, \tilde{\boldsymbol{\phi}}^{2\star}) \tag{27}$$

$$\mathcal{L}(\tilde{\boldsymbol{\phi}}^\star) = \mathcal{F}(\tilde{\boldsymbol{\phi}}^{1\star}) + \mathcal{G}(\tilde{\boldsymbol{\phi}}^{2\star}) \tag{28}$$

where $\boldsymbol{\phi}^1, \tilde{\boldsymbol{\phi}}^{1\star} \in \mathcal{V} = [H^{1/2}(S^1)]^3$ and $\boldsymbol{\phi}^2, \tilde{\boldsymbol{\phi}}^{2\star} \in \mathcal{W} = [H^{1/2}(S^2)]^3$, and with

$$\mathcal{B}(\boldsymbol{\phi}^1, \tilde{\boldsymbol{\phi}}^{1\star}) = \sum_{s\in\mathcal{S}} \sum_{\tilde{s}\in\mathcal{S}} \int_C \int_C B(s\boldsymbol{x}, \tilde{s}\tilde{\boldsymbol{x}}; \boldsymbol{\phi}^1 \circ s, \tilde{\boldsymbol{\phi}}^{1\star} \circ \tilde{s})\, \mathrm{d}S_x\, \mathrm{d}S_{\tilde{x}} \tag{29}$$

$$\mathcal{C}(\boldsymbol{\phi}^1, \tilde{\boldsymbol{\phi}}^{2\star}) = \sum_{s\in\mathcal{S}} \int_{S^2} \int_C B(s\boldsymbol{x}, \tilde{\boldsymbol{x}}; \boldsymbol{\phi}^1 \circ s, \tilde{\boldsymbol{\phi}}^{2\star})\, \mathrm{d}S_x\, \mathrm{d}S_{\tilde{x}} \tag{30}$$

$$\mathcal{D}(\boldsymbol{\phi}^2, \tilde{\boldsymbol{\phi}}^{2\star}) = \sum_{s\in\mathcal{S}} \int_{S^2} \int_{S^2} B(\boldsymbol{x}, \tilde{\boldsymbol{x}}; \boldsymbol{\phi}^2, \tilde{\boldsymbol{\phi}}^{2\star})\, \mathrm{d}S_x\, \mathrm{d}S_{\tilde{x}} \tag{31}$$

and

$$\mathcal{F}(\tilde{\boldsymbol{\phi}}^{1\star}) = \frac{1}{2} \sum_{\tilde{s}\in\mathcal{S}} \int_C \bar{\boldsymbol{p}}(\tilde{s}\tilde{\boldsymbol{x}}).\tilde{\boldsymbol{\phi}}^1(\tilde{s}\tilde{\boldsymbol{x}})\, \mathrm{d}S_{\tilde{x}} \qquad \mathcal{G}(\tilde{\boldsymbol{\phi}}^{2\star}) = \frac{1}{2} \int_{S^2} \bar{\boldsymbol{p}}(\tilde{\boldsymbol{x}}).\tilde{\boldsymbol{\phi}}^2(\tilde{\boldsymbol{x}})\, \mathrm{d}S_{\tilde{x}}$$

Now, inserting the decomposition (25) for both $\boldsymbol{\phi}^1$ and $\tilde{\boldsymbol{\phi}}^1$ in $\mathcal{B}(\boldsymbol{\phi}^1, \tilde{\boldsymbol{\phi}}^{1\star})$ defined by (29), one has:

$$\mathcal{B}(\boldsymbol{\phi}^1, \tilde{\boldsymbol{\phi}}^{1\star}) = \sum_{\mu=1}^{n} \sum_{\nu=1}^{n} \sum_{s\in\mathcal{S}} \sum_{\tilde{s}\in\mathcal{S}} \rho_\nu^\star(\tilde{s}^{-1})\rho_\mu(s^{-1}) \int_C \int_C B(s\boldsymbol{x}, \tilde{s}\tilde{\boldsymbol{x}}; s\boldsymbol{\phi}_\nu, \tilde{s}\tilde{\boldsymbol{\phi}}_\mu^\star)\, \mathrm{d}S_x\, \mathrm{d}S_{\tilde{x}}$$

since $B(\boldsymbol{x},\tilde{\boldsymbol{x}};\boldsymbol{\phi}^1,\tilde{\boldsymbol{\phi}}^{1\star})$ is bilinear in $\boldsymbol{\phi}^1$ and $\tilde{\boldsymbol{\phi}}^{1\star}$. Next, using the change of variable $t=\tilde{s}^{-1}s$ (i.e. $s=\tilde{s}t$) together with property (18), one gets:

$$\mathcal{B}(\boldsymbol{\phi}^1,\tilde{\boldsymbol{\phi}}^{1\star})=\sum_{\mu=1}^{n}\sum_{\nu=1}^{n}\sum_{t\in\mathcal{S}}\sum_{\tilde{s}\in\mathcal{S}}\rho_\nu^\star(\tilde{s}^{-1})\rho_\mu(t^{-1})\rho_\mu(\tilde{s}^{-1})\int_C\int_C B(\tilde{s}t\boldsymbol{x},\tilde{s}\tilde{\boldsymbol{x}};\tilde{s}t\boldsymbol{\phi}_\nu,\tilde{s}\tilde{\boldsymbol{\phi}}_\mu^\star)\,\mathrm{d}S_x\,\mathrm{d}S_{\tilde{x}}$$

The equivariance property (17) implies that:

$$\int_C\int_C B(\tilde{s}t\boldsymbol{x},\tilde{s}\tilde{\boldsymbol{x}};s\boldsymbol{\phi}_\nu,\tilde{s}\tilde{\boldsymbol{\phi}}_\mu^\star)\,\mathrm{d}S_x\,\mathrm{d}S_{\tilde{x}}=\mathcal{B}_t(\boldsymbol{\phi}_\nu,\tilde{\boldsymbol{\phi}}_\mu^\star)\tag{32}$$

having put

$$\mathcal{B}_t(\boldsymbol{u},\boldsymbol{v})=\int_C\int_C B(t\boldsymbol{x},\tilde{\boldsymbol{x}};t\boldsymbol{u},\boldsymbol{v})\,\mathrm{d}S_x\,\mathrm{d}S_{\tilde{x}}\tag{33}$$

Then, by virtue of the orthogonality property (19):

$$\begin{aligned}\mathcal{B}(\boldsymbol{\phi}^1,\tilde{\boldsymbol{\phi}}^{1\star})&=\sum_{\mu=1}^{n}\sum_{\nu=1}^{n}\sum_{t\in\mathcal{S}}\rho_\mu(t^{-1})\Big\{\sum_{\tilde{s}\in\mathcal{S}}\rho_\nu(\tilde{s})\rho_\mu(\tilde{s}^{-1})\Big\}\mathcal{B}_t(\boldsymbol{\phi}_\nu,\tilde{\boldsymbol{\phi}}_\mu^\star)\\&=\sum_{\nu=1}^{n}\Big\{n\sum_{t\in\mathcal{S}}\rho_\nu(t^{-1})\mathcal{B}_t(\boldsymbol{\phi}_\nu,\tilde{\boldsymbol{\phi}}_\nu^\star)\Big\}\\&\equiv\sum_{\nu=1}^{n}\mathcal{B}_\nu(\boldsymbol{\phi}_\nu,\tilde{\boldsymbol{\phi}}_\nu^\star)\end{aligned}\tag{34}$$

The bilinear form $\mathcal{B}(\boldsymbol{\phi}^1,\tilde{\boldsymbol{\phi}}^{1\star})$ is thus seen to have been reduced in *block-diagonal* form.

One establishes in a similar way the decompositions:

$$\begin{aligned}\mathcal{C}(\boldsymbol{\phi}^2,\tilde{\boldsymbol{\phi}}^{1\star})&=\sum_{\nu=1}^{n}\Big\{\sum_{\tilde{s}\in\mathcal{S}}\rho_\nu(\tilde{s})\int_C\int_{S^2}B(\boldsymbol{x},\tilde{s}\tilde{\boldsymbol{x}};\boldsymbol{\phi}^2,\tilde{s}\tilde{\boldsymbol{\phi}}^\star)\,\mathrm{d}S_x\,\mathrm{d}S_{\tilde{x}}\Big\}\\&\equiv\sum_{\nu=1}^{n}\mathcal{C}_\nu(\boldsymbol{\phi}^2,\tilde{\boldsymbol{\phi}}_\nu^\star)\end{aligned}\tag{35}$$

$$\mathcal{F}(\tilde{\boldsymbol{\phi}}^{1\star})=\sum_{\nu=1}^{n}\frac{1}{2}\int_C[\boldsymbol{P}_\nu\bar{\boldsymbol{p}}].\tilde{\boldsymbol{\phi}}_\nu^\star\,\mathrm{d}S\equiv\sum_{\nu=1}^{n}\mathcal{F}_\nu(\tilde{\boldsymbol{\phi}}_\nu^\star)\tag{36}$$

Gathering results (34) (35) and (36), the initial integral equation (9) reduces to a set of SGBIE problems of the form:

$$\text{Find }\boldsymbol{\phi}_\nu\in\mathcal{V}_\nu,\ \boldsymbol{\phi}^2\in\mathcal{W};\ \forall\tilde{\boldsymbol{\phi}}_\nu\in\mathcal{V}_\nu,\tilde{\boldsymbol{\phi}}^2\in\mathcal{W}$$
$$\begin{cases}\mathcal{B}_\nu(\boldsymbol{\phi}_\nu,\tilde{\boldsymbol{\phi}}_\nu^\star)+\mathcal{C}_\nu(\boldsymbol{\phi}^2,\tilde{\boldsymbol{\phi}}_\nu^\star)=\mathcal{F}_\nu(\tilde{\boldsymbol{\phi}}_\nu^\star) & (1\le\nu\le n)\\ \displaystyle\sum_{\nu=1}^{n}\mathcal{C}_\nu^T(\boldsymbol{\phi}_\nu,\tilde{\boldsymbol{\phi}}^{2\star})+\mathcal{D}(\boldsymbol{\phi}^2,\tilde{\boldsymbol{\phi}}^{2\star})=\mathcal{G}(\tilde{\boldsymbol{\phi}}^{2\star})\end{cases}\tag{37}$$

5 Using geometrical symmetry: the non-Abelian case

In this section, $\mathcal{S}$ is a non-Abelian finite group of order n, i.e. there exist $s, t \in \mathcal{S}$ such that $st \neq ts$. This includes the important practical case of the *dihedral symmetry group* D_m, i.e. the group of order $n = 2m$ of the affine transformations that leave a regular m-gon unchanged.

Review of basic results [4,8,9]. Here, the irreducible representations ρ_ν of $\mathcal{S}$ are of integer degree $d_\nu \geq 1$:

$$\rho_\nu \ : \ s \in \mathcal{S} \to \rho_\nu(s) \in \mathrm{GL}(\mathbb{C}^{d_\nu})$$

i.e. each $\rho_\nu(s)$ is a linear endomorphism of a d_ν-dimensional complex vector space; moreover, the number $R(\mathcal{S})$ of such representations and their degrees d_ν are such that at least one of them is ≥ 2 and:

$$\sum_{\nu=1}^{R(\mathcal{S})} d_\nu^2 = n$$

The properties of the irreducible representations ρ_ν include the preservation of group structure:

$$\rho_\nu^{ij}(st) = \sum_{k=1}^{d_\nu} \rho_\nu^{ik}(s)\rho_\nu^{kj}(t) \quad (\forall s, t \in \mathcal{S}) \qquad \rho_\nu^{ij}(1) = \delta_{ij} \tag{38}$$

which implies in particular, since ρ_ν is unitary, that:

$$\rho_\nu^{ij}(s^{-1}) = [\rho_\nu^{ji}(s)]^\star \tag{39}$$

and the 'orthogonality relation':

$$\frac{d_\nu}{n} \sum_{s \in \mathcal{S}} \rho_\nu^{ij}(s)[\rho_\mu^{k\ell}(s)]^\star = \delta_{ik}\delta_{j\ell}\delta_{\mu\nu} \tag{40}$$

	Id	r	r^2	s	sr	sr^2
ρ_1	+1	+1	+1	+1	+1	+1
ρ_2	+1	+1	+1	-1	-1	-1
ρ_3^{11}	1	j	j^2	0	0	0
ρ_3^{21}	0	0	0	1	j	j^2
ρ_3^{12}	0	0	0	1	j^2	j
ρ_3^{22}	1	j^2	j	0	0	0

(with $j = \exp(2i\pi/3)$)

Table 2. Irreducible representations for dihedral symmetry $\mathcal{S} = D_3$.

A space $\mathcal{V}$ of vector functions defined on S_1 is decomposed into orthogonal subspaces [4]:

$$\mathcal{V} = \bigoplus_{j,\nu} \boldsymbol{P}_\nu^{jj} \mathcal{V} \tag{41}$$

with the projectors $\boldsymbol{P}_\nu^{ij}$ defined by:

$$[\boldsymbol{P}_\nu^{ij} \boldsymbol{v}](\boldsymbol{x}) = \frac{d_\nu}{n} \sum_{t \in \mathcal{S}} \rho_\nu^{ji}(t) t^{-1} \boldsymbol{v}(t\boldsymbol{x})$$

From this definition and the properties of the representations, one has for any $s \in \mathcal{S}$ and $\boldsymbol{v} \in \mathcal{V}$:

$$[\boldsymbol{P}_\nu^{ij} \boldsymbol{u}](s\boldsymbol{x}) = \sum_{k=1}^{d_\nu} \rho_\nu^{ki}(s^{-1}) s [\boldsymbol{P}_\nu^{kj} \boldsymbol{u}](\boldsymbol{x}) \tag{42}$$

Hence, for a given $s \in \mathcal{S}$ and a point $\boldsymbol{x} \in C$, the value $\boldsymbol{v}(s\boldsymbol{x})$ of $\boldsymbol{v}$ at the images of $\boldsymbol{x}$ can be expressed by virtue of (41) and (42) in terms of the restriction $\boldsymbol{v}_\nu^{kj}$ on C of the projections $\boldsymbol{P}_\nu^{kj} \boldsymbol{v}$:

$$\boldsymbol{v}(s\boldsymbol{x}) = \sum_{\mu=1}^{R(\mathcal{S})} \sum_{j,k=1}^{d_\mu} \rho_\mu^{kj}(s^{-1}) s \boldsymbol{v}_\mu^{kj}(\boldsymbol{x}) \tag{43}$$

Moreover, let again $I_s = \{\boldsymbol{x} \in \partial C, s\boldsymbol{x} \in \partial C\}$. It is then easy to show, from (42), that the d_ν-uple $\{\boldsymbol{v}_\nu^{ij}, 1 \le i \le d_\nu\}$ of functions defined on C are subject to the following constraints:

$$\boldsymbol{v}_\nu^{ij}(\boldsymbol{x}) - \sum_{k=1}^{d_\nu} \rho_\nu^{ki}(s^{-1}) s \boldsymbol{v}_\nu^{kj}(s^{-1}\boldsymbol{x}) = 0 \qquad (\forall \boldsymbol{x} \in I_s) \tag{44}$$

(note that the constraint does not depend on the rightmost index j). Accordingly, for the non-Abelian case, let $\mathcal{V}_\nu$ denote the set of d_ν-tuples of functions $\boldsymbol{v}^\ell$ $(1 \le \ell \le d_\nu)$ defined on C and such that any pair $(\boldsymbol{v}^i, \boldsymbol{v}^k)$ is linked through the constraints (44) (with the index j omitted).

Exploiting partial symmetry. Again, the decomposition (27) holds. Inserting the decomposition (43) for both ϕ^1 and $\tilde{\phi}^1$ in $\mathcal{B}(\phi^1, \tilde{\phi}^{1\star})$ defined by (27), one obtains:

$$\mathcal{B}(\phi, \tilde{\phi}^\star) = \sum_{\mu=1}^{R(\mathcal{S})} \sum_{\nu=1}^{R(\mathcal{S})} \sum_{j,k=1}^{d_\mu} \sum_{i,\ell=1}^{d_\nu} \sum_{s \in \mathcal{S}} \sum_{\tilde{s} \in \mathcal{S}} \rho_\nu^{\ell i \star}(\tilde{s}^{-1}) \rho_\mu^{kj}(s^{-1}) \int_C \int_C B(s\boldsymbol{x}, \tilde{s}\tilde{\boldsymbol{x}}; s\boldsymbol{\phi}_\nu^{kj}, \tilde{s}\tilde{\boldsymbol{\phi}}_\mu^{\star \ell i}) \, \mathrm{d}S_x \, \mathrm{d}S_{\tilde{x}}$$

Then, making the change of variable $s = \tilde{s}t$ and using (38), (39):

$$\mathcal{B}(\phi, \tilde{\phi}^\star) = \sum_{\mu=1}^{R(\mathcal{S})} \sum_{\nu=1}^{R(\mathcal{S})} \sum_{j,k=1}^{d_\mu} \sum_{\ell,i=1}^{d_\nu} \sum_{m=1}^{d_\nu} \sum_{t\in\mathcal{S}} \sum_{\tilde{s}\in\mathcal{S}} \rho_\nu^{i\ell}(\tilde{s}) \rho_\mu^{km}(t^{-1}) \rho_\mu^{mj}(\tilde{s}^{-1}) \int_C \int_C B(\tilde{s}t\boldsymbol{x}, \tilde{s}\tilde{\boldsymbol{x}}; s\phi_\nu^{kj}, \tilde{s}\tilde{\phi}_\mu^{\star\ell i}) \, \mathrm{d}S_x \, \mathrm{d}S_{\tilde{x}}$$

so that, using (40) and the equivariance property (32), one obtains:

$$\begin{aligned}
\mathcal{B}(\phi, \tilde{\phi}^\star) &= \sum_{\mu=1}^{R(\mathcal{S})} \sum_{\nu=1}^{R(\mathcal{S})} \sum_{j,k=1}^{d_\mu} \sum_{\ell,i=1}^{d_\nu} \sum_{m=1}^{d_\nu} \sum_{t\in\mathcal{S}} \Big\{ \sum_{\tilde{s}\in\mathcal{S}} \rho_\nu^{i\ell}(\tilde{s}) \rho_\mu^{mj}(\tilde{s}^{-1}) \Big\} \\
&\qquad \rho_\mu^{km}(t^{-1}) \mathcal{B}_t(\phi_\nu^{kj}, \tilde{\phi}_\mu^{\star\ell i}) \\
&= \sum_{\mu=1}^{R(\mathcal{S})} \sum_{\nu=1}^{R(\mathcal{S})} \sum_{j,k=1}^{d_\mu} \sum_{\ell,i=1}^{d_\nu} \sum_{m=1}^{d_\nu} \sum_{t\in\mathcal{S}} \frac{n}{d_\nu} \rho_\mu^{km}(t^{-1}) \delta_{ij} \delta_{\ell m} \delta_{\mu\nu} \mathcal{B}_t(\phi_\nu^{kj}, \tilde{\phi}_\mu^{\star\ell i}) \\
&= \sum_{\nu=1}^{R(\mathcal{S})} \sum_{i=1}^{d_\mu} \sum_{k,\ell=1}^{d_\mu} \Big\{ \sum_{t\in\mathcal{S}} \frac{n}{d_\nu} \rho_\nu^{k\ell}(t^{-1}) \Big\} \mathcal{B}_t(\phi_\nu^{kj}, \tilde{\phi}_\nu^{\star\ell i}) \\
&\equiv \sum_{\nu=1}^{R(\mathcal{S})} \sum_{k,\ell=1}^{d_\nu} \sum_{i=1}^{d_\nu} \mathcal{B}_\nu^{k\ell}(\phi_\nu^{ki}, \tilde{\phi}_\nu^{\star\ell i}) \qquad (45)
\end{aligned}$$

One establishes in a similar way the decompositions:

$$\begin{aligned}
\mathcal{C}(\phi^2, \tilde{\phi}^{1\star}) &= \sum_{\nu=1}^{R(\mathcal{S})} \Big\{ \sum_{\ell,i=1}^{d_\nu} \sum_{\tilde{s}\in\mathcal{S}} \rho_\nu^{i\ell}(\tilde{s}) \int_C \int_{S^2} B(\boldsymbol{x}, \tilde{s}\tilde{\boldsymbol{x}}; \phi^2, \tilde{s}\tilde{\phi}_\nu^{\star\ell i}) \, \mathrm{d}S_x \, \mathrm{d}S_{\tilde{x}} \Big\} \\
&\equiv \sum_{\nu=1}^{R(\mathcal{S})} \sum_{\ell,i=1}^{d_\nu} \mathcal{C}_\nu^{\ell i}(\phi^2, \tilde{\phi}_\nu^{\star\ell i}) \qquad (46)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{F}(\tilde{\phi}^{1\star}) &= \frac{1}{2} \sum_{\nu=1}^{R(\mathcal{S})} \sum_{t\in\mathcal{S}} \sum_{i,\ell=1}^{d_\nu} \rho_\nu^{i\ell}(t) \int_C \bar{\boldsymbol{p}}(t\tilde{\boldsymbol{x}}).t\tilde{\phi}_\nu^{\star\ell i}(\tilde{\boldsymbol{x}}) \, \mathrm{d}S_{\tilde{x}} \\
&= \frac{1}{2} \sum_{\nu=1}^{R(\mathcal{S})} \frac{n}{d_\nu} \sum_{i,\ell=1}^{d_\nu} \int_C [\boldsymbol{P}_\nu^{\ell i} \bar{\boldsymbol{p}}](\tilde{\boldsymbol{x}}).\tilde{\phi}_\nu^{\star\ell i}(\tilde{\boldsymbol{x}}) \, \mathrm{d}S_{\tilde{x}} \\
&= \sum_{\nu} \sum_{i,\ell=1}^{d_\nu} \mathcal{F}_\nu^{\ell i}(\tilde{\phi}_\nu^{\star\ell i}) \qquad (47)
\end{aligned}$$

Gathering results (45), (46) and (47), the initial integral equation (9) reduces to a set of SGBIE problems of the form:

$$\text{Find } \{\phi_\nu^{ki}\}_{1\le k\le d_\nu} \in \mathcal{V}_\nu,\ \phi^2 \in \mathcal{W};\ \forall \{\tilde\phi_\nu^{\ell}\}_{1\le \ell\le d_\nu} \in \mathcal{V}_\nu, \tilde\phi^2 \in \mathcal{W}$$

$$\begin{cases} \displaystyle\sum_{k,\ell=1}^{d_\nu} \mathcal{B}_\nu^{k\ell}(\phi_\nu^{ki}, \tilde\phi_\nu^{\star\ell}) + \sum_{\ell=1}^{d_\nu} \mathcal{C}_\nu^{\ell i}(\phi^2, \tilde\phi_\nu^{\star\ell}) = \sum_{\ell=1}^{d_\nu} \mathcal{F}_\nu^{\ell i}(\tilde\phi_\nu^{\star\ell}) \\ \qquad\qquad (1\le \nu\le R(\mathcal{S}), 1\le i\le d_\nu) \\ \displaystyle\sum_{\nu=1}^{R(\mathcal{S})}\sum_{\ell,i=1}^{d_\nu} [\mathcal{C}_\nu^{\ell i}]^T(\phi_\nu^{\ell i}, \tilde\phi^{2\star}) + \mathcal{D}(\phi^2, \tilde\phi^{2\star}) = \mathcal{G}(\tilde\phi^2) \end{cases} \tag{48}$$

6 Calculation of field values at interior points

Displacement values at selected interior points $\tilde{\boldsymbol{x}}$ can be computed explicitly using the representation formula (7) once the density $\boldsymbol{\phi}$ is known, and related quantities (strains, stresses) at $\tilde{\boldsymbol{x}}$ can be easily obtained as well.

Let $\boldsymbol{u} = \boldsymbol{u}^1 + \boldsymbol{u}^2$ in (7), where $\boldsymbol{u}^I$ is the contribution of the integration over S^I. Exploiting symmetry affects the computation of $\boldsymbol{u}^1$. Inserting the decomposition (43) into (7) and following the now usual pattern, one obtains:

$$u_k^1(\tilde s\tilde{\boldsymbol{x}}) = \sum_{s\in\mathcal{S}}\sum_{\nu=1}^{R(\mathcal{S})}\sum_{a,b=1}^{d_\nu} \rho_\nu^{ab}(s^{-1}) \int_C T_i^k(\tilde s\tilde{\boldsymbol{x}}, s\boldsymbol{x}) s_{ij} [\phi_\nu^{ab}]_j(\boldsymbol{x})\, \mathrm{d}S_x$$

Then, putting again $s = \tilde s t$ and using the equivariance property (16), which holds also for the kernel T_i^k, one obtains:

$$u_k^1(\tilde s\tilde{\boldsymbol{x}}) = \sum_{s\in\mathcal{S}}\sum_{\nu=1}^{R(\mathcal{S})}\sum_{a,b,c=1}^{d_\nu} \rho_\nu^{cb}(\tilde s^{-1}) \left\{ \rho_\nu^{ac}(t^{-1}) \int_C s_{k\ell} T_i^\ell(t^{-1}\tilde{\boldsymbol{x}}, \boldsymbol{x}) [\phi_\nu^{ab}]_i(\boldsymbol{x})\, \mathrm{d}S_x \right\} \tag{49}$$

(note that $s^t s = s s^t = Id$). A close examination of (49) thus reveals that, for a given interior point $\boldsymbol{x}$, the same numerical quadrature effort is required by (7) and (49). However, the terms within curly brackets in (49) do not depend on $\tilde s$, so that the same numerical integrations can be reused (with different weights $\rho_\nu^{cb}(\tilde s^{-1})$) to evaluate $\boldsymbol{u}^1$ at all the n images of $\tilde{\boldsymbol{x}}$ under $\mathcal{S}$.

7 Computational implications

7.1 Reduction of numerical quadrature effort

It is obvious from (56) that a reduction of both setup and solution computational efforts results from the block-diagonalization of the operator $\mathcal{B}$. The

numerical quadrature effort consists in evaluating discretized versions of

$$\mathcal{B}_t(\boldsymbol{u},\boldsymbol{v}) \equiv \int_C \int_C B(t\boldsymbol{x},\tilde{\boldsymbol{x}};t\boldsymbol{u},\boldsymbol{v})\,\mathrm{d}S_x\,\mathrm{d}S_{\tilde{x}}$$

for all $t \in \mathcal{S}$ instead of

$$\mathcal{B}(\boldsymbol{u},\boldsymbol{v}) \equiv \int_{S^1} \int_{S^1} B(\boldsymbol{x},\tilde{\boldsymbol{x}};\boldsymbol{u},\boldsymbol{v})\,\mathrm{d}S_x\,\mathrm{d}S_{\tilde{x}}$$

Moreover, a useful consequence of equivariance (16) and the symmetry properties (14) of B is:

$$\mathcal{B}_t(\boldsymbol{u},\boldsymbol{v}) = \mathcal{B}_{t^{-1}}(\boldsymbol{v},\boldsymbol{u}) \tag{50}$$

From this identity and the symmetry of the original bilinear form $\mathcal{B}$, the block-diagonalized $\mathcal{B}$ is seen to entail a numerical quadrature effort n times smaller than the original $\mathcal{B}$.

7.2 Symmetry properties of the matrix equations

Abelian case. The irreducible representations ρ_ν are usually complex-valued functions over $\mathcal{S}$ (e.g. (20) for cyclic groups). In that case, it can be shown that the ρ_ν can be associated by conjugate pairs, i.e. that for any ν such that ρ_ν is complex-valued, there exists $\nu^\star$ such that $\rho_{\nu^\star}(\mathcal{S}) = \rho_\nu^\star(\mathcal{S})$. In that case, from (24), $\boldsymbol{v} \in \mathcal{V}_\nu \Rightarrow \boldsymbol{v}^\star \in \mathcal{V}_{\nu^\star}$. Besides, using (50), one can show that:

$$\mathcal{B}_\nu(\boldsymbol{u},\boldsymbol{v}) = [\mathcal{B}_{\nu^\star}]^T(\boldsymbol{u},\boldsymbol{v}) \qquad \text{complex-valued } \rho_\nu \tag{51}$$

i.e. that, although $\mathcal{B}$ is symmetric, the $\mathcal{B}_\nu(\boldsymbol{u},\boldsymbol{v})$ are not individually symmetric, but have a 'reciprocal symmetry'. In some cases, including the very common one of symmetry with respect to coordinate planes, the ρ_ν are real-valued (see table 1); then, $\boldsymbol{v} \in \mathcal{V}_\nu \Rightarrow \boldsymbol{v}^\star \in \mathcal{V}_\nu$ and the $\mathcal{B}_\nu$ are symmetric:

$$\mathcal{B}_\nu(\boldsymbol{u},\boldsymbol{v}) = [\mathcal{B}_\nu]^T(\boldsymbol{u},\boldsymbol{v}) \qquad \text{real-valued } \rho_\nu \tag{52}$$

Non-Abelian case. The symmetry properties of the matrices associated with degree one representations are as in the Abelian case. Otherwise, one has from (45):

$$\begin{aligned}\mathcal{B}_\nu^{k\ell}(\boldsymbol{u},\boldsymbol{v}) = \frac{n}{d_\nu} \sum_{t\in\mathcal{S}'} \rho_\nu^{k\ell}(t^{-1})\mathcal{B}_t(\boldsymbol{u},\boldsymbol{v}) \\ + \frac{n}{d_\nu} \sum_{t\in\mathcal{S}''} \left\{ \rho_\nu^{k\ell}(t)\mathcal{B}_t(\boldsymbol{v},\boldsymbol{u}) + \rho_\nu^{\star\ell k}(t)\mathcal{B}_t(\boldsymbol{u},\boldsymbol{v}) \right\}\end{aligned} \tag{53}$$

where $\mathcal{S}' = \{t \in \mathcal{S}, t = t^{-1}\}$ and $\mathcal{S}'' \subset \mathcal{S}$ is chosen such that $\mathcal{S}' \cap \mathcal{S}'' = \emptyset$ and $\mathcal{S} = \mathcal{S}' \cup \mathcal{S}'' \cup \{t^{-1}, t \in \mathcal{S}''\}$. First, as a consequence of (50):

$$\mathcal{B}_t(\boldsymbol{u},\boldsymbol{v}) = \mathcal{B}_{t^{-1}}(\boldsymbol{u},\boldsymbol{v}) \qquad (\text{if } t = t^{-1}) \tag{54}$$

Also, whenever the irreducible representations $\rho_\nu^{k\ell}$ and $\rho_\nu^{\ell k}$ are real-valued, one has

$$\rho_\nu^{k\ell}(t)\mathcal{B}_t(\boldsymbol{v},\boldsymbol{u}) + \rho_\nu^{\ell k}(t)\mathcal{B}_t(\boldsymbol{u},\boldsymbol{v}) = [\rho_\nu^{\ell k}(t)\mathcal{B}_t(\boldsymbol{v},\boldsymbol{u}) + \rho_\nu^{k\ell}(t)\mathcal{B}_t(\boldsymbol{u},\boldsymbol{v})]^T$$

Besides, from (44):

$$\{\boldsymbol{v}^\ell\}_{1\le\ell\le d_\nu} \in \mathcal{V}_\nu \Rightarrow \{\boldsymbol{v}^{\ell\star}\}_{1\le\ell\le d_\nu} \in \mathcal{V}_\nu$$

Thus, if all $\rho_\nu^{k\ell}(t)$ are real-valued for a given ν, the bilinear form

$$\sum_{k,\ell=1}^{d_\nu} \mathcal{B}_\nu^{k\ell}(\boldsymbol{u}^k, \boldsymbol{v}^{\star\ell})$$

(where $\{\boldsymbol{u}^k\}_{1\le\ell\le d_\nu} \in \mathcal{V}_\nu$ and $\{\boldsymbol{v}^\ell\}_{1\le\ell\le d_\nu} \in \mathcal{V}_\nu$) is symmetric. On the other hand, if some $\rho_\nu^{k\ell}(t)$ are complex-valued, it is not clear how to establish the symmetry of the above bilinear form from the general properties of the representations.

Besides, it is also important to note that in (48) the same bilinear form $\sum_{k,\ell=1}^{d_\nu} \mathcal{B}_\nu^{k\ell}(\boldsymbol{u}^k, \boldsymbol{v}^{\star\ell})$ appears d_ν times; it should thus be assembled and factored once and then used to solve for all d_ν-uples $\{\boldsymbol{\phi}_\nu^{ki}\}_{1\le k\le d_\nu}$ with $i = 1,\dots,d_\nu$.

Example: the dihedral group $\mathcal{S} = D_3$. Let Σ_r and Σ_s denote two distinct planes in $\mathbb{R}^3$ which intersect along the coordinate line Ox_3 and such that the angle (Σ_r, Σ_s) is $\pi/3$. The dihedral group D_3, which is the simplest non-Abelian one, is generated by the symmetry s w.r.t. Σ_s and the $2\pi/3$ rotation r around Ox_3. Its irreducible representations are shown in Table 2; one has $R(\mathcal{S}) = 3$, $d_1 = d_2 = 1$, $d_3 = 2$.

For the case $\nu = 3$, more explicit expression for the $\mathcal{B}_3^{k\ell}$ are obtained as follows, using Table 2 and (50):

$$\begin{aligned}
\mathcal{B}_3^{11}(\boldsymbol{u}^1, \boldsymbol{v}^{1\star}) &= \mathcal{B}_{Id}(\boldsymbol{u}^1, \boldsymbol{v}^{1\star}) + j\mathcal{B}_r(\boldsymbol{u}^1, \boldsymbol{v}^{1\star}) + j^2\mathcal{B}_r(\boldsymbol{v}^{1\star}, \boldsymbol{u}^1)\\
\mathcal{B}_3^{21}(\boldsymbol{u}^2, \boldsymbol{v}^{1\star}) &= \mathcal{B}_s(\boldsymbol{u}^2, \boldsymbol{v}^{1\star}) + j\mathcal{B}_{sr}(\boldsymbol{u}^2, \boldsymbol{v}^{1\star}) + j^2\mathcal{B}_{sr^2}(\boldsymbol{u}^2, \boldsymbol{v}^{1\star})\\
\mathcal{B}_3^{12}(\boldsymbol{u}^1, \boldsymbol{v}^{2\star}) &= \mathcal{B}_s(\boldsymbol{u}^1, \boldsymbol{v}^{2\star}) + j^2\mathcal{B}_{sr}(\boldsymbol{u}^1, \boldsymbol{v}^{2\star}) + j\mathcal{B}_{sr^2}(\boldsymbol{u}^1, \boldsymbol{v}^{2\star})\\
\mathcal{B}_3^{22}(\boldsymbol{u}^2, \boldsymbol{v}^{2\star}) &= \mathcal{B}_{Id}(\boldsymbol{u}^2, \boldsymbol{v}^{2\star}) + j^2\mathcal{B}_r(\boldsymbol{u}^2, \boldsymbol{v}^{2\star}) + j\mathcal{B}_r(\boldsymbol{v}^{2\star}, \boldsymbol{u}^2)
\end{aligned}$$

It appears that $\mathcal{B}_3^{11}(\boldsymbol{u},\boldsymbol{v}) = .\mathcal{B}_3^{22}(\boldsymbol{v},\boldsymbol{u})$; besides, since $s = s^{-1}$, $sr = (sr)^{-1}$ and $sr^2 = (sr^2)^{-1}$, (54) implies that $\mathcal{B}_3^{21}(\boldsymbol{u},\boldsymbol{v})$ and $\mathcal{B}_3^{12}(\boldsymbol{u},\boldsymbol{v})$ are symmetric (in both cases disregarding for the moment the constraints (44)).

In addition, the constraints (44) reduce to two independent restrictions, as follows. If $\boldsymbol{x} \in \Sigma_s$, $\boldsymbol{x} = s\boldsymbol{x}$, thus:

$$I_s = \partial C \cup \Sigma_s \qquad \text{with} \quad \boldsymbol{u}^2(\boldsymbol{x}) = s\boldsymbol{u}^1(\boldsymbol{x})$$

whereas if $\boldsymbol{x} \in \Sigma_s$, $\boldsymbol{x} = sr\boldsymbol{x}$, which yields:

$$I_{sr} = \partial C \cup \Sigma_r \qquad \text{with} \quad j^2\boldsymbol{u}^2(\boldsymbol{x}) = sr\boldsymbol{u}^1(\boldsymbol{x})$$

From these, it is easy to infer that

$$\{\boldsymbol{v}^1, \boldsymbol{v}^2\} \in \mathcal{V}_3 \Rightarrow \{\boldsymbol{v}^{2\star}, \boldsymbol{v}^{1\star}\} \in \mathcal{V}_3 \tag{55}$$

Hence, the one-to-one substitution $\{\boldsymbol{v}^{2\star}, \boldsymbol{v}^{1\star}\} \in \mathcal{V}_3 = \{\boldsymbol{w}^1, \boldsymbol{w}^2\} \in \mathcal{V}_3$ can be made, and the contributions for $\nu = 3$ in (48) are recast into a form which is symmetric in $(\{\boldsymbol{\phi}^1, \boldsymbol{\phi}^2\}, \{\tilde{\boldsymbol{\phi}}^1, \tilde{\boldsymbol{\phi}}^2\})$:

$$\sum_{k,\ell=1}^{2} \mathcal{B}_3^{k\bar{\ell}}(\boldsymbol{\phi}_3^{ki}, \tilde{\boldsymbol{\phi}}_3^{\ell}) + \sum_{\ell=1}^{2} C_3^{i\bar{\ell}}(\boldsymbol{\phi}^2, \tilde{\boldsymbol{\phi}}_3^{\ell}) = \sum_{\ell=1}^{2} \mathcal{F}_3^{\bar{\ell}}(\tilde{\boldsymbol{\phi}}^{\ell}) \qquad (i = 1, 2;\ \bar{\ell} = 3 - \ell)$$

Similar conclusions can be reached for all dihedral symmetry groups D_m.

7.3 Reduction in solution time

Let N and γN denote the number of degrees of freedom supported by the BEM discretization of S^1 and S^2 respectively. The system of equations (37) or (48) takes the general form:

$$\begin{bmatrix} \boldsymbol{B} & \boldsymbol{C} \\ \boldsymbol{C}^T & \boldsymbol{D} \end{bmatrix} \begin{Bmatrix} \boldsymbol{\phi}^1 \\ \boldsymbol{\phi}^2 \end{Bmatrix} = \begin{bmatrix} \boldsymbol{F} \\ \boldsymbol{G} \end{bmatrix} \tag{56}$$

where the matrix $\boldsymbol{B}$ is block-diagonal: $\boldsymbol{B} = \mathrm{Diag}(\boldsymbol{B}_\nu^i)$ $(1 \le \nu \le R(\mathcal{S}), 1 \le i \le d_\nu)$. Each block $\boldsymbol{B}_\nu^i$ is approximately of size $(d_\nu/n) \times N$ (the constraints (44) causing slight variations in size for the same value of d_ν). Besides, as mentioned before, all blocks $\boldsymbol{B}_\nu^i$ $(1 \le i \le d_\nu)$ are the same for a given ν.

Solving the original (symmetric) SGBEM system thus entails a $T = O((1+\gamma)^3 N^3/6)$ solution time. For solving the system (56), one must first solve the block-diagonal part, whereby each $\{\boldsymbol{\phi}_\nu^{k\ell}\}$ is expressed in terms of $\{\boldsymbol{F}_\nu^\ell\}$ and $\{\boldsymbol{\phi}^2\}$, and then substitute these results into the remaining part

γ	0	0.1	0.2	0.5	1	2
R $(\mathcal{S} = P_1)$	0.25	0.3238	0.3924	0.5556	0.7188	0.8611
R $(\mathcal{S} = P_2)$	0.0625	0.1266	0.1971	0.3889	0.6016	0.7986
R $(\mathcal{S} = D_3)$	0.0463	0.1206	0.1973	0.3964	0.6100	0.8042
R $(\mathcal{S} = P_3)$	0.015625	0.06320	0.1265	0.3194	0.5488	0.7691

Table 3. Expected asymptotic ratios R of solution CPU time with and without exploitation of partial symmetry, for some groups and various values of γ (ratio of numbers of DOFs on the surfaces S^1 and S^2)

of the system in order to build and solve a final system with a (symmetric) $\gamma N \times \gamma N$ matrix. The estimated time T_s for solving (56) (retaining only the $O(N^3)$ contributions) is

$$T_s = O\Big(\frac{N^3}{6}\big[\gamma^3 + 3\gamma^2 + \big(\frac{3\gamma}{n^2} + \frac{1}{n^3}\big)\sum_{\nu=1}^{R(n)} d_\nu^3\big]\Big)$$

assuming that all blocks either are symmetric or have reciprocal symmetry. Let $R = T_s/((1+\gamma)^3 N^3/6)$; for instance, with $\gamma = 0$ (i.e. full symmetry), one has $R = (1/n^3)\sum_{\nu=1}^{R(n)} d_\nu^3$. Table 3 displays R for the groups $P_{1,2,3}$ and D_3 and various values of γ. Obviously, the highest gains in solution time occur for n large (i.e. high degrees of symmetry) and γ small. Also, $\sum_{\nu=1}^{R(n)} d_\nu^3 = n$ if $\mathcal{S}$ is Abelian, hence in that case $R = 1/n^2$ with $\gamma = 0$ as expected.

Elastostatic problems, Abelian case. In the limit of zero frequency (i.e. $k_T = 0$), the problem (1) becomes real-valued, as does the kernel function B. However, when the ρ_ν are complex, Eqs. (22), (34), (35), (36) show that the subproblems (37) are in general complex-valued even in that case. In fact, it is easy to show in this case that:

$$\mathcal{B}_{\nu^\star}(\boldsymbol{u},\boldsymbol{v}) = [\mathcal{B}_\nu]^\star(\boldsymbol{u},\boldsymbol{v}) \qquad \mathcal{C}_{\nu^\star}(\boldsymbol{u},\boldsymbol{v}) = [\mathcal{C}_\nu]^\star(\boldsymbol{u},\boldsymbol{v}) \qquad \mathcal{F}_{\nu^\star}(\boldsymbol{v}) = [\mathcal{F}_\nu]^\star(\boldsymbol{v})$$

Thus, the equations for the ν-subproblem and the $\nu^\star$-subproblem, and hence their solutions $(\boldsymbol{\phi}_\nu, \boldsymbol{\phi}_{\nu^\star})$, are conjugate to each other and thus redundant. It is sufficient to solve (say) the ν-subproblem for u_ν. The contribution of the conjugate pair $(\boldsymbol{\phi}_\nu, \boldsymbol{\phi}_{\nu^\star})$ to the reconstruction of the (real) global solution u is then:

$$[\boldsymbol{P}_\nu\boldsymbol{\phi}](s\boldsymbol{x}) + [\boldsymbol{P}_{\nu^\star}\boldsymbol{\phi}](s\boldsymbol{x}) = \rho_\nu^\star(s)\boldsymbol{\phi}_\nu + \rho_\nu(s)\boldsymbol{\phi}_\nu^\star = 2\mathrm{Re}(\rho_\nu^\star(s)\boldsymbol{\phi}_\nu)$$

In the FEM framework, adequate combinations of the two conjugate equations are known to yield two *coupled* real-valued subproblems defined on the (volumic) symmetry cell. Here, a similar approach could be applied to the symmetry-reduced SGBEM. However, contrarily to the FEM case, this would result in one subproblem of size $2N/n$, and hence would not bring any advantage over solving directly the complex-valued subproblem of size N/n.

8 Conclusion

The analysis, conducted here for the simple case of Neumann boundary-value problems, can be extended to the SGBEM formulations of more general boundary-value problems. This strategy is especially interesting when S^2 is 'small' (in terms of the number of degrees of freedom involved). This is for instance the case for externally symmetric bodies containing holes, cracks or other defects of arbitrary shape and location. This work is expected to be highly beneficial to some computationally intensive problems like defect identification in complex bodies exhibiting geometrical symmetry.

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