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**QUANTIZATION OF FORMAL CLASSICAL DYNAMICAL
r-MATRICES: THE REDUCTIVE CASE**

DAMIEN CALAQUE

ABSTRACT. In this paper we prove the existence of a formal dynamical twist quantization for any triangular and non-modified formal classical dynamical r -matrix in the reductive case. The dynamical twist is constructed as the image of the dynamical r -matrix by a L_∞ -quasi-isomorphism. This quasi-isomorphism also allows us to classify formal dynamical twist quantizations up to gauge equivalence.

INTRODUCTION

In [Fe], Felder introduced dynamical versions of both classical and quantum Yang-Baxter equations which has been generalized to the case of a nonabelian base in [EV] for the classical part and in [X3] for the quantum part. Naturally this leads to quantization problems which have been formulated in terms of twist quantization *à la* Drinfeld ([Dr1]) in [X2, X3, EE1, EE2].

Let us formulate this problem in the general context. Consider an inclusion $\mathfrak{h} \subset \mathfrak{g}$ of Lie algebras equipped with an element $Z \in (\wedge^3 \mathfrak{g})^\mathfrak{g}$. A *(modified) classical dynamical r -matrix* for $(\mathfrak{g}, \mathfrak{h}, Z)$ is a regular (meaning C^∞ , meromorphic, formal, ... depending on the context) \mathfrak{h} -equivariant map $\rho : \mathfrak{h}^* \rightarrow \wedge^2 \mathfrak{g}$ which satisfies the *(modified) classical dynamical Yang-Baxter equation* (CDYBE)

$$(1) \quad \text{CYB}(\rho) - \text{Alt}(d\rho) = Z$$

where $\text{CYB}(\rho) := [\rho^{1,2}, \rho^{1,3}] + [\rho^{1,2}, \rho^{2,3}] + [\rho^{1,3}, \rho^{2,3}] = \frac{1}{2}[\rho, \rho]$ and

$$\text{Alt}(d\rho) := \sum_i \left(h_i^1 \frac{\partial \rho^{2,3}}{\partial \lambda^i} - h_i^2 \frac{\partial \rho^{1,3}}{\partial \lambda^i} + h_i^3 \frac{\partial \rho^{1,2}}{\partial \lambda^i} \right)$$

Here (h_i) and (λ^i) are dual basis of \mathfrak{h} and \mathfrak{h}^* .

Let $\Phi = 1 + O(\hbar^2) \in (U\mathfrak{g}^{\otimes 3})^\mathfrak{g}[[\hbar]]$ be an associator quantizing Z (of which the existence was proved in [Dr2, proposition 3.10]). A *dynamical twist quantization* of a (modified) classical dynamical r -matrix ρ associated to Φ is a regular \mathfrak{h} -equivariant map $J = 1 + O(\hbar) \in \text{Reg}(\mathfrak{h}^*, U\mathfrak{g}^{\otimes 2})[[\hbar]]$ such that $\text{Alt} \frac{J-1}{\hbar} = \rho \bmod \hbar$ and which satisfies the *(modified) dynamical twist equation* (DTE)

$$(2) \quad J^{1,2,3}(\lambda) * J^{1,2}(\lambda + \hbar h^3) = \Phi^{-1} J^{1,2,3}(\lambda) * J^{2,3}(\lambda)$$

where $*$ denotes the PBW star-product of functions on \mathfrak{h}^* and

$$J^{1,2}(\lambda + \hbar h^3) := \sum_{k \geq 0} \frac{\hbar^k}{k!} \sum_{i_1, \dots, i_k} (\partial_{\lambda^{i_1}} \cdots \partial_{\lambda^{i_k}} J)(\lambda) \otimes (h_{i_1} \cdots h_{i_k})$$

Now observe that many (modified) classical dynamical r -matrices can be viewed as formal ones by taking their Taylor expansion at 0. In this paper we are interested in the following conjecture:

Conjecture 0.1 ([EE1]). *Any (modified) formal classical dynamical r -matrix admits a dynamical twist quantization.*

Let us reformulate DTE in the formal framework. A formal (modified) dynamical twist is an element $J(\lambda) = 1 + O(\hbar) \in (U\mathfrak{g}^{\otimes 2} \hat{\otimes} \hat{S}\mathfrak{h})^\hbar[[\hbar]]$ which satisfies DTE, and $J^{1,2}(\lambda + \hbar h^3) \in (U\mathfrak{g}^{\otimes 3} \hat{\otimes} \hat{S}\mathfrak{h})[[\hbar]]$ is equal to $(\text{id}^{\otimes 2} \otimes \tilde{\Delta})(J)$ where $\tilde{\Delta} : \hat{S}\mathfrak{h} \rightarrow (U\mathfrak{g} \hat{\otimes} \hat{S}\mathfrak{h})[[\hbar]]$ is induced by $\mathfrak{h} \ni x \mapsto \hbar x \otimes 1 + 1 \otimes x$. Then define $K := J(\hbar\lambda) \in (U\mathfrak{g}^{\otimes 2} \otimes S\mathfrak{h})^\hbar[[\hbar]]$ which we view as an element of $(U\mathfrak{g}^{\otimes 2} \otimes U\mathfrak{h})^\hbar[[\hbar]]$ using the symmetrization map $S\mathfrak{h} \rightarrow U\mathfrak{h}$. Since J is a solution of DTE K satisfies the (modified) algebraic dynamical twist equation (ADTE)

$$(2) \quad K^{12,3,4} K^{1,2,34} = (\Phi^{-1})^{1,2,3} K^{1,23,4} K^{2,3,4}$$

Moreover and by construction, $K = 1 + \sum_{n \geq 1} \hbar^n K_n$ has the \hbar -adic valuation property. Namely, $U\mathfrak{h}$ is filtered by $(U\mathfrak{h})_{\leq n} = \ker(\text{id} - \eta \circ \varepsilon)^{\otimes n+1} \circ \Delta^{(n)}$ where $\varepsilon : U\mathfrak{h} \rightarrow \mathbf{k}$ and $\eta : \mathbf{k} \rightarrow U\mathfrak{h}$ are the counit and unit maps, and $K_n \in (U\mathfrak{h})_{\leq n-1}$. Conversely, any algebraic dynamical twist having the \hbar -adic valuation property can be obtained from a unique formal dynamical twist by this procedure.

This paper, in which we always assume $Z = 0$ and $\Phi = 1$ (non-modified case), is organized as follow.

In section 1 we define two differential graded Lie algebras (dglas) respectively associated to classical dynamical r -matrices and algebraic dynamical twists. Then we formulate the main theorem of this paper which states that if \mathfrak{h} admits an $\text{ad}\mathfrak{h}$ -invariant complement (the reductive case) then these two dglas are L_∞ -quasi-isomorphic and we prove that it implies Conjecture 0.1 in this case, which generalizes Theorem 5.3 of [X2]:

Theorem 0.2. *In the reductive case, any formal classical dynamical r -matrix for $(\mathfrak{g}, \mathfrak{h}, 0)$ admits a dynamical twist quantization (associated to the trivial associator).*

The second section is devoted to the proof of the main theorem of section 1: using an equivariant formality theorem for homogeneous spaces which is obtain from [Do], we construct a L_∞ -quasi-isomorphism which we then modify in order to obtain the desired one. We use this L_∞ -quasi-isomorphism to classify formal dynamical twist quantizations up to gauge equivalence for the reductive case in section 3. In section 4 we prove that if $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ for \mathfrak{h} abelian and \mathfrak{m} a Lie subalgebra then the results of sections 1 and 2 are still true in this situation. We conclude the paper with some open questions, and recall basic results for L_∞ -algebras in an appendix.

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1. DEFINITIONS AND RESULTS

Let $\mathfrak{h} \subset \mathfrak{g}$ be an inclusion of Lie algebras.

1.1. **Algebraic structures associated to CDYBE.** Let us consider the following graded vector space

$$\underline{\text{CDYB}} := \wedge^* \mathfrak{g} \otimes S\mathfrak{h} = \bigoplus_{k \geq 0} \wedge^k \mathfrak{g} \otimes S\mathfrak{h}$$

equipped with the differential d defined by

$$(4) \quad d(x_1 \wedge \cdots \wedge x_k \otimes h_1 \cdots h_l) := - \sum_{i=1}^l h_i \wedge x_1 \wedge \cdots \wedge x_k \otimes h_1 \cdots h_l \hat{h}_i$$

With the exterior product \wedge it becomes a differential graded commutative associative algebra. Moreover, one can define a graded Lie bracket of degree -1 on $\underline{\text{CDYB}}$ which is the Lie bracket of \mathfrak{g} extended to $\underline{\text{CDYB}}$ in the following way:

$$(5) \quad [a, b \wedge c] = [a, b] \wedge c + (-1)^{(|a|-1)|b|} b \wedge [a, c]$$

Thus one can observe that polynomial solutions to CDYBE are exactly elements $\rho \in \underline{\text{CDYB}}$ of degree 2 such that $d\rho + \frac{1}{2}[\rho, \rho] = 0$. We would like to say that such a ρ is a Maurer-Cartan element but $(\underline{\text{CDYB}}[1], d, [,])$ is not a differential graded Lie algebra ($\text{dgl\mathfrak{a}}$).

Instead, remember that we are interested in \mathfrak{h} -equivariant solutions of CDYBE (i.e., dynamical r -matrices) and thus consider the subspace $\mathfrak{g}_1 = (\underline{\text{CDYB}})^{\mathfrak{h}}$ of \mathfrak{h} -invariants with the same differential and bracket.

Proposition 1.1. $(\mathfrak{g}_1[1], d, [,])$ is a $\text{dgl\mathfrak{a}}$. Moreover $(\mathfrak{g}_1, d, \wedge, [,])$ is a Gerstenhaber algebra.

Proof. Let $a = x_1 \wedge \cdots \wedge x_k \otimes h_1 \cdots h_s$ and $b = y_1 \wedge \cdots \wedge y_l \otimes m_1 \cdots m_t$ be \mathfrak{h} -invariant elements in \mathfrak{g}_1 . We want to show that

$$(6) \quad d[a, b] = [da, b] + (-1)^{k-1} [a, db]$$

The l.h.s. of (6) is equal to

$$\begin{aligned} & - \left(\sum_{i=1}^s h_i \wedge [x_1 \wedge \cdots \wedge x_k, y_1 \wedge \cdots \wedge y_l] \otimes h_1 \cdots h_s m_1 \cdots m_t \hat{h}_i \right. \\ & \left. + \sum_{j=1}^t m_j \wedge [x_1 \wedge \cdots \wedge x_k, y_1 \wedge \cdots \wedge y_l] \otimes h_1 \cdots h_s m_1 \cdots m_t \hat{m}_j \right) \end{aligned}$$

The first term in the r.h.s. of (6) gives

$$\sum_{i=1}^s ((-1)^{k-1} x_1 \wedge \cdots \wedge x_k \wedge [h_i, y_1 \wedge \cdots \wedge y_l] - h_i \wedge [x_1 \wedge \cdots \wedge x_k, y_1 \wedge \cdots \wedge y_l]) \otimes h_1 \cdots h_s m_1 \cdots m_t \hat{h}_i$$

and for the second term we obtain

$$\sum_{j=1}^t ((-1)^{k-1} [m_j, x_1 \wedge \cdots \wedge x_k] \wedge y_1 \wedge \cdots \wedge y_l - m_j \wedge [x_1 \wedge \cdots \wedge x_k, y_1 \wedge \cdots \wedge y_l]) \otimes h_1 \cdots h_s m_1 \cdots m_t \hat{m}_j$$

Thus the difference between the l.h.s. and the r.h.s. of (6) is equal to

$$\begin{aligned} & (-1)^k \left(\sum_{i=1}^k x_1 \wedge \cdots \wedge x_k \wedge [h_i, y_1 \wedge \cdots \wedge y_l] \otimes h_1 \cdots h_s m_1 \cdots m_t \hat{h}_i \right. \\ & \left. + \sum_{j=1}^l [m_j, x_1 \wedge \cdots \wedge x_k] \wedge y_1 \wedge \cdots \wedge y_l \otimes h_1 \cdots h_s m_1 \cdots m_t \hat{m}_j \right) \end{aligned}$$

Then using \mathfrak{h} -invariance of a and b one obtains

$$(-1)^{k-1} \sum_{i,j} x_1 \wedge \cdots \wedge x_k \wedge y_1 \wedge \cdots \wedge y_l \otimes (h_1 \cdots h_s m_1 \cdots m_t ([h_i, m_j] - [m_j, h_i]) \hat{h}_i \hat{m}_j) = 0$$

The second statement of the proposition is obvious from the definition (5) of the bracket. \square

Let $\rho(\lambda) \in (\wedge^2 \mathfrak{g} \hat{\otimes} \hat{S}\mathfrak{h})^{\mathfrak{h}}$ be a formal classical dynamical r -matrix. Since ρ satisfies CDYBE, $\alpha := \hbar \rho(\hbar \lambda) \in \hbar \mathfrak{g}_1[[\hbar]]$ is a Maurer-Cartan element (i.e. $d\alpha + \frac{1}{2}[\alpha, \alpha] = 0$).

1.2. Algebraic structures associated to ADTE. Let us now consider the graded vector space

$$\underline{\text{ADT}} := T^*U\mathfrak{g} \otimes U\mathfrak{h} = \bigoplus_{k \geq 0} \otimes^k U\mathfrak{g} \otimes U\mathfrak{h}$$

equipped with the differential b given by

$$(7) \quad b(P) := P^{2, \dots, k+2} + \sum_{i=1}^{k+1} (-1)^i P^{1, \dots, ii+1, \dots, k+2} \quad \text{for } P \in \otimes^k U\mathfrak{g} \otimes U\mathfrak{h}$$

Remark 1.2. This is just the coboundary operator of Hochschild's cohomology with value in a comodule; and $b^2 = 0$ follows directly from an easy calculation.

One can define on $\underline{\text{ADT}}$ an associative product \cup (the *cup* product) which is given on homogeneous elements $P \in \otimes^k U\mathfrak{g} \otimes U\mathfrak{h}$ and $Q \in \otimes^l U\mathfrak{g} \otimes U\mathfrak{h}$ by

$$P \cup Q := P^{1, \dots, k, k+1 \dots k+l+1} Q^{k+1, \dots, k+l+1}$$

Proposition 1.3. $(\underline{\text{ADT}}, b, \cup)$ is a differential graded associative algebra.

Proof. The cup product is obviously associative. Thus the only thing we have to check is that

$$(8) \quad b(P \cup Q) = bP \cup Q + (-1)^{|P|} P \cup bQ$$

Let $k = |P|$ and $l = |Q|$. The l.h.s. of (8) is equal to

$$\begin{aligned} & P^{2, \dots, k+1, k+2 \dots k+l+2} Q^{k+2, \dots, k+l+2} + \sum_{i=1}^k (-1)^i P^{1, \dots, ii+1, \dots, k+1, k+2 \dots k+l+2} Q^{k+2, \dots, k+l+2} \\ & + \sum_{i=k+1}^{k+l+1} (-1)^i P^{1, \dots, k, k+1 \dots k+l+2} Q^{k+1, \dots, ii+1, \dots, k+l+2} \end{aligned}$$

The first line of this expression is equal to

$$bP \cup Q - (-1)^{k+1} P^{1, \dots, k, k+1 \dots k+l+2} Q^{k+2, \dots, k+l+2}$$

and the last term of the same expression gives

$$(-1)^k (P \cup bQ - P^{1, \dots, k, k+1 \dots k+l+2} Q^{k+2, \dots, k+l+2})$$

The proposition is proved. \square

Recall that in the case $\mathfrak{h} = \{0\}$ one can define a brace algebra structure on $(T^*U\mathfrak{g})[1]$ (see [Ge]). Unfortunately we are not able to extend this structure to $\underline{\text{ADT}}$ in general. Since we deal with \mathfrak{h} -equivariant solutions of ADTE we can consider the subspace $\mathfrak{g}_2 = (\underline{\text{ADT}})^\mathfrak{h}$ of \mathfrak{h} -invariants. Let us now define a collection of linear homogeneous maps of degree zero $\{-|-, \dots, -\} : \mathfrak{g}_2[1] \otimes \mathfrak{g}_2[1]^{\otimes m} \rightarrow \mathfrak{g}_2[1]$ indexed by $m \geq 0$, and $\{P|Q_1, \dots, Q_m\}$ is given by

$$\sum_{\substack{0 \leq i_1, i_m + k_m \leq n \\ i_l + k_l \leq i_{l+1}}} (-1)^\epsilon P^{1, \dots, i_1+1 \dots i_1+k_1, \dots, i_m+1 \dots i_m+k_m, \dots, n+1} \prod_{s=i}^m Q_s^{i_s+1, \dots, i_s+k_s, i_s+k_s+1 \dots n+1}$$

where $k_s = |Q_s|$, $n = |P| + \sum_s k_s - m$ and $\epsilon = \sum_s (k_s - 1)i_s$.

Proposition 1.4. $(\mathfrak{g}_2[1], \{-|-, \dots, -\})$ is a brace algebra.

Proof. Since we work with \mathfrak{h} -invariant elements one can remark that if $i_s + k_s \leq i_t$ then $Q_s^{i_s+1, \dots, i_s+k_s, i_s+k_s+1 \dots n+1}$ and $Q_t^{i_t+1, \dots, i_t+k_t, i_t+k_t+1 \dots n+1}$ commute. Using this the proof becomes identical to the case when $\mathfrak{h} = 0$ (see [Ge] for example). \square

Now observe that since $m = 1^{\otimes 3} \in (\otimes^2 U\mathfrak{g} \otimes U\mathfrak{h})^\mathfrak{h}$ is such that $\{m|m\} = 0$ one obtains a B_∞ -algebra structure ([Ba]) on \mathfrak{g}_2 (see [Kh]). More precisely, we have a differential graded bialgebra structure on the cofree tensorial coalgebra $T(\mathfrak{g}_2[1])$ of which structure maps $a^n, a^{p,q}$ are given by

- $a^1(P) = bP = (-1)^{|P|-1}[m, P]_G$, where

$$[P, Q]_G := \{P|Q\} - (-1)^{(|P|-1)(|Q|-1)}\{Q|P\}$$
- $a^2(P, Q) = \{m|P, Q\} = P \cup Q$
- $a^{0,1} = a^{1,0} = \text{id}$
- $a^{1,n}(P; Q_1, \dots, Q_n) = \{P|Q_1, \dots, Q_n\}$ for $n \geq 1$
- all other maps are zero

In particular, we have

Proposition 1.5. $(\mathfrak{g}_2[1], b, [,]_G)$ is a dgla.

Remark 1.6. Since that for any graded vector space V , dg bialgebra structures on the cofree coassociative coalgebra $T^c V$ are in one-to-one correspondence with dg Lie bialgebra structures on the cofree Lie coalgebra $L^c V$ (see [Ta], section 5), then $L^c(\mathfrak{g}_2[1])$ becomes a dg Lie bialgebra with differential and Lie bracket given by maps $l^n, l^{p,q}$ such that $l^1 = b$ and $l^{1,1} = [,]_G$. Therefore $d_2 := \sum_{i \geq 0} l^i + \sum_{p,q \geq 0} l^{p,q} : C^c(L^c(\mathfrak{g}_2[1])) \rightarrow C^c(L^c(\mathfrak{g}_2[1]))$ defines a G_∞ -algebra structure on \mathfrak{g}_2 ($d_2 \circ d_2 = 0$ since d_2 is just the Chevalley-Eilenberg differential on the dg Lie algebra $L^c(\mathfrak{g}_2[1])$).

1.3. Main result and proof of theorem 0.2. First of all, observe that $\underline{\text{CDYB}}$, \mathfrak{g}_1 and $\mathcal{G}_1 := C^c(\mathfrak{g}_1[2])$ have a natural grading induced by the one of $S\mathfrak{h}$. In the same way $\underline{\text{ADT}}$, \mathfrak{g}_2 and $\mathcal{G}_2 := C^c(\mathfrak{g}_2[2])$ have a natural filtration induced by the one of $U\mathfrak{h}$. Our main goal is to prove the following theorem, which is sufficient to obtain algebraic dynamical twists from formal dynamical r -matrices.

Theorem 1.7. *In the reductive case, there exists a L_∞ -quasi-isomorphism*

$$\Psi : (\mathcal{G}_1, d + [,]_G) \rightarrow (\mathcal{G}_2, b + [,]_G)$$

with the following two filtration properties:

$$\begin{aligned} \text{(F1)} \quad & \forall X \in (\mathfrak{g}_1)_k, \Psi^1(X) = (\text{alt} \otimes \text{sym})(X) \bmod (\mathfrak{g}_2)_{\leq k-1} \\ \text{(F2)} \quad & \forall X \in (\Lambda^n \mathfrak{g}_1)_k, \Psi^n(X) \in (\mathfrak{g}_2)_{\leq n+k-1} \end{aligned}$$

Proof of Theorem 0.2. Now consider a formal solution $\rho(\lambda) \in (\Lambda^2 \mathfrak{g} \hat{\otimes} \hat{S}\mathfrak{h})^{\flat}$ to CDYBE. Let us define $\alpha := \hbar \rho(\hbar \lambda) \in \hbar \mathfrak{g}_1[[\hbar]]$ which is a Maurer-Cartan element in $\hbar \mathfrak{g}_1[[\hbar]]$. The L_∞ -morphism property implies that $\tilde{\alpha} := \sum_{n=1}^{\infty} \frac{1}{n!} \Psi^n(\Lambda^n \alpha)$ is a Maurer-Cartan element in $\hbar \mathfrak{g}_2[[\hbar]]$; this exactly means that $K := 1 + \tilde{\alpha} \in (\otimes^2 U\mathfrak{g} \otimes U\mathfrak{h})^{\flat}[[\hbar]]$ satisfies ADTE. Moreover, due to (F2) the coefficient K_n of \hbar^n in K lies in $(\mathfrak{g}_2)_{\leq n-1}$. It means that there exists $J \in (U\mathfrak{g}^{\otimes 2} \hat{\otimes} \hat{S}\mathfrak{h})^{\flat}[[\hbar]]$ satisfying DTE and such that $K = (\text{id}^{\otimes 2} \otimes \text{sym})(J(\hbar \lambda))$. Finally, property (F1) obviously implies that the semiclassical limit condition $\frac{J - J^{\text{op}}}{\hbar} = \rho \bmod \hbar$ is satisfied. \square

2. PROOF OF THEOREM 1.7

In this section we assume that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ with $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$. Let us denote by $p : \mathfrak{g} \rightarrow \mathfrak{m}$ the projection on \mathfrak{m} along \mathfrak{h} ; it is \mathfrak{h} -equivariant.

2.1. Resolutions. Let us first observe that the bilinear map $[\cdot, \cdot]_{\mathfrak{m}} := (\wedge \cdot p) \circ [\cdot, \cdot]$ defines a graded Lie bracket of degree -1 on $(\wedge^* \mathfrak{m})^{\flat}$. Then we prove

Proposition 2.1. *The natural map $p_1 : (\mathfrak{g}_1[1], d, [\cdot, \cdot]_{\mathfrak{m}}) \rightarrow ((\wedge^* \mathfrak{m})^{\flat}[1], 0, [\cdot, \cdot]_{\mathfrak{m}})$ is a morphism of dgl_a 's. Moreover, there exists an operator $\delta : \mathfrak{g}_1^* \rightarrow \mathfrak{g}_1^{*-1}$ such that $\delta d + d\delta = \text{id} - p_1$, $\delta \circ \delta = 0$ and $\delta((\mathfrak{g}_1)_k) \subset (\mathfrak{g}_1)_{k+1}$. In particular, p_1 induces an isomorphism in cohomology.*

Proof. The projection $p_1 := (\wedge \cdot p) \otimes \varepsilon : (\text{CDYB}, d) \rightarrow (\wedge^* \mathfrak{m}, 0)$ is a \mathfrak{h} -equivariant morphism of complexes, and it obviously restricts to a morphism of (differential) graded Lie algebras at the level of \mathfrak{h} -invariants.

Moreover, $\Lambda^n \mathfrak{g} \otimes S\mathfrak{h} \cong \bigoplus_{p+q=n} \Lambda^p \mathfrak{m} \otimes \Lambda^q \mathfrak{h} \otimes S\mathfrak{h}$ as a \mathfrak{h} -module; and under this identification d becomes $-\text{id} \otimes d_K$, where $d_K : \Lambda^* \mathfrak{h} \otimes S\mathfrak{h} \rightarrow \Lambda^{*+1} \mathfrak{h} \otimes S\mathfrak{h}$ is Koszul's coboundary operator, and p_1 corresponds to the projection on the part of zero antisymmetric and symmetric degrees in \mathfrak{h} . Let us define $\delta = \text{id} \otimes \delta_K$ with $\delta_K : \Lambda^* \mathfrak{h} \otimes S^* \mathfrak{h} \rightarrow \Lambda^{*-1} \mathfrak{h} \otimes S^{*+1} \mathfrak{h}$ defined by

$$\delta_K(x_1 \wedge \cdots \wedge x_n \otimes h_1 \cdots h_m) = \begin{cases} \frac{1}{m+n} \sum_i (-1)^i x_1 \wedge \cdots \wedge \hat{x}_i \cdots \wedge x_n \otimes h_1 \cdots h_m x_i & \text{if } m+n \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

Finally remark δ is a \mathfrak{h} -equivariant homotopy operator: $\delta d + d\delta = \text{id} - p_1$ and $\delta \circ \delta = 0$. The proposition is proved. \square

Now we prove a similar result for \mathfrak{g}_2 . Let us first define $U\mathfrak{m} := \text{sym}(S\mathfrak{m}) \subset U\mathfrak{g}$; this is a sub-coalgebra of $U\mathfrak{g}$ and thus $T^*U\mathfrak{m}$ equipped with its Hochschild's coboundary operator $b_{\mathfrak{m}}$ becomes a cochain subcomplex of the Hochschild complex $(T^*U\mathfrak{g}, b_{\mathfrak{g}})$ of $U\mathfrak{g}$. We also have the following

Lemma 2.2. *$U\mathfrak{g} = U\mathfrak{g} \cdot \mathfrak{h} \oplus U\mathfrak{m}$ as a filtered \mathfrak{h} -module. Moreover $[\cdot, \cdot]_{G, \mathfrak{m}} := (\otimes \cdot p) \circ [\cdot, \cdot]$ defines a graded Lie bracket of degree -1 on $(T^*U\mathfrak{m})^{\flat}$*

Proof. See [He, Ch.II §4.2] for the first statement. The second statement follows from a direct computation. \square

Then we prove the

Proposition 2.3. *The natural map $p_2 : (\mathfrak{g}_2[1], b, [,]_G) \rightarrow ((T^*U\mathfrak{m})^\mathfrak{h}[1], b_m, [,]_{G,m})$ is a morphism of $\mathfrak{dgl}\mathfrak{a}$'s. Moreover, there exists an operator $\kappa : \mathfrak{g}_2^* \rightarrow \mathfrak{g}_2^{*-1}$ such that $\kappa b + b\kappa = \text{id} - p_2$, $\kappa \circ \kappa = 0$ and $\kappa((\mathfrak{g}_2)_{\leq k}) \subset (\mathfrak{g}_2)_{\leq k+1}$. In particular, p_2 induces an isomorphism in cohomology.*

Proof. The projection $p_2 := (\otimes \text{p}) \otimes \varepsilon : (\underline{\text{ADT}}, b) \rightarrow (T^*U\mathfrak{m}, b_m)$ is a \mathfrak{h} -equivariant morphism of complexes, and it obviously restricts to a morphism of $\mathfrak{dgl}\mathfrak{a}$'s at the level of \mathfrak{h} -invariants (by lemma 2.2).

Remember that \mathfrak{g}_2 has a natural filtration induced by the one of $U\mathfrak{h}$. Then one obtains a spectral sequence of which we compute the first terms:

$$\begin{aligned} E_0^{*,*} &= (T^*U\mathfrak{g} \otimes S^*\mathfrak{h})^\mathfrak{h} & d_0 &= b_{\mathfrak{g}} \otimes \text{id} \\ E_1^{*,*} &= (\wedge^*\mathfrak{g} \otimes S^*\mathfrak{h})^\mathfrak{h} & d_1 &= d \\ E_2^{*,*} &= E_2^{*,0} = (\wedge^*\mathfrak{m})^\mathfrak{h} & d_2 &= 0 \end{aligned}$$

Then the proposition follows from proposition 2.1. \square

2.2. Inverting p_2 . In this subsection, taking our inspiration from [Mo, appendix], we prove the following

Proposition 2.4. *There exists a L_∞ -quasi-isomorphism*

$$\mathcal{Q}_2 : (C^c((T^*U\mathfrak{m})^\mathfrak{h}[2]), b_m + [,]_{G,m}) \rightarrow (C^c(\mathfrak{g}_2[2]), b + [,]_G)$$

such that \mathcal{Q}_2^1 is the natural inclusion and \mathcal{Q}_2^n takes values in $(\mathfrak{g}_2)_{\leq n-1}$.

Proof. Let $(N, b_N) \subset (\mathfrak{g}_2, b)$ be the kernel of the surjective morphism of complexes $p_2 : (\mathfrak{g}_2, b) \rightarrow ((T^*U\mathfrak{m})^\mathfrak{h}, b_m)$. It follows from the proofs of propositions 2.1 and 2.3 that there exists an operator $H : N^* \rightarrow N^{*-1}$ such that $H \circ H = 0$, $b_N H + H b_N = \text{id}$ and $H(N_{\leq n}) \subset N_{\leq n+1}$.

Now let us construct a L_∞ -isomorphism

$$\mathcal{F} : (C^c(\mathfrak{g}_2[2]), b + [,]_G) \xrightarrow{\sim} (C^c((T^*U\mathfrak{m})^\mathfrak{h}[2] \oplus N[2]), b_m + b_N + [,]_{G,m})$$

with structure maps $\mathcal{F}^n : \Lambda^n \mathfrak{g}_2 \rightarrow ((T^*U\mathfrak{m})^\mathfrak{h} \oplus N)[1-n]$ such that

- \mathcal{F}^1 is the sum of p_2 with the projection on N along $(T^*U\mathfrak{m})^\mathfrak{h}$ (in some sense \mathcal{F}^1 is the identity),
- for any $n > 1$ and $X \in (\Lambda^n \mathfrak{g}_2)_{\leq k}$, $\mathcal{F}^n(X) \in N_{\leq n+k-1}$.

Let us prove it by induction on n . First \mathcal{F}^1 is a morphism of complexes by definition. Then let us define $\mathcal{K}_2 : \Lambda^2 \mathfrak{g}_2 \rightarrow ((T^*U\mathfrak{m})^\mathfrak{h} \oplus N)[1]$ by

$$\mathcal{K}_2(x\Lambda y) = [\mathcal{F}^1(x), \mathcal{F}^1(y)]_{G,m} - \mathcal{F}^1([x, y]_G)$$

It takes values in $N[1]$ and is such that $b_N \mathcal{K}_2(x, y) + \mathcal{K}_2(bx, y) + \mathcal{K}_2(x, by) = 0$. Consequently $\mathcal{F}^2 := H \circ \mathcal{K}_2 : \Lambda^2 \mathfrak{g}_2 \rightarrow N$ is such that

$$b_N \mathcal{F}^2(x, y) - \mathcal{F}^2(bx, y) - \mathcal{F}^2(x, by) = \mathcal{K}_2(x, y) \quad (L_\infty\text{-condition for } \mathcal{F}^2)$$

and for any $X \in (\Lambda^2 \mathfrak{g}_2)_{\leq k}$, $\mathcal{F}^2(X) \in N_{\leq k+1}$. After this, suppose we have constructed $\mathcal{F}^1, \dots, \mathcal{F}^n$ and let us define

$$\mathcal{K}_{n+1} := [,]_{G,m} \circ \mathcal{F}^{\leq n} - \mathcal{F}^{\leq n} \circ [,]_G : \Lambda^2 \mathfrak{g}_2 \rightarrow ((T^*U\mathfrak{m})^\mathfrak{h} \oplus N)[1]$$

It obviously takes values in $N[1]$ and is such that $b_N \mathcal{K}_{n+1} + \mathcal{K}_{n+1} b = 0$. Consequently $\mathcal{F}^{n+1} := H \circ \mathcal{K}_{n+1}$ satisfies the L_∞ -condition

$$b_N \mathcal{F}^{n+1} - \mathcal{F}^{n+1} b = b_N H \mathcal{K}_{n+1} - H \mathcal{K}_{n+1} b = (b_N H + H b_N) \mathcal{K}_{n+1} = \mathcal{K}_{n+1}$$

and for any $X \in (\Lambda^n \mathfrak{g}_2)_{\leq n+1}$, $\mathcal{F}^{n+1}(X) \in N_{\leq n+k}$ (since $\mathcal{K}_{n+1}(X) \in N_{\leq n+k-1}$).

Now let \mathcal{H} be the inverse of the isomorphism \mathcal{F} , it is such that for any $n \geq 1$ and $X \in (\wedge^n \mathfrak{g}_2)_{\leq k}$, $\mathcal{H}^n(X) \in N_{\leq n+k-1}$. Finally we obtain \mathcal{Q}_2 by composing \mathcal{H} with the inclusion of $\mathfrak{dgl}\mathfrak{a}$'s $(T^*U\mathfrak{m})^\flat[1] \hookrightarrow ((T^*U\mathfrak{m})^\flat \oplus N)[1]$. \square

2.3. End of the proof. Recall from [He, Ch.II §4.2] that $(T^*U\mathfrak{m})^\flat = \text{Diff}^*(G/H)^G$ and $(\wedge^* \mathfrak{m})^\flat = \Gamma(G/H, \wedge^* T(G/H))^G$ as $\mathfrak{dgl}\mathfrak{a}$'s. Remember also from [No, Ch.II §8] that G -invariant connections on G/H are in one-to-one correspondence with \mathfrak{h} -equivariant linear maps $\alpha : \mathfrak{m} \otimes \mathfrak{m} \rightarrow \mathfrak{m}$, and that the torsion tensor is given by $\alpha - \alpha^{21} - \mathfrak{p} \circ [\cdot, \cdot]$. Thus G/H is equipped with a G -invariant torsion free connection ∇ , corresponding to the map $\alpha := \frac{1}{2}\mathfrak{p} \circ [\cdot, \cdot]$. Then using a theorem of Dolgushev, see [Do, theorem 5], we obtain a G -equivariant L_∞ -quasi-isomorphism $\phi : \Gamma(G/H, \wedge^* T(G/H)) \rightarrow \text{Diff}^*(G/H)$ with first structure map $\phi^1 = \text{alt}$, which restricts to a L_∞ -quasi-isomorphism at the level of G -invariants. Let us define $\psi := \mathcal{Q}_2 \circ \phi \circ p_1 : (C^c(\mathfrak{g}_1[2]), \mathfrak{d} + [\cdot, \cdot]) \rightarrow (C^c(\mathfrak{g}_2[2]), b + [\cdot, \cdot]_G)$; it is a L_∞ -quasi-isomorphism with first structure map $\psi^1 = (\text{alt} \otimes 1) \circ (\wedge^* \mathfrak{p} \otimes \varepsilon)$.

Finally define $V := (\text{alt} \otimes \text{sym}) \circ \delta : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2[-1]$ and use lemma A.3 to construct a L_∞ -quasi-morphism $\Psi : (C^c(\mathfrak{g}_1[2]), \mathfrak{d} + [\cdot, \cdot]) \rightarrow (C^c(\mathfrak{g}_2[2]), b + [\cdot, \cdot]_G)$ with first structure map $\Psi^1 = \psi^1 + b \circ V + V \circ \mathfrak{d}$. Since for any $X \in (\mathfrak{G}_1)_k$, then

$$b \circ (\text{alt} \otimes \text{sym})(X) = (\text{alt} \otimes \text{sym}) \circ \mathfrak{d}(X) \text{ mod } (\mathfrak{g}_2)_{\leq k-1}$$

$$\begin{aligned} \text{and thus } \Psi^1(X) &= \psi^1(X) + bV(X) + V(\mathfrak{d}X) \\ &= (\text{alt} \otimes \text{sym}) \circ (p_1 + \mathfrak{d}\delta + \delta\mathfrak{d})(X) \text{ mod } (\mathfrak{g}_2)_{\leq k-1} \\ &= (\text{alt} \otimes \text{sym})(X) \text{ mod } (\mathfrak{g}_2)_{\leq k-1} \end{aligned}$$

Consequently Ψ satisfies (F1). Moreover, it follows from remark A.4 that Ψ also satisfies (F2). \square

3. CLASSIFICATION

Theorem 1.7 implies a stronger result than just the existence of the twist quantization. Namely, since Ψ is a L_∞ -quasi-isomorphism there is a bijection between the moduli spaces of Maurer-Cartan elements of $\mathfrak{dgl}\mathfrak{a}$'s $(\mathfrak{g}_1[1])[[\hbar]]$ and $(\mathfrak{g}_2[1])[[\hbar]]$.

3.1. Classification of algebraic and formal dynamical twists. Following [EE1], two dynamical twists $J(\lambda)$ and $J'(\lambda)$ are said to be *gauge equivalent* if there exists a regular \mathfrak{h} -equivariant map $T(\lambda) = \exp(q) + O(\hbar) \in \text{Reg}(\mathfrak{h}^*, U\mathfrak{g})^\flat[[\hbar]]$, with $q \in \text{Reg}(\mathfrak{h}^*, \mathfrak{g})^\flat$ such that $q(0) = 0$, and satisfying

$$(9) \quad J'(\lambda) = T^{12}(\lambda) * J(\lambda) * T^2(\lambda)^{-1} * T^1(\lambda + \hbar h^2)^{-1}$$

Dealing with formal functions one can easily derive an equivalence relation for the corresponding algebraic dynamical twists $K = J(\hbar\lambda)$ and $K' = J'(\hbar\lambda)$:

$$(10) \quad K' = Q^{12,3} K (Q^{2,3})^{-1} (Q^{1,23})^{-1}$$

in $(U\mathfrak{g}^{\otimes 2} \otimes U\mathfrak{h})^\flat[[\hbar]]$, with $Q = 1 + O(\hbar) \in (U\mathfrak{g} \otimes U\mathfrak{h})^\flat[[\hbar]]$ given by $T(\hbar\lambda)$.

Assume now we are in the reductive case.

Since the composition $\mathcal{Q}_2 \circ \phi : (C^c((\wedge^* \mathfrak{m})^\flat[2]), [\cdot, \cdot]_{\mathfrak{m}}) \rightarrow (C^c(\mathfrak{g}_2[2]), b + [\cdot, \cdot]_G)$ in the previous section is a L_∞ -quasi-isomorphism then we have a bijective correspondance

$$(11) \quad \frac{\{\pi \in \hbar(\wedge^2 \mathfrak{m})^\flat[[\hbar]] \text{ s.t. } [\pi, \pi]_{\mathfrak{m}} = 0\}}{G_0} \longleftrightarrow \frac{\{\text{algebraic dynamical twists}\}}{\text{gauge equivalence (10)}}$$

where G_0 is the prounipotent group corresponding to the Lie algebra $\hbar \mathfrak{m}^{\flat}[[\hbar]]$. Moreover, since the structure maps \mathcal{Q}_2^n take values in $(\mathfrak{g}_2)_{\leq n-1}$ then it appears that any algebraic dynamical twist is gauge equivalent to a one with the \hbar -adic valuation property and thus we have a bijection

$$(12) \quad \frac{\{\text{algebraic dynamical twists}\}}{\text{gauge equivalence (10)}} \longleftrightarrow \frac{\{\text{formal dynamical twists}\}}{\text{gauge equivalence (9)}}$$

3.2. Classical counterpart. Assume that we are in the reductive case. Since p_1 is a L_∞ -quasi-isomorphism by proposition 2.1 then we have a bijection

$$\frac{\{\alpha \in \hbar(\wedge^2 \mathfrak{g} \otimes S\mathfrak{h})^{\flat}[[\hbar]] \text{ s.t. } d\alpha + \frac{1}{2}[\alpha, \alpha] = 0\}}{G_1} \longleftrightarrow \frac{\{\pi \in \hbar(\wedge^2 \mathfrak{m})^{\flat}[[\hbar]] \text{ s.t. } [\pi, \pi]_{\mathfrak{m}} = 0\}}{G_0}$$

where G_1 is a prounipotent group and its action (by affine transformations) is given by the exponentiation of the infinitesimal action of its Lie algebra $\hbar(\mathfrak{g} \otimes S\mathfrak{h})^{\flat}[[\hbar]]$:

$$(13) \quad q \cdot \alpha = dq + [q, \alpha] \quad (q \in \hbar(\mathfrak{g} \otimes S\mathfrak{h})^{\flat}[[\hbar]])$$

Then going along the lines of subsection 2.2 one can prove the following

Proposition 3.1. *There exists a L_∞ -quasi-isomorphism*

$$\mathcal{Q}_1 : (C^c((\wedge^* \mathfrak{m})^{\flat}[2]), [,]_{\mathfrak{m}}) \rightarrow (C^c(\mathfrak{g}_1[2]), d + [,])$$

such that \mathcal{Q}_1^1 is the natural inclusion and \mathcal{Q}_1^n takes values in $(\mathfrak{g}_1)_{\leq n-1}$.

Consequently any Maurer-Cartan element in $(\mathfrak{g}_1[1])[[\hbar]]$ is equivalent to a one of the form $\hbar \rho_{\hbar}(\hbar \lambda)$, where $\rho_{\hbar} \in (\wedge^2 \mathfrak{g} \hat{\otimes} \hat{S}\mathfrak{h})^{\flat}[[\hbar]]$ satisfies CDYBE. In other words ρ_{\hbar} is \hbar -dependant formal dynamical r -matrix. On such a ρ_{\hbar} the infinitesimal action (13) becomes

$$(14) \quad q \cdot \rho_{\hbar} = - \sum_i h_i \wedge \frac{\partial q}{\partial \lambda^i} + [q, \rho_{\hbar}] \quad (q \in \mathfrak{g} \hat{\otimes} \hat{S}\mathfrak{h})^{\flat}[[\hbar]]$$

This action integrates in an affine action of some group \widetilde{G}_1 of \mathfrak{h} -equivariant formal maps with values in the Lie group G of \mathfrak{g} . And then we have a bijection

$$(15) \quad \frac{\{\pi \in \hbar(\wedge^2 \mathfrak{m})^{\flat}[[\hbar]] \text{ s.t. } [\pi, \pi]_{\mathfrak{m}} = 0\}}{G_0} \longleftrightarrow \frac{\{\text{form. dynam. } r\text{-matrices}/\mathbb{R}[[\hbar]]\}}{\widetilde{G}_1}$$

Remark 3.2. This bijection has to be compared with Proposition 2.13 in [X2] and section 3 of [ES]

Finally, combining (15), (11) and (12) we obtain the following generalization of Theorem 6.11 in [X2] to the case of a nonabelian base:

Theorem 3.3. *Let $\pi \in (\wedge^2 \mathfrak{m})^{\flat}$ such that $[\pi, \pi]_{\mathfrak{m}} = 0$. Then there are bijective correspondances between*

- (1) *the set of \hbar -dependant and G -invariant Poisson structures $\pi_{\hbar} = \hbar \pi \text{ mod } \hbar^2$ on G/H , modulo the action of G_0 ,*
- (2) *the set of \hbar -dependant formal dynamical r -matrices $\rho_{\hbar}(\lambda)$ such that $\rho_{\hbar}(0) = \pi \text{ mod } \hbar$ in $\wedge^2(\mathfrak{g}/\mathfrak{h})[[\hbar]]$, modulo the action (14) of \widetilde{G}_1 ,*
- (3) *the set of formal dynamical twists $J(\lambda)$ satisfying $\text{Alt} \frac{J(0)-1}{\hbar} = \pi \text{ mod } \hbar$ in $\wedge^2(\mathfrak{g}/\mathfrak{h})[[\hbar]]$, modulo gauge equivalence (9).*

4. ANOTHER CASE WHEN THE TWIST QUANTIZATION EXISTS

In this section we assume that \mathfrak{h} is abelian and admits a Lie subalgebra \mathfrak{m} as complement.

Note that since \mathfrak{h} is abelian and \mathfrak{m} a Lie subalgebra, the projection $p : \mathfrak{g} \rightarrow \mathfrak{g}$ on \mathfrak{m} along \mathfrak{h} extends to a morphism of graded Lie algebras $\wedge p : (\wedge \mathfrak{g})^{\mathfrak{h}} \rightarrow (\wedge \mathfrak{g})^{\mathfrak{h}}$ at the level of \mathfrak{h} -invariants. And thus $\wedge p \otimes \varepsilon : (\mathfrak{g}_1[1], d, [,]) \rightarrow ((\wedge \mathfrak{g})^{\mathfrak{h}}[1], 0, [,])$ is a morphism of $\mathfrak{dgl}\mathfrak{a}$'s. Then the natural inclusion $\text{id} \otimes 1 : (T^*U\mathfrak{g})^{\mathfrak{h}} \rightarrow \mathfrak{g}_2$ obviously allows one to consider $(T^*U\mathfrak{g})^{\mathfrak{h}}[1]$ as a sub- $\mathfrak{dgl}\mathfrak{a}$ of $\mathfrak{g}_2[1]$. Finally recall from [Ca, section 3.3] that there exists a L_∞ -quasi-isomorphism $\mathcal{F} : C^c((\wedge^* \mathfrak{g})^{\mathfrak{h}}[2]) \rightarrow C^c((T^*U\mathfrak{g})^{\mathfrak{h}}[2])$ with $\mathcal{F}^1 = \text{alt}$. By composing these maps one obtains a L_∞ -morphism

$$\tilde{\mathcal{F}} : (\mathcal{G}_1, d + [,]) \rightarrow (\mathcal{G}_2, b + [,]_G)$$

with values in $(\mathcal{G}_2)_{\leq 0}$ and first structure map $\tilde{\mathcal{F}}^1 = (\text{alt} \otimes 1) \circ (\wedge p \otimes \varepsilon)$.

Theorem 4.1. *There exists a L_∞ -quasi-isomorphism*

$$\Psi : (\mathcal{G}_1, d + [,]) \rightarrow (\mathcal{G}_2, b + [,]_G)$$

with properties (F1) and (F2) of Theorem 1.7.

Proof. First observe that since \mathfrak{h} is abelian then $\mathfrak{g}_1 \cong ((\wedge \mathfrak{g})^{\mathfrak{h}} \cap \wedge \mathfrak{m}) \otimes \wedge \mathfrak{h} \otimes S\mathfrak{h}$ as a vector space. Thus if δ_K is as in the proof of proposition 2.1 then $\delta := \text{id} \otimes \delta_K$ is a homotopy operator: $\delta d + d\delta = \text{id} - \wedge p \otimes \varepsilon$ and $\delta \circ \delta = 0$.

Now we proceed like in subsection 2.3: use lemma A.3 to construct a L_∞ -morphism Ψ with first structure map $\Psi^1 = \tilde{\mathcal{F}}^1 + b \circ V + V \circ d$, where $V := (\text{alt} \otimes \text{sym}) \circ \delta : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2[-1]$.

It remains to prove that Ψ is a quasi-isomorphism. It follows from the first observation in this proof that $H^*(\mathfrak{g}_1, d) = (\wedge \mathfrak{g})^{\mathfrak{h}} \cap \wedge \mathfrak{m}$, which also equals $H^*(\mathfrak{g}_2, b)$ due to the spectral sequence argument. Consequently $\tilde{\mathcal{F}}^1$ is a quasi-isomorphism of complexes, and so is Ψ^1 . \square

Finally using the same argumentation as in the proof of theorem 0.2 (subsection 1.3) one obtains the

Theorem 4.2. *If \mathfrak{h} is an abelian subalgebra of \mathfrak{g} with a Lie subalgebra as a complement, then any formal classical dynamical r -matrix for $(\mathfrak{g}, \mathfrak{h}, 0)$ admits a dynamical twist quantization (associated to the trivial associator).*

Example 4.3. In particular, this allows us to quantize dynamical r -matrices arising from semi-direct products $\mathfrak{g} = \mathfrak{m} \ltimes \mathbb{C}^n$ like in [EN, example 3.7].

CONCLUDING REMARKS

Let us first observe that if \mathfrak{h} is abelian then $(\wedge^* \mathfrak{g})^{\mathfrak{h}} \cap \wedge^* \mathfrak{m}[1]$ (resp. $(T^*U\mathfrak{g})^{\mathfrak{h}} \cap T^*\text{sym}(S\mathfrak{m})[1]$) inherits a $\mathfrak{dgl}\mathfrak{a}$ structure from the one of $\mathfrak{g}_1[1]$ (resp. $\mathfrak{g}_2[1]$) and $H^*(\mathfrak{g}_1, d) = (\wedge^* \mathfrak{g})^{\mathfrak{h}} \cap \wedge^* \mathfrak{m} = H^*(\mathfrak{g}_2, b)$, for any complement \mathfrak{m} of \mathfrak{h} . Thus I conjecture that there exists a L_∞ -quasi-isomorphism between $(\wedge^* \mathfrak{g})^{\mathfrak{h}} \cap \wedge^* \mathfrak{m}[1]$ and $(T^*U\mathfrak{g})^{\mathfrak{h}} \cap T^*\text{sym}(S\mathfrak{m})[1]$ which generalizes together ϕ of subsection 2.3 and \mathcal{F} of section 4. In particular this would imply conjecture 0.1 in the abelian (and non-modified) case.

Let us then mention that one can consider a non-triangular (i.e., non-antisymmetric) version of non-modified classical dynamical r -matrices. Namely, \mathfrak{h} -equivariant maps

$r \in \text{Reg}(\mathfrak{h}^*, \mathfrak{g} \otimes \mathfrak{g})$ such that $\text{CYB}(r) - \text{Alt}(dr) = 0$. According to [X3], a quantization of such a r is a \mathfrak{h} -equivariant map $R = 1 + \hbar r + O(\hbar^2) \in \text{Reg}(\mathfrak{h}^*, U\mathfrak{g}^{\otimes 2})[[\hbar]]$ that satisfies the *quantum dynamical Yang-Baxter equation* (QDYBE)

$$(16) \quad R^{1,2}(\lambda) * R^{1,3}(\lambda + \hbar h^2) * R^{2,3}(\lambda) = R^{2,3}(\lambda + \hbar h^1) * R^{1,3}(\lambda) * R^{1,2}(\lambda + \hbar h^3)$$

Question 4.4. *Does such a quantization always exist?*

The most famous example of non-triangular dynamical r -matrices was found in [AM] by Alekseev and Meinrenken, then extended successively to a more general context in [EV, ES, EE1], and quantized in [EE1].

Following [EE1], remark that for any non-triangular dynamical r -matrix r such that $r + r^{\text{op}} = t \in (S^2\mathfrak{g})^{\mathfrak{g}}$ (quasi-triangular case) one can define $\rho := r - t/2$ and $Z := \frac{1}{4}[t^{1,2}, t^{2,3}]$. Then ρ is a modified dynamical r -matrix for $(\mathfrak{g}, \mathfrak{h}, Z)$; moreover the assignment $r \mapsto \rho$ is a bijective correspondence between quasi-triangular dynamical r -matrices for $(\mathfrak{g}, \mathfrak{h}, t)$ and modified dynamical r -matrices for $(\mathfrak{g}, \mathfrak{h}, Z)$. Now observe that if $J(\lambda)$ is a dynamical twist quantizing ρ , then $R(\lambda) = J^{\text{op}}(\lambda)^{-1} * e^{\hbar t/2} * J(\lambda)$ is a quantum dynamical R -matrix quantizing r .

In this paper we have constructed such a dynamical twist in the triangular case $t = 0$. One can ask

Question 4.5. *Does such a dynamical twist exist for any quasi-triangular dynamical r -matrix? At least in the reductive and abelian cases?*

This question seems to be more reasonable than the previous one. More generally one can ask if conjecture 0.1 (and its smooth and meromorphic versions) is true in general. A positive answer was given in [EE1] when $\mathfrak{h} = \mathfrak{g}$; but unfortunately it is not known in general, even for the non-dynamical case $\mathfrak{h} = \{0\}$ (which is the last problem of Drinfeld [Dr1]: quantization of coboundary Lie bialgebras).

Finally let me mention that if $r(\lambda)$ is a triangular dynamical r -matrix for $(\mathfrak{g}, \mathfrak{h})$, then the bivector field

$$\pi := \overrightarrow{r(\lambda)} + \sum_i \frac{\partial}{\partial \lambda^i} \wedge \overrightarrow{h_i} + \pi_{\mathfrak{h}^*}$$

is a $G \times H$ -biinvariant Poisson structure on $G \times \mathfrak{h}^*$ and the projection $p : G \times \mathfrak{h}^* \rightarrow \mathfrak{h}^*$ is a momentum map. Moreover, according to [X3] any dynamical twist quantization $J(\lambda)$ of $r(\lambda)$ allows us to define a $G \times H$ -biinvariant star-product $*$ quantizing π on $G \times \mathfrak{h}^*$ as follows:

$$\begin{array}{ll} f * g = f *_{PBW} g & \text{if } f, g \in C^\infty(\mathfrak{h}^*) \\ f * g = fg & \text{if } f \in C^\infty(G), g \in C^\infty(\mathfrak{h}^*) \\ f * g = \overrightarrow{\exp}(\hbar \sum_i \frac{\partial}{\partial \lambda^i} \otimes \overrightarrow{h_i}) \cdot (f \otimes g) & \text{if } f \in C^\infty(\mathfrak{h}^*), g \in C^\infty(G) \\ f * g = \overrightarrow{J(\lambda)}(f \otimes g) & \text{if } f, g \in C^\infty(G) \end{array}$$

This way the map $p^* : (\text{Fct}(\mathfrak{h}^*)[[\hbar]], *_PBW) \rightarrow (\text{Fct}(G \times \mathfrak{h}^*)[[\hbar]], *)$ becomes a quantum momentum map in the sens of [X1].

So there may be a way to see momentum maps and their quantum analogues as Maurer-Cartan elements in dgl_a 's.

APPENDIX A. HOMOTOPY LIE ALGEBRAS

See [HS] for a detailed discussion of the theory.

Recall that a L_∞ -algebra structure on a graded vector space \mathfrak{g} is a degree 1 coderivation Q on the cofree cocommutative coalgebra $C^c(\mathfrak{g}[1])$ such that $Q \circ Q = 0$. By cofreeness, such a coderivation Q is uniquely determined by structure maps $Q^n : \Lambda^n \mathfrak{g} \rightarrow \mathfrak{g}[2-n]$ which satisfy an infinite collection of equations. In particular (\mathfrak{g}, Q^1) is a cochain complex.

Example A.1. Any $\mathfrak{dgl}_a(\mathfrak{g}, d, [,])$ is canonically a L_∞ -algebra. Namely, Q is given by structure maps $Q^1 = d$, $Q^2 = [,]$ and $Q^n = 0$ for $n > 2$.

A L_∞ -morphism between two L_∞ -algebras (\mathfrak{g}_1, Q_1) and (\mathfrak{g}_2, Q_2) is a degree 0 morphism of coalgebras $F : C^c(\mathfrak{g}_1[1]) \rightarrow C^c(\mathfrak{g}_2[1])$ such that $F \circ Q_1 = Q_2 \circ F$. Again by cofreeness, such a morphism is uniquely determined by structure maps $F^n : \Lambda^n \mathfrak{g}_1 \rightarrow \mathfrak{g}_2[1-n]$ which satisfy an infinite collection of equations. In particular $F^1 : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ is a morphism of complexes; when it induces an isomorphism in cohomology we say that F is a L_∞ -quasi-isomorphism.

Example A.2. Any morphism of \mathfrak{dgl}_a 's is a L_∞ -morphism with all structure maps equal to zero except the first one.

In this paper we use many times the following

Lemma A.3 ([Do]). *Let $F : C^c(\mathfrak{g}_1[1]) \rightarrow C^c(\mathfrak{g}_2[1])$ be a L_∞ -morphism. For any linear map $V : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2[-1]$ there exists a L_∞ -morphism $\Psi : C^c(\mathfrak{g}_1[1]) \rightarrow C^c(\mathfrak{g}_2[1])$ with first structure map $\Psi^1 = F^1 + Q_2^1 \circ V + V \circ Q_1^1$. Moreover, if F is a L_∞ -quasi-isomorphism then Ψ is also.*

Proof. First remark that V extends uniquely to a linear map $C^c(\mathfrak{g}_1[1]) \rightarrow C^c(\mathfrak{g}_2[1])$ of degree -1 such that

$$\Delta_2 \circ V = (F \otimes V + V \otimes F + \frac{1}{2} V \otimes (Q_2 \circ V + V \circ Q_1) + \frac{1}{2} (Q_2 \circ V + V \circ Q_1) \otimes V) \circ \Delta_1$$

where Δ_1 and Δ_2 denote comultiplications in $C^c(\mathfrak{g}_1[1])$ and $C^c(\mathfrak{g}_2[1])$, respectively. Then define $\Psi := F + Q_2 \circ V + V \circ Q_1$. \square

Remark A.4. Assume that in the previous lemma \mathfrak{g}_1 and \mathfrak{g}_2 are filtrated, F is such that F^n takes values in $(\mathfrak{g}_2)_{\leq n-1}$, and $V((\mathfrak{g}_1)_{\leq k}) \subset (\mathfrak{g}_2)_{\leq k+1}$. Then one can obviously check that for any $X \in (\Lambda^n \mathfrak{g}_1)_{\leq k}$, $F^n(X) \in (\mathfrak{g}_2)_{\leq n+k-1}$.

REFERENCES

- [AM] A. Alekseev, E. Meinrenken, The non-commutative Weil algebra, *Invent. Math.* **139** (2000), 135-172.
- [Ba] J. H. Baues, The double bar and cobar construction, *Compos. Math.* **43** (1981), 331-341.
- [Ca] D. Calaque, Formality for Lie algebroids, to appear in *Comm. Math. Phys.*
- [Do] V. Dolgushev, Covariant and equivariant formality theorem, *Adv. Math.* **191** (2005), 147-177.
- [Dr1] V. Drinfeld, On some unsolved problems in quantum group theory, *Lecture Notes Math.* **1510** (1992), 1-8.
- [Dr2] V. Drinfeld, Quasi-Hopf algebras, *Leningrad Math. J.* **1** (1990), 1419-1457.
- [EE1] B. Enriquez, P. Etingof, Quantization of Alekseev-Meinrenken dynamical r -matrices, *Trans. Am. Math. Soc. (ser. 2)* **210** (2003), 81-98.

- [EE2] B. Enriquez, P. Etingof, Quantization of classical dynamical r -matrices with nonabelian base, to appear in Comm. Math. Phys.
- [EN] P. Etingof, D. Nikshych, Vertex-IRF transformations and quantization of dynamical r -matrices, Math. Res. Lett. **8** (2001), 331-345.
- [ES] P. Etingof, O. Schiffmann, On the moduli space of classical dynamical r -matrices, Math. Res. Lett. **8** (2001), 157-170.
- [EV] P. Etingof, A. Varchenko, Geometry and classification of solutions of the classical dynamical Yang-Baxter equation, Comm. Math. Phys. **192** (1998), 77-120.
- [Fe] G. Felder, Conformal field theory and integrable systems associated to elliptic curves, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zurich, 1994), 1247-1255, Birkhäuser, Basel, 1995.
- [Ge] E. Getzler, Cartan homotopy formula and the Gauss-Manin connection in cyclic homology, Israel Math. Conf. Proc. **102** (1993), 256-283.
- [He] S. Helgason, *Groups and geometric analysis*, Pure Appl. Math., Vol. 113, Orlando, 1984.
- [HS] V. Hinich, V. Schechtman, Homotopy Lie algebras, I.M. Gelfand Seminar, Adv. Sov. Math. **16** (1993), no. 2, 1-28.
- [Kh] M. Khalkhali, Operations on cyclic homology, the X complex, and a conjecture of Deligne, Comm. Math. Phys. **202** (1999), no. 2, 309-323.
- [Ko] M. Kontsevich, Deformation quantization of Poisson manifolds, Lett. Math. Phys. **66** (2003), no. 3, 157-216.
- [Mo] T. Mochizuki, An application of formality theorem to a quantization of dynamical r -matrices, unpublished preprint.
- [No] K. Nomizu, Invariant affine connections on homogeneous spaces, Amer. J. Math. **76** (1954), 33-65.
- [Ta] D. Tamarkin, Another proof of M. Kontsevich formality theorem for \mathbb{R}^n , preprint math.QA/9803025 (1998).
- [X1] P. Xu, Fedosov $*$ -products and quantum momentum maps, Comm. Math. Phys. **197** (1998), 167-197.
- [X2] P. Xu, Triangular dynamical r -matrices and quantization, Adv. Math. **166** (2002), no. 1, 1-49.
- [X3] P. Xu, Quantum dynamical Yang-Baxter equation over a nonabelian base, Comm. Math. Phys. **226** (2002), no. 3, 475-495.

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