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## QUANTIZATION OF FORMAL CLASSICAL DYNAMICAL $r$ -MATRICES: THE REDUCTIVE CASE

DAMIEN CALAQUE

**ABSTRACT.** In this paper we prove the existence of a formal dynamical twist quantization for any triangular and non-modified formal classical dynamical  $r$ -matrix in the reductive case. The dynamical twist is constructed as the image of the dynamical  $r$ -matrix by a  $L_\infty$ -quasi-isomorphism. This quasi-isomorphism also allows us to classify formal dynamical twist quantizations up to gauge equivalence.

### INTRODUCTION

In [Fe], Felder introduced dynamical versions of both classical and quantum Yang-Baxter equations which has been generalized to the case of a nonabelian base in [EV] for the classical part and in [X3] for the quantum part. Naturally this leads to quantization problems which have been formulated in terms of twist quantization *à la* Drinfeld ([Dr1]) in [X2, X3, EE1, EE2].

Let us formulate this problem in the general context. Consider an inclusion  $\mathfrak{h} \subset \mathfrak{g}$  of Lie algebras equipped with an element  $Z \in (\wedge^3 \mathfrak{g})^{\mathfrak{g}}$ . A (*modified*) *classical dynamical  $r$ -matrix* for  $(\mathfrak{g}, \mathfrak{h}, Z)$  is a regular (meaning  $C^\infty$ , meromorphic, formal, ... depending on the context)  $\mathfrak{h}$ -equivariant map  $\rho : \mathfrak{h}^* \rightarrow \wedge^2 \mathfrak{g}$  which satisfies the (*modified*) *classical dynamical Yang-Baxter equation* (CDYBE)

$$(1) \quad \text{CYB}(\rho) - \text{Alt}(d\rho) = Z$$

where  $\text{CYB}(\rho) := [\rho^{1,2}, \rho^{1,3}] + [\rho^{1,2}, \rho^{2,3}] + [\rho^{1,3}, \rho^{2,3}] = \frac{1}{2}[\rho, \rho]$  and

$$\text{Alt}(d\rho) := \sum_i \left( h_i^1 \frac{\partial \rho^{2,3}}{\partial \lambda^i} - h_i^2 \frac{\partial \rho^{1,3}}{\partial \lambda^i} + h_i^3 \frac{\partial \rho^{1,2}}{\partial \lambda^i} \right)$$

Here  $(h_i)$  and  $(\lambda^i)$  are dual basis of  $\mathfrak{h}$  and  $\mathfrak{h}^*$ .

Let  $\Phi = 1 + O(\hbar^2) \in (U\mathfrak{g}^{\otimes 3})^{\mathfrak{g}}[[\hbar]]$  be an associator quantizing  $Z$  (of which the existence was proved in [Dr2, proposition 3.10]). A *dynamical twist quantization* of a (modified) classical dynamical  $r$ -matrix  $\rho$  associated to  $\Phi$  is a regular  $\mathfrak{h}$ -equivariant map  $J = 1 + O(\hbar) \in \text{Reg}(\mathfrak{h}^*, U\mathfrak{g}^{\otimes 2})[[\hbar]]$  such that  $\text{Alt} \frac{J-1}{\hbar} = \rho \bmod \hbar$  and which satisfies the (*modified*) *dynamical twist equation* (DTE)

$$(2) \quad J^{1,2,3}(\lambda) * J^{1,2}(\lambda + \hbar h^3) = \Phi^{-1} J^{1,2,3}(\lambda) * J^{2,3}(\lambda)$$

where  $*$  denotes the PBW star-product of functions on  $\mathfrak{h}^*$  and

$$J^{1,2}(\lambda + \hbar h^3) := \sum_{k \geq 0} \frac{\hbar^k}{k!} \sum_{i_1, \dots, i_k} (\partial_{\lambda^{i_1}} \cdots \partial_{\lambda^{i_k}} J)(\lambda) \otimes (h_{i_1} \cdots h_{i_k})$$

Now observe that many (modified) classical dynamical  $r$ -matrices can be viewed as formal ones by taking their Taylor expansion at 0. In this paper we are interested in the following conjecture:

**Conjecture 0.1** ([EE1]). *Any (modified) formal classical dynamical  $r$ -matrix admits a dynamical twist quantization.*

Let us reformulate DTE in the formal framework. A formal (modified) dynamical twist is an element  $J(\lambda) = 1 + O(\hbar) \in (U\mathfrak{g}^{\otimes 2} \hat{\otimes} \hat{S}\mathfrak{h})^{\flat}[[\hbar]]$  which satisfies DTE, and  $J^{1,2}(\lambda + \hbar h^3) \in (U\mathfrak{g}^{\otimes 3} \hat{\otimes} \hat{S}\mathfrak{h})[[\hbar]]$  is equal to  $(\text{id}^{\otimes 2} \otimes \tilde{\Delta})(J)$  where  $\tilde{\Delta} : \hat{S}\mathfrak{h} \rightarrow (U\mathfrak{g} \hat{\otimes} \hat{S}\mathfrak{h})[[\hbar]]$  is induced by  $\mathfrak{h} \ni x \mapsto \hbar x \otimes 1 + 1 \otimes x$ . Then define  $K := J(\hbar\lambda) \in (U\mathfrak{g}^{\otimes 2} \otimes S\mathfrak{h})^{\flat}[[\hbar]]$  which we view as an element of  $(U\mathfrak{g}^{\otimes 2} \otimes U\mathfrak{h})^{\flat}[[\hbar]]$  using the symmetrization map  $S\mathfrak{h} \rightarrow U\mathfrak{h}$ . Since  $J$  is a solution of DTE  $K$  satisfies the (modified) algebraic dynamical twist equation (ADTE)

$$(2) \quad K^{12,3,4} K^{1,2,34} = (\Phi^{-1})^{1,2,3} K^{1,23,4} K^{2,3,4}$$

Moreover and by construction,  $K = 1 + \sum_{n \geq 1} \hbar^n K_n$  has the  $\hbar$ -adic valuation property. Namely,  $U\mathfrak{h}$  is filtered by  $(U\mathfrak{h})_{\leq n} = \ker(\text{id} - \eta \circ \varepsilon)^{\otimes n+1} \circ \Delta^{(n)}$  where  $\varepsilon : U\mathfrak{h} \rightarrow \mathbf{k}$  and  $\eta : \mathbf{k} \rightarrow U\mathfrak{h}$  are the counit and unit maps, and  $K_n \in (U\mathfrak{h})_{\leq n-1}$ . Conversely, any algebraic dynamical twist having the  $\hbar$ -adic valuation property can be obtained from a unique formal dynamical twist by this procedure.

This paper, in which we always assume  $Z = 0$  and  $\Phi = 1$  (non-modified case), is organized as follow.

In section 1 we define two differential graded Lie algebras (dglas) respectively associated to classical dynamical  $r$ -matrices and algebraic dynamical twists. Then we formulate the main theorem of this paper which states that if  $\mathfrak{h}$  admits an  $\text{ad}\mathfrak{h}$ -invariant complement (the reductive case) then these two dglas are  $L_{\infty}$ -quasi-isomorphic and we prove that it implies Conjecture 0.1 in this case, which generalizes Theorem 5.3 of [X2]:

**Theorem 0.2.** *In the reductive case, any formal classical dynamical  $r$ -matrix for  $(\mathfrak{g}, \mathfrak{h}, 0)$  admits a dynamical twist quantization (associated to the trivial associator).*

The second section is devoted to the proof of the main theorem of section 1: using an equivariant formality theorem for homogeneous spaces which is obtain from [Do], we construct a  $L_{\infty}$ -quasi-isomorphism which we then modify in order to obtain the desired one. We use this  $L_{\infty}$ -quasi-isomorphism to classify formal dynamical twist quantizations up to gauge equivalence for the reductive case in section 3. In section 4 we prove that if  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$  for  $\mathfrak{h}$  abelian and  $\mathfrak{m}$  a Lie subalgebra then the results of sections 1 and 2 are still true in this situation. We conclude the paper with some open questions, and recall basic results for  $L_{\infty}$ -algebras in an appendix.

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## 1. DEFINITIONS AND RESULTS

Let  $\mathfrak{h} \subset \mathfrak{g}$  be an inclusion of Lie algebras.

1.1. **Algebraic structures associated to CDYBE.** Let us consider the following graded vector space

$$\underline{\text{CDYB}} := \wedge^* \mathfrak{g} \otimes S\mathfrak{h} = \bigoplus_{k \geq 0} \wedge^k \mathfrak{g} \otimes S\mathfrak{h}$$

equipped with the differential  $d$  defined by

$$(4) \quad d(x_1 \wedge \cdots \wedge x_k \otimes h_1 \cdots h_l) := - \sum_{i=1}^l h_i \wedge x_1 \wedge \cdots \wedge x_k \otimes h_1 \cdots \widehat{h}_i \cdots h_l$$

With the exterior product  $\wedge$  it becomes a differential graded commutative associative algebra. Moreover, one can define a graded Lie bracket of degree  $-1$  on  $\underline{\text{CDYB}}$  which is the Lie bracket of  $\mathfrak{g}$  extended to  $\underline{\text{CDYB}}$  in the following way:

$$(5) \quad [a, b \wedge c] = [a, b] \wedge c + (-1)^{(|a|-1)|b|} b \wedge [a, c]$$

Thus one can observe that polynomial solutions to CDYBE are exactly elements  $\rho \in \underline{\text{CDYB}}$  of degree 2 such that  $d\rho + \frac{1}{2}[\rho, \rho] = 0$ . We would like to say that such a  $\rho$  is a Maurer-Cartan element but  $(\underline{\text{CDYB}}[1], d, [,])$  is not a differential graded Lie algebra ( $\text{dglA}$ ).

Instead, remember that we are interested in  $\mathfrak{h}$ -equivariant solutions of CDYBE (i.e., dynamical  $r$ -matrices) and thus consider the subspace  $\mathfrak{g}_1 = (\underline{\text{CDYB}})^{\mathfrak{h}}$  of  $\mathfrak{h}$ -invariants with the same differential and bracket.

**Proposition 1.1.**  $(\mathfrak{g}_1[1], d, [,])$  is a  $\text{dglA}$ . Moreover  $(\mathfrak{g}_1, d, \wedge, [,])$  is a Gerstenhaber algebra.

*Proof.* Let  $a = x_1 \wedge \cdots \wedge x_k \otimes h_1 \cdots h_s$  and  $b = y_1 \wedge \cdots \wedge y_l \otimes m_1 \cdots m_t$  be  $\mathfrak{h}$ -invariant elements in  $\mathfrak{g}_1$ . We want to show that

$$(6) \quad d[a, b] = [da, b] + (-1)^{k-1} [a, db]$$

The l.h.s. of (6) is equal to

$$\begin{aligned} & - \left( \sum_{i=1}^s h_i \wedge [x_1 \wedge \cdots \wedge x_k, y_1 \wedge \cdots \wedge y_l] \otimes h_1 \cdots \widehat{h}_i \cdots h_s m_1 \cdots m_t \right. \\ & \left. + \sum_{j=1}^t m_j \wedge [x_1 \wedge \cdots \wedge x_k, y_1 \wedge \cdots \wedge y_l] \otimes h_1 \cdots h_s m_1 \cdots \widehat{m}_j \cdots m_t \right) \end{aligned}$$

The first term in the r.h.s. of (6) gives

$$\sum_{i=1}^s ((-1)^{k-1} x_1 \wedge \cdots \wedge x_k \wedge [h_i, y_1 \wedge \cdots \wedge y_l] - h_i \wedge [x_1 \wedge \cdots \wedge x_k, y_1 \wedge \cdots \wedge y_l]) \otimes h_1 \cdots \widehat{h}_i \cdots h_s m_1 \cdots m_t$$

and for the second term we obtain

$$\sum_{j=1}^t ((-1)^{k-1} [m_j, x_1 \wedge \cdots \wedge x_k] \wedge y_1 \wedge \cdots \wedge y_l - m_j \wedge [x_1 \wedge \cdots \wedge x_k, y_1 \wedge \cdots \wedge y_l]) \otimes h_1 \cdots h_s m_1 \cdots \widehat{m}_j \cdots m_t$$

Thus the difference between the l.h.s. and the r.h.s. of (6) is equal to

$$\begin{aligned} & (-1)^k \left( \sum_{i=1}^k x_1 \wedge \cdots \wedge x_k \wedge [h_i, y_1 \wedge \cdots \wedge y_l] \otimes h_1 \cdots h_s m_1 \cdots m_t \hat{h}_i \right. \\ & \left. + \sum_{j=1}^l [m_j, x_1 \wedge \cdots \wedge x_k] \wedge y_1 \wedge \cdots \wedge y_l \otimes h_1 \cdots h_s m_1 \cdots m_t \hat{m}_j \right) \end{aligned}$$

Then using  $\mathfrak{h}$ -invariance of  $a$  and  $b$  one obtains

$$(-1)^{k-1} \sum_{i,j} x_1 \wedge \cdots \wedge x_k \wedge y_1 \wedge \cdots \wedge y_l \otimes (h_1 \cdots h_s m_1 \cdots m_t ([h_i, m_j] - [m_j, h_i]) \hat{h}_i \hat{m}_j) = 0$$

The second statement of the proposition is obvious from the definition (5) of the bracket.  $\square$

Let  $\rho(\lambda) \in (\wedge^2 \mathfrak{g} \hat{\otimes} \hat{S}\mathfrak{h})^{\mathfrak{h}}$  be a formal classical dynamical  $r$ -matrix. Since  $\rho$  satisfies CDYBE,  $\alpha := \hbar \rho(\hbar \lambda) \in \hbar \mathfrak{g}_1[[\hbar]]$  is a Maurer-Cartan element (i.e.  $d\alpha + \frac{1}{2}[\alpha, \alpha] = 0$ ).

**1.2. Algebraic structures associated to ADTE.** Let us now consider the graded vector space

$$\underline{\text{ADT}} := T^*U\mathfrak{g} \otimes U\mathfrak{h} = \bigoplus_{k \geq 0} \otimes^k U\mathfrak{g} \otimes U\mathfrak{h}$$

equipped with the differential  $b$  given by

$$(7) \quad b(P) := P^{2, \dots, k+2} + \sum_{i=1}^{k+1} (-1)^i P^{1, \dots, ii+1, \dots, k+2} \quad \text{for } P \in \otimes^k U\mathfrak{g} \otimes U\mathfrak{h}$$

**Remark 1.2.** This is just the coboundary operator of Hochschild's cohomology with value in a comodule; and  $b^2 = 0$  follows directly from an easy calculation.

One can define on  $\underline{\text{ADT}}$  an associative product  $\cup$  (the *cup* product) which is given on homogeneous elements  $P \in \otimes^k U\mathfrak{g} \otimes U\mathfrak{h}$  and  $Q \in \otimes^l U\mathfrak{g} \otimes U\mathfrak{h}$  by

$$P \cup Q := P^{1, \dots, k, k+1 \dots k+l+1} Q^{k+1, \dots, k+l+1}$$

**Proposition 1.3.**  $(\underline{\text{ADT}}, b, \cup)$  is a differential graded associative algebra.

*Proof.* The cup product is obviously associative. Thus the only thing we have to check is that

$$(8) \quad b(P \cup Q) = bP \cup Q + (-1)^{|P|} P \cup bQ$$

Let  $k = |P|$  and  $l = |Q|$ . The l.h.s. of (8) is equal to

$$\begin{aligned} & P^{2, \dots, k+1, k+2 \dots k+l+2} Q^{k+2, \dots, k+l+2} + \sum_{i=1}^k (-1)^i P^{1, \dots, ii+1, \dots, k+1, k+2 \dots k+l+2} Q^{k+2, \dots, k+l+2} \\ & + \sum_{i=k+1}^{k+l+1} (-1)^i P^{1, \dots, k, k+1 \dots k+l+2} Q^{k+1, \dots, ii+1, \dots, k+l+2} \end{aligned}$$

The first line of this expression is equal to

$$bP \cup Q - (-1)^{k+1} P^{1, \dots, k, k+1 \dots k+l+2} Q^{k+2, \dots, k+l+2}$$

and the last term of the same expression gives

$$(-1)^k (P \cup bQ - P^{1, \dots, k, k+1 \dots k+l+2} Q^{k+2, \dots, k+l+2})$$

The proposition is proved.  $\square$

Recall that in the case  $\mathfrak{h} = \{0\}$  one can define a brace algebra structure on  $(T^*U\mathfrak{g})[1]$  (see [Ge]). Unfortunately we are not able to extend this structure to  $\underline{\text{ADT}}$  in general. Since we deal with  $\mathfrak{h}$ -equivariant solutions of ADTE we can consider the subspace  $\mathfrak{g}_2 = (\underline{\text{ADT}})^\mathfrak{h}$  of  $\mathfrak{h}$ -invariants. Let us now define a collection of linear homogeneous maps of degree zero  $\{-|-, \dots, -\} : \mathfrak{g}_2[1] \otimes \mathfrak{g}_2[1]^{\otimes m} \rightarrow \mathfrak{g}_2[1]$  indexed by  $m \geq 0$ , and  $\{P|Q_1, \dots, Q_m\}$  is given by

$$\sum_{\substack{0 \leq i_1, i_m + k_m \leq n \\ i_l + k_l \leq i_{l+1}}} (-1)^\epsilon P^{1, \dots, i_1+1 \dots i_1+k_1, \dots, i_m+1 \dots i_m+k_m, \dots, n+1} \prod_{s=i}^m Q_s^{i_s+1, \dots, i_s+k_s, i_s+k_s+1 \dots n+1}$$

where  $k_s = |Q_s|$ ,  $n = |P| + \sum_s k_s - m$  and  $\epsilon = \sum_s (k_s - 1)i_s$ .

**Proposition 1.4.**  $(\mathfrak{g}_2[1], \{-|-, \dots, -\})$  is a brace algebra.

*Proof.* Since we work with  $\mathfrak{h}$ -invariant elements one can remark that if  $i_s + k_s \leq i_t$  then  $Q_s^{i_s+1, \dots, i_s+k_s, i_s+k_s+1 \dots n+1}$  and  $Q_t^{i_t+1, \dots, i_t+k_t, i_t+k_t+1 \dots n+1}$  commute. Using this the proof becomes identical to the case when  $\mathfrak{h} = 0$  (see [Ge] for example).  $\square$

Now observe that since  $m = 1^{\otimes 3} \in (\otimes^2 U\mathfrak{g} \otimes U\mathfrak{h})^\mathfrak{h}$  is such that  $\{m|m\} = 0$  one obtains a  $B_\infty$ -algebra structure ([Ba]) on  $\mathfrak{g}_2$  (see [Kh]). More precisely, we have a differential graded bialgebra structure on the cofree tensorial coalgebra  $T(\mathfrak{g}_2[1])$  of which structure maps  $a^n, a^{p,q}$  are given by

- $a^1(P) = bP = (-1)^{|P|-1}[m, P]_G$ , where
 
$$[P, Q]_G := \{P|Q\} - (-1)^{(|P|-1)(|Q|-1)}\{Q|P\}$$
- $a^2(P, Q) = \{m|P, Q\} = P \cup Q$
- $a^{0,1} = a^{1,0} = \text{id}$
- $a^{1,n}(P; Q_1, \dots, Q_n) = \{P|Q_1, \dots, Q_n\}$  for  $n \geq 1$
- all other maps are zero

In particular, we have

**Proposition 1.5.**  $(\mathfrak{g}_2[1], b, [, ]_G)$  is a dgla.

**Remark 1.6.** Since that for any graded vector space  $V$ ,  $\text{dg}$  bialgebra structures on the cofree coassociative coalgebra  $T^c V$  are in one-to-one correspondence with  $\text{dg}$  Lie bialgebra structures on the cofree Lie coalgebra  $L^c V$  (see [Ta], section 5), then  $L^c(\mathfrak{g}_2[1])$  becomes a  $\text{dg}$  Lie bialgebra with differential and Lie bracket given by maps  $l^n, l^{p,q}$  such that  $l^1 = b$  and  $l^{1,1} = [, ]_G$ . Therefore  $d_2 := \sum_{i \geq 0} l^i + \sum_{p,q \geq 0} l^{p,q} : C^c(L^c(\mathfrak{g}_2[1])) \rightarrow C^c(L^c(\mathfrak{g}_2[1]))$  defines a  $G_\infty$ -algebra structure on  $\mathfrak{g}_2$  ( $d_2 \circ d_2 = 0$  since  $d_2$  is just the Chevalley-Eilenberg differential on the  $\text{dg}$  Lie algebra  $L^c(\mathfrak{g}_2[1])$ ).

**1.3. Main result and proof of theorem 0.2.** First of all, observe that  $\underline{\text{CDYB}}$ ,  $\mathfrak{g}_1$  and  $\mathcal{G}_1 := C^c(\mathfrak{g}_1[2])$  have a natural grading induced by the one of  $S\mathfrak{h}$ . In the same way  $\underline{\text{ADT}}$ ,  $\mathfrak{g}_2$  and  $\mathcal{G}_2 := C^c(\mathfrak{g}_2[2])$  have a natural filtration induced by the one of  $U\mathfrak{h}$ . Our main goal is to prove the following theorem, which is sufficient to obtain algebraic dynamical twists from formal dynamical  $r$ -matrices.

**Theorem 1.7.** *In the reductive case, there exists a  $L_\infty$ -quasi-isomorphism*

$$\Psi : (\mathcal{G}_1, d + [, ]_G) \rightarrow (\mathcal{G}_2, b + [, ]_G)$$

*with the following two filtration properties:*

$$\begin{aligned} \text{(F1)} \quad & \forall X \in (\mathfrak{g}_1)_k, \Psi^1(X) = (\text{alt} \otimes \text{sym})(X) \bmod (\mathfrak{g}_2)_{\leq k-1} \\ \text{(F2)} \quad & \forall X \in (\Lambda^n \mathfrak{g}_1)_k, \Psi^n(X) \in (\mathfrak{g}_2)_{\leq n+k-1} \end{aligned}$$

*Proof of Theorem 0.2.* Now consider a formal solution  $\rho(\lambda) \in (\Lambda^2 \mathfrak{g} \hat{\otimes} \hat{S}\mathfrak{h})^{\flat}$  to CDYBE. Let us define  $\alpha := \hbar \rho(\hbar \lambda) \in \hbar \mathfrak{g}_1[[\hbar]]$  which is a Maurer-Cartan element in  $\hbar \mathfrak{g}_1[[\hbar]]$ . The  $L_\infty$ -morphism property implies that  $\tilde{\alpha} := \sum_{n=1}^{\infty} \frac{1}{n!} \Psi^n(\Lambda^n \alpha)$  is a Maurer-Cartan element in  $\hbar \mathfrak{g}_2[[\hbar]]$ ; this exactly means that  $K := 1 + \tilde{\alpha} \in (\otimes^2 U\mathfrak{g} \otimes U\mathfrak{h})^{\flat}[[\hbar]]$  satisfies ADTE. Moreover, due to (F2) the coefficient  $K_n$  of  $\hbar^n$  in  $K$  lies in  $(\mathfrak{g}_2)_{\leq n-1}$ . It means that there exists  $J \in (U\mathfrak{g}^{\otimes 2} \hat{\otimes} \hat{S}\mathfrak{h})^{\flat}[[\hbar]]$  satisfying DTE and such that  $K = (\text{id}^{\otimes 2} \otimes \text{sym})(J(\hbar \lambda))$ . Finally, property (F1) obviously implies that the semiclassical limit condition  $\frac{J - J^{\text{op}}}{\hbar} = \rho \bmod \hbar$  is satisfied.  $\square$

## 2. PROOF OF THEOREM 1.7

In this section we assume that  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$  with  $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$ . Let us denote by  $p : \mathfrak{g} \rightarrow \mathfrak{m}$  the projection on  $\mathfrak{m}$  along  $\mathfrak{h}$ ; it is  $\mathfrak{h}$ -equivariant.

**2.1. Resolutions.** Let us first observe that the bilinear map  $[\cdot, \cdot]_{\mathfrak{m}} := (\wedge \cdot p) \circ [\cdot, \cdot]$  defines a graded Lie bracket of degree  $-1$  on  $(\wedge^* \mathfrak{m})^{\flat}$ . Then we prove

**Proposition 2.1.** *The natural map  $p_1 : (\mathfrak{g}_1[1], d, [\cdot, \cdot]_{\mathfrak{m}}) \rightarrow ((\wedge^* \mathfrak{m})^{\flat}[1], 0, [\cdot, \cdot]_{\mathfrak{m}})$  is a morphism of  $\text{dgl}_a$ 's. Moreover, there exists an operator  $\delta : \mathfrak{g}_1^* \rightarrow \mathfrak{g}_1^{*-1}$  such that  $\delta d + d\delta = \text{id} - p_1$ ,  $\delta \circ \delta = 0$  and  $\delta((\mathfrak{g}_1)_k) \subset (\mathfrak{g}_1)_{k+1}$ . In particular,  $p_1$  induces an isomorphism in cohomology.*

*Proof.* The projection  $p_1 := (\wedge \cdot p) \otimes \varepsilon : (\text{CDYB}, d) \rightarrow (\wedge^* \mathfrak{m}, 0)$  is a  $\mathfrak{h}$ -equivariant morphism of complexes, and it obviously restricts to a morphism of (differential) graded Lie algebras at the level of  $\mathfrak{h}$ -invariants.

Moreover,  $\Lambda^n \mathfrak{g} \otimes S\mathfrak{h} \cong \bigoplus_{p+q=n} \Lambda^p \mathfrak{m} \otimes \Lambda^q \mathfrak{h} \otimes S\mathfrak{h}$  as a  $\mathfrak{h}$ -module; and under this identification  $d$  becomes  $-\text{id} \otimes d_K$ , where  $d_K : \Lambda^* \mathfrak{h} \otimes S\mathfrak{h} \rightarrow \Lambda^{*+1} \mathfrak{h} \otimes S\mathfrak{h}$  is Koszul's coboundary operator, and  $p_1$  corresponds to the projection on the part of zero antisymmetric and symmetric degrees in  $\mathfrak{h}$ . Let us define  $\delta = \text{id} \otimes \delta_K$  with  $\delta_K : \Lambda^* \mathfrak{h} \otimes S^* \mathfrak{h} \rightarrow \Lambda^{*-1} \mathfrak{h} \otimes S^{*+1} \mathfrak{h}$  defined by

$$\delta_K(x_1 \wedge \cdots \wedge x_n \otimes h_1 \cdots h_m) = \begin{cases} \frac{1}{m+n} \sum_i (-1)^i x_1 \wedge \cdots \wedge \hat{x}_i \cdots \wedge x_n \otimes h_1 \cdots h_m x_i & \text{if } m+n \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

Finally remark  $\delta$  is a  $\mathfrak{h}$ -equivariant homotopy operator:  $\delta d + d\delta = \text{id} - p_1$  and  $\delta \circ \delta = 0$ . The proposition is proved.  $\square$

Now we prove a similar result for  $\mathfrak{g}_2$ . Let us first define  $U\mathfrak{m} := \text{sym}(S\mathfrak{m}) \subset U\mathfrak{g}$ ; this is a sub-coalgebra of  $U\mathfrak{g}$  and thus  $T^*U\mathfrak{m}$  equipped with its Hochschild's coboundary operator  $b_{\mathfrak{m}}$  becomes a cochain subcomplex of the Hochschild complex  $(T^*U\mathfrak{g}, b_{\mathfrak{g}})$  of  $U\mathfrak{g}$ . We also have the following

**Lemma 2.2.**  *$U\mathfrak{g} = U\mathfrak{g} \cdot \mathfrak{h} \oplus U\mathfrak{m}$  as a filtered  $\mathfrak{h}$ -module. Moreover  $[\cdot, \cdot]_{G, \mathfrak{m}} := (\otimes \cdot p) \circ [\cdot, \cdot]$  defines a graded Lie bracket of degree  $-1$  on  $(T^*U\mathfrak{m})^{\flat}$*

*Proof.* See [He, Ch.II §4.2] for the first statement. The second statement follows from a direct computation.  $\square$

Then we prove the

**Proposition 2.3.** *The natural map  $p_2 : (\mathfrak{g}_2[1], b, [, ]_G) \rightarrow ((T^*U\mathfrak{m})^\mathfrak{h}[1], b_m, [, ]_{G,m})$  is a morphism of  $\mathfrak{dgl}\mathfrak{a}$ 's. Moreover, there exists an operator  $\kappa : \mathfrak{g}_2^* \rightarrow \mathfrak{g}_2^{*-1}$  such that  $\kappa b + b\kappa = \text{id} - p_2$ ,  $\kappa \circ \kappa = 0$  and  $\kappa((\mathfrak{g}_2)_{\leq k}) \subset (\mathfrak{g}_2)_{\leq k+1}$ . In particular,  $p_2$  induces an isomorphism in cohomology.*

*Proof.* The projection  $p_2 := (\otimes \text{p}) \otimes \varepsilon : (\underline{\text{ADT}}, b) \rightarrow (T^*U\mathfrak{m}, b_m)$  is a  $\mathfrak{h}$ -equivariant morphism of complexes, and it obviously restricts to a morphism of  $\mathfrak{dgl}\mathfrak{a}$ 's at the level of  $\mathfrak{h}$ -invariants (by lemma 2.2).

Remember that  $\mathfrak{g}_2$  has a natural filtration induced by the one of  $U\mathfrak{h}$ . Then one obtains a spectral sequence of which we compute the first terms:

$$\begin{aligned} E_0^{*,*} &= (T^*U\mathfrak{g} \otimes S^*\mathfrak{h})^\mathfrak{h} & d_0 &= b_{\mathfrak{g}} \otimes \text{id} \\ E_1^{*,*} &= (\wedge^*\mathfrak{g} \otimes S^*\mathfrak{h})^\mathfrak{h} & d_1 &= d \\ E_2^{*,*} &= E_2^{*,0} = (\wedge^*\mathfrak{m})^\mathfrak{h} & d_2 &= 0 \end{aligned}$$

Then the proposition follows from proposition 2.1.  $\square$

**2.2. Inverting  $p_2$ .** In this subsection, taking our inspiration from [Mo, appendix], we prove the following

**Proposition 2.4.** *There exists a  $L_\infty$ -quasi-isomorphism*

$$\mathcal{Q}_2 : (C^c((T^*U\mathfrak{m})^\mathfrak{h}[2]), b_m + [, ]_{G,m}) \rightarrow (C^c(\mathfrak{g}_2[2]), b + [, ]_G)$$

such that  $\mathcal{Q}_2^1$  is the natural inclusion and  $\mathcal{Q}_2^n$  takes values in  $(\mathfrak{g}_2)_{\leq n-1}$ .

*Proof.* Let  $(N, b_N) \subset (\mathfrak{g}_2, b)$  be the kernel of the surjective morphism of complexes  $p_2 : (\mathfrak{g}_2, b) \rightarrow ((T^*U\mathfrak{m})^\mathfrak{h}, b_m)$ . It follows from the proofs of propositions 2.1 and 2.3 that there exists an operator  $H : N^* \rightarrow N^{*-1}$  such that  $H \circ H = 0$ ,  $b_N H + H b_N = \text{id}$  and  $H(N_{\leq n}) \subset N_{\leq n+1}$ .

Now let us construct a  $L_\infty$ -isomorphism

$$\mathcal{F} : (C^c(\mathfrak{g}_2[2]), b + [, ]_G) \xrightarrow{\sim} (C^c((T^*U\mathfrak{m})^\mathfrak{h}[2] \oplus N[2]), b_m + b_N + [, ]_{G,m})$$

with structure maps  $\mathcal{F}^n : \Lambda^n \mathfrak{g}_2 \rightarrow ((T^*U\mathfrak{m})^\mathfrak{h} \oplus N)[1-n]$  such that

- $\mathcal{F}^1$  is the sum of  $p_2$  with the projection on  $N$  along  $(T^*U\mathfrak{m})^\mathfrak{h}$  (in some sense  $\mathcal{F}^1$  is the identity),
- for any  $n > 1$  and  $X \in (\Lambda^n \mathfrak{g}_2)_{\leq k}$ ,  $\mathcal{F}^n(X) \in N_{\leq n+k-1}$ .

Let us prove it by induction on  $n$ . First  $\mathcal{F}^1$  is a morphism of complexes by definition. Then let us define  $\mathcal{K}_2 : \Lambda^2 \mathfrak{g}_2 \rightarrow ((T^*U\mathfrak{m})^\mathfrak{h} \oplus N)[1]$  by

$$\mathcal{K}_2(x\Lambda y) = [\mathcal{F}^1(x), \mathcal{F}^1(y)]_{G,m} - \mathcal{F}^1([x, y]_G)$$

It takes values in  $N[1]$  and is such that  $b_N \mathcal{K}_2(x, y) + \mathcal{K}_2(bx, y) + \mathcal{K}_2(x, by) = 0$ . Consequently  $\mathcal{F}^2 := H \circ \mathcal{K}_2 : \Lambda^2 \mathfrak{g}_2 \rightarrow N$  is such that

$$b_N \mathcal{F}^2(x, y) - \mathcal{F}^2(bx, y) - \mathcal{F}^2(x, by) = \mathcal{K}_2(x, y) \quad (L_\infty\text{-condition for } \mathcal{F}^2)$$

and for any  $X \in (\Lambda^2 \mathfrak{g}_2)_{\leq k}$ ,  $\mathcal{F}^2(X) \in N_{\leq k+1}$ . After this, suppose we have constructed  $\mathcal{F}^1, \dots, \mathcal{F}^n$  and let us define

$$\mathcal{K}_{n+1} := [, ]_{G,m} \circ \mathcal{F}^{\leq n} - \mathcal{F}^{\leq n} \circ [, ]_G : \Lambda^2 \mathfrak{g}_2 \rightarrow ((T^*U\mathfrak{m})^\mathfrak{h} \oplus N)[1]$$

It obviously takes values in  $N[1]$  and is such that  $b_N \mathcal{K}_{n+1} + \mathcal{K}_{n+1} b = 0$ . Consequently  $\mathcal{F}^{n+1} := H \circ \mathcal{K}_{n+1}$  satisfies the  $L_\infty$ -condition

$$b_N \mathcal{F}^{n+1} - \mathcal{F}^{n+1} b = b_N H \mathcal{K}_{n+1} - H \mathcal{K}_{n+1} b = (b_N H + H b_N) \mathcal{K}_{n+1} = \mathcal{K}_{n+1}$$

and for any  $X \in (\Lambda^n \mathfrak{g}_2)_{\leq n+1}$ ,  $\mathcal{F}^{n+1}(X) \in N_{\leq n+k}$  (since  $\mathcal{K}_{n+1}(X) \in N_{\leq n+k-1}$ ).

Now let  $\mathcal{H}$  be the inverse of the isomorphism  $\mathcal{F}$ , it is such that for any  $n \geq 1$  and  $X \in (\wedge^n \mathfrak{g}_2)_{\leq k}$ ,  $\mathcal{H}^n(X) \in N_{\leq n+k-1}$ . Finally we obtain  $\mathcal{Q}_2$  by composing  $\mathcal{H}$  with the inclusion of  $\mathfrak{dgl}\mathfrak{a}$ 's  $(T^*U\mathfrak{m})^\flat[1] \hookrightarrow ((T^*U\mathfrak{m})^\flat \oplus N)[1]$ .  $\square$

**2.3. End of the proof.** Recall from [He, Ch.II §4.2] that  $(T^*U\mathfrak{m})^\flat = \text{Diff}^*(G/H)^G$  and  $(\wedge^* \mathfrak{m})^\flat = \Gamma(G/H, \wedge^* T(G/H))^G$  as  $\mathfrak{dgl}\mathfrak{a}$ 's. Remember also from [No, Ch.II §8] that  $G$ -invariant connections on  $G/H$  are in one-to-one correspondence with  $\mathfrak{h}$ -equivariant linear maps  $\alpha : \mathfrak{m} \otimes \mathfrak{m} \rightarrow \mathfrak{m}$ , and that the torsion tensor is given by  $\alpha - \alpha^{21} - \mathfrak{p} \circ [\cdot, \cdot]$ . Thus  $G/H$  is equipped with a  $G$ -invariant torsion free connection  $\nabla$ , corresponding to the map  $\alpha := \frac{1}{2}\mathfrak{p} \circ [\cdot, \cdot]$ . Then using a theorem of Dolgushev, see [Do, theorem 5], we obtain a  $G$ -equivariant  $L_\infty$ -quasi-isomorphism  $\phi : \Gamma(G/H, \wedge^* T(G/H)) \rightarrow \text{Diff}^*(G/H)$  with first structure map  $\phi^1 = \text{alt}$ , which restricts to a  $L_\infty$ -quasi-isomorphism at the level of  $G$ -invariants. Let us define  $\psi := \mathcal{Q}_2 \circ \phi \circ p_1 : (C^c(\mathfrak{g}_1[2]), \mathfrak{d} + [\cdot, \cdot]) \rightarrow (C^c(\mathfrak{g}_2[2]), b + [\cdot, \cdot]_G)$ ; it is a  $L_\infty$ -quasi-isomorphism with first structure map  $\psi^1 = (\text{alt} \otimes 1) \circ (\wedge^* \mathfrak{p} \otimes \varepsilon)$ .

Finally define  $V := (\text{alt} \otimes \text{sym}) \circ \delta : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2[-1]$  and use lemma A.3 to construct a  $L_\infty$ -quasi-morphism  $\Psi : (C^c(\mathfrak{g}_1[2]), \mathfrak{d} + [\cdot, \cdot]) \rightarrow (C^c(\mathfrak{g}_2[2]), b + [\cdot, \cdot]_G)$  with first structure map  $\Psi^1 = \psi^1 + b \circ V + V \circ \mathfrak{d}$ . Since for any  $X \in (\mathfrak{G}_1)_k$ , then

$$b \circ (\text{alt} \otimes \text{sym})(X) = (\text{alt} \otimes \text{sym}) \circ \mathfrak{d}(X) \text{ mod } (\mathfrak{g}_2)_{\leq k-1}$$

$$\begin{aligned} \text{and thus } \Psi^1(X) &= \psi^1(X) + bV(X) + V(\mathfrak{d}X) \\ &= (\text{alt} \otimes \text{sym}) \circ (p_1 + \mathfrak{d}\delta + \delta\mathfrak{d})(X) \text{ mod } (\mathfrak{g}_2)_{\leq k-1} \\ &= (\text{alt} \otimes \text{sym})(X) \text{ mod } (\mathfrak{g}_2)_{\leq k-1} \end{aligned}$$

Consequently  $\Psi$  satisfies (F1). Moreover, it follows from remark A.4 that  $\Psi$  also satisfies (F2).  $\square$

### 3. CLASSIFICATION

Theorem 1.7 implies a stronger result than just the existence of the twist quantization. Namely, since  $\Psi$  is a  $L_\infty$ -quasi-isomorphism there is a bijection between the moduli spaces of Maurer-Cartan elements of  $\mathfrak{dgl}\mathfrak{a}$ 's  $(\mathfrak{g}_1[1])[[\hbar]]$  and  $(\mathfrak{g}_2[1])[[\hbar]]$ .

**3.1. Classification of algebraic and formal dynamical twists.** Following [EE1], two dynamical twists  $J(\lambda)$  and  $J'(\lambda)$  are said to be *gauge equivalent* if there exists a regular  $\mathfrak{h}$ -equivariant map  $T(\lambda) = \exp(q) + O(\hbar) \in \text{Reg}(\mathfrak{h}^*, U\mathfrak{g})^\flat[[\hbar]]$ , with  $q \in \text{Reg}(\mathfrak{h}^*, \mathfrak{g})^\flat$  such that  $q(0) = 0$ , and satisfying

$$(9) \quad J'(\lambda) = T^{12}(\lambda) * J(\lambda) * T^2(\lambda)^{-1} * T^1(\lambda + \hbar h^2)^{-1}$$

Dealing with formal functions one can easily derive an equivalence relation for the corresponding algebraic dynamical twists  $K = J(\hbar\lambda)$  and  $K' = J'(\hbar\lambda)$ :

$$(10) \quad K' = Q^{12,3} K (Q^{2,3})^{-1} (Q^{1,23})^{-1}$$

in  $(U\mathfrak{g}^{\otimes 2} \otimes U\mathfrak{h})^\flat[[\hbar]]$ , with  $Q = 1 + O(\hbar) \in (U\mathfrak{g} \otimes U\mathfrak{h})^\flat[[\hbar]]$  given by  $T(\hbar\lambda)$ .

Assume now we are in the reductive case.

Since the composition  $\mathcal{Q}_2 \circ \phi : (C^c((\wedge^* \mathfrak{m})^\flat[2]), [\cdot, \cdot]_{\mathfrak{m}}) \rightarrow (C^c(\mathfrak{g}_2[2]), b + [\cdot, \cdot]_G)$  in the previous section is a  $L_\infty$ -quasi-isomorphism then we have a bijective correspondance

$$(11) \quad \frac{\{\pi \in \hbar(\wedge^2 \mathfrak{m})^\flat[[\hbar]] \text{ s.t. } [\pi, \pi]_{\mathfrak{m}} = 0\}}{G_0} \longleftrightarrow \frac{\{\text{algebraic dynamical twists}\}}{\text{gauge equivalence (10)}}$$

where  $G_0$  is the pronipotent group corresponding to the Lie algebra  $\hbar \mathfrak{m}^{\flat}[[\hbar]]$ . Moreover, since the structure maps  $\mathcal{Q}_2^n$  take values in  $(\mathfrak{g}_2)_{\leq n-1}$  then it appears that any algebraic dynamical twist is gauge equivalent to a one with the  $\hbar$ -adic valuation property and thus we have a bijection

$$(12) \quad \frac{\{\text{algebraic dynamical twists}\}}{\text{gauge equivalence (10)}} \longleftrightarrow \frac{\{\text{formal dynamical twists}\}}{\text{gauge equivalence (9)}}$$

**3.2. Classical counterpart.** Assume that we are in the reductive case. Since  $p_1$  is a  $L_\infty$ -quasi-isomorphism by proposition 2.1 then we have a bijection

$$\frac{\{\alpha \in \hbar(\wedge^2 \mathfrak{g} \otimes S\mathfrak{h})^{\flat}[[\hbar]] \text{ s.t. } d\alpha + \frac{1}{2}[\alpha, \alpha] = 0\}}{G_1} \longleftrightarrow \frac{\{\pi \in \hbar(\wedge^2 \mathfrak{m})^{\flat}[[\hbar]] \text{ s.t. } [\pi, \pi]_{\mathfrak{m}} = 0\}}{G_0}$$

where  $G_1$  is a pronipotent group and its action (by affine transformations) is given by the exponentiation of the infinitesimal action of its Lie algebra  $\hbar(\mathfrak{g} \otimes S\mathfrak{h})^{\flat}[[\hbar]]$ :

$$(13) \quad q \cdot \alpha = dq + [q, \alpha] \quad (q \in \hbar(\mathfrak{g} \otimes S\mathfrak{h})^{\flat}[[\hbar]])$$

Then going along the lines of subsection 2.2 one can prove the following

**Proposition 3.1.** *There exists a  $L_\infty$ -quasi-isomorphism*

$$\mathcal{Q}_1 : (C^c((\wedge^* \mathfrak{m})^{\flat}[2]), [ , ]_{\mathfrak{m}}) \rightarrow (C^c(\mathfrak{g}_1[2]), d + [ , ])$$

such that  $\mathcal{Q}_1^1$  is the natural inclusion and  $\mathcal{Q}_1^n$  takes values in  $(\mathfrak{g}_1)_{\leq n-1}$ .

Consequently any Maurer-Cartan element in  $(\mathfrak{g}_1[1])[[\hbar]]$  is equivalent to a one of the form  $\hbar \rho_{\hbar}(\hbar \lambda)$ , where  $\rho_{\hbar} \in (\wedge^2 \mathfrak{g} \hat{\otimes} \hat{S}\mathfrak{h})^{\flat}[[\hbar]]$  satisfies CDYBE. In other words  $\rho_{\hbar}$  is  $\hbar$ -dependant formal dynamical  $r$ -matrix. On such a  $\rho_{\hbar}$  the infinitesimal action (13) becomes

$$(14) \quad q \cdot \rho_{\hbar} = - \sum_i h_i \wedge \frac{\partial q}{\partial \lambda^i} + [q, \rho_{\hbar}] \quad (q \in \mathfrak{g} \hat{\otimes} \hat{S}\mathfrak{h})^{\flat}[[\hbar]]$$

This action integrates in an affine action of some group  $\widetilde{G}_1$  of  $\mathfrak{h}$ -equivariant formal maps with values in the Lie group  $G$  of  $\mathfrak{g}$ . And then we have a bijection

$$(15) \quad \frac{\{\pi \in \hbar(\wedge^2 \mathfrak{m})^{\flat}[[\hbar]] \text{ s.t. } [\pi, \pi]_{\mathfrak{m}} = 0\}}{G_0} \longleftrightarrow \frac{\{\text{form. dynam. } r\text{-matrices}/\mathbb{R}[[\hbar]]\}}{\widetilde{G}_1}$$

**Remark 3.2.** This bijection has to be compared with Proposition 2.13 in [X2] and section 3 of [ES]

Finally, combining (15), (11) and (12) we obtain the following generalization of Theorem 6.11 in [X2] to the case of a nonabelian base:

**Theorem 3.3.** *Let  $\pi \in (\wedge^2 \mathfrak{m})^{\flat}$  such that  $[\pi, \pi]_{\mathfrak{m}} = 0$ . Then there are bijective correspondances between*

- (1) *the set of  $\hbar$ -dependant and  $G$ -invariant Poisson structures  $\pi_{\hbar} = \hbar \pi \text{ mod } \hbar^2$  on  $G/H$ , modulo the action of  $G_0$ ,*
- (2) *the set of  $\hbar$ -dependant formal dynamical  $r$ -matrices  $\rho_{\hbar}(\lambda)$  such that  $\rho_{\hbar}(0) = \pi \text{ mod } \hbar$  in  $\wedge^2(\mathfrak{g}/\mathfrak{h})[[\hbar]]$ , modulo the action (14) of  $\widetilde{G}_1$ ,*
- (3) *the set of formal dynamical twists  $J(\lambda)$  satisfying  $\text{Alt} \frac{J(0)-1}{\hbar} = \pi \text{ mod } \hbar$  in  $\wedge^2(\mathfrak{g}/\mathfrak{h})[[\hbar]]$ , modulo gauge equivalence (9).*

## 4. ANOTHER CASE WHEN THE TWIST QUANTIZATION EXISTS

In this section we assume that  $\mathfrak{h}$  is abelian and admits a Lie subalgebra  $\mathfrak{m}$  as complement.

Note that since  $\mathfrak{h}$  is abelian and  $\mathfrak{m}$  a Lie subalgebra, the projection  $p : \mathfrak{g} \rightarrow \mathfrak{g}$  on  $\mathfrak{m}$  along  $\mathfrak{h}$  extends to a morphism of graded Lie algebras  $\wedge p : (\wedge \mathfrak{g})^{\mathfrak{h}} \rightarrow (\wedge \mathfrak{g})^{\mathfrak{h}}$  at the level of  $\mathfrak{h}$ -invariants. And thus  $\wedge p \otimes \varepsilon : (\mathfrak{g}_1[1], d, [,]) \rightarrow ((\wedge \mathfrak{g})^{\mathfrak{h}}[1], 0, [,])$  is a morphism of  $\mathfrak{dgl}\mathfrak{a}$ 's. Then the natural inclusion  $\text{id} \otimes 1 : (T^*U\mathfrak{g})^{\mathfrak{h}} \rightarrow \mathfrak{g}_2$  obviously allows one to consider  $(T^*U\mathfrak{g})^{\mathfrak{h}}[1]$  as a sub- $\mathfrak{dgl}\mathfrak{a}$  of  $\mathfrak{g}_2[1]$ . Finally recall from [Ca, section 3.3] that there exists a  $L_\infty$ -quasi-isomorphism  $\mathcal{F} : C^c((\wedge^* \mathfrak{g})^{\mathfrak{h}}[2]) \rightarrow C^c((T^*U\mathfrak{g})^{\mathfrak{h}}[2])$  with  $\mathcal{F}^1 = \text{alt}$ . By composing these maps one obtains a  $L_\infty$ -morphism

$$\tilde{\mathcal{F}} : (\mathcal{G}_1, d + [,]) \rightarrow (\mathcal{G}_2, b + [,]_G)$$

with values in  $(\mathcal{G}_2)_{\leq 0}$  and first structure map  $\tilde{\mathcal{F}}^1 = (\text{alt} \otimes 1) \circ (\wedge p \otimes \varepsilon)$ .

**Theorem 4.1.** *There exists a  $L_\infty$ -quasi-isomorphism*

$$\Psi : (\mathcal{G}_1, d + [,]) \rightarrow (\mathcal{G}_2, b + [,]_G)$$

with properties (F1) and (F2) of Theorem 1.7.

*Proof.* First observe that since  $\mathfrak{h}$  is abelian then  $\mathfrak{g}_1 \cong ((\wedge \mathfrak{g})^{\mathfrak{h}} \cap \wedge \mathfrak{m}) \otimes \wedge \mathfrak{h} \otimes S\mathfrak{h}$  as a vector space. Thus if  $\delta_K$  is as in the proof of proposition 2.1 then  $\delta := \text{id} \otimes \delta_K$  is a homotopy operator:  $\delta d + d\delta = \text{id} - \wedge p \otimes \varepsilon$  and  $\delta \circ \delta = 0$ .

Now we proceed like in subsection 2.3: use lemma A.3 to construct a  $L_\infty$ -morphism  $\Psi$  with first structure map  $\Psi^1 = \tilde{\mathcal{F}}^1 + b \circ V + V \circ d$ , where  $V := (\text{alt} \otimes \text{sym}) \circ \delta : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2[-1]$ .

It remains to prove that  $\Psi$  is a quasi-isomorphism. It follows from the first observation in this proof that  $H^*(\mathfrak{g}_1, d) = (\wedge \mathfrak{g})^{\mathfrak{h}} \cap \wedge \mathfrak{m}$ , which also equals  $H^*(\mathfrak{g}_2, b)$  due to the spectral sequence argument. Consequently  $\tilde{\mathcal{F}}^1$  is a quasi-isomorphism of complexes, and so is  $\Psi^1$ .  $\square$

Finally using the same argumentation as in the proof of theorem 0.2 (subsection 1.3) one obtains the

**Theorem 4.2.** *If  $\mathfrak{h}$  is an abelian subalgebra of  $\mathfrak{g}$  with a Lie subalgebra as a complement, then any formal classical dynamical  $r$ -matrix for  $(\mathfrak{g}, \mathfrak{h}, 0)$  admits a dynamical twist quantization (associated to the trivial associator).*

**Example 4.3.** In particular, this allows us to quantize dynamical  $r$ -matrices arising from semi-direct products  $\mathfrak{g} = \mathfrak{m} \ltimes \mathbb{C}^n$  like in [EN, example 3.7].

## CONCLUDING REMARKS

Let us first observe that if  $\mathfrak{h}$  is abelian then  $(\wedge^* \mathfrak{g})^{\mathfrak{h}} \cap \wedge^* \mathfrak{m}[1]$  (resp.  $(T^*U\mathfrak{g})^{\mathfrak{h}} \cap T^*\text{sym}(S\mathfrak{m})[1]$ ) inherits a  $\mathfrak{dgl}\mathfrak{a}$  structure from the one of  $\mathfrak{g}_1[1]$  (resp.  $\mathfrak{g}_2[1]$ ) and  $H^*(\mathfrak{g}_1, d) = (\wedge^* \mathfrak{g})^{\mathfrak{h}} \cap \wedge^* \mathfrak{m} = H^*(\mathfrak{g}_2, b)$ , for any complement  $\mathfrak{m}$  of  $\mathfrak{h}$ . Thus I conjecture that there exists a  $L_\infty$ -quasi-isomorphism between  $(\wedge^* \mathfrak{g})^{\mathfrak{h}} \cap \wedge^* \mathfrak{m}[1]$  and  $(T^*U\mathfrak{g})^{\mathfrak{h}} \cap T^*\text{sym}(S\mathfrak{m})[1]$  which generalizes together  $\phi$  of subsection 2.3 and  $\mathcal{F}$  of section 4. In particular this would imply conjecture 0.1 in the abelian (and non-modified) case.

Let us then mention that one can consider a non-triangular (i.e., non-antisymmetric) version of non-modified classical dynamical  $r$ -matrices. Namely,  $\mathfrak{h}$ -equivariant maps

$r \in \text{Reg}(\mathfrak{h}^*, \mathfrak{g} \otimes \mathfrak{g})$  such that  $\text{CYB}(r) - \text{Alt}(dr) = 0$ . According to [X3], a quantization of such a  $r$  is a  $\mathfrak{h}$ -equivariant map  $R = 1 + \hbar r + O(\hbar^2) \in \text{Reg}(\mathfrak{h}^*, U\mathfrak{g}^{\otimes 2})[[\hbar]]$  that satisfies the *quantum dynamical Yang-Baxter equation* (QDYBE)

$$(16) \quad R^{1,2}(\lambda) * R^{1,3}(\lambda + \hbar h^2) * R^{2,3}(\lambda) = R^{2,3}(\lambda + \hbar h^1) * R^{1,3}(\lambda) * R^{1,2}(\lambda + \hbar h^3)$$

**Question 4.4.** *Does such a quantization always exist?*

The most famous example of non-triangular dynamical  $r$ -matrices was found in [AM] by Alekseev and Meinrenken, then extended successively to a more general context in [EV, ES, EE1], and quantized in [EE1].

Following [EE1], remark that for any non-triangular dynamical  $r$ -matrix  $r$  such that  $r + r^{\text{op}} = t \in (S^2\mathfrak{g})^{\mathfrak{g}}$  (quasi-triangular case) one can define  $\rho := r - t/2$  and  $Z := \frac{1}{4}[t^{1,2}, t^{2,3}]$ . Then  $\rho$  is a modified dynamical  $r$ -matrix for  $(\mathfrak{g}, \mathfrak{h}, Z)$ ; moreover the assignment  $r \mapsto \rho$  is a bijective correspondence between quasi-triangular dynamical  $r$ -matrices for  $(\mathfrak{g}, \mathfrak{h}, t)$  and modified dynamical  $r$ -matrices for  $(\mathfrak{g}, \mathfrak{h}, Z)$ . Now observe that if  $J(\lambda)$  is a dynamical twist quantizing  $\rho$ , then  $R(\lambda) = J^{\text{op}}(\lambda)^{-1} * e^{\hbar t/2} * J(\lambda)$  is a quantum dynamical  $R$ -matrix quantizing  $r$ .

In this paper we have constructed such a dynamical twist in the triangular case  $t = 0$ . One can ask

**Question 4.5.** *Does such a dynamical twist exist for any quasi-triangular dynamical  $r$ -matrix? At least in the reductive and abelian cases?*

This question seems to be more reasonable than the previous one. More generally one can ask if conjecture 0.1 (and its smooth and meromorphic versions) is true in general. A positive answer was given in [EE1] when  $\mathfrak{h} = \mathfrak{g}$ ; but unfortunately it is not known in general, even for the non-dynamical case  $\mathfrak{h} = \{0\}$  (which is the last problem of Drinfeld [Dr1]: quantization of coboundary Lie bialgebras).

Finally let me mention that if  $r(\lambda)$  is a triangular dynamical  $r$ -matrix for  $(\mathfrak{g}, \mathfrak{h})$ , then the bivector field

$$\pi := \overrightarrow{r(\lambda)} + \sum_i \frac{\partial}{\partial \lambda^i} \wedge \overrightarrow{h_i} + \pi_{\mathfrak{h}^*}$$

is a  $G \times H$ -biinvariant Poisson structure on  $G \times \mathfrak{h}^*$  and the projection  $p : G \times \mathfrak{h}^* \rightarrow \mathfrak{h}^*$  is a momentum map. Moreover, according to [X3] any dynamical twist quantization  $J(\lambda)$  of  $r(\lambda)$  allows us to define a  $G \times H$ -biinvariant star-product  $*$  quantizing  $\pi$  on  $G \times \mathfrak{h}^*$  as follows:

$$\begin{array}{ll} f * g = f *_{PBW} g & \text{if } f, g \in C^\infty(\mathfrak{h}^*) \\ f * g = fg & \text{if } f \in C^\infty(G), g \in C^\infty(\mathfrak{h}^*) \\ f * g = \overrightarrow{\exp}(\hbar \sum_i \frac{\partial}{\partial \lambda^i} \otimes \overrightarrow{h_i}) \cdot (f \otimes g) & \text{if } f \in C^\infty(\mathfrak{h}^*), g \in C^\infty(G) \\ f * g = \overrightarrow{J(\lambda)}(f \otimes g) & \text{if } f, g \in C^\infty(G) \end{array}$$

This way the map  $p^* : (\text{Fct}(\mathfrak{h}^*)[[\hbar]], *_PBW) \rightarrow (\text{Fct}(G \times \mathfrak{h}^*)[[\hbar]], *)$  becomes a quantum momentum map in the sens of [X1].

So there may be a way to see momentum maps and their quantum analogues as Maurer-Cartan elements in  $\text{dgl}_a$ 's.

## APPENDIX A. HOMOTOPY LIE ALGEBRAS

See [HS] for a detailed discussion of the theory.

Recall that a  $L_\infty$ -algebra structure on a graded vector space  $\mathfrak{g}$  is a degree 1 coderivation  $Q$  on the cofree cocommutative coalgebra  $C^c(\mathfrak{g}[1])$  such that  $Q \circ Q = 0$ . By cofreeness, such a coderivation  $Q$  is uniquely determined by structure maps  $Q^n : \Lambda^n \mathfrak{g} \rightarrow \mathfrak{g}[2-n]$  which satisfy an infinite collection of equations. In particular  $(\mathfrak{g}, Q^1)$  is a cochain complex.

**Example A.1.** Any  $\mathfrak{dgl}_a(\mathfrak{g}, d, [,])$  is canonically a  $L_\infty$ -algebra. Namely,  $Q$  is given by structure maps  $Q^1 = d$ ,  $Q^2 = [,]$  and  $Q^n = 0$  for  $n > 2$ .

A  $L_\infty$ -morphism between two  $L_\infty$ -algebras  $(\mathfrak{g}_1, Q_1)$  and  $(\mathfrak{g}_2, Q_2)$  is a degree 0 morphism of coalgebras  $F : C^c(\mathfrak{g}_1[1]) \rightarrow C^c(\mathfrak{g}_2[1])$  such that  $F \circ Q_1 = Q_2 \circ F$ . Again by cofreeness, such a morphism is uniquely determined by structure maps  $F^n : \Lambda^n \mathfrak{g}_1 \rightarrow \mathfrak{g}_2[1-n]$  which satisfy an infinite collection of equations. In particular  $F^1 : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$  is a morphism of complexes; when it induces an isomorphism in cohomology we say that  $F$  is a  $L_\infty$ -quasi-isomorphism.

**Example A.2.** Any morphism of  $\mathfrak{dgl}_a$ 's is a  $L_\infty$ -morphism with all structure maps equal to zero except the first one.

In this paper we use many times the following

**Lemma A.3** ([Do]). *Let  $F : C^c(\mathfrak{g}_1[1]) \rightarrow C^c(\mathfrak{g}_2[1])$  be a  $L_\infty$ -morphism. For any linear map  $V : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2[-1]$  there exists a  $L_\infty$ -morphism  $\Psi : C^c(\mathfrak{g}_1[1]) \rightarrow C^c(\mathfrak{g}_2[1])$  with first structure map  $\Psi^1 = F^1 + Q_2^1 \circ V + V \circ Q_1^1$ . Moreover, if  $F$  is a  $L_\infty$ -quasi-isomorphism then  $\Psi$  is also.*

*Proof.* First remark that  $V$  extends uniquely to a linear map  $C^c(\mathfrak{g}_1[1]) \rightarrow C^c(\mathfrak{g}_2[1])$  of degree  $-1$  such that

$$\Delta_2 \circ V = (F \otimes V + V \otimes F + \frac{1}{2} V \otimes (Q_2 \circ V + V \circ Q_1) + \frac{1}{2} (Q_2 \circ V + V \circ Q_1) \otimes V) \circ \Delta_1$$

where  $\Delta_1$  and  $\Delta_2$  denote comultiplications in  $C^c(\mathfrak{g}_1[1])$  and  $C^c(\mathfrak{g}_2[1])$ , respectively. Then define  $\Psi := F + Q_2 \circ V + V \circ Q_1$ .  $\square$

**Remark A.4.** Assume that in the previous lemma  $\mathfrak{g}_1$  and  $\mathfrak{g}_2$  are filtrated,  $F$  is such that  $F^n$  takes values in  $(\mathfrak{g}_2)_{\leq n-1}$ , and  $V((\mathfrak{g}_1)_{\leq k}) \subset (\mathfrak{g}_2)_{\leq k+1}$ . Then one can obviously check that for any  $X \in (\Lambda^n \mathfrak{g}_1)_{\leq k}$ ,  $F^n(X) \in (\mathfrak{g}_2)_{\leq n+k-1}$ .

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