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# CYCLES OF FREE WORDS IN SEVERAL INDEPENDENT RANDOM PERMUTATIONS WITH RESTRICTED CYCLE LENGTHS

FLORENT BENAYCH-GEORGES

ABSTRACT. In this text, we consider random permutations which can be written as free words in several independent random permutations: firstly, we fix a non trivial word  $w$  in letters  $g_1, g_1^{-1}, \dots, g_k, g_k^{-1}$ , secondly, for all  $n$ , we introduce a  $k$ -tuple  $s_1(n), \dots, s_k(n)$  of independent random permutations of  $\{1, \dots, n\}$ , and the random permutation  $\sigma_n$  we are going to consider is the one obtained by replacing each letter  $g_i$  in  $w$  by  $s_i(n)$ . For example, for  $w = g_1 g_2 g_3 g_2^{-1}$ ,  $\sigma_n = s_1(n) \circ s_2(n) \circ s_3(n) \circ s_2(n)^{-1}$ . Moreover, we restrict the set of possible lengths of the cycles of the  $s_i(n)$ 's: we fix sets  $A_1, \dots, A_k$  of positive integers and suppose that for all  $n$ , for all  $i$ ,  $s_i(n)$  is uniformly distributed on the set of permutations of  $\{1, \dots, n\}$  which have all their cycle lengths in  $A_i$ . For all positive integer  $l$ , we are going to give asymptotics, as  $n$  goes to infinity, on the number  $N_l(\sigma_n)$  of cycles of length  $l$  of  $\sigma_n$ . We shall also consider the joint distribution of the random vectors  $(N_1(\sigma_n), \dots, N_l(\sigma_n))$ . We first prove that the order of  $w$  in a certain quotient of the free group with generators  $g_1, \dots, g_k$  determines the asymptotic order of the random variables  $N_l(\sigma_n)$  as  $n$  goes to infinity. We also prove that in many cases, the distribution of  $N_l(\sigma_n)$  converges to a Poisson law with parameter  $1/l$  and that the random variables  $N_1(\sigma_n), N_2(\sigma_n), \dots$  are asymptotically independent. We notice the surprising fact that from this point of view, many things happen as if  $\sigma_n$  were uniformly distributed on the  $n$ -th symmetric group.

## 1. INTRODUCTION

**1.1. General introduction.** It is well known that if  $\sigma_n$  is a uniformly distributed random permutation of  $\{1, \dots, n\}$ , then if for all  $l \geq 1$ , one denotes the number of cycles of  $\sigma_n$  with length  $l$  by  $N_l(\sigma_n)$ , the joint distribution of  $(N_1(\sigma_n), N_2(\sigma_n), \dots)$  converges in distribution, as  $n$  goes to infinity, to  $\text{Poisson}(1/1) \otimes \text{Poisson}(1/2) \otimes \dots$  (see [ABT05]). The law of “large cycles” can be recovered using the Poisson-Dirichlet distribution, but in this paper, we are only going to deal with “small cycles”, i.e. cycles with a size which does not vary with  $n$ .

In [Ni94], Nica was the first to consider the case where  $\sigma_n$  is a free word in several independent random permutations: he fixed  $k \geq 1$  and a free word  $w$  in the letters  $g_1, g_1^{-1}, \dots, g_k, g_k^{-1}$ , he introduced, for all  $n$ , a  $k$ -tuple  $s_1(n), \dots, s_k(n)$  of independent random permutations uniformly distributed on the  $n$ -th symmetric group, and he proved that under the hypothesis that  $w$  is not a power of another word, if one defines  $\sigma_n$  to be the random permutation obtained by replacing each letter  $g_i$  in  $w$  by  $s_i(n)$ , then for all positive integer  $l$ ,  $N_l(\sigma_n)$  converges in distribution, as  $n$  goes to infinity, to a Poisson distribution with mean  $1/l$ . His result was reproved recently by Linial and Puder in [LP08], where it appeared that free words in random permutations are relevant to analyze spectrums of  $n$ -lifts of graphs. None of these papers considered the joint distribution of the  $N_l(\sigma_n)$ 's.

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Independently, in [Y07] and in [BG07], the authors considered random permutations whose distribution is not uniform on the whole symmetric group, but on the set of permutations of  $\{1, \dots, n\}$  which have all their cycle lengths in a fixed set  $A$  of positive integers. This kind of random permutation has been the subject of study by a number of authors for more than thirty years (for a review of the literature in this area, see [Y05]). It was proved (see [BG07]) that if  $A$  is finite with greatest element  $d$ , then as  $n$  goes to infinity, such a random permutation tends to get not far away from having order  $d$ : the cardinality of the subset of  $\{1, \dots, n\}$  covered by the supports of cycles with length  $d$  in such a random permutation is asymptotic to  $n$ . It was also proved ([Y07, BG07]) that if  $A$  is infinite, then the order of such a random permutation goes to infinity as  $n$  does.

In the present paper, we shall somehow mix both of the previously presented problems: we shall fix  $k \geq 1$ , a free word  $w$  in the letters  $g_1, g_1^{-1}, \dots, g_k, g_k^{-1}$ , and a list  $A_1, \dots, A_k$  of sets of positive integers satisfying a technical assumption:

$$(1) \quad \forall i, \quad A_i \text{ is finite or satisfies } \sum_{j \geq 1, j \notin A_i} \frac{1}{j} < \infty.$$

Then for all  $n$  such that it is possible, we shall introduce a  $k$ -tuple  $s_1(n), \dots, s_k(n)$  of independent random permutations such that for all  $i$ ,  $s_i(n)$  is uniformly distributed on the set of permutations of  $\{1, \dots, n\}$  having all of their cycle lengths in  $A_i$  and define  $\sigma_n$  to be the random permutation obtained by replacing each letter  $g_i$  in  $w$  by  $s_i(n)$ .

**1.2. Link with a quotient of the free group generated by  $g_1, \dots, g_k$ .** Since, as explained above, for  $i$  such that  $A_i$  is finite,  $s_i(n)$  is not far away from having order  $d_i := \sup A_i$ , it is natural to expect that for large values of  $n$ , the distribution of  $\sigma_n$  will depend firstly on the word obtained from  $w$  by removing all sequences of the type  $g_i^{\pm d_i}$  for  $i$  such that  $A_i$  is finite. This is what has led us to introduce the group  $F_k/[g_1^{d_1}, \dots, g_k^{d_k}]$  generated by free elements  $g_1, \dots, g_k$  and quotiented by the relations  $g_i^{d_i} = 1$  for  $i \in [k]$ , with  $d_i = \sup A_i$  (when  $d_i = +\infty$ , the relation  $g_i^{d_i} = 1$  is not included). Our more general results, respectively Theorems 3.6 and 3.4, are the followings:

- If the element of  $F_k/[g_1^{d_1}, \dots, g_k^{d_k}]$  represented by  $w$  has finite order  $d \geq 1$ , then, as  $n$  tends to infinity,  $N_d(\sigma_n)/n$  converges to  $1/d$  and for all  $l \neq d$ ,  $N_l(\sigma_n)/n$  converges to 0. This means that  $\sigma_n$  is not far away from having order  $d$ : the cardinality of the subset of  $\{1, \dots, n\}$  covered by the supports of the cycles with length  $d$  in such a random permutation is asymptotic to  $n$ .
- If the element of  $F_k/[g_1^{d_1}, \dots, g_k^{d_k}]$  represented by  $w$  has infinite order, then two cases can occur:
  - (i) The element of  $F_k/[g_1^{d_1}, \dots, g_k^{d_k}]$  represented by  $w$  is not conjugate to an element represented by a word of the type  $g_i^\alpha$ , with  $i \in [k]$ ,  $\alpha$  an integer: then for all  $l \geq 1$ ,

$$\liminf_{n \rightarrow \infty} (\mathbb{E}(N_l(\sigma_n))) \geq 1/l.$$

- (ii) There is  $i \in [k]$  such that  $A_i$  is infinite and  $\alpha$  a nonzero integer such that  $w$  represents the same element, in  $F_k/[g_1^{d_1}, \dots, g_k^{d_k}]$ , as the word  $g_i^\alpha$ : then for all  $l \geq 1$  such that  $l|\alpha| \in A_i$ ,

$$\liminf_{n \rightarrow \infty} (\mathbb{E}(N_l(\sigma_n))) \geq 1/l.$$

This means that the cycles of the letters of  $w$  are going to mix sufficiently well to give rise to cycles of most lengths, at least as much as in a uniform random permutation, even when the letters of  $w$  have very restricted cycle lengths.

**1.3. Weak limit of the distributions of the  $N_l(\sigma_n)$ 's.** Then, we are going to prove more precise results: under certain hypotheses on  $w$ , for all  $l \geq 1$ , the joint distribution of the random vector

$$(N_1(\sigma_n), \dots, N_l(\sigma_n))$$

converges weakly, as  $n$  goes to infinity, to

$$\text{Poisson}(1/1) \otimes \text{Poisson}(1/2) \otimes \dots \otimes \text{Poisson}(1/l),$$

just as if the distribution of  $\sigma_n$  would have been uniform on  $\mathfrak{S}_n$ . When all  $A_i$ 's are infinite (Theorem 3.7), our hypotheses are that  $w$  is neither a single letter nor a power of another word. If some of the  $A_i$ 's are finite, we prove this result for  $w = g_1 \cdots g_k$  (Theorem 3.8), with the exception of the case  $k = 2$  and  $A_1 \cup A_2 \subset \{1, 2\}$  (i.e. of the product of two random involutions), where we prove that the  $N_l(\sigma_n)$ 's are still asymptotically independent, but that their asymptotic distributions are slightly different (Theorem 3.12).

These results extend on the one hand a result of Nica (reproved by Linial and Puder in [LP08]), who proved in [Ni94] that in the case where all  $A_i$ 's are equal to the set of all positive integer and where  $w$  is not a power of another word, for all  $l \geq 1$ , the distribution of  $N_l(\sigma_n)$  converges weakly to  $\text{Poisson}(1/l)$  (without considering the joint distribution), and on the other hand a result of Neagu, who proved in [Ne07] that the number of fixed points of  $\sigma_n$  is  $o(n)$ . Here, we shall mention that much of the methods we use in this paper are inspired by the ones invented in [Ni94, Ne07], and that we use many of their results.

**1.4. Comments on these results and open questions.** a) In this paper, only small cycles (i.e. cycles which size does not depend on  $n$ ) are considered by our investigations. It would be interesting to know if it is possible to prove a limit theorem for the whole random partition of  $\{1, \dots, n\}$  defined by the cycles of  $\sigma_n$ . It is natural to expect a Poisson-Dirichlet distribution (see [P02] or [ABT05]).

b) Another interesting question arising from the similarity between  $\sigma_n$  and a uniform random permutation on  $\mathfrak{S}_n$  is the following: do we have a characterization of the words  $w$  such that for  $n$  large enough,  $\sigma_n$  is uniformly distributed ?

**1.5. Notations.** In this text, for  $n$  an integer, we shall denote by  $[n]$  the set  $\{1, \dots, n\}$  and by  $\mathfrak{S}_n$  the group of permutations of  $[n]$ . For  $A$  a set of positive integers,  $\mathfrak{S}_n(A)$  denotes the set of permutations of  $[n]$  whose cycles have length in  $A$ . For  $\sigma \in \mathfrak{S}_n$  and  $l \geq 1$ , we shall denote by  $N_l(\sigma)$  the number of cycles of length  $l$  in the decomposition of  $\sigma$  as a product of cycles with disjoint supports. For  $\lambda > 0$ ,  $\text{Poisson}(\lambda)$  will denote the Poisson distribution with parameter  $\lambda$ .

## 2. COMBINATORIAL PRELIMINARIES TO THE STUDY OF WORDS IN RANDOM PERMUTATIONS

### 2.1. Words and groups generated by relations.

2.1.1. *Words.* Let, for  $k \geq 1$ ,  $\mathbb{M}_k$  be the set of *words* in the letters  $g_1, g_1^{-1}, \dots, g_k, g_k^{-1}$ , i.e. the set of sequences  $g_{i_1}^{\alpha_1} \cdots g_{i_n}^{\alpha_n}$ , with  $n \geq 0$ ,  $i_1, \dots, i_n \in [k]$ ,  $\alpha_1, \dots, \alpha_n = \pm 1$ . A word  $w \in \mathbb{M}_k$  is said to be *reduced* if in its writing, no letter is followed by its inverse. It is said to be *cyclically reduced* if moreover, the first and the last letters are not the inverses of each other.

A cyclically reduced word is said to be *primitive* if it is not the concatenation of  $d \geq 2$  times the same word.

For  $w = g_{i_1}^{\alpha_1} \cdots g_{i_{|w|}}^{\alpha_{|w|}} \in \mathbb{M}_k$  and  $s = (s_1, \dots, s_k)$  a  $k$ -tuple of elements of a group,  $w(s)$  denotes  $s_{i_1}^{\alpha_1} \cdots s_{i_{|w|}}^{\alpha_{|w|}}$ .

2.1.2. *The quotient of the free group with  $k$  generators by the relations  $g_1^{d_1} = 1, \dots, g_k^{d_k} = 1$ .* Let  $F_k$  be the free group generated by  $g_1, \dots, g_k$ . This is the set of reduced words of  $\mathbb{M}_k$  endowed with the operation of concatenation-reduction via the relations  $g_i g_i^{-1} = 1, g_i^{-1} g_i = 1, i \in [k]$ . For  $w, w_0 \in \mathbb{M}_k$  with  $w_0$  reduced,  $w$  is said to *represent* or to be a *writing* of the element  $w_0$  of  $F_k$  if one can reduce  $w$  to  $w_0$  via the previous relations. For example,  $g_1 g_3 g_3^{-1} g_1$  is a writing of  $g_1^2$ .

Consider  $d_1, \dots, d_k \in \{2, 3, 4, \dots\} \cup \{\infty\}$ . Let  $F_k/[g_1^{d_1}, \dots, g_k^{d_k}]$  be the group  $F_k$  quotiented by its normal subgroup generated by the set  $\{g_i^{d_i}; i \in [k], d_i < \infty\}$ . A word  $w \in \mathbb{M}_k$  is said to *represent* or to be a *writing* of an element  $C$  of  $F_k/[g_1^{d_1}, \dots, g_k^{d_k}]$  if it is a writing of an element of  $C$  (seen as a subset of  $F_k$ ).

Theorem 1.4 of Section 1.4 of [MKS66] states the following facts.

**Theorem 2.1.** (a) *Any element of  $F_k/[g_1^{d_1}, \dots, g_k^{d_k}]$  has a writing of the type  $g_{i_1}^{\alpha_1} \cdots g_{i_n}^{\alpha_n}$  with  $n \geq 1$ ,  $i_1 \neq i_2 \neq \dots \neq i_n \in [k]$ ,  $\alpha_1, \dots, \alpha_n$  integers such that  $0 < |\alpha_1| < d_{i_1}, \dots, 0 < |\alpha_n| < d_{i_n}$ , and this writing is unique up to replacements of the type*

$$g_{i_j}^{\alpha_j} \rightarrow \begin{cases} g_{i_j}^{d_{i_j} + \alpha_j} & \text{if } \alpha_j < 0 \text{ and } d_{i_j} < \infty, \\ -d_{i_j} + \alpha_j & \\ g_{i_j} & \text{if } \alpha_j > 0 \text{ and } d_{i_j} < \infty, \end{cases}$$

with  $j \in [n]$ .

(b) *In any conjugacy class of the group  $F_k/[g_1^{d_1}, \dots, g_k^{d_k}]$ , there is an element represented by a word of the previous type such that moreover,  $i_n \neq i_1$ , and such a word is unique up to replacements of the previous type and to transformations of the type  $g_{i_1}^{\alpha_1} \cdots g_{i_n}^{\alpha_n} \rightarrow g_{i_n}^{\alpha_n} g_{i_1}^{\alpha_1} \cdots g_{i_{n-1}}^{\alpha_{n-1}}$ .*

Let us define the  $(d_1, \dots, d_k)$ -*cyclically reduced words* to be the words of the type  $g_{i_1}^{\alpha_1} \cdots g_{i_n}^{\alpha_n}$  with  $n \geq 0$ ,  $i_1 \neq i_2 \neq \dots \neq i_n \neq i_1 \in [k]$ ,  $\alpha_1, \dots, \alpha_n$  integers such that  $0 < |\alpha_1| < d_{i_1}, \dots, 0 < |\alpha_n| < d_{i_n}$ .

Consider a cyclically reduced word  $w \in \mathbb{M}_k$ . We shall call a *partial  $(d_1, \dots, d_k)$ -cyclic reduction of  $w$*  a word which can be obtained from  $w$  in a finite number of steps of the following types:

- (a)  $wv \rightarrow vu$ , with  $u, v \in \mathbb{M}_k$ ,
- (b)  $u g_i^{\alpha d_i} u^{-1} v \rightarrow v$ , with  $i \in [k]$ ,  $\alpha$  a nonzero integer,  $u$  a nonempty word and  $v \in \mathbb{M}_k$  a cyclically reduced word,
- (c)  $g_i^{\alpha d_i} v \rightarrow v$ , with  $i \in [k]$ ,  $\alpha$  a nonzero integer, and  $v$  a word which can be written  $v = g_i^\beta v'$ , with  $\beta$  an integer,  $|\beta| < d_i$  and  $v'$  a word whose first and last letters do not belong to  $\{g_i, g_i^{-1}\}$ .

A partial  $(d_1, \dots, d_k)$ -cyclic reduction of  $w$  which is  $(d_1 \dots, d_k)$ -cyclically reduced is called a  $(d_1 \dots, d_k)$ -cyclic reduction of  $w$ . In general, there is more than one  $(d_1 \dots, d_k)$ -cyclic reduction of  $w$ , but if there is only one, we call it *the*  $(d_1 \dots, d_k)$ -cyclic reduction of  $w$ .

For example, for  $d_1 = 4, d_2 = 5$ , the following words are partial  $(d_1, \dots, d_k)$ -cyclic reductions of  $w = g_2^2 g_1 g_2^6 g_3 g_1^{-4} g_3^{-1} g_1^{-1} g_2^3$ :

$$g_2^5 g_1 g_2^6 g_3 g_1^{-4} g_3^{-1} g_1^{-1}, \quad g_2^6 g_3 g_1^{-4} g_3^{-1}, \quad g_2 g_3 g_1^{-4} g_3^{-1}, \quad g_3 g_1^{-4} g_3^{-1} g_2, \quad g_2.$$

The last one is a  $(d_1 \dots, d_k)$ -cyclic reduction of  $w$ .

## 2.2. Admissible graphs and partitions. Colored graphs associated to words and permutations.

**2.2.1. Basic graph theoretic definitions.** In this text, we shall consider *oriented edge-colored graphs with color set*  $[k]$ . These are families  $G = (V; E_1, \dots, E_k)$ , where  $V$  is a finite set (its elements are called the *vertices* of  $G$ ) and for all  $r \in [k]$ ,  $E_r$  is a subset of  $V^2$  (the set of *edges with color*  $r$  of  $G$ ). For  $e = (u, v)$  edge of  $G$ ,  $u$  (resp.  $v$ ) is called the *beginning vertex* of  $e$  (resp. the *ending vertex* of  $e$ ) and is denoted by  $\text{Beg}(e)$  (resp.  $\text{End}(e)$ ).  $e$  is often denoted by  $u \rightarrow v$ . Note that the  $E_r$ 's are not supposed to be pairwise disjoint. Throughout this paper, the color set of the edges will always be  $[k]$ , so it will often be implicit.

For  $r \in [k]$ ,  $G = (V; E_1, \dots, E_k)$  is said to be *monochromatic with color*  $r$  if for all  $i \neq r$ ,  $E_i = \emptyset$ . If moreover,

$$V = \{v_1, \dots, v_l\} \text{ and } E_r = \{v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_l\} \text{ (resp. } E_r = \{v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_l \rightarrow v_1\}),$$

with  $l \geq 1$  and  $v_1, \dots, v_l$  pairwise distinct,  $G$  is said to be a *monochromatic directed path* (resp. *monochromatic directed cycle*) with *length*  $l - 1$  (resp.  $l$ ).

A *subgraph* of a graph  $(V; E_1, \dots, E_k)$  is a graph of the type  $(W; F_1, \dots, F_k)$ , with  $W \subset V, F_1 \subset E_1, \dots, F_k \subset E_k$ .

A *monochromatic directed cycle* (resp. a *monochromatic directed path*) of an oriented edge-colored graph is a subgraph which is a monochromatic directed cycle (resp. a monochromatic directed path).

A graph  $(V; E_1, \dots, E_k)$  is said to be the *disjoint union* of the graphs  $(V'; E'_1, \dots, E'_k)$  and  $(V''; E''_1, \dots, E''_k)$  if  $V = V' \cup V''$ ,  $E_1 = E'_1 \cup E''_1, \dots, E_k = E'_k \cup E''_k$  and all of these unions are disjoint.

An *isomorphism* between two oriented edge-colored graphs  $(V; E_1, \dots, E_k)$ ,  $(V'; E'_1, \dots, E'_k)$  is a bijection  $\varphi : V \rightarrow V'$  such that for all  $u, v \in V$ , for all  $r \in [k]$ , one has

$$u \rightarrow v \in E_r \iff \varphi(u) \rightarrow \varphi(v) \in E'_r.$$

**2.2.2. Admissible graphs and partitions.** Recall that a *partition*  $\Delta$  of a set  $X$  is a set of pairwise disjoint, nonempty subsets of  $X$  (called the *classes* of  $\Delta$ ) whose union is  $X$ . In this case, for  $x, y \in X$ , “ $x = y \pmod{\Delta}$ ” and “ $\Delta$  links  $x$  and  $y$ ” both mean that  $x, y$  are in the same class of  $\Delta$ . Since  $\Delta$  is a set,  $|\Delta|$  denotes its cardinality.

For any function  $\gamma$  defined on a set  $X$ , we shall denote by  $\text{Part}(\gamma)$  the partition of  $X$  by the level sets of  $\gamma$ .

Let  $G$  be an oriented edge-colored graph with color set  $[k]$  and with vertices set  $V$ .

$G$  is said to be *admissible* if two different edges with the same color cannot have the same beginning or the same ending vertex.

We are going to use the notion of *quotient graph*, that we define now. Let us define, for  $\Delta$  a partition of  $V$ ,  $G/\Delta$  to be the oriented edge-colored graph whose vertices are the classes of  $\Delta$  and such that for all  $C, C'$  classes of  $\Delta$ , for all  $r \in [k]$ , there is an edge with color  $r$  from  $C$  to  $C'$  in  $G/\Delta$  when there is an edge with color  $r$  in  $G$  from a vertex of  $C$  to a vertex of  $C'$ .

**Remark 2.2.** Note that clearly, if  $\Delta_1$  is a partition of  $V$  and  $\Delta_2$  is a partition of  $\Delta_1$ , then the oriented edge-colored graph  $(G/\Delta_1)/\Delta_2$  is isomorphic to  $G/\Gamma$ , where  $\Gamma$  is the partition of  $V$  such that for all  $x, y \in V$ ,  $x = y \pmod{\Gamma}$  if and only if the classes of  $x$  and  $y$  in  $\Delta_1$  are in the same class of  $\Delta_2$ .

A partition  $\Delta$  of  $V$  is said to be an *admissible partition* of  $G$  when the graph  $G/\Delta$  is admissible, i.e. when for all pair  $(e, f)$  of edges of  $G$  with the same color,

$$\text{Beg}(e) = \text{Beg}(f) \pmod{\Delta} \iff \text{End}(e) = \text{End}(f) \pmod{\Delta}.$$

In the following proposition, whose proof is obvious, we define the operator  $\text{Adm}$  on the set of oriented edge-colored graphs.

**Proposition 2.3.** Let  $G$  be an oriented edge-colored graph. Let  $\Delta$  be the partition of the vertex set of  $G$  which links two vertices  $r, s$  if and only if there are  $n \geq 0$ ,  $t_0 = r, t_1, \dots, t_n = s$  some vertices of  $G$ ,  $i_1, \dots, i_n \in [k]$ , and  $\varepsilon_1, \dots, \varepsilon_n \in \{\pm 1\}$  such that:

- the reduction of the word  $g_{i_1}^{\varepsilon_1} \dots g_{i_n}^{\varepsilon_n}$  is the empty word,
- for all  $l \in [n]$ , the  $i_l$ -colored edge  $t_{l-1} \rightarrow t_l$  (resp.  $t_{l-1} \leftarrow t_l$ ) belongs to  $G$  if  $\varepsilon_l = 1$  (resp. if  $\varepsilon_l = -1$ ).

Then  $\Delta$  is the minimal partition  $P$  (with respect to the refinement order) such that  $G/P$  is admissible.

With the notations of the previous proposition, the partition  $\Delta$  will be called the *minimal admissible partition* of  $G$  and  $G/\Delta$  will be denoted by  $\text{Adm}(G)$ .

**Remark 2.4.** Let  $H$  be an oriented edge-colored graph with vertex set  $W$  and let  $G$  be a subgraph of  $H$  with vertex set  $V$ . Then  $\text{Adm}(H)$  can be obtained (up to an isomorphism) from  $\text{Adm}(G)$ , by adding to  $\text{Adm}(G)$  the vertices and edges of  $H$  which are not in  $G$ , and then quotienting the obtained graph by its minimal admissible partition. More specifically, if, for each vertex  $v$  of  $G$ , one denotes by  $\bar{v}$  the class of  $v$  in the minimal admissible partition of  $G$ , then  $\text{Adm}(H)$  can be identified, via an isomorphism, with  $\text{Adm}(H')$ , where  $H'$  is the oriented edge-colored graph whose vertex set is the union of the vertex set of  $\text{Adm}(G)$  with the set  $W \setminus V$  and whose set of edges of color  $i$ , for each  $i \in [k]$ , is the union of the set of edges of color  $i$  of  $\text{Adm}(G)$  with the set

$$\begin{aligned} & \{w \rightarrow w'; w, w' \in W \setminus V, w \rightarrow w' \text{ } i\text{-colored edge of } H\} \\ & \cup \{\bar{v} \rightarrow w; v \in V, w \in W \setminus V, v \rightarrow w \text{ } i\text{-colored edge of } H\} \\ & \cup \{w \rightarrow \bar{v}; v \in V, w \in W \setminus V, w \rightarrow v \text{ } i\text{-colored edge of } H\}. \end{aligned}$$

**Lemma 2.5.** Consider an oriented edge-colored graph  $G = (V; E_1, \dots, E_k)$ ,  $r, s, t \in V$ ,  $i \in [k]$  such that  $s \rightarrow t \in E_i$ . Then for  $H$  the oriented edge-colored graph obtained from  $G$  by adding, if it is not already in  $G$ , the  $i$ -colored edge  $r \rightarrow t$  (resp.  $s \rightarrow r$ ),  $\text{Adm}(H)$  is isomorphic to  $\text{Adm}(G/\{r = s\})$  (resp.  $\text{Adm}(G/\{r = t\})$ ), where for  $u, v \in V$ ,  $\{u = v\}$  denotes the partition of  $V$  whose classes are all singletons, except one:  $\{u, v\}$ .

**Proof.** The proof is immediate with Remark 2.2 and the following observation: for  $\Gamma$  an admissible partition of  $G$ , if, for any  $v \in V$ , one denotes by  $\bar{v}$  the class of  $v$  in  $\Gamma$ , then

$$\bar{r} = \bar{s} \text{ (resp. } \bar{r} = \bar{t}) \iff \bar{r} \rightarrow \bar{t} \text{ (resp. } \bar{s} \rightarrow \bar{r}) \text{ is an } i\text{-colored edge of } G/\Gamma.$$

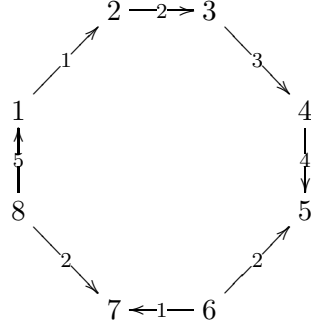
□

2.2.3. *The graph  $G(\sigma, w)$ .* Fix  $k, p \geq 1$ ,  $\sigma \in \mathfrak{S}_p$  and  $w = g_{i_1}^{\alpha_1} \cdots g_{i_{|w|}}^{\alpha_{|w|}} \in \mathbb{M}_k$  a nonempty word. In [Ni94], Nica defined  $G(\sigma, w)$  (denoted by  $\mathcal{H}_{\sigma^{-1}} \star w$  in his paper) to be the directed, edge-colored graph with vertex set  $V := [p] \times [|w|]$  and whose edges are the followings: for all  $(m, l) \in [p] \times [|w|]$ ,

$$\begin{aligned} \alpha_l = 1 &\Rightarrow (m, l) \rightarrow \begin{cases} (m, l+1) & \text{if } l \neq |w|, \\ (\sigma^{-1}(m), 1) & \text{if } l = |w|, \end{cases} \text{ is an } i_l\text{-colored edge of } G(\sigma, w), \\ \alpha_l = -1 &\Rightarrow (m, l) \leftarrow \begin{cases} (m, l+1) & \text{if } l \neq |w|, \\ (\sigma^{-1}(m), 1) & \text{if } l = |w|, \end{cases} \text{ is an } i_l\text{-colored edge of } G(\sigma, w). \end{aligned}$$

In the case where  $p = 1$  and  $\sigma = Id$ , we shall denote  $G(\sigma, w)$  by  $G(w)$  and identify its vertex set with  $[|w|]$  by  $(1, l) \simeq l$  for all  $l \in [|w|]$ . We will also use the convention that if  $w$  is the empty word, then  $G(w)$  is the graph having 1 for only vertex and no edge.

For example, for  $w = g_1 g_2 g_3 g_4 g_2^{-1} g_1 g_2^{-1} g_5$ ,  $G(w)$  is the graph



where the colors of the edges appear on them. We shall give examples of graphs of the type  $G(\sigma, w)$  after the following lemma, which expresses  $G(\sigma, w)$  as a disjoint union of graphs isomorphic to  $G(w^\alpha)$ , with  $\alpha \geq 1$ .

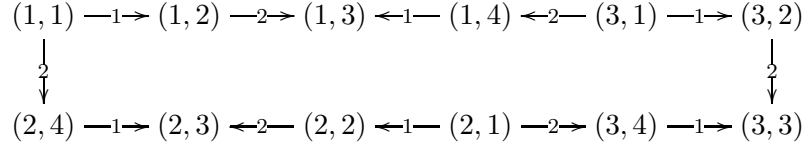
**Lemma 2.6.**  $G(\sigma, w)$  is the disjoint union of  $N_1(\sigma)$  graphs isomorphic to  $G(w^1)$ ,  $N_2(\sigma)$  graphs isomorphic to  $G(w^2)$ ,  $\dots$ ,  $N_p(\sigma)$  graphs isomorphic to  $G(w^p)$ .

**Proof.** Let us denote  $G(\sigma, w) = (V; E_1, \dots, E_k)$ . If  $I, J$  are disjoint subsets of  $[p]$  stabilized by  $\sigma$ , then in  $G(\sigma, w)$ , there is no edge between elements of  $I \times [|w|]$  and  $J \times [|w|]$ . Hence  $G(\sigma, w)$  is the disjoint union of the graphs  $(V(c); E_1(c), \dots, E_k(c))$ , where  $c$  varies over the set of cycles of  $\sigma$ , and where for any such cycle  $c$ , with support  $C \subset [p]$ ,  $V(c) = C \times [|w|]$  and for any  $i \in [k]$ ,  $E_i(c) = E_i \cap (V(c)^2)$ .

Hence it suffices to prove that for any  $d \in [p]$ , for any cycle  $c = (m_1 m_2 \cdots m_d)$  of  $\sigma$  of length  $d$ , there is an isomorphism between  $(V(c); E_1(c), \dots, E_k(c))$  and  $G(w^d)$ . The function which maps  $(m_i, l) \in \{m_1, m_2, \dots, m_d\} \times [|w|]$  to  $(d-i)|w| + l \in [d|w|]$  is such an isomorphism. □

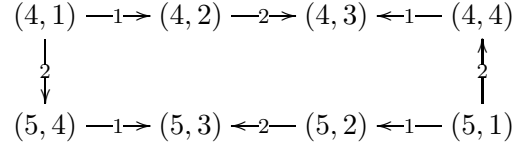


For example, when  $w = g_1 g_2 g_1^{-1} g_2^{-1}$ ,  $p = 3$  and  $\sigma$  is the cycle (123),  $G(\sigma, w)$  is the graph



where the colors of the edges appear on them.

When  $w$  is still  $g_1 g_2 g_1^{-1} g_2^{-1}$ , but  $p = 5$  and  $\sigma$  is the product of disjoint cycles (123)(45),  $G(\sigma, w)$  is the disjoint union of the previous graph and of



#### 2.2.4. Admissible graphs with restricted monochromatic directed cycle and directed path lengths.

Let us fix  $A_1, \dots, A_k$  nonempty sets of positive integers, none of them being  $\{1\}$ . Let  $d_1, \dots, d_k$  denote respectively  $\sup A_1, \dots, \sup A_k$  (which can be infinite).

If an oriented edge-colored graph  $G$  is admissible, it is easy to see that for all  $i \in [k]$ , the graph obtained from  $G$  by erasing all edges whose color is not  $i$  is a disjoint union of directed paths and directed cycles. If for each  $i \in [k]$ , all these directed cycles have length in  $A_i$  (resp. length equal to  $d_i$ ) and all these directed paths have length  $< d_i$ ,  $G$  will be said to be  $(A_1, \dots, A_k)$ -admissible (resp.  $(d_1, \dots, d_k)$ -strongly admissible). A partition  $\Delta$  of the vertex set of an oriented edge-colored graph  $G$  will be said to be an  $(A_1, \dots, A_k)$ -admissible partition (resp. a  $(d_1, \dots, d_k)$ -strongly admissible partition) if  $G/\Delta$  is  $(A_1, \dots, A_k)$ -admissible (resp.  $(d_1, \dots, d_k)$ -strongly admissible).

We define the *Neagu characteristic* of an admissible oriented edge-colored graph  $G$  to be

$$\chi(G) = |\{\text{vertices of } G\}| - \sum_{r=1}^k |\{\text{edges of } G \text{ with color } r\}| + \sum_{r=1}^k \sum_{\substack{L \text{ directed cycle} \\ \text{of } G \text{ with color } r}} \frac{\text{length of } L}{d_r},$$

with the convention  $l/\infty = 0$ .

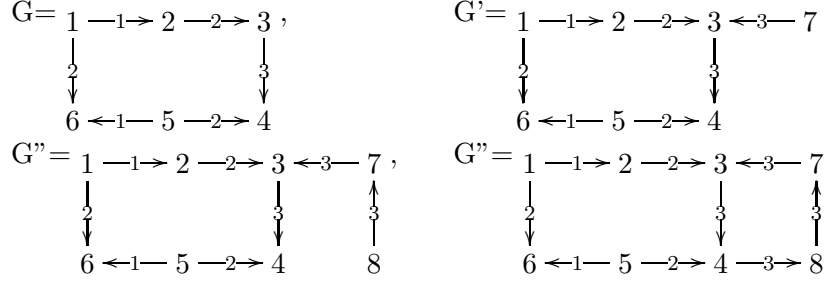
Let  $G = (V; E_1, \dots, E_k)$  be an admissible oriented edge-colored graph. A *direct extension* of  $G$  is a graph  $G'$  of one of the following types:

- $G' = (V \cup \{r\}; E_1, \dots, E_{i-1}, E_i \cup \{s \rightarrow r\}, E_{i+1}, \dots, E_k)$ , with  $r \notin V$ ,  $i \in [k]$ , and  $s \in V$  such that no edge of  $G$  with color  $i$  has  $s$  for beginning vertex,
- $G' = (V \cup \{r\}; E_1, \dots, E_{i-1}, E_i \cup \{r \rightarrow s\}, E_{i+1}, \dots, E_k)$ , with  $r \notin V$ ,  $i \in [k]$ , and  $s \in V$  such that no edge of  $G$  with color  $i$  has  $s$  for ending vertex,
- $G' = (V; E_1, \dots, E_{i-1}, E_i \cup \{r \rightarrow s\}, E_{i+1}, \dots, E_k)$ , with  $i \in [k]$  such that  $d_i < \infty$ ,  $r, s \in V$  such that no edge of  $G$  with color  $i$  has  $r$  for beginning vertex, no edge of  $G$  with color  $i$  has  $s$  for ending vertex, and there exists  $t_1, \dots, t_{d_i} \in V$  pairwise distinct vertices such that  $t_1 = s, t_{d_i} = r$  and  $t_1 \rightarrow t_2, t_2 \rightarrow t_3, \dots, t_{d_i-1} \rightarrow t_{d_i} \in E_i$ .

In other words, a direct extension of  $G$  is an admissible oriented edge-colored graph which can be obtained from  $G$  either by adding a vertex and connecting this vertex to a vertex of  $G$  by an edge of any sense and color, or by adding an edge which closes a monochromatic directed

cycle with color  $i \in [k]$  of length  $d_i$ . In the first case, the direct extension  $G \subset G'$  is said to be *vertex-adding*, whereas in the second one, it is said to be *cycle-closing*.

For example, for  $d_3 = 4$ , the graphs  $G, G', G'', G'''$  below (where the colors of the edges appear on them) are such that  $G \subset G' \subset G'' \subset G'''$  are direct extensions, the first ones being vertex-adding, whereas the last one is cycle-closing.



An *extension* of  $G$  is an oriented edge-colored graph  $G'$  such that there is  $n \geq 0$ ,  $G_0 = G, \dots, G_n = G'$  such that for all  $i = 1, \dots, n$ ,  $G_i$  is a direct extension of  $G_{i-1}$ . Such an integer  $n$  is unique (it is the number of edges of  $G'$  minus the number of edges of  $G$ ) and will be called the *degree* of the extension.

In other words, an extension of  $G$  is an admissible oriented edge-colored graph which can be obtained from  $G$  by successively adding vertices and/or edges without adding either any connected component or any monochromatic directed cycle whose length is not  $d_i$  when its color is  $i$ .

**Remark 2.7.** *It can easily be proved, using admissibility, that a vertex or an edge of an extension of  $G$  which is not a vertex of  $G$  cannot belong to a monochromatic directed cycle with color  $i \in [k]$  whose length is not equal to  $d_i$ .*

**Lemma 2.8.** *The Neagu characteristic is preserved by extension.*

**Proof.** It suffices to prove that the Neagu characteristic is preserved by direct extension. In the case of a vertex-adding direct extension, it is obvious. In the case where the direct extension is monochromatic directed cycle-closing, it suffices to notice that the admissibility implies that exactly one monochromatic directed cycle is closed.  $\square$

The following lemma is the key result of this section. Recall that for  $u, v$  in a set  $V$ ,  $\{u = v\}$  denotes the partition of  $V$  whose classes are all singletons, except one:  $\{u, v\}$ .

**Lemma 2.9.** *Consider an oriented edge-colored graph  $G$  which is an extension of the graph  $G(w)$ , with  $w \in \mathbb{M}_k$  a cyclically reduced word such that the order, in  $F_k/[g_1^{d_1}, \dots, g_k^{d_k}]$ , of the element represented by  $w$  is either infinite or equal to one. Suppose that for a certain  $i \in [k]$  such that  $d_i < \infty$ ,  $G$  contains an  $i$ -colored directed path  $t_0 \rightarrow t_1 \rightarrow \dots \rightarrow t_{d_i}$  with length  $d_i$ . Then there is a partial  $(d_1, \dots, d_k)$ -reduction  $\tilde{w}$  of  $w$  such that  $\text{Adm}(G/\{t_0 = t_{d_i}\})$  is isomorphic to an extension of  $G(\tilde{w})$ .*

**Remark 2.10.** *Note that by definition of the operator  $\text{Adm}$  and by Remark 2.2, with the notations of the previous lemma, there is a partition  $\Delta$  of the vertex set of  $G$  such that  $\text{Adm}(G/\{t_0 = t_{d_i}\})$  can be identified with  $G/\Delta$ . Clearly, by definition of  $\{t_0 = t_{d_i}\}$  and of  $\text{Adm}$ , if two vertices of  $r, s$  of  $G$  are in the same class of  $\Delta$ , then there are  $n \geq 0$ ,  $x_0 = r, x_1, \dots, x_n = s$  some vertices of  $G$ ,  $i_1, \dots, i_n \in [k]$ , and  $\varepsilon_1, \dots, \varepsilon_n \in \{\pm 1\}$  such that:*  
- the  $(d_1, \dots, d_k)$ -reduction of the word  $g_{i_1}^{\varepsilon_1} \dots g_{i_n}^{\varepsilon_n}$  is the empty word,

- for all  $l \in [n]$ , the  $i_l$ -colored edge  $x_{l-1} \rightarrow x_l$  (resp.  $x_{l-1} \leftarrow x_l$ ) belongs to  $G$  if  $\varepsilon_l = 1$  (resp. if  $\varepsilon_l = -1$ ).

**Proof of the lemma.** The lemma is a consequence of the following proposition, that we shall prove by induction on  $(|w|, n)$  (for the lexical order  $\preceq$ ): for all  $w \in \mathbb{M}_k$  a cyclically reduced word such that the order, in  $F_k/[g_1^{d_1}, \dots, g_k^{d_k}]$ , of the element represented by  $w$  is either infinite or equal to one, for all extensions  $G$  of degree  $n$  of  $G(w)$ , if for a certain  $i \in [k]$  such that  $d_i < \infty$ ,  $G$  contains an  $i$ -colored directed path  $t_0 \rightarrow t_1 \rightarrow \dots \rightarrow t_{d_i}$  with length  $d_i$ , then  $\text{Adm}(G/\{t_0 = t_{d_i}\})$  is either isomorphic to an extension of  $G(w)$  with degree  $\leq n$  or isomorphic to an extension of  $G(\tilde{w})$  for a certain strict partial  $(d_1, \dots, d_k)$ -reduction  $\tilde{w}$  of  $w$ .

For  $|w| = 0$  and  $n = 0$ , then  $G = G(w)$  is a single point, thus there is nothing to prove.

Now, let us consider a cyclically reduced word  $w$  such that the order, in  $F_k/[g_1^{d_1}, \dots, g_k^{d_k}]$ , of the element represented by  $w$  is either infinite or equal to one, and an extension  $G$  of  $G(w)$  with degree  $n$  such that  $(0, 0) \prec (|w|, n)$ , and let us suppose the result to be proved for any word  $w'$  and any extension  $G(w') \subset G'$  with degree  $n'$  such that  $(|w'|, n') \prec (|w|, n)$ . Suppose that  $G$  contains an  $i$ -colored directed path  $t_0 \rightarrow t_1 \rightarrow \dots \rightarrow t_{d_i}$  with length  $d_i$ .

• If  $n = 0$ , then  $G = G(w)$ , and up to a cyclic permutation of the letters of  $w$  (which, up to an isomorphism, does not change  $G(w)$ ), one can suppose that one of the following cases occurs:

Case 1:  $w = u g_i^{\alpha d_i} u^{-1} \tilde{w}$ , with  $\alpha$  a nonzero integer,  $u$  nonempty word and  $\tilde{w} \in \mathbb{M}_k$  a cyclically reduced word, and there is an integer  $j$ ,  $1 + |u| \leq j \leq 1 + |u| + (|\alpha| - 1)d_i$ , such that

$$\begin{cases} t_0 = j, t_1 = j + 1, \dots, t_{d_i} = j + d_i & \text{if } \alpha > 0, \\ t_0 = j + d_i, t_1 = j + d_i - 1, \dots, t_{d_i} = j & \text{if } \alpha < 0. \end{cases}$$

Case 2:  $w = g_i^{\alpha d_i} \tilde{w}$ , with  $\alpha$  a nonzero integer and  $\tilde{w}$  a word which can be written  $\tilde{w} = g_i^\beta v$ , with  $\beta$  an integer,  $|\beta| < d_i$  and  $v$  a nonempty (except possibly if  $\beta = 0$ ) word whose first and last letters do not belong to  $\{g_i, g_i^{-1}\}$ , and there is an integer  $j$ ,  $1 \leq j \leq 1 + (|\alpha| - 1)d_i + |\beta|$ , such that

$$\begin{cases} t_0 = j, t_1 = j + 1, \dots, t_{d_i} = j + d_i & \text{if } \alpha > 0, \\ t_0 = j + d_i, t_1 = j + d_i - 1, \dots, t_{d_i} = j & \text{if } \alpha < 0. \end{cases}$$

Note that in both cases,  $\tilde{w}$  is a strict partial  $(d_1, \dots, d_k)$ -reduction of  $w$ . We are going to prove that  $\text{Adm}(G(w)/\{t_0 = t_{d_i}\})$  is isomorphic to an extension of  $G(\tilde{w})$ .

In Case 1, let us write  $u = g_{i_1}^{\varepsilon_1} \dots g_{i_{|u|}}^{\varepsilon_{|u|}}$ , with  $i_1, \dots, i_{|u|} \in [k]$ ,  $\varepsilon_1, \dots, \varepsilon_{|u|} \in \{\pm 1\}$ . Let us define the sequence of direct extensions  $G(\tilde{w}) = G_0 \subset G_1 \subset \dots \subset G_{|u|+d_i}$  in the following way (we consider a copy  $1', 2', \dots$  of the set of positive integers):

- 1) First add the vertex  $1'$  and the  $i_1$ -colored edge  $1 \rightarrow 1'$  or  $1 \leftarrow 1'$  according to whether  $\varepsilon_1 = 1$  or  $-1$ . This is a direct extension, because since  $w$  is cyclically reduced, we know that if the last (resp. first) letter of  $\tilde{w}$  is  $g_{i_1}^\varepsilon$  with  $\varepsilon \in \{\pm 1\}$ , then  $\varepsilon = \varepsilon_1$  (resp.  $-\varepsilon_1$ ).
- 2) Then add the vertex  $2'$  and the  $i_2$ -colored edge  $1' \rightarrow 2'$  or  $1' \leftarrow 2'$  according to whether  $\varepsilon_2 = 1$  or  $-1$ . This is also a direct extension, because since  $w$  is cyclically reduced, we know that if  $i_1 = i_2$ , then  $\varepsilon_1 = \varepsilon_2$ .
- $\vdots$
- $|u|$ ) Then add the vertex  $|u|'$  and the  $i_{|u|}$ -colored edge  $(|u| - 1)' \rightarrow |u|'$  or  $(|u| - 1)' \leftarrow |u|'$  according to whether  $\varepsilon_{|u|} = 1$  or  $-1$ .

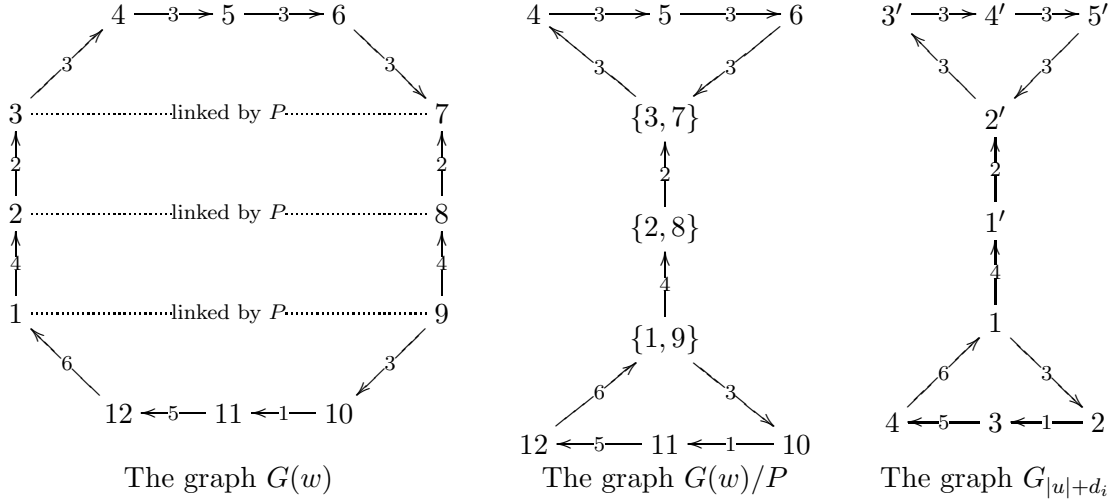


FIGURE 1.

$|u| + 1$ ) Then add the vertex  $(|u| + 1)'$  and the  $i$ -colored edge  $|u|' \rightarrow (|u| + 1)'$ . This is a direct extension, because we know that  $i_{|u|} \neq i$ : indeed,  $w$  is cyclically reduced, and on the one hand, the letter  $g_{i_{|u|}}^{\varepsilon_{|u|}}$  appears in before  $g_i^{\alpha d_i}$  in  $w$ , whereas on the other hand, the letter  $g_{i_{|u|}}^{-\varepsilon_{|u|}}$  appears in after  $g_i^{\alpha d_i}$  in  $w$ .

$|u| + 2$ ) Then add the vertex  $(|u| + 2)'$  and the  $i$ -colored edge  $(|u| + 1)' \rightarrow (|u| + 2)'$ .

$\vdots$

$|u| + d_i - 1$ ) Then add the vertex  $(|u| + d_i - 1)'$  and the  $i$ -colored edge  $(|u| + d_i - 2)' \rightarrow (|u| + d_i - 1)'$ .

$|u| + d_i$ ) Then add the  $i$ -colored edge  $(|u| + d_i - 1)' \rightarrow |u|'$ . This is a direct extension because as explained at the  $(|u| + 1)$ th step,  $i_{|u|} \neq i$ .

Claim: the last graph of this sequence,  $G_{|u|+d_i}$ , is isomorphic to  $\text{Adm}(G(w)/\{t_0 = t_{d_i}\})$ . To prove this, let us first notice that by Remark 2.2,  $\text{Adm}(G/\{t_0 = t_{d_i}\})$  is isomorphic to  $G(w)/P$ , where  $P$  is the partition of the vertex set of  $G$  whose classes are singletons, except that:

- In the “ $g_i^{\alpha d_i}$  part” of  $G(w)$ , vertices are linked every  $d_i$  edges: for all  $x, y \in \{|u| + 1, \dots, |u| + |\alpha|d_i + 1\}$ ,  $x = y \pmod P$  if and only if  $x = y \pmod{d_i}$ .
- In the “ $u g_i^{\alpha d_i} u^{-1}$  part” of  $G(w)$ , symmetric vertices of “the  $u$  and  $u^{-1}$  parts” are linked by  $P$ : we have, modulo  $P$ ,  $1 = 2|u| + |\alpha|d_i + 1$ ,  $2 = 2|u| + |\alpha|d_i$ ,  $\dots$ ,  $|u| = |u| + |\alpha|d_i + 2$ .

As an illustration, on Figure 1, we draw the graphs  $G(w)$ ,  $G(w)/P$  and  $G_{|u|+d_i}$  for  $w = g_4 g_2 g_3^4 g_2^{-1} g_4^{-1} g_3 g_1 g_5 g_6$ ,  $\tilde{w} = g_3 g_1 g_5 g_6$  and  $d_3 = 4$ , linking by edges of the type  $\dots\dots\dots$  the vertices of  $G(w)$  which are in the same class of  $P$  (but these edges do not belong to  $G(w)$ ).

Now, to be precise, let us give a bijective map  $\varphi$  from the vertex set of  $G_{|u|+d_i}$  to the one of  $G(w)/P$  which is an isomorphism of edge-colored oriented graphs. First, recall that the vertex set of  $G_{|u|+d_i}$  is  $[|\tilde{w}|] \cup \{1', \dots, (|u| + d_i - 1)'\}$  and that the vertex set of  $G(w)/P$  is

$$\begin{aligned} & \{ \{1, 2|u| + |\alpha|d_i + 1\}, \{2, 2|u| + |\alpha|d_i\}, \{3, 2|u| + |\alpha|d_i - 1\}, \dots, \{|u|, |u| + |\alpha|d_i + 2\} \} \cup \\ & \{ \{|u| + 1 + ld_i; 0 \leq l \leq |\alpha|\} \} \cup \{ \{|u| + j + ld_i; 0 \leq l < |\alpha|\}; j = 2, \dots, d_i \} \cup \\ & \{ \{2|u| + |\alpha|d_i + 2\}, \dots, \{2|u| + |\alpha|d_i + |\tilde{w}|\} \}. \end{aligned}$$

The bijection  $\varphi$  is defined by

$$\begin{aligned} \forall x \in [|\tilde{w}|], \quad \varphi(x) &= \begin{cases} \{1, 2|u| + |\alpha|d_i + 1\} & \text{if } x = 1, \\ \{2|u| + |\alpha|d_i + x\} & \text{if } x > 1, \end{cases} \\ \forall x \in [|\tilde{w}| + d_i - 1], \quad \varphi(x') &= \begin{cases} \{x + 1, 2|u| + |\alpha|d_i + 1 - x\} & \text{if } x < |u|, \\ \{|u| + 1 + ld_i; 0 \leq l \leq |\alpha|\} & \text{if } x = |u|, \\ \{x + 1 + ld_i; 0 \leq l < |\alpha|\} & \text{if } x > |u| \text{ and } \alpha > 0, \\ \{2|u| + d_i + 1 - x + ld_i; 0 \leq l < |\alpha|\} & \text{if } x > |u| \text{ and } \alpha < 0. \end{cases} \end{aligned}$$

In Case 2, let us define the sequence of direct extensions  $G(v) = G_0 \subset G_1 \subset \dots \subset G_{d_i - |\beta|}$  in the following way (again, we consider a copy  $1', 2', \dots$  of the set of positive integers):

- 1) First add the vertex  $(|\beta| + 2)'$  and the  $i$ -colored edge  $|\beta| + 1 \rightarrow (|\beta| + 2)'$  or  $|\beta| + 1 \leftarrow (|\beta| + 2)'$  according to whether  $\beta \geq 0$  or  $< 0$ . This is a direct extension, because  $v$  is a nonempty (except possibly if  $\beta = 0$ ) word whose first and last letters do not belong to  $\{g_i, g_i^{-1}\}$ .
- 2) Then add the vertex  $(|\beta| + 3)'$  and the  $i$ -colored edge  $(|\beta| + 2)' \rightarrow (|\beta| + 3)'$  or  $(|\beta| + 2)' \leftarrow (|\beta| + 3)'$  according to whether  $\beta \geq 0$  or  $< 0$ .
- $\vdots$
- $d_i - |\beta| - 1$ ) Then add the vertex  $d_i'$  and the  $i$ -colored edge  $(d_i - 1)' \rightarrow d_i'$  or  $(d_i - 1)' \leftarrow d_i'$  according to whether  $\beta \geq 0$  or  $< 0$ .
- $d_i - |\beta|$ ) Then add the  $i$ -colored edge  $d_i' \rightarrow 1$  or  $d_i' \leftarrow 1$  according to whether  $\beta \geq 0$  or  $< 0$ .

As in Case 1, one proves that the last graph of this sequence of direct extensions, namely  $G_{d_i - |\beta|}$ , is isomorphic to  $\text{Adm}(G(w)/\{t_0 = t_{d_i}\})$  (which, by Remark 2.2, is isomorphic to  $G(w)/P$ , where  $P$  is the partition of the vertex set of  $G$  whose classes are singletons, except that in the “ $g_i^{\alpha d_i + \beta}$  part” of  $G$ , vertices are linked every  $d_i$  edges). As an illustration, on Figure 2, we draw the graphs  $G(w)$ ,  $G(w)/P$  and  $G_{d_i - |\beta|}$ , with  $w = g_1^5 g_2 g_3 g_4 g_1^{-1} g_4 g_2^{-2}$ ,  $\tilde{w} = g_1 g_2 g_3 g_4 g_1^{-1} g_4 g_2^{-2}$  and  $d_1 = 4$ , linking by edges of the type ..... the vertices of  $G(w)$  which are in the same class of  $P$  (but these edges do not belong to  $G(w)$ ).

- If the degree  $n$  of the extension  $G$  of  $G(w)$  is  $\geq 1$ , let us introduce a sequence  $G_0 = G(w) \subset G_1 \subset \dots \subset G_n = G$  of direct extensions. Again, two cases will be to consider.

- If  $t_0, \dots, t_{d_i}$  are not all vertices of  $G_{n-1}$ :

Since  $G_n$  is a direct extension of  $G_{n-1}$ , at most one vertex of  $G_n$  is not a vertex of  $G_{n-1}$ . Hence exactly one vertex of  $G_n$  is not a vertex of  $G_{n-1}$ , the direct extension  $G_{n-1} \subset G_n$  is vertex-adding and the vertex added is at the extremity of exactly one edge. Thus this vertex is either  $t_0$  or  $t_{d_i}$ . We suppose that it is  $t_{d_i}$ : the case where it is  $t_0$  can be treated analogously.

- If no  $i$ -colored edge of  $G_{n-1}$  has  $t_0$  for ending vertex, then  $\text{Adm}(G_n/\{t_0 = t_{d_i}\})$  can simply be identified with the graph obtained from  $G_{n-1}$  by adding the  $i$ -colored edge  $t_{d_i-1} \rightarrow t_0$ : this graph is a cycle-closing direct extension of  $G_{n-1}$ , hence an extension of degree  $n$  of  $G(w)$ , so the result holds.
- If  $t_0$  is the ending vertex of a certain  $i$ -colored edge  $t_{-1} \rightarrow t_0$  of  $G_{n-1}$ , then notice first that by Lemma 2.5,  $\text{Adm}(G_n/\{t_0 = t_{d_i}\})$  is isomorphic to  $\text{Adm}(G_{n-1}/\{t_{-1} = t_{d_i-1}\})$ . Hence by the induction hypothesis, it suffices to prove that  $t_{-1}, \dots, t_{d_i-1}$  are pairwise distinct: if it wasn't the case, since  $t_0, \dots, t_{d_i-1}$  are pairwise distinct, then we would have  $t_{-1} = t_l$  for some  $l \in \{0, \dots, d_i - 1\}$ , but it would imply that  $t_l$  is the beginning

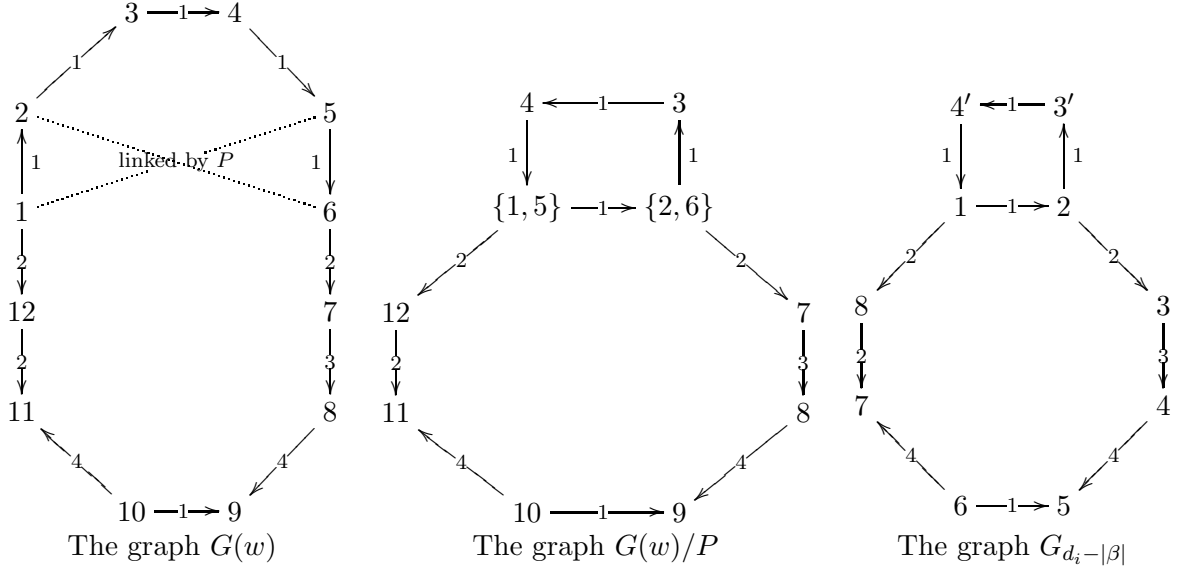


FIGURE 2.

vertex of two distinct  $i$ -colored edges of  $G_n$  (the edges  $t_{-1} \rightarrow t_0$  and  $t_l \rightarrow t_{l+1}$ ), which is impossible since  $G_n$ , as an extension of  $G(w)$ , is admissible.

•• If  $t_0, \dots, t_{d_i}$  are all vertices of  $G_{n-1}$ :

There will be no ambiguity for us to also denote by  $\{t_0 = t_{d_i}\}$  the restriction of the partition  $\{t_0 = t_{d_i}\}$  to the vertex set of  $G_{n-1}$ . By definition of the operator  $\text{Adm}$  and by Remark 2.2, there is a partition  $\Gamma$  of the vertex set of  $G_{n-1}$  such that  $\text{Adm}(G_{n-1}/\{t_0 = t_{d_i}\})$  is isomorphic to  $G_{n-1}/\Gamma$ . Let, for  $x$  vertex of  $G_{n-1}$ ,  $\bar{x}$  denote the class of  $x$  in  $\Gamma$ . We know, by the induction hypothesis, that  $G_{n-1}/\Gamma$  is either isomorphic to an extension of  $G(w)$  with degree  $\leq n-1$  or isomorphic to an extension of  $G(\tilde{w})$  for a certain strict partial  $(d_1, \dots, d_k)$ -reduction  $\tilde{w}$  of  $w$ . Hence it suffices to prove one of the following properties:

- (P1)  $\text{Adm}(G_n/\{t_0 = t_{d_i}\})$  is isomorphic to  $G_{n-1}/\Gamma$ ,
- (P2)  $\text{Adm}(G_n/\{t_0 = t_{d_i}\})$  is isomorphic to a direct extension of  $G_{n-1}/\Gamma$ ,
- (P3) there is  $j \in [k]$  such that  $d_j < \infty$  and  $G_{n-1}/\Gamma$  contains a  $j$ -colored directed path with length  $d_j$  and extremity vertices  $x, y$  and such that  $\text{Adm}(G_n/\{t_0 = t_{d_i}\})$  is isomorphic to  $\text{Adm}((G_{n-1}/\Gamma)/\{x = y\})$ .

(The conclusion, in the (P3) case, requires to use the induction hypothesis again.)

Note that for all  $\tilde{w}$  partial  $(d_1, \dots, d_k)$ -cyclic reductions of  $w$ , since the order of the element of  $F_k/[g_1^{d_1}, \dots, g_k^{d_k}]$  represented by  $w$  (hence by  $\tilde{w}$ ) cannot be finite and nonzero, we have the following property:

- (Q) for all  $j \in [k]$  such that  $d_j < \infty$ ,  $G(\tilde{w})$  does not contain any  $j$ -colored directed cycle.

Now, again, we have two sub-cases to consider:

- If the extension  $G_{n-1} \subset G_n$  is vertex-adding: there is  $j \in [k]$ , a vertex  $r$  of  $G_n$ , a vertex  $s$  of  $G_{n-1}$  such that  $G_n$  can be obtained from  $G_{n-1}$  by the addition of the vertex  $r$  and of the

$j$ -colored edge  $s \rightarrow r$  (or  $r \rightarrow s$ ). Now, if no edge with color  $j$  has  $\bar{s}$  for beginning vertex (or respectively ending vertex) in  $G_{n-1}/\Gamma$ , then by Remark 2.4,  $\text{Adm}(G_n/\{t_0 = t_{d_i}\})$  is isomorphic to the vertex-adding direct extension of  $G_{n-1}/\Gamma$  obtained by adding the vertex  $r$  and the edge  $\bar{s} \rightarrow r$  (or respectively  $r \rightarrow \bar{s}$ ). On the other hand, if an edge with color  $j$  has  $\bar{s}$  for beginning vertex (or respectively ending vertex) in  $G_{n-1}/\Gamma$ , then  $\text{Adm}(G_n/\{t_0 = t_{d_i}\})$  is isomorphic to  $G_{n-1}/\Gamma$ . In both cases, (P1) or (P2) holds.

- If the extension  $G_{n-1} \subset G_n$  is cycle-closing: there is  $j \in [k]$  such that  $G_n$  can be obtained from  $G_{n-1}$  by adding an edge with color  $j$  between two vertices of  $G_{n-1}$  which are the extremities of a  $j$ -colored directed path with length  $d_j - 1$ . Let us denote by  $v_1, \dots, v_{d_j}$  the successive vertices of this directed path: the edge added in the direct extension  $G_{n-1} \subset G_n$  is  $v_{d_j} \rightarrow v_1$ , with color  $j$ . Notice that  $\bar{v}_1, \dots, \bar{v}_{d_j}$  are pairwise distinct vertices of  $G_{n-1}/\Gamma$ . Indeed, if it wasn't the case, since  $G_{n-1}$  contains the  $j$ -colored directed path  $v_1 \rightarrow \dots \rightarrow v_{d_j}$ ,  $G_{n-1}/\Gamma$  would contain a  $j$ -colored monochromatic directed cycle with length  $< d_j$ , which is impossible by property (Q) and Remark 2.7. Again, two cases will be to consider.

- a) If no  $j$ -colored edge of  $G_{n-1}/\Gamma$  has  $\bar{v}_{d_j}$  for beginning vertex or  $\bar{v}_1$  for ending vertex: then, by Remark 2.4,  $\text{Adm}(G_n/\{t_0 = t_{d_i}\})$  is isomorphic to the graph obtained from  $G_{n-1}/\Gamma$  by adding the vertex  $\bar{v}_{d_j} \rightarrow \bar{v}_1$ , which is a cycle-closing extension of  $G_{n-1}/\Gamma$ : (P2) holds.
- b) If a  $j$ -colored edge of  $G_{n-1}/\Gamma$  has  $\bar{v}_{d_j}$  for beginning vertex or  $\bar{v}_1$  for ending vertex. We can suppose that a  $j$ -colored edge of  $G_{n-1}/\Gamma$  has  $\bar{v}_1$  for ending vertex: the other case can be treated analogously. Hence there is a vertex  $V_0$  of  $G_{n-1}/\Gamma$  such that  $G_{n-1}/\Gamma$  contains the  $j$ -colored edge  $V_0 \rightarrow \bar{v}_1$ . Now, notice that by Lemma 2.5,  $G_n/\{t_0 = t_{d_i}\}$  is isomorphic to  $(G_{n-1}/\Gamma)/\{\bar{v}_{d_j} = V_0\}$ : (P3) holds.

□

The following lemma is a consequence of the previous one, by induction on the sum, over  $j \in [k]$ , on the number of  $j$ -colored paths of  $G$  with length  $\geq d_j$ .

**Lemma 2.11.** *Consider a cyclically reduced word  $w \in \mathbb{M}_k$  such that if  $w$  admits a  $(d_1, \dots, d_k)$ -cyclic reduction of the type  $g_i^\alpha$ , with  $i \in [k]$  and  $\alpha$  a nonzero integer, then  $d_i = \infty$  and  $|\alpha| \in A_i$ . Consider an oriented edge-colored graph  $G$  which is an extension of the graph  $G(w)$ . Then  $G$  admits an  $(A_1, \dots, A_k)$ -admissible partition  $\Delta$  such that  $G/\Delta$  is isomorphic to an extension of  $G(w_{\text{red}})$ , for a certain  $(d_1, \dots, d_k)$ -reduction  $w_{\text{red}}$  of  $w$  and such that if two vertices of  $r, s$  of  $G$  are in the same class of  $\Delta$ , then there are  $n \geq 0$ ,  $t_0 = r, t_1, \dots, t_n = s$  some vertices of  $G$ ,  $i_1, \dots, i_n \in [k]$ , and  $\varepsilon_1, \dots, \varepsilon_n \in \{\pm 1\}$  such that:*

- the  $(d_1, \dots, d_k)$ -reduction of the word  $g_{i_1}^{\varepsilon_1} \dots g_{i_n}^{\varepsilon_n}$  is the empty word,
- for all  $l \in [n]$ , the  $i_l$ -colored edge  $t_{l-1} \rightarrow t_l$  (resp.  $t_{l-1} \leftarrow t_l$ ) belongs to  $G$  if  $\varepsilon_l = 1$  (resp. if  $\varepsilon_l = -1$ ).

The previous lemma allows us to prove the following Theorem, which is the main result of this section.

**Theorem 2.12.** *Consider a cyclically reduced word  $w = g_{i_1}^{\alpha_1} \dots g_{i_{|w|}}^{\alpha_{|w|}} \in \mathbb{M}_k$  such that if  $w$  admits a  $(d_1, \dots, d_k)$ -cyclic reduction of the type  $g_i^\alpha$ , with  $i \in [k]$  and  $\alpha$  a nonzero integer, then  $d_i = \infty$  and  $|\alpha| \in A_i$ . Then  $G(w)$  admits an  $(A_1, \dots, A_k)$ -admissible partition  $\Delta$  such that:*

$$(i) \quad \chi(G(w)/\Delta) = \begin{cases} 1 & \text{if } w \text{ admits the empty word for } (d_1, \dots, d_k)\text{-cyclic reduction,} \\ 0 & \text{in the other case,} \end{cases}$$

(ii) for all  $r < s \in [|w|]$ ,  $r = s \pmod{\Delta}$  implies that one of the words

$$g_{i_r}^{\alpha_r} \cdots g_{i_{s-1}}^{\alpha_{s-1}}, \quad g_{i_s}^{\alpha_s} \cdots g_{i_{|w|}}^{\alpha_{|w|}} g_{i_1}^{\alpha_1} \cdots g_{i_{r-1}}^{\alpha_{r-1}}$$

admits the empty word as a  $(d_1, \dots, d_k)$ -cyclic reduction.

**Proof.** Note that by hypothesis, for any  $(d_1, \dots, d_k)$ -cyclic reduction  $w_{\text{red}}$  of  $w$ ,  $G(w_{\text{red}})$  is an  $(A_1, \dots, A_k)$ -admissible graph with Neagu characteristic equal to 1 if the only  $(d_1, \dots, d_k)$ -cyclic reduction of  $w$  is the empty word, and to 0 in the other case. Hence by Lemma 2.8, for any partition  $\Delta$  of the vertex set of  $G(w)$ , to have (i), it suffices to prove that  $G(w)/\Delta$  is an extension of  $G(w_{\text{red}})$ . Hence this Theorem is an immediate consequence of the previous lemma, applied for  $G = G(w)$ .  $\square$

We shall use the following Corollary later. For all  $l \geq 1$ ,  $(1 \cdots l)$  denotes the cyclic permutation of  $[l]$  which maps 1 to 2, 2 to 3, ...,  $l-1$  to  $l$  and  $l$  to 1.

**Corollary 2.13.** Consider a positive integer  $l$  and a cyclically reduced word  $v \in \mathbb{M}_k$ .

a) Suppose that the order, in  $F_k/[g_1^{d_1}, \dots, g_k^{d_k}]$ , of the element represented by  $v$  is infinite and that in the case where  $v$  admits a  $(d_1, \dots, d_k)$ -cyclic reduction of the type  $g_i^\alpha$ , with  $i \in [k]$  and  $\alpha$  an integer, we have  $l|\alpha| \in A_i$ . Then  $G((1 \cdots l), v)$  admits an  $(A_1, \dots, A_k)$ -admissible partition  $\Delta$  such that:

- (i)  $\chi(G((1 \cdots l), v)/\Delta) = 0$ ,
- (ii) for all  $m \neq m' \in [l]$ ,  $(m, 1) \neq (m', 1) \pmod{\Delta}$ .

b) Suppose that the order, in  $F_k/[g_1^{d_1}, \dots, g_k^{d_k}]$ , of the element represented by  $w$  is equal to  $l$ . Then  $G((1 \cdots l), v)$  admits an  $(A_1, \dots, A_k)$ -admissible partition  $\Delta$  such that:

- (i)  $\chi(G((1 \cdots l), v)/\Delta) = 1$ ,
- (ii) for all  $m \neq m' \in [l]$ ,  $(m, 1) \neq (m', 1) \pmod{\Delta}$ .

**Proof.** Note that by Lemma 2.6, for all  $l \geq 1$ , the function  $(m, i) \in [l] \times [|v|] \mapsto (l-m)|v| + i \in [l|v|]$  realizes an isomorphism between  $G((1 \cdots l), v)$  and  $G(v^l)$ , hence we are going to work with  $G(v^l)$  instead of  $G((1 \cdots l), v)$  (and the condition “for all  $m \neq m' \in [l]$ ,  $(m, 1) \neq (m', 1) \pmod{\Delta}$ ” gets “for all  $m \neq m' \in [l]$ ,  $(m-1)|v| + 1 \neq (m'-1)|v| + 1 \pmod{\Delta}$ ”). Then it suffices to apply Theorem 2.12.  $\square$

### 3. WORDS IN RANDOM PERMUTATIONS

We fix, until the end of the article,  $k \geq 1$  and  $A_1, \dots, A_k$  nonempty sets of positive integers, none of them being  $\{1\}$ , satisfying (1). We also fix a cyclically reduced word  $w \in \mathbb{M}_k$ . For all  $n$  such that  $\mathfrak{S}_n(A_1), \dots, \mathfrak{S}_n(A_k)$  are all nonempty (which is equivalent, for  $n$  large enough, to the fact that for all  $i \in [k]$ ,  $n$  is divisible by the greatest common divisor of  $A_i$  [Ne07, Lem. 2.3]), we consider an independent  $k$ -tuple  $s_1(n), \dots, s_k(n)$  of random permutations chosen uniformly in respectively  $\mathfrak{S}_n(A_1), \dots, \mathfrak{S}_n(A_k)$  and define  $\sigma_n = w(s_1(n), \dots, s_k(n))$ : the permutation obtained by replacing any  $g_i$  (or  $g_i^{-1}$ ) in  $w$  by  $s_i(n)$  (or  $s_i(n)^{-1}$ ).

#### 3.1. A key preliminary result.

**Proposition 3.1.** Consider  $p \geq 1$  and  $\sigma \in \mathfrak{S}_p$ . The probability of the event

$$(2) \quad \{\forall m = 1, \dots, p, \sigma_n(m) = \sigma(m)\}$$



is equivalent, as  $n$  goes to infinity in such a way that  $\mathfrak{S}_n(A_1), \dots, \mathfrak{S}_n(A_k)$  are all nonempty, to

$$(3) \quad \frac{1}{n^p} \sum_{\Delta \in C(\sigma, w, A_1, \dots, A_k)} n^{\chi(G(\sigma, w)/\Delta)},$$

where  $C(\sigma, w, A_1, \dots, A_k)$  is the set of  $(A_1, \dots, A_k)$ -admissible partitions  $\Delta$  of  $G(\sigma, w)$  such that for all  $m \neq m' \in [p]$ ,  $(m, 1) \not\equiv (m', 1) \pmod{\Delta}$ .

**Proof.** Set  $w = g_{i_1}^{\alpha_1} \dots g_{i_{|w|}}^{\alpha_{|w|}}$ , with  $i_1, \dots, i_{|w|} \in [k]$  and  $\alpha_1, \dots, \alpha_{|w|} \in \{-1, 1\}$ . We denote by  $V$  the vertex set  $[p] \times [|w|]$  of  $G(\sigma, w)$ , fix  $n \geq 1$ , and define, for any  $s = (s_1, \dots, s_k) \in (\mathfrak{S}_n)^k$ , the function

$$\gamma_s : (m, l) \in V \mapsto s_{i_l}^{\alpha_l} \dots s_{i_1}^{\alpha_1}(m) \in [n].$$

Note that since all  $s_i$ 's are one-to-one,  $\text{Part}(\gamma_s)$  is an admissible partition of  $G(\sigma, w)$ . Note also that if  $\gamma$  is a fixed function from  $V$  to  $[n]$  such that for all  $m = 1, \dots, p$ ,  $\gamma(m, 1) = \sigma(m)$ , then for all  $s = (s_1, \dots, s_k) \in (\mathfrak{S}_n)^k$ ,  $\gamma_s = \gamma$  if and only if

$$(4) \quad \text{for all } r \in [k], \text{ for all edge } e \text{ of } G(\sigma, w) \text{ with color } r, s_r(\gamma(\text{End}(e))) = \gamma(\text{Beg}(e)).$$

Note at last that since the joint distributions of the  $s_i(n)$ 's is invariant under conjugation, for such a function  $\gamma$ , the probability of the event  $\{\gamma_{(s_1(n), \dots, s_k(n))} = \gamma\}$  only depends on  $\text{Part}(\gamma)$ .

Hence probability of the event of (2) is the sum, over all admissible partitions  $\Delta$  of  $G(\sigma, w)$ , of the number of functions  $\gamma : V \rightarrow [n]$  whose level set partition is  $\Delta$  and which satisfy  $\gamma(m, 1) = \sigma(m)$  for all  $m \in [p]$ , times the probability that  $\gamma_{(s_1(n), \dots, s_k(n))}$  is a certain (fixed, but the choice is irrelevant) of these functions.

Now, note that for  $\Delta$  admissible partition of  $G(\sigma, w)$ , the number of such functions is

$$\begin{cases} n(n-1) \dots (n - |\Delta| + p + 1) & \text{if } \forall m \neq m' \in [p], (m, 1) \not\equiv (m', 1) \pmod{\Delta}, \\ 0 & \text{in the other case.} \end{cases}$$

Suppose this number to be nonzero. Then by (4), the probability that  $\gamma_{s_1(n), \dots, s_k(n)}$  is a certain (fixed) of these functions is

$$\begin{cases} \prod_{r=1}^k p_n^{(A_r)}((G(\sigma, w)/\Delta)[r]) & \text{if } \Delta \text{ is an } (A_1, \dots, A_k)\text{-admissible partition of } G(\sigma, w), \\ 0 & \text{in the other case,} \end{cases}$$

where for all  $r \in [k]$ ,  $(G(\sigma, w)/\Delta)[r]$  is the graph obtained from  $G(\sigma, w)/\Delta$  by removing all edges which do not have color  $r$  and all vertices which are not the extremity of an  $r$ -colored edge, and where for all set  $A$  of positive integers and  $F$  monochromatic oriented graph,  $p_n^{(A)}(F)$  is the number defined in 3.8(a) of [Ne07].

So by definition of  $C(\sigma, w, A_1, \dots, A_k)$ , the probability of the event of (2) is equal to

$$\sum_{\Delta \in C(\sigma, w, A_1, \dots, A_k)} n(n-1) \dots (n - |\Delta| + p + 1) \prod_{r=1}^k p_n^{(A_r)}((G(\sigma, w)/\Delta)[r]),$$

and Proposition 3.8 of [Ne07] allows one to conclude the proof.  $\square$

The following result is a direct application of Corollary 1.7 of [BG07].

**Corollary 3.2.** *If  $w$  cyclically reduced is such that for all  $p \geq 1$ , for all  $\sigma \in \mathfrak{S}_p$ , for all  $\Delta \in C(\sigma, w, A_1, \dots, A_k)$ , one has  $\chi(G(\sigma, w)/\Delta) \leq 0$ , with equality for exactly one  $\Delta \in C(\sigma, w, A_1, \dots, A_k)$ ,*

then for all  $l \geq 1$ , the law of  $(N_1(\sigma_n), \dots, N_l(\sigma_n))$  converges weakly, as  $n$  goes to infinity in such a way that  $\mathfrak{S}_n(A_1), \dots, \mathfrak{S}_n(A_k)$  are all nonempty, to

$$\text{Poisson}(1/l) \otimes \dots \otimes \text{Poisson}(1/l).$$

**Remark 3.3.** *The hypotheses of this corollary do not always hold: for  $w = g_1^3 g_2$ ,  $p = 1$ , with  $A_1 = \{3, 4\}$ ,  $A_2 = \{1, 2\}$ , the quotient of the graph  $G(w)$  by the partition  $\{1 = 4\}$  has Neagu characteristic  $3 - 4 + 3/4 + 1/2 > 0$ .*

### 3.2. Asymptotic order of the $N_l(\sigma_n)$ 's.

3.2.1. *Case of a word with infinite order in  $F_k/[g_1^{d_1}, \dots, g_k^{d_k}]$ : existence of cycles of most lengths.*

Here, we shall see that even though the lengths of the cycles of  $s_1(n), \dots, s_k(n)$  are supposed to belong to the specific sets  $A_1, \dots, A_k$  of positive integers, these cycles are going to mix sufficiently well to give birth to cycles of most lengths in  $\sigma_n$ , at least as much as in a uniform random permutation.

**Theorem 3.4.** *Suppose that the order, in  $F_k/[g_1^{d_1}, \dots, g_k^{d_k}]$ , of the element represented by  $w$  is infinite. Then two cases can occur:*

- (i) *None of the  $(d_1, \dots, d_k)$ -cyclic reductions of  $w$  is of the type  $g_i^\alpha$ , with  $i \in [k]$ ,  $\alpha$  an integer: then for all  $l \geq 1$ , as  $n$  goes to infinity in such a way that for all  $r = 1, \dots, k$ ,  $\mathfrak{S}_n(A_r) \neq \emptyset$ ,*

$$\liminf(\mathbb{E}(N_l(\sigma_n))) \geq \frac{1}{l}.$$

- (ii) *There is  $i \in [k]$  such that  $A_i$  is infinite and the  $(d_1, \dots, d_k)$ -cyclic reduction of  $w$  is  $g_i^\alpha$ , with  $\alpha$  nonzero integer: then for all  $l \geq 1$  such that  $l|\alpha| \in A_i$ , as  $n$  goes to infinity in such a way that for all  $r = 1, \dots, k$ ,  $\mathfrak{S}_n(A_r) \neq \emptyset$ ,*

$$\liminf(\mathbb{E}(N_l(\sigma_n))) \geq \frac{1}{l}.$$

**Remark 3.5.** *In fact, we prove the more general result: if  $c_l$  denotes the cycle  $(1 \dots l)$ , then*

$$(5) \quad \forall \Delta \in C(c_l, w, A_1, \dots, A_k), \quad \liminf \left( \frac{\mathbb{E}(N_l(\sigma_n))}{n^{\chi(G(c_l, w)/\Delta)}} \right) \geq \frac{1}{l}.$$

**Proof.** We have

$$\mathbb{E}(N_l(\sigma_n)) = \frac{1}{l} \mathbb{E} \left[ \sum_{i=1}^n \mathbf{1}_{i \text{ belongs to a cycle of length } l \text{ of } \sigma_n} \right] = \frac{1}{l} \sum_{i=1}^n P(\{i \text{ belongs to a cycle of length } l\}).$$

But in the last sum, since the law of  $\sigma_n$  is invariant under conjugation, all terms are equal. Moreover, by this invariance principle again, each term is equal to the number of cycles of length  $l$  containing 1 times the probability that  $\sigma_n$  contains the cycle  $c_l := (1 \dots l)$ . So

$$\mathbb{E}(N_l(\sigma_n)) = \frac{n}{l} \binom{n}{l-1} (l-1)! P(\{\forall m = 1, \dots, l-1, \sigma_n(m) = c_l(m)\}).$$

Thus, by Proposition 3.1, (5) follows, and then Corollary 2.13 allows us to conclude.  $\square$

3.2.2. *Case of a word with finite order in  $F_k/[g_1^{d_1}, \dots, g_k^{d_k}]$ .* The following theorem states that in the case where the element of  $F_k/[g_1^{d_1}, \dots, g_k^{d_k}]$  represented by  $w$  has finite order  $d \geq 1$ ,  $\sigma_n$  is not far away from having order  $d$ : the cardinality of the subset of  $\{1, \dots, n\}$  covered by the supports of the cycles with length  $d$  in such a random permutation is asymptotic to  $n$ .

**Theorem 3.6.** *Suppose that the element of  $F_k/[g_1^{d_1}, \dots, g_k^{d_k}]$  represented by  $w$  has finite order  $d \geq 1$ . Then, as  $n$  goes to infinity in such a way that for all  $r = 1, \dots, k$ ,  $\mathfrak{S}_n(A_r) \neq \emptyset$ , for all  $p \in [1, +\infty)$ ,  $N_d(\sigma_n)/n$  converges to  $1/d$  in  $L^p$  and for all  $l \neq d$ ,  $N_l(\sigma_n)/n$  converges to 0 in  $L^p$ .*

**Proof.** Notice first that for all  $n$ ,

$$(6) \quad \sum_{l \geq 1} l N_l(\sigma_n)/n = 1,$$

which implies that for all positive integer  $l$ , the sequence  $N_l(\sigma_n)/n$  is bounded by 0 and  $1/l$ . Now, recall that for a sequence  $(X_n)$  of random variables, if there exists  $M < \infty$  such that for all  $n$ ,  $0 \leq X_n \leq M$ , the convergence of  $X_n$  to a limit  $X$  in all  $L^p$  spaces is implied by this result for  $p = 1$  (indeed, under this hypothesis, if for a certain  $p \geq 1$ ,  $X_n$  does not converge to  $X$  in  $L^p$ , then a subsequence of  $(X_n)$  converges to  $X$  almost surely but not in  $L^p$ , which is impossible by the Dominated Convergence Theorem).

Hence by (6), it suffices to prove that  $\liminf(\mathbb{E}(N_d(\sigma_n)/n)) \geq 1/d$ . This follows from (5) (which has been established in the proof of Theorem 3.4 without using the specific hypothesis of this theorem) and of Corollary 2.13.  $\square$

### 3.3. Case when all $A_i$ 's are infinite.

**Theorem 3.7.** *Suppose that all  $A_i$ 's are infinite, that  $|w| > 1$  and that  $w$  is not a power of another word. Then as  $n$  goes to infinity in such a way that for all  $r = 1, \dots, k$ ,  $\mathfrak{S}_n(A_r) \neq \emptyset$ , for all  $l \geq 1$ , the law of  $(N_1(\sigma_n), \dots, N_l(\sigma_n))$  converges weakly to*

$$\text{Poisson}(1/1) \otimes \dots \otimes \text{Poisson}(1/l).$$

**Proof.** This result is a direct consequence of Corollary 3.2. Note first that since all  $A_i$ 's are infinite, for all  $i$ ,  $d_i = \infty$ . Hence for all  $p \geq 1$ , for all  $\sigma \in \mathfrak{S}_p$ , for all  $\Delta$  an admissible partition of  $G(\sigma, w)$ ,  $\chi(G(\sigma, w)/\Delta)$  is the number of classes of  $\Delta$  minus the number of edges of  $G(\sigma, w)/\Delta$ , hence is not positive by Proposition 2.4.3 of [Ni94]. The last sentence of Corollary 2.6.6 of the same article also says that there is only one admissible partition  $\Delta$  of  $G(\sigma, w)$  such that  $\chi(G(\sigma, w)/\Delta)$  is null and such that for all  $m \neq m' \in [p]$ ,  $(m, 1)$  and  $(m', 1)$  are not in the same class: it is the singletons partition. It remains only to prove that the singletons partition is in  $C(\sigma, w, A_1, \dots, A_k)$ :

- this partition is admissible and does not link  $(m, 1)$  and  $(m', 1)$  for  $m \neq m'$ ,
- there is no monochromatic directed cycle in  $G(\sigma, w)$ : indeed, by Lemma 2.6,  $G(\sigma, w)$  is a disjoint union of graphs of the type  $G(w^d)$  ( $d \geq 1$ ), where there is no monochromatic directed cycle since  $w$  is primitive and  $|w| > 1$ .  $\square$

The previous theorem, with the fact that for any permutation  $\sigma$  and  $l, p \geq 1$ , we have  $N_l(\sigma^p) = \sum N_{lp/h}(\sigma)p/h$ , where the sum runs over the positive integers  $h$  such that  $h|p$  and  $\gcd(h, l) = 1$ , allows us to compute easily the limit distribution of  $(N_1(\sigma_n), \dots, N_l(\sigma_n))$  whenever  $w$  is not a power of a  $g_i$ .

**3.4. Case where  $w = g_1 \cdots g_k$ .** Now, we are not going to make the hypothesis that all  $A_i$ 's are infinite anymore, but we are going to suppose that  $w$  is a particular word:  $w = g_1 \cdots g_k$ , with  $k \geq 2$  (the case  $k = 1$  has already been treated in [BG07]).

3.4.1. *Case where  $k > 2$  or  $A_1 \cup A_2 \not\subseteq \{1, 2\}$ .*

**Theorem 3.8.** *Under this hypothesis, as  $n$  goes to infinity in such a way that for all  $r = 1, \dots, k$ ,  $\mathfrak{S}_n(A_r) \neq \emptyset$ , for all  $l \geq 1$ , the law of  $(N_1(\sigma_n), \dots, N_l(\sigma_n))$  converges weakly to*

$$\text{Poisson}(1/1) \otimes \cdots \otimes \text{Poisson}(1/l).$$

In order to prove the Theorem, we shall need the following lemmas.

**Lemma 3.9.** *Let  $\mathcal{X}$  be a finite set, let  $\mathcal{B}$  be a set of subsets of  $\mathcal{X}$  which have all cardinality 2. Let  $\Delta$  be a partition of  $\mathcal{X}$  such that for all  $\{x, y\} \in \mathcal{B}$ ,  $x = y \pmod{\Delta}$ . Then  $|\Delta| \leq |\mathcal{X}| - |\mathcal{B}|$ .*

**Proof.** Let us define the set  $\Gamma$  of subsets of  $\mathcal{X}$  by  $\Gamma = \mathcal{B} \cup \{\{z\}; z \in \mathcal{X}, z \notin \cup_{\{x,y\} \in \mathcal{B}} \{x, y\}\}$ . By hypothesis, any class of  $\Delta$  is a union of elements of  $\Gamma$ , so  $|\Delta| \leq |\Gamma| = |\mathcal{X}| - |\mathcal{B}|$ .  $\square$

**Lemma 3.10.** *Consider  $w = g_1 \cdots g_k$ , with  $k \geq 2$ , and  $\sigma \in \mathfrak{S}_p$ , with  $p \geq 1$ . Let  $\Delta$  be an admissible partition of  $G(\sigma, w)$  such that for all  $i \neq j \in [p]$ ,  $(i, 1) \neq (j, 1) \pmod{\Delta}$ . Then*

- (i) *two different edges of  $G(\sigma, w)$  with the same color cannot have their beginning vertices in the same class of  $\Delta$ , and the same holds for ending vertices,*
- (ii) *the following inequalities hold:*

$$(a) \text{ if } k > 2, \text{ then } |\Delta| \leq pk - \sum_{r=1}^k \sum_{\substack{L \text{ directed cycle of} \\ G(\sigma, w)/\Delta \text{ with color } r}} \text{length of } L,$$

$$(b) \text{ if } k = 2, \text{ then for all } r = 1, 2, |\Delta| \leq pk - \sum_{\substack{L \text{ directed cycle of} \\ G(\sigma, w)/\Delta \text{ with color } r}} \text{length of } L.$$

**Remark 3.11.** *Note that (i) implies that there is a canonical identification between the set of edges of the graph  $G(\sigma, w)$  and the set of edges of the graph  $G/\Delta$ : if, for all vertex  $x$  of  $G(\sigma, w)$ , one denotes by  $\bar{x}$  the class of  $x$  in  $\Delta$ , then for all  $r \in [k]$ , the function from the set of  $r$ -colored edges of  $G(\sigma, w)$  to the set of  $r$ -colored edges of  $G(\sigma, w)/\Delta$  which maps any edge  $x \rightarrow y$  to  $\bar{x} \rightarrow \bar{y}$  is bijective.*

**Proof.** Note first that since for all  $i \neq j \in [p]$ ,  $(i, 1) \neq (j, 1) \pmod{\Delta}$ , and by definition of an admissible partition, one has (by an obvious induction on  $l$ ):

$$(7) \quad \forall i \neq j \in [p], \forall l \in [k], (i, l) \neq (j, l) \pmod{\Delta}.$$

Since two edges of  $G(\sigma, w)$  with the same color have beginning vertices (and ending vertices) with the same second coordinate, it implies (i).

Now, note that, as observed in Remark 3.11, the edges of  $G(\sigma, w)/\Delta$  can be identified with the ones of  $G(\sigma, w)$ . Let, for  $r \in [k]$ ,  $\mathcal{L}[r]$  be the set of edges of  $G(\sigma, w)$  which, via this identification, belong to an  $r$ -colored directed cycle of  $G(\sigma, w)/\Delta$  and define  $\mathcal{L} := \cup_{r=1}^k \mathcal{L}[r]$ .

Claim: no edge of  $G(\sigma, w)/\Delta$  can belong to more than one monochromatic directed cycle of  $G(\sigma, w)/\Delta$ .

Indeed, if it were the case, there would be  $r \in [k]$ ,  $e, e', e'' \in \mathcal{L}[r]$  such that  $e'$  and  $e''$  both follow  $e$  in  $r$ -colored directed cycles of  $G(\sigma, w)/\Delta$  and  $e' \neq e''$ . Since  $e'$  and  $e''$  follow  $e$  in directed

cycles of  $G(\sigma, w)/\Delta$ , we have  $\text{Beg}(e') = \text{End}(e) = \text{Beg}(e'') \pmod{\Delta}$ , which implies  $e' = e''$  by (i). Contradiction.

The first consequences of this claim are that

$$(8) \quad |\mathcal{L}| = \sum_{r=1}^k \sum_{\substack{L \text{ directed cycle of} \\ G(\sigma, w)/\Delta \text{ with color } r}} \text{length of } L,$$

and that for all  $r = 1, \dots, k$ ,

$$(9) \quad |\mathcal{L}[r]| = \sum_{\substack{L \text{ directed cycle of} \\ G(\sigma, w)/\Delta \text{ with color } r}} \text{length of } L.$$

Another consequence of the claim is that one can define a permutation  $\varphi$  of  $\mathcal{L}$  which maps any edge  $e \in \mathcal{L}$  to the edge which follows  $e$  in the monochromatic directed cycle  $e$  belongs to. Let us define, for  $e \in \mathcal{L}$ ,

$$S(e) := \{\text{End}(e), \text{Beg}(\varphi(e))\} \subset [p] \times [|w|].$$

Note that since  $k \geq 2$ , for all edge  $e$  of  $G(\sigma, w)$ , the color of  $e$  is not the same as has the one of the edge whose beginning is the end of  $e$ . This allows us to claim that for all  $e \in \mathcal{L}$ ,  $|S(e)| = 2$ .

For all  $e \in \mathcal{L}$ , since  $\varphi(e)$  follows  $e$  in  $G(\sigma, w)/\Delta$ , we have  $\text{End}(e) = \text{Beg}(\varphi(e)) \pmod{\Delta}$ . So, in order to apply the previous lemma, we have to minor the cardinality of

$$\mathcal{A} := \{S(e); e \in \mathcal{L}\} \quad (\text{to prove (a)})$$

or, for  $r \in [k]$ , of

$$\mathcal{A}[r] := \{S(e); e \in \mathcal{L}, e \text{ has color } r\} \quad (\text{to prove (b)}).$$

Suppose that there are  $e \neq f \in \mathcal{L}$  such that  $S(e) = S(f)$ . One has either

$$(\text{End}(e), \text{Beg}(\varphi(e))) = (\text{End}(f), \text{Beg}(\varphi(f)))$$

or

$$(\text{End}(e), \text{Beg}(\varphi(e))) = (\text{Beg}(\varphi(f)), \text{End}(f)).$$

But  $(\text{End}(e), \text{Beg}(\varphi(e))) = (\text{End}(f), \text{Beg}(\varphi(f)))$  is impossible because two different edges of  $G(\sigma, w)$  cannot have the same end, since no letter of  $w$  has the exponent  $-1$ . So one has  $(\text{End}(e), \text{Beg}(\varphi(e))) = (\text{Beg}(\varphi(f)), \text{End}(f))$ .

If  $k > 2$ : let us prove that  $S(e) = S(f)$  with  $e \neq f$  is impossible. We have  $\text{End}(e) = \text{Beg}(\varphi(f))$ , so the color following the one of  $e$  in the cyclic order  $1, 2, \dots, k, 1, \dots$  is the one of  $\varphi(f)$ , i.e. of  $f$ . In the same way, the relation  $\text{End}(f) = \text{Beg}(\varphi(e))$  implies that the color following the one of  $f$  in the same cyclic order is the one of  $e$ . To sum up, in this cyclic order, one has the following direct sequence:

$$\dots, \text{color of } e, \text{color of } f, \text{color of } e, \dots$$

which is impossible, since  $k > 2$ .

So the cardinality of  $\mathcal{A}$  is the one of  $\mathcal{L}$ , and by (8), the result (a) of the lemma is an immediate application of the previous lemma, for  $\mathcal{X} = [p] \times [|w|]$  and  $\mathcal{B} = \mathcal{A}$ .

If  $k = 2$ :  $\text{End}(f) = \text{Beg}(\varphi(e))$  implies that the color of  $f$  is different from the color of  $\varphi(e)$ , i.e. of  $e$ . So for all  $r = 1, 2$ , the cardinality of  $\mathcal{A}[r]$  is the one of  $\mathcal{L}[r]$ , and by (9), the result (b) is an immediate application of the previous lemma, for  $\mathcal{X} = [p] \times [|w|]$  and  $\mathcal{B} = \mathcal{A}[r]$ .  $\square$

**Proof of Theorem 3.8.** Again, we are going to apply Corollary 3.2. Let us fix  $p \geq 1$  and  $\sigma \in \mathfrak{S}_p$ . Note that the singletons partition, denoted by  $\Delta_s$ , is in  $C(\sigma, w, A_1, \dots, A_k)$  (the proof is the same one as in the proof of Theorem 3.7). Moreover, using Lemma 2.6, one easily sees that the Neagu characteristic of  $G(\sigma, w)$  is 0. Hence it suffices to prove that for all  $\Delta \in C(\sigma, w, A_1, \dots, A_k)$  such that  $\Delta \neq \Delta_s$ , the Neagu characteristic of  $G(\sigma, w)/\Delta$  is negative. Let us fix such a partition  $\Delta$ . By Remark 3.11, we have to prove that

$$(10) \quad |\Delta| < pk - \sum_{r=1}^k \sum_{\substack{L \text{ directed cycle of} \\ G(\sigma, w)/\Delta \text{ with color } r}} \frac{\text{length of } L}{d_r}.$$

• If there is no monochromatic directed cycle in  $G(\sigma, w)/\Delta$ , then, since  $\Delta \neq \Delta_s$ ,  $|\Delta| < [p] \times [w] = pk$ , and hence (10) holds.

• If there is a monochromatic directed cycle in  $G(\sigma, w)/\Delta$  and  $k > 2$ , then since for all  $r$ ,  $d_r \geq 2 > 1$ , by (ii) (a) of Lemma 3.10, (10) holds.

• If there is a monochromatic directed cycle in  $G(\sigma, w)/\Delta$  and  $k = 2$ . First note that adding (ii) (b) of Lemma 3.10 for  $r = 1$  and  $r = 2$ , and then dividing by 2, one gets

$$(11) \quad |\Delta| \leq pk - \frac{1}{2} \sum_{r=1}^2 \sum_{\substack{L \text{ directed cycle of} \\ G(\sigma, w)/\Delta \text{ with color } r}} \text{length of } L.$$

By hypothesis, one has either  $d_1 > 2$  or  $d_2 > 2$ . By symmetry, we will suppose that  $d_1 > 2$ :

- If there is at least one directed cycle of color 1 in  $G(\sigma, w)/\Delta$ . Then (10) holds, because since  $d_1 > 2$ , the right hand term of (11) is strictly less than

$$pk - \sum_{r=1}^2 \sum_{\substack{L \text{ directed cycle of} \\ G(\sigma, w)/\Delta \text{ with color } r}} \frac{\text{length of } L}{d_r}.$$

- If there is no directed cycle of color 1 in  $G(\sigma, w)/\Delta$ . Then there is at least one directed cycle of color 2, and, since  $d_2 > 1$ , the right hand term in (ii) (b) of Lemma 3.10 is strictly less than

$$pk - \sum_{\substack{L \text{ directed cycle of} \\ G(\sigma, w)/\Delta \text{ with color } 2}} \frac{\text{length of } L}{d_2} = pk - \sum_{r=1}^2 \sum_{\substack{L \text{ directed cycle of} \\ G(\sigma, w)/\Delta \text{ with color } r}} \frac{\text{length of } L}{d_r},$$

i.e. than the right hand term of (10). So (10) holds. □

**3.4.2. Case where  $k = 2$  and  $A_1 \cup A_2 \subset \{1, 2\}$ .** Here, for  $n$  positive integer, we consider  $\sigma_n = s_1(n)s_2(n)$  with  $s_1(n), s_2(n)$  independent random involutions. There are three cases:

- (i)  $s_1(n), s_2(n)$  have both uniform distribution on the set of involutions of  $[n]$ :  $A_1 = A_2 = \{1, 2\}$ ,
- (ii)  $s_1(n), s_2(n)$  have both uniform distribution on the set of involutions of  $[n]$  without any fixed point:  $A_1 = A_2 = \{2\}$ ,

- (iii) one of the random involutions has uniform distribution on the set of involutions of  $[n]$  and the other one has uniform distribution on the set of involutions of  $[n]$  without any fixed point: since for  $s, s'$  involutions,  $ss'$  and  $s's$  are conjugate, one can suppose that  $A_1 = \{2\}$ ,  $A_2 = \{1, 2\}$ .

**Theorem 3.12.** (i) If  $A_1 = A_2 = \{1, 2\}$ , then as  $n$  goes to infinity, for all  $q \geq 1$ , the law of  $(N_1(\sigma_n), \dots, N_q(\sigma_n))$  converges weakly to  $\mu_1 \otimes \dots \otimes \mu_q$ , where for all  $l \geq 1$ ,  $\mu_l$  is the law of  $P_1 + 2P_{\frac{1}{2l}}$ , with  $P_1, P_{\frac{1}{2l}}$  independent random variables with distributions Poisson(1), Poisson( $\frac{1}{2l}$ ).

- (ii) If  $A_1 = A_2 = \{2\}$ , then as  $n$  even goes to infinity, for all  $q \geq 1$ , the law of

$$\frac{1}{2}(N_1(\sigma_n), \dots, N_q(\sigma_n))$$

converges weakly to Poisson(1/2)  $\otimes$  Poisson(1/4)  $\otimes \dots \otimes$  Poisson(1/2q).

- (iii) If  $A_1 = \{2\}$ ,  $A_2 = \{1, 2\}$ , then as  $n$  even goes to infinity, for all  $q \geq 1$ , the law of  $(N_1(\sigma_n), \dots, N_q(\sigma_n))$  converges weakly to  $\mu_1 \otimes \dots \otimes \mu_q$ , where for all  $l$  odd,  $\mu_l$  is the law of  $2P_{\frac{1}{2l}}$  and for all  $l$  even,  $\mu_l$  is the law of  $P_{\frac{1}{2}} + 2P_{\frac{1}{2l}}$ , with  $P_{\frac{1}{2}}, P_{\frac{1}{2l}}$  independent random variables with distributions Poisson( $\frac{1}{2}$ ), Poisson( $\frac{1}{2l}$ ).

To prove the Theorem, we shall need the following lemmas.

**Lemma 3.13.** Consider  $a, b$  positive real numbers and define the measure

$$\nu_{a,b} := e^{-\frac{1+2ab}{2b^2}} \sum_{r=0}^{+\infty} \frac{\mathbb{E}[(X+a)^r]}{r!b^r} \delta_r,$$

with  $X$  standard Gaussian random variable. Then  $\nu_{a,b}$  is the distribution of  $P_{\frac{a}{b}} + 2P_{\frac{1}{2b^2}}$ , with  $P_{\frac{a}{b}}, P_{\frac{1}{2b^2}}$  independent random variables with respective distributions Poisson( $a/b$ ), Poisson( $1/(2b^2)$ ).

**Lemma 3.14.** Consider  $A_1, A_2$  as in (i), (ii) or (iii) of Theorem 3.12. Fix  $p \geq 1$  and  $\sigma \in \mathfrak{S}_p$ . Then the following holds:

- for all  $\Delta \in C(\sigma, g_1 g_2, A_1, A_2)$ ,  $\chi(G(\sigma, g_1 g_2)/\Delta) = 0$ ,
- the cardinality of  $C(\sigma, g_1 g_2, A_1, A_2)$  is

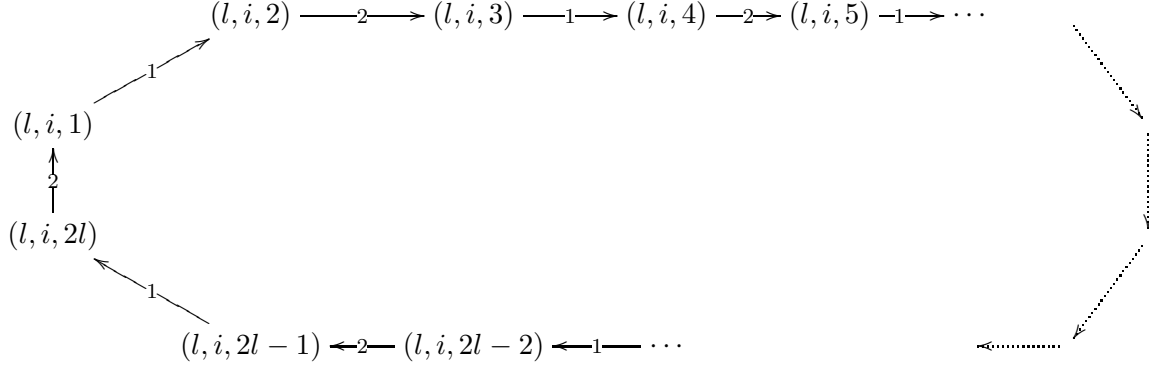
$$\begin{cases} \prod_{l=1}^p \mathbb{E}((\sqrt{l}X + l + 1)^{N_l(\sigma)}) & \text{if } A_1 = A_2 = \{1, 2\}, \\ \prod_{l=1}^p \mathbb{E}((1 + \sqrt{l}X)^{N_l(\sigma)}) & \text{if } A_1 = A_2 = \{2\}, \\ \prod_{\substack{1 \leq l \leq p \\ l \text{ odd}}} \mathbb{E}[(\sqrt{l}X + 1)^{N_l(\sigma)}] \prod_{\substack{1 \leq l \leq p \\ l \text{ even}}} \mathbb{E}[(\sqrt{l}X + l/2 + 1)^{N_l(\sigma)}] & \text{if } A_1 = \{2\}, A_2 = \{1, 2\}, \end{cases}$$

where  $X$  is a standard Gaussian random variable.

**Proof.** Let, for  $l$  positive integer,  $c_{2l}$  be the cycle  $(12 \dots 2l)$ .

First, we are going to use Lemma 2.6: up to a renaming of its vertices and edges,  $G(\sigma, g_1 g_2)$  is the disjoint union, for  $(l, i)$  such that  $l \in [p], i \in [N_l(\sigma)]$ , of the graphs  $H(l, i)$ , where each  $H(l, i)$  is the oriented edge-colored graph with color set  $\{1, 2\}$ , with vertex set  $\{(l, i, j); j \in [2l]\}$ , with 1-colored edge set  $\{(l, i, j) \rightarrow (l, i, c_{2l}(j)); j \in [2l] \text{ odd}\}$  and with 2-colored edge set  $\{(l, i, j) \rightarrow (l, i, c_{2l}(j)); j \in [2l] \text{ even}\}$ . An illustration is given in Figure 3. With this renaming, for  $\Delta$  a partition of the vertex set of  $G(\sigma, g_1 g_2)$ , the condition

$$\forall m \neq m' \in [p], (m, 1) \neq (m', 1) \pmod{\Delta}$$

FIGURE 3. The graph  $H(l, i)$ .

gets

$$\forall l, l' \in [p], \forall (i, i') \in [N_i(\sigma)] \times [N_{l'}(\sigma)], \forall (j, j') \in [2l] \times [2l'],$$

$$(12) \quad [(l, i, j) \neq (l', i', j') \text{ and } j = j' \pmod{2}] \Rightarrow [(l, i, j) \neq (l', i', j') \pmod{\Delta}].$$

Now, let us fix  $\Delta \in C(\sigma, g_1 g_2, A_1, A_2)$ .

a) By Remark 3.11,  $G(\sigma, w)/\Delta$  has  $2p$  edges.

b) By (12), the classes of  $\Delta$  cannot have more than two elements.

c)  $\Delta$  is an  $(A_1, A_2)$ -admissible partition, so no directed path of color either 1 or 2 of  $G(\sigma, g_1 g_2)/\Delta$  can have length at least two. This implies that if two distinct edges of  $G(\sigma, g_1 g_2)/\Delta$  have the same color and an extremity in common, then these edges are the ones of a directed cycle of length two, i.e. that the beginning of each of them is the end of the other one and vice versa.

d) Let us consider two distinct vertices  $(l, i, j) \neq (l', i', j')$  of  $G(\sigma, g_1 g_2)$  which form together a class of  $\Delta$ .

( $\alpha$ ) By (12),  $j$  and  $j'$  do not have the same parity, hence the edge which begins at  $(l, i, j)$  has the color of the edge which ends at  $(l', i', j')$ , and vice versa. So by an immediate induction on  $|m|$  (using c)), one has:

$$(13) \quad \forall m \text{ positive or negative integer, } (l, i, c_{2l}^m(j)) = (l', i', c_{2l'}^{-m}(j')) \pmod{\Delta}.$$

( $\beta$ ) Suppose that  $l \neq l'$ . One can suppose that  $l < l'$ . Then by (13) for  $m = 2l$ ,  $(l, i, j) = (l, i, c_{2l}^{2l}(j))$  is in the same time linked, by  $\Delta$ , with  $(l', i', c_{2l'}^{-2l}(j'))$  and  $(l', i', j')$  which are not the same. By b), it is impossible. Hence  $l = l'$ .

( $\gamma_1$ ) Suppose that  $i \neq i'$ . Then by (13) for  $m = -j + 1, -j + 2, \dots, -j + 2l$ , there is a unique  $q \in [l]$  such that the classes, in  $\Delta$ , of

$$(l, i, 1), (l, i, 2), \dots, (l, i, 2l), (l, i', 1), (l, i', 2), \dots, (l, i', 2l)$$

are

$$(14) \quad \{(l, i, 1), (l, i', 2q)\}, \{(l, i, 2), (l, i', c_{2l}^{-1}(2q))\}, \dots, \{(l, i, 2l), (l, i', c_{2l}^{-(2l-1)}(2q))\}.$$

Note that  $q$  depends on  $i$  and  $i'$  in a symmetric way, since the set of (14) can also be written

$$\{(l, i', 1), (l, i, 2q)\}, \{(l, i', 2), (l, i, c_{2l}^{-1}(2q))\}, \dots, \{(l, i', 2l), (l, i, c_{2l}^{-(2l-1)}(2q))\}.$$



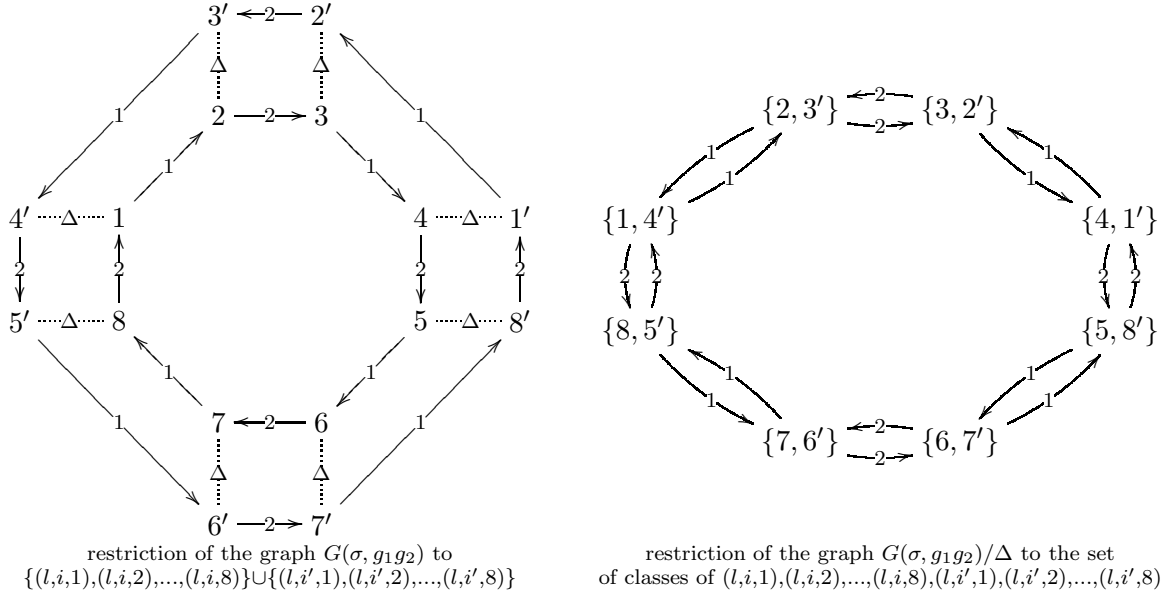


FIGURE 4.

On Figure 4, we give an illustration of the case  $(\gamma_1)$  with  $l = 4$ ,  $q = 2$ . In the inner (resp. outer) octagon, the vertices were not denoted by  $(l, i, j)$  (resp.  $(l, i', j)$ ) but by  $j$  (resp.  $j'$ ), for  $j \in [8]$ , in order to lighten the figure. In  $G(\sigma, g_1g_2)$ , vertices linked by  $\Delta$  are linked by edges of the type  $\cdots\Delta\cdots$  (but these edges do not belong to  $G(\sigma, g_1g_2)$ ). On this figure, it appears clearly that if two vertices of the left graph linked by an edge of the type  $\cdots\Delta\cdots$  are in the same class of  $\Delta$ , then, since no directed path of color either 1 or 2 of  $G(\sigma, g_1g_2)/\Delta$  can have length at least two, all the other pairs of vertices linked by an edge of the type  $\cdots\Delta\cdots$  must form classes of  $\Delta$ .

$(\gamma_2)$  Let us prove that if  $i = i'$ , then there is  $j_0 \in [2l]$  such that the classes, in  $\Delta$ , of the elements  $(l, i, 1), (l, i, 2), \dots, (l, i, 2l)$  are

$$(15) \quad \{(l, i, j_0), (l, i, c_{2l}(j_0))\}, \{(l, i, c_{2l}^{-1}(j_0)), (l, i, c_{2l}^2(j_0))\}, \{(l, i, c_{2l}^{-2}(j_0)), (l, i, c_{2l}^3(j_0))\}, \\ \dots, \{(l, i, c_{2l}^{-(l-1)}(j_0)), (l, i, c_{2l}^l(j_0))\},$$

and that this  $j_0$  is unique up to a replacement by  $c_{2l}^l(j_0)$ .

Note first that if such a  $j_0$  exists, then its uniqueness up to a replacement by  $c_{2l}^l(j_0)$  is obvious. Indeed, if one denotes  $c_{2l}^l(j_0)$  by  $j'_0$ , then since  $c_{2l}$  is a cycle of length  $2l$ , the partition of (15) is equal to

$$\{(l, i, j'_0), (l, i, c_{2l}(j'_0))\}, \{(l, i, c_{2l}^{-1}(j'_0)), (l, i, c_{2l}^2(j'_0))\}, \{(l, i, c_{2l}^{-2}(j'_0)), (l, i, c_{2l}^3(j'_0))\}, \\ \dots, \{(l, i, c_{2l}^{-(l-1)}(j'_0)), (l, i, c_{2l}^l(j'_0))\},$$

and  $\{(l, i, j_0), (l, i, c_{2l}(j_0))\}, \{(l, i, c_{2l}^{-(l-1)}(j_0)), (l, i, c_{2l}^l(j_0))\}$  ( $= \{(l, i, j'_0), (l, i, c_{2l}(j'_0))\}$ ) are the only classes of the type  $\{(l, i, x), (l, i, c_{2l}(x))\}$  in the partition of (15).

To prove the existence of such a  $j_0$ , it suffices to notice that by (13) and b),  $j_0 = (j' + j - 1)/2$  is convenient.

On Figure 5, give an illustration of the case  $(\gamma_2)$  with  $l = 4$ ,  $j_0 = 2$ . The comments on this figure are the same as the ones on Figure 4.

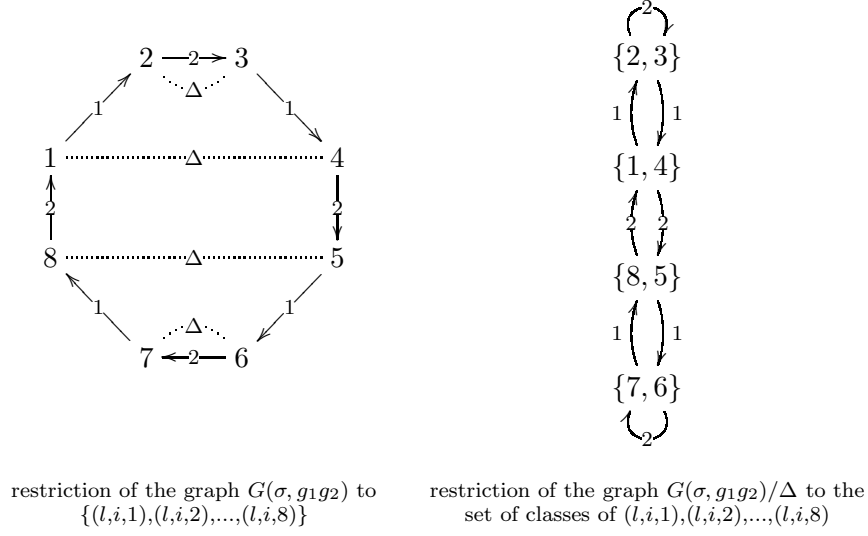


FIGURE 5.

( $\gamma_3$ ) Now, notice that the case ( $\gamma_2$ ) implies the existence of two monochromatic directed cycles with length 1 in  $G(\sigma, g_1 g_2)/\Delta$ : indeed, the edges

$$(l, i, j_0) \rightarrow (l, i, c_{2l}(j_0)), \quad (l, i, c_{2l}^l(j_0)) \rightarrow (l, i, c_{2l}^{l+1}(j_0))$$

of  $G(\sigma, g_1 g_2)$  give rise to the edges

$$\{(l, i, j_0), (l, i, c_{2l}(j_0))\} \curvearrowright \{(l, i, c_{2l}^l(j_0)), (l, i, c_{2l}^{l+1}(j_0))\} \curvearrowright$$

in  $G(\sigma, g_1 g_2)/\Delta$ . The respective colors of these edges are

$$\begin{cases} 1, 1 & \text{if } j_0 \text{ is odd and } l \text{ is even,} \\ 1, 2 & \text{if } j_0 \text{ is odd and } l \text{ is odd,} \\ 2, 2 & \text{if } j_0 \text{ is even and } l \text{ is even,} \\ 2, 1 & \text{if } j_0 \text{ is even and } l \text{ is odd.} \end{cases}$$

Hence this case is excluded if  $A_1 = A_2 = \{2\}$ , and if  $A_1 = \{2\}$ ,  $A_2 = \{1, 2\}$ , then  $j_0$  and  $l$  have to be even.

e) So we have proved that the only non singleton classes of  $\Delta$  are of two types

I.  $\{(l, i, j), (l, i', j')\}$  with  $l \in [p]$ ,  $i \neq i' \in [N_l(\sigma)]$ ,  $j, j' \in [2l]$ , and if there is such a class  $\{(l, i, j), (l, i', j')\}$  in  $\Delta$ , then there is a unique  $q \in [l]$  (depending symmetrically on  $i$  and  $i'$ ) such that the classes, in  $\Delta$ , of  $(l, i, 1), (l, i, 2), \dots, (l, i, 2l), (l, i', 1), (l, i', 2), \dots, (l, i', 2l)$  are given by (14). We shall denote this integer  $q$  by  $q_\Delta(l, \{i, i'\})$ .

II.  $\{(l, i, j), (l, i, j')\}$  with  $l \in [p]$ ,  $i \in [N_l(\sigma)]$ ,  $j, j' \in [2l]$ , and that if there is such a class  $\{(l, i, j), (l, i, j')\}$  in  $\Delta$ , then there is  $j_0 \in [2l]$ , unique up to a replacement by  $c_{2l}^l(j_0)$ , such that the classes, in  $\Delta$ , of  $(l, i, 1), (l, i, 2), \dots, (l, i, 2l)$  are given by (15). We denote by  $T_l(\Delta)$  the set of such  $i$ 's in  $[N_l(\sigma)]$ . If  $A_1 = A_2 = \{2\}$ , then this case cannot occur, hence  $T_l(\Delta) = \emptyset$ . If  $A_1 = A_2 = \{1, 2\}$ , then there is no restriction on  $l$  and  $j_0$ , and we shall denote the unique element of  $\{j_0, c_{2l}^l(j_0)\} \cap [l]$  by  $q_\Delta(l, \{i\})$ . If  $A_1 = \{2\}$ ,  $A_2 = \{1, 2\}$ , then the only restriction on  $l, j_0$  is that they have to be even, and we shall denote the unique element of  $\{\frac{1}{2}j_0, \frac{1}{2}c_{2l}^l(j_0)\} \cap [\frac{l}{2}]$  by  $q_\Delta(l, \{i\})$ .

If  $i \in [N_l(\sigma)]$  is such that for all  $j \in [2l]$ , the singleton  $\{(l, i, j)\}$  is a class of  $\Delta$ , then we define  $q_\Delta(l, \{i\})$  to be 0.

f) Let us define, for  $l \in [p]$ ,  $P_l(\Delta)$  to be the partition of  $[N_l(\sigma)]$  which links to elements  $i, i'$  if and only if there is  $j, j' \in [2l]$  such that  $(l, i, j)$  and  $(l, i', j')$  are linked by  $\Delta$  (we use the convention that the empty set has a unique partition, that this partition has cardinality zero and that if  $N_l(\sigma) = 0$ ,  $P_l(\Delta)$  is this partition).

g) It is easily seen that the number of vertices of  $G(\sigma, w)/\Delta$  is  $\sum_{l=1}^p 2l|P_l(\Delta)| - l|T_l(\Delta)|$ , that the sum of the number of 1-colored directed cycles of length two and of the number of 2-colored directed cycles of length two in  $G(\sigma, w)/\Delta$  is  $\sum_{l=1}^p 2l|\{C \in P_l(\Delta); |C| = 2\}| + (l-1)|T_l(\Delta)|$  and that the sum of the number of 1-colored directed cycles of length one and of the number of 2-colored directed cycles of length one in  $G(\sigma, w)/\Delta$  is  $\sum_{l=1}^p 2|T_l(\Delta)|$ . Hence by a), the Neagu characteristic of  $G(\sigma, w)/\Delta$  is

$$\begin{aligned} & (\sum_{l=1}^p 2l|P_l(\Delta)| - l|T_l(\Delta)|) - 2p + (\sum_{l=1}^p 2l|\{C \in P_l(\Delta); |C| = 2\}| + (l-1+1)|T_l(\Delta)|) \\ & = -2p + 2 \sum_{l=1}^p l(|P_l(\Delta)| + |\{C \in P_l(\Delta); |C| = 2\}|). \end{aligned}$$

But since for all  $l$ ,  $P_l(\Delta)$  is a partition of  $[N_l(\sigma)]$  where all classes have cardinality one or two,  $|P_l(\Delta)| + |\{C \in P_l(\Delta); |C| = 2\}| = |[N_l(\sigma)]| = N_l(\sigma)$ . Hence since  $\sum_{l=1}^p lN_l(\sigma) = p$ , the Neagu characteristic of  $G(\sigma, w)/\Delta$  is null.

i) In the case where  $A_1 = A_2 = \{1, 2\}$ , let us prove that cardinality of  $C(\sigma, g_1g_2, \{1, 2\}, \{1, 2\})$  is  $\prod_{l=1}^p \mathbb{E}[(\sqrt{l}X + l + 1)^{N_l(\sigma)}]$ , for  $X$  normal standard random variable. Note first that if  $C(\sigma)$  denotes  $\{l \in [p]; N_l(\sigma) \neq 0\}$ , then this reduces to

$$\prod_{l \in C(\sigma)} \sum_{i=0}^{\lfloor N_l(\sigma)/2 \rfloor} \binom{N_l(\sigma)}{2i} l^i \mathbb{E}(X^{2i})(l+1)^{N_l(\sigma)-2i}.$$

Let us define, for  $N \geq 1$ ,  $\text{Part}_{2,1}(N)$  to be the set of partitions of  $[N]$  in which all classes have cardinality 1 or 2. It is easy to see that the function

$$\begin{aligned} \varphi : C(\sigma, g_1g_2, \{1, 2\}, \{1, 2\}) & \rightarrow \{(P_l, (q(l, A))_{A \in P_l})_{l \in C(\sigma)}; \forall l \in C(\sigma), P_l \in \text{Part}_{2,1}(N_l(\sigma)), \\ & \quad \forall A \in P_l \text{ such that } |A| = 2, q(l, A) \in [l] \text{ and} \\ & \quad \forall A \in P_l \text{ such that } |A| = 1, q(l, A) \in \{0\} \cup [l]\} \\ \Delta & \mapsto (P_l(\Delta), (q_\Delta(l, A))_{A \in P_l(\Delta)})_{l \in C(\sigma)} \end{aligned}$$

is a bijection. Hence it suffices to prove that for all  $l \geq 1$ ,  $N \geq 1$ , the cardinality of the set of pairs  $(P, (q(A))_{A \in P})$  such that  $P \in \text{Part}_{2,1}(N)$  and for all  $A \in P$ ,  $q(A) \in [l]$  if  $|A| = 2$  and  $q(A) \in \{0\} \cup [l]$  in the other case, is equal to

$$\sum_{i=0}^{\lfloor N/2 \rfloor} \binom{N}{2i} l^i \mathbb{E}(X^{2i})(l+1)^{N-2i}.$$

It follows easily from the well known fact (called Wick's formula) that for all  $i \geq 1$ , the number of partitions of a set with cardinality  $2i$  in which all classes have two elements is equal to  $\mathbb{E}(X^{2i})$ .

To prove that cardinalities of  $C(\sigma, g_1g_2, \{2\}, \{2\})$  and of  $C(\sigma, g_1g_2, \{2\}, \{1, 2\})$  are the ones given in the statement of the Lemma, we use the same technique, replacing the bijection  $\varphi$  by

$$\begin{aligned}
\phi : C(\sigma, g_1 g_2, \{2\}, \{2\}) &\rightarrow \{(P_l, (q(l, A))_{A \in P_l})_{l \in C(\sigma)} ; \forall l \in C(\sigma), P_l \in \text{Part}_{2,1}(N_l(\sigma)) \text{ and} \\
&\quad \forall A \in P_l \text{ such that } |A| = 2, q(l, A) \in [l], \\
&\quad \forall A \in P_l \text{ such that } |A| = 1, q(l, A) = 1\} \\
\Delta &\mapsto (P_l(\Delta), (q_\Delta(l, A))_{A \in P_l(\Delta)})_{l \in C(\sigma)} \\
\psi : C(\sigma, g_1 g_2, \{2\}, \{1, 2\}) &\rightarrow \{(P_l, (q(l, A))_{A \in P_l})_{l \in C(\sigma)} ; \forall l \in C(\sigma), P_l \in \text{Part}_{2,1}(N_l(\sigma)), \\
&\quad \forall l \in C(\sigma), \forall A \in P_l \text{ such that } |A| = 2, q(l, A) \in [l], \\
&\quad \forall l \text{ even}, \forall A \in P_l \text{ such that } |A| = 1, q(l, A) \in \{0\} \cup [l/2]\} \text{ and} \\
&\quad \forall l \text{ odd}, \forall A \in P_l \text{ such that } |A| = 1, q(l, A) = 1\} \\
\Delta &\mapsto (P_l(\Delta), (q_\Delta(l, A))_{A \in P_l(\Delta)})_{l \in C(\sigma)}
\end{aligned}$$

□

**Proof of Theorem 3.12.** Let us prove (i). (ii) and (iii) can be treated analogously. We are going to apply Theorem 1.5 of [BG07]. For  $q \geq 1$ ,  $k_1, \dots, k_q \geq 1$ , for  $p = k_1 1 + \dots + k_q q$  and  $\sigma \in \mathfrak{S}_p$  such that  $N_1(\sigma) = k_1, \dots, N_q(\sigma) = k_q$ , by Proposition 3.1 and Lemma 3.14,

$$\frac{n^p}{1^{k_1} \dots q^{k_q} k_1! \dots k_q!} P(\{\forall i = 1, \dots, p, \sigma_n(i) = \sigma(i)\})$$

converges, as  $n$  goes to infinity, to  $\prod_{l=1}^q \frac{1}{k_l!} \mathbb{E}[(\sqrt{l}X + l + 1)/l]^{k_l}$ , where  $X$  is a standard Gaussian random variable. Hence we have to prove that for all  $r_1, \dots, r_q \geq 0$ , the series

$$\sum_{k_1 \geq r_1} \dots \sum_{k_q \geq r_q} (-1)^{k_1 - r_1 + \dots + k_q - r_q} \binom{k_1}{r_1} \dots \binom{k_q}{r_q} \prod_{l=1}^q \frac{1}{k_l!} \mathbb{E}[(\sqrt{l}X + l + 1)/l]^{k_l}$$

converges to  $\prod_{l=1}^q \mu_l(r_l)$ , which is equal, by Lemma 3.13, to  $\prod_{l=1}^q e^{-1 - \frac{1}{2l}} \frac{\mathbb{E}[(X + \sqrt{l})^{r_l}]}{r_l! l^{r_l/2}}$ . Since this series factorize and by Remark 1.6 of [BG07], it suffices to prove that for  $l \geq 1$ ,  $r \geq 1$ , the series  $\sum_{k \geq r} (-1)^{k-r} \binom{k}{r} \frac{1}{k!} \mathbb{E}[(\sqrt{l}X + l + 1)/l]^k$  converges to  $e^{-1 - \frac{1}{2l}} \frac{\mathbb{E}[(X + \sqrt{l})^r]}{r! l^{r/2}}$ . This follows from an application of the Dominated Convergence Theorem. □

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