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► **To cite this version:**

Baohua Fu. Wreath products, nilpotent orbits and symplectic deformations. *International Journal of Mathematics*, 2007, 18 (5), pp.473–481. hal-00114165

**HAL Id: hal-00114165**

**<https://hal.science/hal-00114165>**

Submitted on 15 Nov 2006

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# Wreath products, nilpotent orbits and symplectic deformations

Baohua Fu

November 15, 2006

## Abstract

We recover the wreath product  $X := \text{Sym}^2(\mathbb{C}^2/\pm 1)$  as a transversal slice to a nilpotent orbit in  $\mathfrak{sp}_6$ . By using deformations of Springer resolutions, we construct a symplectic deformation of symplectic resolutions of  $X$ . AMS Classification: 14E15, 14M17

## 0. Introduction

Let  $H \subset \text{Sp}(2n)$  be a finite sub-group and  $X := \mathbb{C}^{2n}/H$  the quotient symplectic variety. Given a projective symplectic resolution

$$Z \rightarrow X, \tag{1}$$

it was shown in [GK] that there exists a symplectic deformation of (1) over  $B := H^2(Z, \mathbb{C})$ , i.e. a morphism  $\Pi : \mathcal{Z} \rightarrow \mathcal{X}$  over  $B$  such that over the origin  $0 \in B$ ,  $\Pi_0 : \mathcal{Z}_0 \rightarrow \mathcal{X}_0$  is the resolution (1), and over a generic point  $b \in B$ ,  $\mathcal{Z}_b, \mathcal{X}_b$  are symplectic smooth varieties isomorphic under  $\Pi_b$ , where  $\Pi_b$  is the restriction of  $\Pi$  to the fibers over  $b$ . The proof of this theorem is based on the infinitesimal and formal deformations of  $\pi$  developed in [KV] and the globalization is obtained by using the expanding  $\mathbb{C}^*$ -action on  $X$ . As noted already in [GK], this deformation is very similar to the deformation of the Springer resolution of nilpotent cones given by Grothendieck's simultaneous resolution ([Slo]). However, the construction of symplectic deformations in general is rather implicit. The purpose of this note is to provide some explicit examples of such deformations.

A class of important examples of symplectic resolutions is given by Hilbert-Chow morphisms ([Wan]):  $\text{Hilb}^n(\mathbb{C}^2/\Gamma) \rightarrow \text{Sym}^n(\mathbb{C}^2/\Gamma)$ , where  $\Gamma \subset SL(2)$

is a finite sub-group and  $\mathbb{C}^2//\Gamma \rightarrow \mathbb{C}^2/\Gamma$  is the minimal resolution. The simplest case is  $n = 1$ . It can be shown ([Slo]) that a transverse slice of the sub-regular nilpotent orbit in the nilpotent cone has *ADE* singularities, then Grothendieck's simultaneous resolution provides symplectic deformations of the minimal resolution (see also [GK] section 3).

The next simple case is  $n = 2$  and  $\Gamma = \pm 1$ , i.e. the resolution  $\pi : \text{Hilb}^2(T^*\mathbb{P}^1) \rightarrow \text{Sym}^2(\mathbb{C}^2/\pm 1)$ . Our aim of this note is to construct a symplectic deformation of the resolution  $\pi$ . The key idea is to recover  $\pi$  as a slice of some Springer resolution. More precisely, let us consider the following two nilpotent orbits in  $\mathfrak{sp}_6$ :

$$\mathcal{O}_{[2,2,2]} := \text{Sp}_6 \cdot \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{O}_{[4,2]} := \text{Sp}_6 \cdot \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix},$$

then their closures in  $\mathfrak{sp}_6$  are given by:

$$\overline{\mathcal{O}}_{[2,2,2]} := \{A \in \mathfrak{sp}_6 \mid A^2 = 0\}, \quad \overline{\mathcal{O}}_{[4,2]} := \{A \in \mathfrak{sp}_6 \mid A^4 = 0\}.$$

We will prove that the wreath product  $\text{Sym}^2(\mathbb{C}^2/\pm 1)$  is in fact isomorphic to the intersection of a transverse slice of the nilpotent orbit  $\mathcal{O}_{[2,2,2]}$  with the nilpotent orbit closure  $\overline{\mathcal{O}}_{[4,2]}$  in  $\mathfrak{sp}_6$ . The singular variety  $\overline{\mathcal{O}}_{[4,2]}$  admits exactly two symplectic resolutions. By restricting them to the transverse slice, we recover exactly the two symplectic resolutions of  $\text{Sym}^2(\mathbb{C}^2/\pm 1)$ . Using deformations of Springer resolutions (e.g. [Fu]), we construct a symplectic deformation of  $\pi$ .

It is somewhat surprising that we can recover the wreath product  $\text{Sym}^2(\mathbb{C}^2/\pm 1)$  from nilpotent orbits, although the interplay between nilpotent orbits and Hilbert schemes has been noticed in [Man], where a transverse slice to the nilpotent orbit  $\mathcal{O}_{[2m-n,n]} (n \leq m)$  in the nilpotent cone of  $\mathfrak{sl}_{2m}$  is recovered as an open subset (whose complement is of codimension 1 when  $n \geq 2$ ) of the Hilbert scheme  $\text{Hilb}^n(A_{2m})$ , for some singular surface  $A_{2m}$ . Here  $\mathcal{O}_{[2m-n,n]}$  consists of nilpotent matrices  $A \in \mathfrak{sl}_{2m}$  whose Jordan form has only two blocks, with sizes  $2m - n$  and  $n$  respectively. It would be very interesting to recover other wreath products as a transverse slice to nilpotent orbits, which would in turn reveal more the mysterious relationships between Hilbert-Chow

resolutions and Springer resolutions, although the two objects are studied in usual separately.

### 1. The transverse slice

Let  $\mathfrak{g}$  be a simple Lie algebra and  $G$  its adjoint group. For any nilpotent element  $x \in \mathfrak{g}$ , by the theorem of Jacobson-Morozov, there exists an  $\mathfrak{sl}_2$ -triplet  $(x, y, h)$ . Then  $S = x + \mathfrak{g}^y$  is a transverse slice to the nilpotent orbit  $G \cdot x$  in  $\mathfrak{g}$ , and the morphism  $G \times S \rightarrow \mathfrak{g}$  is smooth ([Slo], Section 7.4). Here  $\mathfrak{g}^y := \{z \in \mathfrak{g} \mid [z, y] = 0\}$ .

From now on, let  $\mathfrak{g} = \mathfrak{sp}_6$ , and consider the following  $\mathfrak{sl}_2$ -triplet associated to the nilpotent orbit  $\mathcal{O}_{[2,2,2]}$ :

$$x_0 = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}, \quad y_0 = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}, \quad h_0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad (2)$$

where  $I$  is the  $3 \times 3$  identity matrix. Note that  $\mathcal{O}_{[2,2,2]} = \mathrm{Sp}_6 \cdot x_0$ .

The transverse slice to the orbit  $\mathcal{O}_{[2,2,2]}$  is given by

$$S = x_0 + \mathfrak{g}^{y_0} = \left\{ \begin{pmatrix} Z_1 & I \\ Z_2 & Z_1 \end{pmatrix} \mid Z_1 + Z_1^T = 0, Z_2 = Z_2^T \right\} \subset \mathfrak{sp}_6.$$

We choose the following parameters for  $Z_1$  and  $Z_2$ :

$$Z_1 = \begin{pmatrix} 0 & a_3/2 & -a_2/2 \\ -a_3/2 & 0 & a_1/2 \\ a_2/2 & -a_1/2 & 0 \end{pmatrix}, \quad Z_2 = \begin{pmatrix} x_1 & y_1 & y_2 \\ y_1 & x_2 & y_3 \\ y_2 & y_3 & x_3 \end{pmatrix}.$$

Note that  $\mathcal{O}_{[2,2,2]} \subset \overline{\mathcal{O}}_{[4,2]}$ , and the codimension is 4. Let  $T$  be the scheme intersection  $S \cap \overline{\mathcal{O}}_{[4,2]}$ . The variety  $\overline{\mathcal{O}}_{[4,2]}$  is normal and the morphism  $G \times T \rightarrow \overline{\mathcal{O}}_{[4,2]}$  is smooth. It follows that  $T$  is normal. As easily seen, a matrix  $A \in S$  is in  $T$  if and only if  $\mathrm{rk}(A) \leq 4$  and  $\mathrm{tr}(A) = \mathrm{tr}(A^2) = \mathrm{tr}(A^3) = \mathrm{tr}(A^4) = 0$ .

Notice that  $\mathrm{rk}(A) \leq 4$  is equivalent to  $\mathrm{rk}(Z_2 - Z_1^2) \leq 1$ . The matrix  $Z_2 - Z_1^2$  is symmetric, so this is equivalent to the existence of  $u = (u_1, u_2, u_3) \in \mathbb{C}^3$  such that  $Z_2 - Z_1^2 = u^T u$ , from which we can substitute the variables  $x_i, y_j$  by  $u_k$ . Remark that  $u$  and  $-u$  give the same  $Z_2$ , so we should quotient by the following action of  $\mathbb{Z}_2$ :  $u \mapsto -u$ .

Now a direct calculus shows that  $\mathrm{tr}(A) = \mathrm{tr}(A^3) = 0$ . That  $\mathrm{tr}(A^2) = 0$  is equivalent to  $\sum_{i=1}^3 u_i^2 = \sum_{i=1}^3 a_i^2$ , and  $\mathrm{tr}(A^4) = 2 \mathrm{tr}(Z_1^4) + 2 \mathrm{tr}(Z_2^2) +$

$12 \operatorname{tr}(Z_1^2 Z_2) = 0$  is equivalent to  $\sum_{i=1}^3 a_i u_i = 0$ , which gives:

$$T = \{(a_1, a_2, a_3, u_1, u_2, u_3) \mid \sum_i u_i^2 = \sum_i a_i^2, \sum_i a_i u_i = 0\} / \mathbb{Z}_2,$$

where the action of  $\mathbb{Z}_2$  is given by

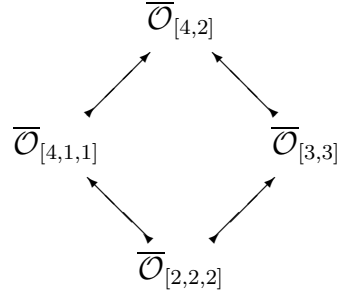
$$(a_1, a_2, a_3, u_1, u_2, u_3) \mapsto (a_1, a_2, a_3, -u_1, -u_2, -u_3).$$

Consider the following two nilpotent orbits in  $\mathfrak{sp}_6$ :

$$\mathcal{O}_{[3,3]} = \{A \in \mathfrak{sp}_6 \mid A^3 = 0, \operatorname{rk}(A) = 4\},$$

$$\mathcal{O}_{[4,1,1]} = \{A \in \mathfrak{sp}_6 \mid A^4 = 0, \operatorname{rk}(A) = 3, A^2 \neq 0\}.$$

Then  $\mathcal{O}_{[2,2,2]} = \overline{\mathcal{O}}_{[3,3]} \cap \overline{\mathcal{O}}_{[4,1,1]}$  and  $\overline{\mathcal{O}}_{[3,3]} \subset \overline{\mathcal{O}}_{[4,2]} \supset \overline{\mathcal{O}}_{[4,1,1]}$ . The relationship of inclusions can be resumed in the following diagram:



The intersection of  $T$  with the two orbit closures  $\overline{\mathcal{O}}_{[3,3]}, \overline{\mathcal{O}}_{[4,1,1]}$  is exactly the singular locus of  $T$ , which is defined by the following

$$T \cap \overline{\mathcal{O}}_{[3,3]} = \left\{ \sum_i a_i^2 = 0, u^T u = -4Z_1^2 \right\},$$

$$T \cap \overline{\mathcal{O}}_{[4,1,1]} = \left\{ \sum_i a_i^2 = 0, u_1 = u_2 = u_3 = 0 \right\}.$$

Both are isomorphic to the surface  $\mathbb{C}^2/\pm 1$  with an isolated  $A_1$ -singularity. The intersection  $T \cap \mathcal{O}_{[2,2,2]} = x_0$  is just a point.

## 2. The wreath product

Now we consider the simplest wreath product  $\operatorname{Sym}^2(\mathbb{C}^2/\pm 1) = \mathbb{C}^4/H$ , where  $H$  is the subgroup of  $\operatorname{Sp}(4)$  generated by the following elements:

$$\sigma(x_1, x_2, y_1, y_2) = (y_1, y_2, x_1, x_2), \quad \tau(x_1, x_2, y_1, y_2) = (-x_1, -x_2, y_1, y_2).$$

To write down equations for this affine normal variety, we put

$$a_1 = -i(x_1^2 + y_1^2 + x_2^2 + y_2^2)/2, a_2 = (x_1^2 + y_1^2 - x_2^2 - y_2^2)/2, a_3 = x_1x_2 + y_1y_2,$$

$$u_1 = x_1y_1 + x_2y_2, u_2 = i(x_1y_1 - x_2y_2), u_3 = i(x_1y_2 + x_2y_1).$$

The functions  $a_1, a_2, a_3$  are  $H$ -invariant, and the action of  $H$  restricts to a  $\mathbb{Z}_2$ -action on  $(u_1, u_2, u_3)$  given by  $(u_1, u_2, u_3) \mapsto (-u_1, -u_2, -u_3)$ . Now it is straight-ward to check that

$$\mathbb{C}^4/H \simeq \{(a_1, a_2, a_3, u_1, u_2, u_3) \mid \sum a_i^2 = \sum u_i^2, \sum_i a_i u_i = 0\}/\mathbb{Z}_2,$$

where the  $\mathbb{Z}_2$ -action is given by  $(a_1, a_2, a_3, u_1, u_2, u_3) \mapsto (a_1, a_2, a_3, -u_1, -u_2, -u_3)$ . This gives the following proposition.

**Proposition 1.** *The transverse slice  $T$  is isomorphic to the wreath product  $X := \text{Sym}^2(\mathbb{C}^2/\pm 1)$ .*

The singular locus of the wreath product  $X$  has two components: one is the diagonal  $\Delta$  and the other will be denoted by  $\Xi$ . One sees that the isomorphism between  $T$  and  $X$  sends  $T \cap \overline{\mathcal{O}}_{[3,3]}$  to  $\Delta$  and  $T \cap \overline{\mathcal{O}}_{[4,1,1]}$  to  $\Xi$ .

A symplectic resolution of  $X$  is given by the composition:

$$\pi : \text{Hilb}^2(T^*\mathbb{P}^1) \rightarrow \text{Sym}^2(T^*\mathbb{P}^1) \rightarrow \text{Sym}^2(\mathbb{C}^2/\pm 1) = X.$$

The central fiber of  $\pi$  contains a  $\mathbb{P}^2$ , so we can blow up this  $\mathbb{P}^2$  and then blow down along the other direction, i.e. we can perform a Mukai flop, which gives another symplectic resolution  $\pi^+ : \text{Hilb}^2(T^*\mathbb{P}^1) \rightarrow X$ . One sees that  $\pi^{-1}(0) \subset \overline{\pi^{-1}(\Delta - \{0\})}$ , but  $\pi^{-1}(0)$  is not contained in  $\overline{\pi^{-1}(\Xi - \{0\})}$ , so  $\Delta$  and  $\Xi$  are not symmetric with respect to  $\pi$ . For  $\pi^+$ , it changes the role of  $\Delta$  and  $\Xi$ . It is known that any projective symplectic resolutions of  $X$  is isomorphic to  $\pi$  or  $\pi^+$  (for details, see ([FN], [Fuj])).

### 3. Springer resolutions

The nilpotent orbit closure  $\overline{\mathcal{O}}_{[4,2]}$  admits exactly two symplectic resolutions, given by Springer maps:

$$T^*(G/P_1) \xrightarrow{\phi_1} \overline{\mathcal{O}}_{[4,2]} \xleftarrow{\phi_2} T^*(G/P_2), \quad (3)$$

where  $P_1$  (resp.  $P_2$ ) is the standard parabolic sub-group of  $G$  with flag type  $[1,2,2,1]$  (resp.  $[2,1,1,2]$ ). The matrix forms of  $P_1$  and  $P_2$  are as follows:

$$P_1 = \left\{ \begin{pmatrix} * & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * & * \end{pmatrix} \right\}, \quad P_2 = \left\{ \begin{pmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & * \end{pmatrix} \right\}.$$

The restrictions of  $\phi_1, \phi_2$  to the pre-image of the transverse slice  $T$  give two projective symplectic resolutions of  $T$ .

$$\text{Hilb}^2(T^*\mathbb{P}^1) \simeq Z_1 \xrightarrow{\pi_1} T \xleftarrow{\pi_2} Z_2 \simeq \text{Hilb}^2(T^*\mathbb{P}^1).$$

**Proposition 2.** *The two symplectic resolutions  $\pi_1, \pi_2$  are related by a Mukai flop, in particular, they are not isomorphic. Furthermore  $\pi_1 = \pi$  and  $\pi_2 = \pi^+$ .*

*Proof.* We will calculate the central fiber over the point  $x_0 = T \cap \mathcal{O}_{[2,2,2]}$  (c.f. (2)) under the maps  $\pi_1$  and  $\pi_2$ . Let  $\{e_i, 1 \leq i \leq 6\}$  be the natural basis of  $\mathbb{C}^6$  and the symplectic form is  $\omega = \sum_{i=1}^3 e_i^* \wedge e_{i+3}^*$ . Notice that  $\text{Im}(x_0) = \text{Ker}(x_0) = \mathbb{C}\langle e_1, e_2, e_3 \rangle =: K$  is Lagrangian.

It is easy to see that

$$\pi_1^{-1}(x_0) = \{\text{flags } (F_1 \subset F_2) \mid x_0 F_2 \subset F_1 \subset K, F_2 = F_2^\perp, \dim F_1 = 1\}.$$

Since  $x_0 F_2 \subset F_1$  is of dimension 1, one has two possibilities:

(i).  $\dim(K \cap F_2) = 2$ , then  $F_1 = x_0 F_2 \subset x_0 F_1^\perp$ . Suppose that  $F_1$  is generated by  $\sum_{i=1}^3 a_i e_i$ , then  $x_0 F_1^\perp = \{\sum_{i=1}^3 b_i e_i \mid \sum_i a_i b_i = 0\}$ . The condition  $F_1 \subset x_0 F_1^\perp$  is equivalent to  $\sum_{i=1}^3 a_i^2 = 0$ , which is a  $\mathbb{P}^1$  inside  $\mathbb{P}(K)$ . The condition for  $F_2$  is just  $x_0 F_1^\perp \subset F_2 \subset x_0^{-1} F_1$  which is a  $\mathbb{P}^1$ . So finally this component is a  $\mathbb{P}^1$ -bundle over  $\mathbb{P}^1$ .

(ii).  $F_2 = K$ , then this is isomorphic to  $\mathbb{P}(K)$ . The two components intersect at a curve  $C_1 \simeq \mathbb{P}^1$  inside  $\mathbb{P}(K)$ .

The fiber of  $x_0$  under  $\pi_2$  consists of flags  $(F_1 \subset F_2)$  such that  $x_0 F_2 \subset F_1 \subset K, F_2 = F_2^\perp$  and  $\dim F_1 = 2$ . Since  $F_1 \subset K \cap F_2$ , so  $\dim(K \cap F_2) \geq 2$ . There are two cases:

(i).  $\dim(K \cap F_2) = 2$ , then  $F_1 = K \cap F_2$  and  $x_0 F_2 \subset F_1$ . This gives that  $x_0 F_2 = x_0 F_1^\perp \subset F_1$ . Suppose  $F_1$  is generated by  $\sum_{i=1}^3 a_i e_i, \sum_{i=1}^3 b_i e_i$ . Then

we have  $x_0 F_1^\perp = \{\sum_i c_i e_i \mid \sum_i a_i c_i = \sum_i b_i c_i = 0\}$ . The condition  $x_0 F_1^\perp \subset F_1$  is equivalent to the existence of  $(y, y') \neq (0, 0)$  such that  $y(\sum_i a_i^2) + y'(\sum_i a_i b_i) = 0$  and  $y(\sum_i a_i b_i) + y'(\sum_i b_i^2) = 0$ . So the condition for  $F_1$  is  $(\sum_i a_i^2)(\sum_i b_i^2) = (\sum_i a_i b_i)^2$ . Under the Plücker embedding  $\mathbb{P}(\wedge^2 F_1) \rightarrow \mathbb{P}(\wedge^2 K) \simeq \mathbb{P}(K^*)$ , one sees that this is a conic in  $\mathbb{P}^2$ . The condition for  $F_2$  turns to be  $F_1 \subset F_2 \subset F_1^\perp$ . So this component is a  $\mathbb{P}^1$ -bundle over  $\mathbb{P}^1$ .

(ii).  $K = F_2$ , then  $F_1 \subset K$ , this component is just  $\mathbb{P}(K^*)$ . The two components intersect at a  $C_2 \simeq \mathbb{P}^1$  inside  $\mathbb{P}(K^*)$ .

Now it is clear that the two resolutions are different and are related by the Mukai flop along the component  $\mathbb{P}(K^*)$ , and  $C_1, C_2$  are dual conics.

Now we will identify  $\pi_1, \pi_2$  with  $\pi, \pi^+$ . By definition, we have

$$\pi_1^{-1}(T \cap \overline{\mathcal{O}}_{[3,3]}) = \{(F_1 \subset F_2, z) \mid zF_2 \subset F_1 \subset \text{Ker}(z), F_2 = F_1^\perp, \dim F_1 = 1\},$$

where  $z$  is in  $T \cap \overline{\mathcal{O}}_{[3,3]}$ . Consider the elements  $z_t \in T \cap \overline{\mathcal{O}}_{[3,3]}$  ( $t \in \mathbb{C}$ ) given by

$$z_t = \begin{pmatrix} tB & I \\ -3t^2 B^2 & tB \end{pmatrix}, \text{ with } B = \begin{pmatrix} 0 & \sqrt{-1} & 1 \\ -\sqrt{-1} & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

One has  $\text{Ker}(z_t) = \mathbb{C}\langle e_1 + \sqrt{-1}te_5 + te_6, e_2 - \sqrt{-1}e_3 \rangle$ . When  $t$  goes to 0,  $\text{Ker}(z_t)$  goes to  $\mathbb{C}\langle e_1, e_2 - \sqrt{-1}e_3 \rangle$ , thus the limit of  $\pi_1^{-1}(z_t)$  will be  $\mathbb{P}(\mathbb{C}\langle e_1, e_2 - \sqrt{-1}e_3 \rangle) \subset \mathbb{P}(K)$ , which is not a point. This shows that  $\pi_1 = \pi$  by the description of  $\pi$  and  $\pi^+$  in section 2.  $\square$

#### 4. Symplectic deformations

A deformation of the symplectic resolutions  $\phi_i, i = 1, 2$  (cf. (3)) can be constructed as follows ([Fu]). Let  $\mathfrak{c}_i$  be the center of the Levi sub-algebra of  $\mathfrak{p}_i := \text{Lie}(P_i)$  and  $\mathfrak{u}_i$  the nil-radical of  $\mathfrak{p}_i$ . The vector space  $V_i := \mathfrak{c}_i + \mathfrak{u}_i$  is a flat family over  $\mathfrak{c}_i$ . Let  $\mathcal{Y}_i$  be the closed sub-variety

$$\mathcal{Y}_i := \{(z, v) \in \mathfrak{c}_i \times G \cdot V_i \mid v \in G \cdot (z + \mathfrak{u}_i)\}.$$

Now consider the morphism  $\Phi_i : G \times^{P_i} V_i \rightarrow \mathcal{Y}_i$  given by

$$g * (z + u) \mapsto (z, g \cdot (z + u)),$$

where  $g \in G, z \in \mathfrak{c}_i$  and  $u \in \mathfrak{u}_i$ . Notice that if  $z \neq 0$ , then  $z + \mathfrak{u}_i = P_i \cdot z$ , so this morphism is well-defined. One can show that  $\Phi_i$  is birational and it



gives a family of morphisms over  $\mathfrak{c}_i$ . When  $z \in \mathfrak{c}_i$  is generic, in the sense that the stabilizer  $G^z$  of  $z$  is exactly the Levi sub-group  $L_i$  of  $P_i$ , then

$$\Phi_i^z : G \times^{P_i} (z + \mathfrak{u}_i) \simeq G \times^{P_i} (P_i \cdot z) \rightarrow \mathcal{Y}_i^z = G \cdot z \simeq G/L_i$$

is an isomorphism. Notice that  $\mathcal{Y}_i^z$  is a semi-simple orbit, thus it is symplectic. When  $z = 0$ , the map  $\Phi_i^0$  is just the Springer resolution  $\phi_i$ . In other words,  $\Phi_i$  is a symplectic deformation of  $\phi_i$  with base  $\mathfrak{c}_i$ .

Let  $\mathcal{T}_i$  be the intersection  $(\mathfrak{c}_i \times S) \cap \mathcal{Y}_i$  and  $\mathcal{Z}_i$  its pre-image under the morphism  $\Phi_i$ , which gives a map  $\mathcal{Z}_i \xrightarrow{\Pi_i} \mathcal{T}_i$  over  $\mathfrak{c}_i$ . Now we will show that the family  $\psi_i : \mathcal{Z}_i \rightarrow \mathfrak{c}_i$  is smooth. Recall that the map  $G \times S \rightarrow \mathfrak{g}$  is smooth, so is  $G \times (S \cap (G \cdot V_i)) \rightarrow G \cdot V_i$  ([Slo] Section 5.1). The morphism  $\Phi_i$  is  $G$ -equivariant, so  $G \times \mathcal{Z}_i \rightarrow G \times^{P_i} V_i$  is smooth. Notice that the map  $G \times^{P_i} V_i \rightarrow \mathfrak{c}_i$  is smooth, so is the composition  $G \times \mathcal{Z}_i \rightarrow \mathfrak{c}_i$ . The projection  $G \times \mathcal{Z}_i \rightarrow \mathcal{Z}_i$  is smooth, which implies the smoothness of  $\mathcal{Z}_i \rightarrow \mathfrak{c}_i$ .

An immediately corollary is that  $\mathcal{Z}_i$  is smooth and for any  $0 \neq z \in \mathfrak{c}_i$  generic, the intersection  $S \cap G \cdot z$  is smooth and symplectic, which deforms  $\text{Sym}^2(\mathbb{C}^2/\pm 1)$ . So  $\mathcal{Z}_i \xrightarrow{\Pi_i} \mathcal{T}_i$  gives a symplectic deformation of the symplectic resolution  $\pi_i : \text{Hilb}^2(T^*\mathbb{P}^1) \rightarrow \text{Sym}^2(\mathbb{C}^2/\pm 1)$ .

## 5. Universal Poisson deformations

Now we will show that our picture is similar to that of Brieskorn (see section 3 [GK]). We will only consider  $\phi_1$ . To simplify the notations, we will write  $P$  (resp.  $\phi$ ,  $L$  etc.) instead of  $P_1$  (resp.  $\phi_1$ ,  $L_1$  etc.).

Fix a maximal torus  $U$  in  $G$  and a Cartan sub-algebra  $\mathfrak{h}$  in  $\mathfrak{g} = \mathfrak{sp}_6$ . Coordinates in  $\mathfrak{h}$  are denoted by  $(h_1, h_2, h_3)$ . We define the Weyl group of  $L$  to be  $W(L) := N_L(U)/U$ , where  $N_L(U)$  is the normalizer of  $U$  in  $L$ . The partial Weyl group of  $P$  is  $W^P := N_G(L)/L$ . Then  $W^P$  is naturally isomorphic to the quotient  $N_{W(G)}(W(L))/W(L)$ , where  $W(G)$  is the Weyl group of  $G$ .

It is easy to see that  $W(L)$  is isomorphic to  $\mathbb{Z}_2$ , acting on  $\mathfrak{h}$  by  $(h_1, h_2, h_3) \mapsto (h_1, h_3, h_2)$ . The center  $\mathfrak{c}$  of  $\text{Lie}(L)$  is naturally identified with the fixed point set  $\mathfrak{h}^{W(L)}$ . The group  $W^P$  is isomorphic to  $\mathbb{Z}_2 \times \mathbb{Z}_2$ , which acts on  $\mathfrak{h}^{W(L)}$  by  $(h_1, h_2, h_2) \mapsto (-h_1, h_2, h_2)$  and  $(h_1, h_2, h_2) \mapsto (h_1, -h_2, -h_2)$ , i.e. it is the sum of two copies of the sign representation of  $\mathbb{Z}_2$ .

Let  $\mathcal{T}'$  be the intersection  $S \cap G \cdot (\mathfrak{c} + \mathfrak{u})$ , then we have a natural projection

$p : \mathcal{T} \rightarrow \mathcal{T}'$  and the following diagram is commutative:

$$\begin{array}{ccccc}
 \mathcal{Z} & \xrightarrow{\Pi} & \mathcal{T} & \xrightarrow{p} & \mathcal{T}' \\
 \psi \downarrow & & \psi' \downarrow & & \beta \downarrow \\
 \mathfrak{h}^{W(L)} & \xrightarrow{id} & \mathfrak{h}^{W(L)} & \xrightarrow{\eta} & \mathfrak{h}^{W(L)}/W^P,
 \end{array} \tag{4}$$

where  $\eta$  is the natural quotient map and  $\beta$  is the restriction to  $\mathcal{T}'$  of the adjoint quotient map  $\mathfrak{g} \rightarrow \mathfrak{h}/W(G)$ .

*Claim:* The second Poisson cohomology  $HP^2(T)$  can be naturally identified with  $\mathfrak{h}^{W(L)}/W^P$ .

Let  $H'$  be the semi-direct product of  $\mathbb{Z}_2$  with  $W^P$ . Let  $\mathbb{Z}_2$  acts on  $\mathfrak{h}^{W(L)}$  by  $(h_1, h_2, h_2) \mapsto (h_2, h_1, h_1)$ , then it is easy to see that  $(\mathfrak{h}^{W(L)} \oplus (\mathfrak{h}^{W(L)})^*)/H'$  is isomorphic to  $T \simeq \text{Sym}^2(\mathbb{C}^2/\pm 1)$ . By [GK] (section 4), we have  $HP^2(T)$  is naturally isomorphic to  $HP^2(T \cap \overline{\mathcal{O}}_{[3,3]}) \oplus HP^2(T \cap \overline{\mathcal{O}}_{[4,1,1]})$ . By Lemma 3.1 [GK],  $HP^2(T \cap \overline{\mathcal{O}}_{[3,3]})$  is naturally identified with  $\mathbb{C}v_1/\mathbb{Z}_2$  and  $HP^2(T \cap \overline{\mathcal{O}}_{[4,1,1]})$  is identified with  $\mathbb{C}v_2/\mathbb{Z}_2$ , where  $v_1 = (0, 1, 1)$ ,  $v_2 = (1, 0, 0)$  are two points in  $\mathfrak{h}^{W(L)}$ , and the group  $\mathbb{Z}_2$  acts by sign representation.

Note that under this identification,  $\mathfrak{h}^{W(L)}$  is identified with  $H^2(\text{Hilb}^2(T^*\mathbb{P}^1))$ . The first square in (4) is the symplectic deformation of  $\pi$ , the second square is Cartesian. For the vertical morphisms,  $\psi$  is a universal Poisson deformation of  $\text{Hilb}^2(T^*\mathbb{P}^1)$ ,  $\psi'$  is similar to the Calogero-Moser deformation of  $T \simeq (\mathfrak{h}^{W(L)} \oplus (\mathfrak{h}^{W(L)})^*)/H'$ , and  $\beta$  is the universal Poisson deformation of  $T$ .

**Remark 1.** A diagram analogous to (4) can be constructed using the same method for any Springer resolution.

*Acknowledgments:* This note comes out from very helpful discussions with S.-S. Roan. I want to thank him and W. Wang for their interest in my work, and M.S.R.I.(Berkeley) and Max-Planck-Institut für Mathematik (Bonn) for the hospitality.

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