# Quantum field theory meets Hopf algebra

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Received 15 November 2003, revised 30 November 2003, accepted 2 December 2003
Published online 3 December 2003

**Key words** Quantum field theory, renormalization, Hopf algebra, infinitesimal algebra
**MSC (2000)** 81T15, 81T18, 81T75,16W30

This paper provides a primer in quantum field theory (QFT) based on Hopf algebra and describes new Hopf algebraic constructions inspired by QFT concepts. The following QFT concepts are introduced: chronological products, S-matrix, Feynman diagrams, connected diagrams, Green functions, renormalization. The use of Hopf algebra for their definition allows for simple recursive derivations and lead to a correspondence between Feynman diagrams and semi-standard Young tableaux. Reciprocally, these concepts are used as models to derive Hopf algebraic constructions such as a connected coregular action or a group structure on the linear maps from $\mathcal{S}(V)$ to $V$. In most cases, noncommutative analogues are derived.





## 1 Introduction

Although Hopf algebraic concepts were used in quantum field theory (QFT) as early as 1969 [1], the real boom in the collaboration between Hopf algebra and QFT started with the work of Connes and Kreimer in 1998 [2], that spurred an enthousiastic activity partly reviewed by Figueroa and Gracia-Bondia [3]. In these works, Hopf algebraic structures were discovered in QFT and used to reinterpret some aspects of renormalization theory.

The aim of the present paper is a bit different. As a first purpose, it tries to convince the reader that Hopf algebra is a natural language for QFT. For that purpose, it uses Hopf algebraic techniques to express important concepts of QFT: chronological products, S-matrix, Feynman diagrams, connected diagrams, Green functions, renormalization. The power of Hopf algebra manifests itself through the ease and economy with which complete proofs can be given, with full combinatorial factors.

The second purpose of this paper is to demonstrate that QFT concepts can help designing new Hopf algebraic objects. As a first example, the connected Feynman diagrams lead us to the definition of two coproducts on $\mathcal{S}(\mathcal{C})$ and $\mathcal{T}(\mathcal{C})$ (where $\mathcal{C}$ is a coalgebra, $\mathcal{S}(\mathcal{C})$ the symmetric algebra over $\mathcal{C}$ and $\mathcal{T}(\mathcal{C})$ the tensor algebra over $\mathcal{C}$). These two coproducts are in a comodule coalgebra relation and enable us to define "connected" coregular actions on $\mathcal{S}(\mathcal{C})$ and $\mathcal{T}(\mathcal{C})$. As a second example, the Bogoliubov approach to renormalization leads to a group structure on the linear maps from $\mathcal{S}(V)$ to $V$ and from $\mathcal{T}(V)$ to $V$, where $V$ is a vector space. There, the infinitesimal bialgebraic structure of $\mathcal{T}(V)$ plays an essential role [4]. As a last example we recall that renormalization can be considered as a functor on bialgebras.

It might be useful to explain why Hopf algebra is so powerful to deal with quantum field theory. The first reason was given by Joni and Rota a long time ago [5]: the coproduct splits an object into subobjects and the product merges two objects into a new one. These operations are fundamental in most combinatorial problems. Therefore, Hopf algebra is a convenient framework to deal with combinatorics in general and with the combinatorial problems of QFT in particular. The second reason has to do with the fact that Hopf algebraic techniques exploit efficiently the recursive nature of perturbative QFT. If we express this property in terms of Feynman diagrams, Hopf algebra makes it very easy to add a new vertex to a diagram and to deduce properties of diagrams with $n$ vertices from properties of diagrams with $n-1$ vertices. Such recursive procedure can also be carried out directly on Feynman diagrams, without using Hopf algebra, but it is much harder and is the source "egregious

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errors by distinguished savants", as Wightman put it [6]. As a consequence, many textbooks give detailed proofs for very simple diagrams and leave as an exercise to the the proof of the general case. No such thing happens with Hopf algebras: the present paper makes clear that a proof for one million vertices is as simple as for two vertices. Finally, the Hopf algebraic approach uses naturally the fact that the unrenormalized chronological product is an associative product. This property is usually neglected.

The use of Hopf algebraic techniques reveals also that many quantum field concepts can be defined on any cocommutative coalgebra. Sometimes, natural noncommutative analogues of the commutative constructions of quantum field theory can be found.

The Hopf algebra background of this paper can be learnt from the first chapters of any book on the subject, but Majid's monograph [7] is particularly well suited.

## 2 A primer in quantum field theory

This section provides a self-contained introduction to QFT using Hopf algebraic tools. Some aspects of this section were already published in a conference proceedings [8], but complete proofs are given here for the first time. We deliberately avoid the delicate analytical problems of quantum field theory.

For some well-known physicists [9], Feynman diagrams are the essence of QFT. Indeed, Feynman diagrams contain a complete description of the perturbative aspects of QFT, providing its most spectacular success: the calculation of the gyromagnetic factor of the electron [10, 11]. Therefore, this primer goes all the way to the derivation of Feynman diagrams. However, it is not restricted to a particular quantum field theory but is valid for any cocommutative coalgebra $\mathcal{C}$ over $\mathbb{C}$. By extending the coproduct and counit of $\mathcal{C}$ to the symmetric algebra $\mathcal{S}(\mathcal{C})$, we equip $\mathcal{S}(\mathcal{C})$ with the structure of a commutative and cocommutative bialgebra. Then, we twist the product of $\mathcal{S}(\mathcal{C})$ using a Laplace pairing (or coquasitriangular structure) to define the chronological product $\circ$. This chronological product enables us to describe the S-matrix of the theory, that contains all the measurable quantities. The S-matrix is then expanded over Feynman diagrams and the Green functions are defined.

### 2.1 The coalgebra $\mathcal{C}$

In the QFT of the scalar field, the coalgebra $\mathcal{C}$ is generated as a vector space over $\mathbb{C}$ by the symbols $\varphi^n(x_i)$ where $n$ runs over the nonnegative integers and $x_i$ runs over a finite number of points in $\mathbb{R}^4$. The choice of a finite number of points is meant to avoid some analytical problems and is consistent with the framework of perturbative QFT. The coproduct $\Delta'$ of $\mathcal{C}$ is

$$\Delta'\varphi^n(x_i) \quad = \quad \sum_{j=0}^{n} \binom{n}{j} \varphi^j(x_i) \otimes \varphi^{n-j}(x_i),$$

the counit $\varepsilon'$ of $\mathcal{C}$ is $\varepsilon'\big(\varphi^n(x_i)\big) = \delta_{n0}$. The coalgebra $\mathcal{C}$ is cocommutative (it is a direct sum of binomial coalgebras). Moreover, $\mathcal{C}$ is a pointed coalgebra because all its simple subcoalgebras are one-dimensional (each simple subcoalgebra is generated by a $\varphi^0(x_i)$).

This coalgebra is chosen for comparison with QFT, but the following construction is valid for any cocommutative coalgebra. From the coalgebra $\mathcal{C}$ we now build a commutative and cocommutative bialgebra $\mathcal{S}(\mathcal{C})$.

### 2.2 The bialgebra $\mathcal{S}(\mathcal{C})$

The symmetric algebra $\mathcal{S}(\mathcal{C}) = \bigoplus_{n=0}^{\infty} \mathcal{S}^n(\mathcal{C})$ can be equipped with the structure of a bialgebra over $\mathbb{C}$. The product of the bialgebra $\mathcal{S}(\mathcal{C})$ is the symmetric product (denoted by juxtaposition) and its coproduct $\Delta$ is defined on $\mathcal{S}^1(\mathcal{C}) \cong \mathcal{C}$ by $\Delta a = \Delta' a$ and extended to $\mathcal{S}(\mathcal{C})$ by algebra morphism: $\Delta 1 = 1 \otimes 1$ and $\Delta(uv) = \sum u_{(1)}v_{(1)} \otimes u_{(2)}v_{(2)}$. The elements of $\mathcal{S}^n(\mathcal{C})$ are said to be of degree $n$. The counit $\varepsilon$ of $\mathcal{S}(\mathcal{C})$ is defined to be equal to $\varepsilon'$ on $\mathcal{S}^1(\mathcal{C}) \cong \mathcal{C}$ and extended to $\mathcal{S}(\mathcal{C})$ by algebra morphism: $\varepsilon(1) = 1$ and $\varepsilon(uv) = \varepsilon(u)\varepsilon(v)$. It can be checked recursively that $\Delta$ is coassociative and cocommutative. Thus, $\mathcal{S}(\mathcal{C})$ is a commutative and cocommutative

bialgebra which is graded as an algebra. In the case of the coalgebra of the scalar field, we have

$$\Delta\big(\varphi^{n_1}(x_1)\dots\varphi^{n_k}(x_k)\big) \quad = \quad \sum_{j_1=0}^{n_1}\cdots\sum_{j_k=0}^{n_k}\binom{n_1}{j_1}\cdots\binom{n_k}{j_k} \varphi^{j_1}(x_1)\cdots\varphi^{j_k}(x_k)\otimes\varphi^{n_1-j_1}(x_1)\cdots\varphi^{n_k-j_k}(x_k).$$

The powers $\Delta^k$ of the coproduct are called *iterated coproducts* and are defined by $\Delta^0 = \mathrm{Id}$, $\Delta^1 = \Delta$ and $\Delta^{k+1} = (\mathrm{Id}^{\otimes k}\otimes\Delta)\Delta^k$. Their action on an element $u$ of $\mathcal{S}(\mathcal{C})$ is denoted by $\Delta^k u = \sum u_{(1)}\otimes\cdots\otimes u_{(k+1)}$. In the case of the scalar field, we have

**Lemma 2.1** *If $k$ is a positive integer, the $k$-th iterated coproduct of $\varphi^n(x)$ is*

$$\Delta^{k-1}\varphi^n(x) \quad = \quad \sum_{\mathbf{m}}\frac{n!}{m_1!\dots m_k!}\varphi^{m_1}(x)\otimes\cdots\otimes\varphi^{m_k}(x),$$

*where $\mathbf{m} = (m_1,\dots,m_k)$ runs over all $k$-tuples of nonnegative integers $m_1,\dots,m_k$ such that $\sum_{i=1}^k m_i = n$.*

Proof. For $k = 1$, we have $\Delta^0 = \mathrm{Id}$. Thus, the left hand side of the equality is $\varphi^n(x)$. On the other hand, we have only one integer $m_1$ that must be equal to $n$ because of the constraint $\sum_{i=1}^k m_i = n$. Therefore, the right hand side is also $\varphi^n(x)$ and the lemma is true for $k = 1$. Assume that the lemma is true up $\Delta^{k-1}$. From the definition of $\Delta^k$ and the recursion hypothesis, we have

$$\Delta^k\varphi^n(x) \quad = \quad \sum_{\mathbf{m}}\sum_{i=0}^{m_k}\frac{n!}{m_1!\dots m_k!}\binom{m_k}{i}\varphi^{m_1}(x)\otimes\cdots\otimes\varphi^{m_{k-1}}(x)\otimes\varphi^{i}(x)\otimes\varphi^{m_k-i}(x).$$

If we define the tuple $\mathbf{m}' = (m'_1,\dots,m'_{k+1})$ with $m'_j = m_j$ for $j < k$, $m'_k = i$ and $m'_{k+1} = m_k - i$, then we see that that $\mathbf{m}'$ runs over all tuples of $k+1$ nonnegative integers such that $\sum_{j=1}^{k+1} m'_j = n$ and we can rewrite

$$\Delta^k\varphi^n(x) \quad = \quad \sum_{\mathbf{m}'}\frac{n!}{m'_1!\dots m'_{k+1}!}\varphi^{m_1}(x)\otimes\cdots\otimes\varphi^{m'_{k+1}}(x),$$

and the lemma is proved for $\Delta^k$. □

Now, we equip $\mathcal{S}(\mathcal{C})$ with a Laplace pairing that will be used to twist the commutative product of $\mathcal{S}(\mathcal{C})$.

### 2.3 Laplace pairing

The concept of Laplace pairing was introduced by Rota and collaborators [12, 13]. In the quantum group literature, it is called a coquasitriangular structure [7].

**Definition 2.2** A Laplace pairing is a linear map $(|)$ from $\mathcal{S}(\mathcal{C})\otimes\mathcal{S}(\mathcal{C})$ to the complex numbers such that $(1|u) = (u|1) = \varepsilon(u)$, $(uv|w) = \sum(u|w_{(1)})(v|w_{(2)})$ and $(u|vw) = \sum(u_{(1)}|v)(u_{(2)}|w)$ for any $u$, $v$ and $w$ in $\mathcal{S}(\mathcal{C})$.

The Laplace pairing of products of elements of $\mathcal{S}(\mathcal{C})$ is calculated with the following lemma

**Lemma 2.3** *For $u^i$ and $v^j$ in $\mathcal{S}(\mathcal{C})$, we have*

$$(u^1\cdots u^k|v^1\cdots v^l) \quad = \quad \sum\prod_{i=1}^{k}\prod_{j=1}^{l}(u^i_{(j)}|v^j_{(i)}),$$

*where $u^i_{(j)}$ is the term at position $j$ in the iterated coproduct $\Delta^{l-1}u^i$ and $v^j_{(i)}$ is the term at position $i$ in the iterated coproduct $\Delta^{k-1}v^j$.*

For example $(uvw|st) = \sum(u_{(1)}|s_{(1)})(u_{(2)}|t_{(1)})(v_{(1)}|s_{(2)})(v_{(2)}|t_{(2)})(w_{(1)}|s_{(3)})(w_{(2)}|t_{(3)})$.

Proof. The proof is recursive. If $k = l = 1$, we have $\Delta^0 u^1 = u^1$ and $\Delta^0 v^1 = v^1$, so that lemma 2.3 becomes $(u^1|v^1) = (u^1|v^1)$, which is true. Now assume that the lemma is true up to $k$ and $l$ and write $u^k = st$ with $s$ and $t$ in $\mathcal{S}(\mathcal{C})$. Lemma 2.3 becomes

$$(u^1 \cdots u^k | v^1 \cdots v^l) = \sum \prod_{i=1}^{k-1} \prod_{j=1}^{l} (u^i_{(j)}|v^j_{(i)}) \prod_{j=1}^{l} ((st)_{(j)}|v^j_{(k)}). \tag{2.1}$$

By algebra morphism $(st)_{(j)} = s_{(j)}t_{(j)}$ and by the definition of the Laplace pairing and by the coassociativity of the coproduct $(s_{(j)}t_{(j)}|v^j_{(k)}) = \sum (s_{(j)}|v^j_{(k)})(t_{(j)}|v^j_{(k+1)})$. If we introduce this equation in (2.1) and redefine $u^k = s$ and $u^{k+1} = t$, we obtain Lemma 2.3 for $k+1$ and $l$. If we apply the same reasoning to $v^l = st$, we see that the lemma is true for $k$ and $l+1$. Thus, it is true for all $k$ and $l$. □

If we write lemma 2.3 with all $u^i$ and $v^j$ in $\mathcal{C}$, we see that the Laplace pairing is entirely determined by its value on $\mathcal{C}$. In other words, once we know $(a|b)$ for all $a$ and $b$ in $\mathcal{C}$, lemma 2.3 enables us to calculate the Laplace pairing on $\mathcal{S}(\mathcal{C})$. In the case of the algebra of the scalar field, we can use an additional structure to determine the usual QFT expression for $(\varphi^n(x)|\varphi^m(y))$.

**Lemma 2.4** *For nonnegative integers $n$ and $m$,*

$$(\varphi^n(x)|\varphi^m(y)) = \delta_{nm} n! g(x,y)^n,$$

*where $g(x,y) = (\varphi(x)|\varphi(y))$.*

Proof. The coalgebra $\mathcal{C}$ was not supposed to be a bialgebra. However, the algebra of the scalar field can be equipped with the structure of a bialgebra. This additional structure will be described in section 4.5. At this point, we only need the obvious product structure $\phi^n(x) \cdot \phi^m(x) = \phi^{n+m}(x)$. We do not need to know what is the product of fields at different points and the product is only used here as a heuristic to determine $(\varphi^n(x)|\varphi^m(y))$. We consider that the Laplace pairing satisfies its defining properties for the product $\cdot$ of $\mathcal{C}$: $(a \cdot b|c) = \sum (a|c_{(1)})(b|c_{(2)})$ and $(a|b \cdot c) = \sum (a_{(1)}|b)(a_{(2)}|c)$ for $a$, $b$ and $c$ in $\mathcal{C}$. The lemma is clearly true for $n = m = 1$. Assume that it is true up to $n$ and $m$ and calculate $(\varphi^{n+1}(x)|\varphi^m(y)) = (\varphi(x) \cdot \varphi^n(x)|\varphi^m(y))$. From the definition of a Laplace pairing, we have

$$(\varphi(x) \cdot \varphi^n(x)|\varphi^m(y)) = \sum_{j=0}^{m} \binom{m}{j} (\varphi(x)|\varphi^j(y))(\varphi^n(x)|\varphi^{m-j}(y)).$$

The recursion hypothesis gives us $j = 1$ and $n = m - 1$, so that

$$(\varphi^{n+1}(x)|\varphi^m(y)) = mn!\delta_{n,m-1}(\varphi(x)|\varphi(y))(\varphi^n(x)|\varphi^n(y)) = \delta_{m,n+1}(n+1)!(\varphi(x)|\varphi(y))^{n+1},$$

and the lemma is proved for $n+1$. The same reasoning leads to the lemma for $m+1$. Note that the lemma is also true for $n = 0$ or $m = 0$ if we put $(\phi^0(x)|\phi^n(y)) = (\phi^n(x)|\phi^0(y)) = \delta_{0,n}$. □

In QFT, the function $g(x,y)$ is a distribution [14]. Two distributions are commonly used: the Wightman function and the Feynman propagator. The product $g(x,y)^n$ is well-defined for Wightman functions but not for Feynman propagators [15]. The solution of this problem is the first step of the renormalization theory. In the following, we assume that $g(x,y)$ was regularized to make it a smooth function, so that $g(x,y)^n$ is well defined.

### 2.4 Twisted product

The Laplace pairing induces a twisted product $\circ$ on $\mathcal{S}(\mathcal{C})$.

**Definition 2.5** If $u$ and $v$ are elements of $\mathcal{S}(\mathcal{C})$, the twisted product of $u$ and $v$ is denoted by $u \circ v$ and defined by $u \circ v = \sum (u_{(1)}|v_{(1)})u_{(2)}v_{(2)}$.

This product was introduced by Sweedler [16] as a crossed product in Hopf algebra cohomology theory because a Laplace pairing is a 2-cocycle. It was defined independently by Rota and Stein as a *circle product* [17]. To become familiar with this twisted product, we first prove a useful relation

**Lemma 2.6** *For $u$ and $v$ in $\mathcal{S}(\mathcal{C})$, we have*

$$\varepsilon(u \circ v) \quad = \quad (u|v)$$

P r o o f. The proof is straightforward. By linearity and algebra morphism property of the counit

$$\varepsilon(u \circ v) \quad = \quad \sum (u_{(1)}|v_{(1)})\varepsilon(u_{(2)}v_{(2)}) = \sum (u_{(1)}|v_{(1)})\varepsilon(u_{(2)})\varepsilon(v_{(2)}).$$

Now, by linearity of the Laplace pairing and the definition of the counit

$$\varepsilon(u \circ v) \quad = \quad \big(\sum u_{(1)}\varepsilon(u_{(2)})|\sum v_{(1)}\varepsilon(v_{(2)})\big) = (u|v).$$

□

For completeness, we now prove the classical

**Proposition 2.7** *The twisted product $\circ$ endows $\mathcal{S}(\mathcal{C})$ with the structure of an associative and unital algebra with unit 1.*

The proof is the consequence of several lemmas. The first lemma is

**Lemma 2.8** *For $u$ and $v$ in $\mathcal{S}(\mathcal{C})$,*

$$\Delta(u \circ v) \quad = \quad \sum u_{(1)} \circ v_{(1)} \otimes u_{(2)}v_{(2)}.$$

P r o o f. By the definition of the twisted product,

$$\Delta(u \circ v) \quad = \quad \sum (u_{(1)}|v_{(1)})\Delta(u_{(2)}v_{(2)}) = \sum (u_{(1)}|v_{(1)})u_{(2)}v_{(2)} \otimes u_{(3)}v_{(3)} = \sum u_{(1)} \circ v_{(1)} \otimes u_{(2)}v_{(2)},$$

where we used the coassociativity of the coproduct. □

The second lemma is

**Lemma 2.9** *For $u$, $v$ and $w$ in $\mathcal{S}(\mathcal{C})$,*

$$(u|v \circ w) \quad = \quad (u \circ v|w).$$

P r o o f. From the definitions of the twisted product and of the Laplace pairing we find

$$\begin{aligned}(u|v \circ w) \quad &= \quad \sum (v_{(1)}|w_{(1)})(u|v_{(2)}w_{(2)}) = \sum (v_{(1)}|w_{(1)})(u_{(1)}|v_{(2)})(u_{(2)}|w_{(2)}) \\ &= \quad \sum (v_{(1)}|w_{(2)})(u_{(2)}|v_{(2)})(u_{(1)}|w_{(1)}),\end{aligned}$$

where we used the cocommutativity of the coproduct of $u$ and $w$. The definition of the Laplace pairing gives now

$$(u|v \circ w) \quad = \quad \sum (u_{(1)}v_{(1)}|w)(u_{(2)}|v_{(2)}) = (u \circ v|w).$$

□

These two lemmas enable us to prove the associtavity of the twisted product as follows

$$u \circ (v \circ w) \quad = \quad \sum (u_{(1)}|(v \circ w)_{(1)})u_{(2)}(v \circ w)_{(2)} = \sum (u_{(1)}|v_{(1)} \circ w_{(1)})u_{(2)}v_{(2)}w_{(2)},$$

by lemma 2.8. Lemma 2.9 is used to transform $(u_{(1)}|v_{(1)} \circ w_{(1)})$ into $(u_{(1)} \circ v_{(1)}|w_{(1)})$, so that

$$u \circ (v \circ w) \quad = \quad \sum (u_{(1)} \circ v_{(1)}|w_{(1)})u_{(2)}v_{(2)}w_{(2)} = \sum ((u \circ v)_{(1)}|w_{(1)})(u \circ v)_{(2)}w_{(2)} = (u \circ v) \circ w,$$

where we used again lemma 2.8 and the definition of the twisted product. Finally, the fact that 1 is the unit of the twisted product follows from the condition $(u|1) = \varepsilon(u)$ by

$$u \circ 1 \quad = \quad \sum (u_{(1)}|1)u_{(1)} = \sum \varepsilon(u_{(1)})u_{(2)} = u.$$

### 2.5 Iterated twisted products

In quantum field theory, the Lagrangian is an element of $\mathcal{C}$ and the S-matrix is defined (in the sense of formal power series in the complex number $\lambda$) as

$$S = \exp_\circ \lambda a = 1 + \lambda a + \frac{\lambda^2}{2!} a \circ a + \frac{\lambda^3}{6!} a \circ a \circ a + \dots \tag{2.2}$$

To compare this expression with that given in QFT textbooks [14, 18], take $\lambda = i$. Therefore, it is important to investigate the iterated twisted products. We shall see that the Feynman diagrams arise from these iterated products. The main properties of the iterated twisted product are consequences of the three following lemmas

**Lemma 2.10** *For $u^1, \dots, u^k$ in $\mathcal{S}(\mathcal{C})$ we have*

$$\Delta(u^1 \circ \cdots \circ u^k) = \sum u^1_{(1)} \circ \cdots \circ u^k_{(1)} \otimes u^1_{(2)} \cdots u^k_{(2)}.$$

Proof. The lemma is true for $k = 2$ by lemma 2.8. Assume that it is true up to $k$ and put $u^k = v \circ w$. Then, lemma 2.8 and the associativity of the twisted product imply that lemma 2.10 is true for $k + 1$. □

The next lemma gives an explicit expression for $\varepsilon(u^1 \circ \cdots \circ u^k)$

**Lemma 2.11** *For $u^1, \dots, u^k$ in $\mathcal{S}(\mathcal{C})$ we have*

$$\varepsilon(u^1 \circ \cdots \circ u^k) = \sum \prod_{i=1}^{k-1} \prod_{j=i+1}^{k} (u^i_{(j-1)} | u^j_{(i)}),$$

*where $u^j_{(i)}$ is the term at position $i$ in the iterated coproduct $\Delta^{k-2} u^j$.*

For example, $\varepsilon(u \circ v) = (u|v)$ and $\varepsilon(u \circ v \circ w) = \sum (u_{(1)}|v_{(1)})(u_{(2)}|w_{(1)})(v_{(2)}|w_{(2)})$.

Proof. For $k = 2$, lemma 2.11 is true because of lemma 2.6. Assume that it is true up to $k$ and write $U = u^1 \circ \cdots \circ u^k$. From lemma 2.6 and $U = \sum \varepsilon(U_{(1)}) U_{(2)}$ we find

$$\begin{aligned} \varepsilon(U \circ u^{k+1}) &= (U|u^{k+1}) = \sum \varepsilon(U_{(1)})(U_{(2)}|u^{k+1}) = \sum \varepsilon(u^1_{(1)} \circ \cdots \circ u^k_{(1)})(u^1_{(2)} \cdots u^k_{(2)}|u^{k+1}) \\ &= \sum \varepsilon(u^1_{(1)} \circ \cdots \circ u^k_{(1)}) \prod_{n=1}^{k} (u^n_{(2)}|u^{k+1}_{(n)}), \end{aligned}$$

where we used lemmas 2.10 and 2.3. By the recursion hypothesis, we have

$$\begin{aligned} \varepsilon(u^1 \circ \cdots \circ u^{k+1}) &= \sum \prod_{i=1}^{k-1} \prod_{j=i+1}^{k} (u^i_{(j-1)}|u^j_{(i)}) \prod_{n=1}^{k} (u^n_{(k)}|u^{k+1}_{(n)}) \\ &= \sum \prod_{i=1}^{k-1} \prod_{j=i+1}^{k+1} (u^i_{(j-1)}|u^j_{(i)})(u^k_{(k)}|u^{k+1}_{(k)}) = \sum \prod_{i=1}^{k} \prod_{j=i+1}^{k+1} (u^i_{(j-1)}|u^j_{(i)}) \end{aligned}$$

and the identity is proved for the twisted product of $k + 1$ elements. □

The last lemma completes the calculation of $u^1 \circ \cdots \circ u^k$ by expressing it as a linear combination of elements of $\mathcal{S}(\mathcal{C})$.

**Lemma 2.12** *For $u^1, \dots, u^k$ in $\mathcal{S}(\mathcal{C})$ we have*

$$u^1 \circ \cdots \circ u^k = \sum \varepsilon(u^1_{(1)} \circ \cdots \circ u^k_{(1)}) u^1_{(2)} \cdots u^k_{(2)}.$$

Proof. To show lemma 2.12 recursively, we observe that it is true for $k = 2$ by the definition of the twisted product. We write $U = u^1 \circ \cdots \circ u^k$ and we assume that the property is true up $k$. Since, by definition, $U \circ v = \sum (U_{(1)}|v_{(1)}) U_{(2)} v_{(2)}$, lemma 2.10 yields $U \circ v = \sum (u^1_{(1)} \circ \cdots \circ u^k_{(1)}|v_{(1)}) u^1_{(2)} \cdots u^k_{(2)} v_{(2)}$ and the result follows for the twisted product of $k + 1$ terms because of lemma 2.6. □

### 2.6 Application to the scalar field

If we apply these results to the coalgebra of the scalar field, we obtain the following expression for the iterated twisted product

**Proposition 2.13**

$$\begin{aligned}\phi^{n_1}(x_1)\circ\cdots\circ\phi^{n_k}(x_k) &= \sum_{p_1=0}^{n_1}\cdots\sum_{p_k=0}^{n_k}\binom{n_1}{p_1}\dots\binom{n_k}{p_k}\varepsilon\big(\phi^{p_1}(x_1)\circ\cdots\circ\phi^{p_k}(x_k)\big)\\ &\qquad \phi^{n_1-p_1}(x_1)\dots\phi^{n_k-p_k}(x_k),,\end{aligned} \tag{2.3}$$

*with*

$$\varepsilon\big(\phi^{p_1}(x_1)\circ\cdots\circ\phi^{p_k}(x_k)\big) = p_1!\dots p_k!\sum_M\prod_{i=1}^{k-1}\prod_{j=i+1}^{k}\frac{g(x_i,x_j)^{m_{ij}}}{m_{ij}!}, \tag{2.4}$$

*where the sum is over all symmetric $k\times k$ matrices $M$ of nonnegative integers $m_{ij}$ such that $\sum_{j=1}^k m_{ij}=p_i$ with $m_{ii}=0$ for all $i$.*

P r o o f. Equation (2.3) is a simple rewriting of lemma 2.12 for $u^1=\phi^{(n_1)}(x_1)$,…,$u^k=\phi^{(n_k)}(x_k)$. For the proof of (2.4), we first recall from lemma (2.1) that

$$\Delta^{k-2}\phi^{p_i}(x_i) = \sum\frac{p_i!}{q_{i1}!\dots q_{ik-1}!}\phi^{q_{i1}}(x_i)\otimes\cdots\otimes\phi^{q_{ik-1}}(x_i),$$

where the sum is over all nonnegative integers $q_{ij}$ such that $\sum_{j=1}^{k-1}q_{ij}=p_i$. Thus, lemma 2.11 becomes

$$\varepsilon\big(\phi^{p_1}(x_1)\circ\cdots\circ\phi^{p_k}(x_k)\big) = \sum_Q\Big(\prod_{i=1}^k\frac{p_i!}{\prod_{j=1}^{k-1}q_{ij}!}\Big)\prod_{i=1}^{k-1}\prod_{j=i+1}^{k}\big(\varphi^{q_{ij-1}}(x_i)|\varphi^{q_{ji}}(x_j)\big),$$

where the sum is over all $k\times(k-1)$ matrices $Q$ with matrix elements $q_{ij}$ such that $\sum_{j=1}^{k-1}q_{ij}=p_i$. Lemma 2.4 implies that $q_{ij-1}=q_{ji}$. It is very convenient to associate to each matrix $Q$ a $k\times k$ matrix $M$ with matrix elements $m_{ji}=q_{ji}$ if $i<j$, $m_{ii}=0$ and $m_{ji}=q_{ji-1}$ if $j<i$. With this definition, the condition $q_{ij-1}=q_{ji}$ implies that the matrix $M$ is symmetric. The condition $\sum_{j=1}^{k-1}q_{ij}=p_i$ implies $\sum_{j=1}^k m_{ji}=p_i$ and we recover the condition of proposition 2.13. It remains to show that the combinatorial factors come out correctly. Lemma 2.4 gives us the combinatorial factor $\prod_{i=1}^{k-1}\prod_{j=i+1}^k m_{ij}!$. On the other hand, we have

$$\begin{aligned}\prod_{i=1}^{k}\prod_{j=1}^{k-1}q_{ij}! &= \Big(\prod_{i=2}^{k}\prod_{j=1}^{i-1}q_{ij}!\Big)\Big(\prod_{i=1}^{k-1}\prod_{j=i}^{k-1}q_{ij}!\Big)=\Big(\prod_{i=2}^{k}\prod_{j=1}^{i-1}m_{ij}!\Big)\Big(\prod_{i=1}^{k-1}\prod_{j=i}^{k-1}m_{ij+1}!\Big)\\ &= \Big(\prod_{j=1}^{k-1}\prod_{i=j+1}^{k}m_{ij}!\Big)\Big(\prod_{i=1}^{k-1}\prod_{j=i+1}^{k}m_{ij}!\Big)=\Big(\prod_{i=1}^{k-1}\prod_{j=i+1}^{k}m_{ij}!\Big)^2,\end{aligned}$$

by symmetry of $M$. This completes the proof □

Each matrix $M$ is the adjacency matrix of a graph called a Feynman diagram. To clarify this matter, it is convenient to give a few examples.

**Example 2.14** We consider the case of $\varepsilon\big(\varphi^3(x)\varphi^3(y)\big)$. If we compare with proposition 2.13, we have $k=2$, $p_1=3$ and $p_2=3$. The only nonnegative integer 2x2 symmetric matrix $M$ with zero diagonal such that $\sum_{j=1}^k m_{ij}=p_i$ is

$$M=\begin{pmatrix}0&3\\3&0\end{pmatrix}.$$

Thus, according to the general formula, $\varepsilon\big(\varphi^3(x)\varphi^3(y)\big)=3!g(x,y)^3$.

The matrix $M$ is the adjacency matrix of the following Feynman diagram $\gamma$, which is called the *setting sun*.

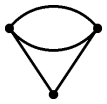
$\gamma \quad =$ 

**Example 2.15** We consider the case of $\varepsilon\big(\varphi^3(x)\varphi^3(y)\varphi^2(z)\big)$. We have now $k = 3$, $p_1 = 3$, $p_2 = 3$ and $p_3 = 2$. The only matrix that satisfies all the conditions is

$$M = \begin{pmatrix} 0 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Thus, according to the general formula, $\varepsilon\big(\varphi^3(x)\varphi^3(y)\varphi^2(z)\big) = 3!3!g(x,y)^2g(x,z)g(y,z)$.

The matrix $M$ is the adjacency matrix of the following Feynman diagram $\gamma$, which is called the *ice cream*.

$\gamma \quad =$ 

**Example 2.16** Finally, we consider the case of $\varepsilon\big(\varphi^2(x_1)\varphi^2(x_2)\varphi^2(x_3)\varphi^2(x_4)\big)$. Six matrices $M$ satisfy the conditions:

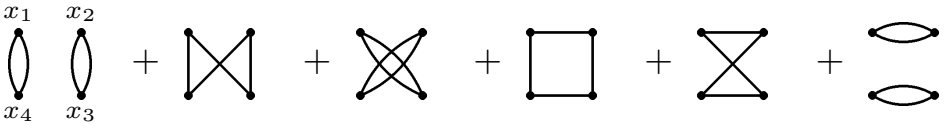
$$M_1 = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix},$$

$$M_4 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}, \quad M_5 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \quad M_6 = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{pmatrix}.$$

The matrices $M_i$ enable us to write the value of $\varepsilon\big(\varphi^2(x_1)\varphi^2(x_2)\varphi^2(x_3)\varphi^2(x_4)\big)$ as

$$\begin{aligned} &4g(x_1,x_4)^2g(x_2,x_3)^2 + 8g(x_1,x_3)g(x_1,x_4)g(x_2,x_3)g(x_2,x_4) + 4g(x_1,x_3)^2g(x_2,x_4)^2 \\ &+8g(x_1,x_2)g(x_1,x_4)g(x_2,x_3)g(x_3,x_4) + 8g(x_1,x_2)g(x_1,x_3)g(x_2,x_4)g(x_3,x_4) \\ &+4g(x_1,x_2)^2g(x_3,x_4)^2. \end{aligned}$$

The matrices $M_i$ are the adjacency matrices of the following Feynman diagrams $\gamma$.

$\sum_\gamma \gamma \quad =$ 


Note that the first, third and last Feynman diagrams are disconnected. This will be important in the following.

### 2.7 Enumeration of Feynman diagrams

Various authors [19, 20, 21, 22] studied the number of matrices $M$ (or of Feynman diagrams) corresponding to $\varepsilon\big(\varphi^{n_1}(x_1) \circ \cdots \circ \varphi^{n_k}(x_k)\big)$.

**Proposition 2.17** *The number of Feynman diagrams generated by* $\varepsilon\big(\varphi^{n_1}(x_1)\circ\cdots\circ\varphi^{n_k}(x_k)\big)$ *is the coefficient of* $z_1^{n_1}\dots z_k^{n_k}$ *in* $\prod_{i<j}(1-z_iz_j)^{-1}$.

Since we do not know of any published simple proof of this proposition, we provide the following one.

P r o o f. Let $N_k(n_1,\dots,n_k)$ denote the number of (labelled) Feynman diagrams of $\varepsilon\big(\varphi^{n_1}(x_1)\circ\cdots\circ\varphi^{n_k}(x_k)\big)$ and $f_k(z_1,\dots,z_k)$ the generating function

$$f_k(z_1,\dots,z_k) \quad = \quad \sum_{n_1\dots n_k} N_k(n_1,\dots,n_k) z_1^{n_1}\dots z_k^{n_k}.$$

When $k=2$, we have $\varepsilon\big(\varphi^{n_1}(x_1)\circ\varphi^{n_2}(x_2)\big) = \delta_{n_1,n_2} n_1!(\varphi(x_1)|\varphi(x_2))^{n_1}$ so there is no diagram if $n_1\neq n_2$ and 1 if $n_1=n_2$ ($n_1$ lines linking $x_1$ and $x_2$). Therefore

$$f_2(z_1,z_2) \quad = \quad \sum_{n_1,n_2} N_2(n_1,n_2) z_1^{n_1} z_2^{n_2} = \sum_{n=0}^{\infty} z_1^n z_2^n = \frac{1}{1-z_1z_2}.$$

Assume that you know $N_l(n_1,\dots,n_l)$ up to $l=k-1$. To calculate it for $k$, take a matrix representing a diagram for $k$ and call $i_1,\dots,i_{k-1}$ its last line (recall that the diagonal is zero so that $i_k=0$). The matrix obtained by removing the last line and the last column encodes a diagram for $\varepsilon\big(\varphi^{n_1-i_1}(x_1)\dots\varphi^{n_{k-1}-i_{k-1}}(x_{k-1})\big)$. Therefore,

$$N_k(n_1,\dots,n_k) \quad = \quad \sum_{i_1+\dots+i_{k-1}=n_k} N_{k-1}(n_1-i_1,\dots,n_{k-1}-i_{k-1}).$$

This gives us

$$\begin{aligned} f_k(z_1,\dots,z_k) &= \sum_{n_1,\dots,n_k} N_k(n_1,\dots,n_k) z_1^{n_1}\dots z_k^{n_k} \\ &= \sum_{n_1,\dots,n_k} z_1^{n_1}\dots z_k^{n_k} \sum_{i_1+\dots+i_{k-1}=n_k} N_{k-1}(n_1-i_1,\dots,n_{k-1}-i_{k-1}) \\ &= \sum_{n_k} z_k^{n_k} \sum_{i_1+\dots+i_{k-1}=n_k} z_1^{i_1}\dots z_{k-1}^{i_{k-1}} \sum_{j_1,\dots,j_{k-1}} N_{k-1}(j_1,\dots,j_{k-1}) z_1^{j_1}\dots z_{k-1}^{j_{k-1}}, \end{aligned}$$

where we put $n_l = i_l + j_l$ for $l=1,\dots,k-1$. The sum over $j_l$ is the generating function $f_{k-1}$ so

$$\begin{aligned} f_k(z_1,\dots,z_k) &= \sum_{n_k} z_k^{n_k} \sum_{i_1+\dots+i_{k-1}=n_k} z_1^{i_1}\dots z_{k-1}^{i_{k-1}} f_{k-1}(z_1,\dots,z_{k-1}). \\ &= \sum_{i_1,\dots,i_{k-1}} (z_1z_k)^{i_1}\dots(z_{k-1}z_k)^{i_{k-1}} f_{k-1}(z_1,\dots,z_{k-1}). \\ &= \frac{f_{k-1}(z_1,\dots,z_{k-1})}{(1-z_1z_k)\dots(1-z_{k-1}z_k)}. \end{aligned}$$

So finally

$$f_k(z_1,\dots,z_k) \quad = \quad \prod_{j=2}^{k}\prod_{i=1}^{j-1} \frac{1}{1-z_iz_j}.$$

□

Note that, by a classical identity of the theory of symmetric functions (see [23], p. 77),

$$\prod_{i<j}(1-z_iz_j)^{-1} \quad = \quad \sum_{\lambda} s_\lambda,$$

where $s_\lambda$ is the Schur function for partition $\lambda$ and where the sum runs over all partitions $\lambda$ having an even number of parts of any given magnitude, for example $\{1^2\}$, $\{2^2\}$, $\{1^4\}$, $\{2^21^2\}$, $\{3^2\}$, $\{1^6\}$.

Burge [22] proposed algorithms to generate different types of graphs based on a correspondence with semi-standard Young tableaux. We recall that a Young diagram is a collection of boxes arranged in left-justified rows, with a weakly decreasing number of boxes in each row [24]. For example  is a Young diagram. A Young

tableau is a Young diagram where each box contains a strictly positive integer. A Young tableau is semi-standard if the numbers contained in the boxes are weakly increasing from left to right in each row and strictly increasing down each column. For example, $T$=  is a semi-standard tableau. The number of semi-standard Young tableaux of a given shape and content is given by the Kostka numbers [24].

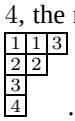
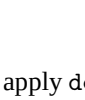
To obtain all the Feynman diagrams contributing to $t\big(\varphi^{n_1}(x_1)\dots\varphi^{n_k}(x_k)\big)$, list all the semi-standard Young tableaux with $n_1+\dots+n_k=2n$ boxes, where all columns of $T$ have an even number of boxes, filled with $n_1$ times the symbol 1, …, $n_k$ times the symbol $k$. For instance, our example tableau $T$ corresponds to $t\big(\varphi^2(x_1)\varphi^2(x_2)\varphi^2(x_3)\varphi^2(x_4)\big)$. There are efficient algorithms to generate this list [25].

Then, for each tableau $T$ of the list, use the Robinson-Schensted-Knuth correspondence [26] for the pair $(T,T)$ to generate the Feynman diagram. This relation between Feynman diagrams and semi-standard Young tableaux is not widely known. Thus, it is useful to give it here in some detail.

A box in a Young tableau is a *boundary box* if it is the rightmost box of a row and if it has no box below it. If a box is referenced by a pair $(r,c)$ where $r$ is the row number and $c$ the column number (counted from the upper-left corner), then the boundary boxes of our example tableau $T$ are the boxes $(2,3)$ and $(4,1)$ containing the numbers 4 and 4, respectively. We define now the algorithm `delete` taking a tableau $T$ and a boundary box $(r,c)$ and building a tableau $T'$ and a number $k$. If $r=1$, then $c$ is the last column of the first row. The result of `delete`$(T;(1,c))$ is the tableau $T'$ obtained from the tableau $T$ by removing box $(1,c)$ and number $k$ is the number contained in box $(1,c)$ of $T$. If $r>1$, box $(r,c)$ is removed from $T$ and the number $i$ contained in box $(r,c)$ replaces the first number in row $r-1$ (from right to left) that is strictly smaller than $i$. The number $j$ it replaces is then moved to row $r-2$ with the same rule, until row 1 is reached. The tableau thus obtained is $T'$ and the number replaced in the first row is $k$ (see Knuth's paper [26] for a more computer-friendly algorithm).

In our example $T$, if we choose the boundary box $(2,3)$, the number 4 of box $(2,3)$ is moved to the first row, where it replaces the number 3, so that $T'$=  and $k=3$. If we choose the boundary box $(4,1)$, we have successively  , where 4 replaces a 3; moving this 3 to the second row gives  , where 3 replaces a 2; moving this 2 to the first row gives $T'$=  and the replaced number in the first row is $k=1$.

To generate the Feynman diagram corresponding to a given tableau $T$ (with $2n$ boxes), we first need to define $n$ pairs of integers $(u_k,v_k)$, where $k=1,\dots,n$ (each pair represents an edge of the diagram). Let $u_n$ be the largest number contained in the boxes of $T$ and, among the boxes of $T$ containing the number $u_n$, let $(r,c)$ be the one with largest $c$. Let $T_1$ be the tableau obtained from $T$ by removing box $(r,c)$. Calculate `delete`$(T_1;(r-1,c))$ to obtain a tableau $T_1'$ and a number $k$. Assign $v_n=k$ and repeat the procedure on $T_1'$. This gives $n$ pairs $(u_1,v_1),\dots,(u_n,v_n)$, with $u_n>v_n$.

Let us apply this procedure to our example tableau $T$. It contains 8 boxes, so that $n=4$. Its largest number is 4, the rightmost box containing it is $(2,3)$ and the Young tableau obtained by removing this box from $T$ is $T_1$=  . If we apply `delete` to $T_1$ and boundary box $(1,3)$ we obtain $T_1'$=  and $k=3$. Thus, $(u_4,v_4)=(4,3)$. We have now $u_3=4$ and the box $(4,1)$. We apply the same procedure to get $T_2'$=  and $k=1$, so that $(u_3,v_3)=(4,1)$. Doing this again gives us $T_3'$=  and $(u_2,v_2)=(3,1)$. Finally $(u_1,v_1)=(2,1)$.

Then, the Feynman diagram is generated as follows: let $k$ be the largest number contained in $T$, draw $k$ points labelled $x_1,\dots,x_k$. Then, for each pair $(u_i,v_i)$, draw an edge between $x_a$ and $x_b$, where $a=u_i$ and $b=v_i$. Note that our example tableau $T$ corresponds to the fourth diagram of example 2.16.

Similarly, the adjacency matrix $M$ is built as follows: let $k$ be the largest number contained in $T$ and $M$ the $k\times k$ null matrix. For $i=1$ to $n$ increase $m_{ab}$ and $m_{ba}$, by one, where $a=u_i$ and $b=v_i$.

The semi-standard Young tableaux corresponding to the diagrams of examples 2.14, 2.15 and 2.16 are, respectively, $\begin{array}{|c|c|c|}\hline 1&1&1\\\hline 2&2&2\\\hline\end{array}$, $\begin{array}{|c|c|c|c|c|}\hline 1&1&1&1&2\\\hline 2&2&3&3\\\cline{1-4}\end{array}$ and $\begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline 3&3\\\hline 4&4\\\hline\end{array}$, $\begin{array}{|c|c|c|}\hline 1&1&2\\\hline 2&3&4\\\hline\end{array}$, $\begin{array}{|c|c|c|c|}\hline 1&1&2&2\\\hline 3&3&4&4\\\hline\end{array}$, $\begin{array}{|c|c|c|}\hline 1&1&3\\\hline 2&2&4\\\hline\end{array}$, $\begin{array}{|c|c|c|c|}\hline 1&1&2&3\\\hline 2&3&4&4\\\hline\end{array}$, $\begin{array}{|c|c|c|c|}\hline 1&1&3&3\\\hline 2&2&4&4\\\hline\end{array}$.

### 2.8 Chronological product and Green functions

If the Laplace coupling is symmetric, i.e. if $(u|v) = (v|u)$ for all $u$ and $v$ in $\mathcal{S}(\mathcal{C})$, then the twisted product is commutative and we can define a map $T$ from $\mathcal{S}(\mathcal{C})$ to $\mathcal{S}(\mathcal{C})$ by $T(uv) = T(u) \circ T(v)$. In particular, if $a_1, \ldots, a_k$ are elements of $\mathcal{C}$, then $T(a_1 \ldots a_k) = a_1 \circ \cdots \circ a_k$. Using lemma 2.12, we see that $T(u) = \sum t(u_{(1)})u_{(2)}$, where $t$ is a linear map from $\mathcal{S}(\mathcal{C})$ to $\mathbb{C}$ defined by $t(u) = \varepsilon\big(T(u)\big)$. For historical reasons, the map $T$ is called the *chronological product* or the *time-ordered product*. The map $t$ is an element of $\mathcal{S}(\mathcal{C})^*$, the dual of $\mathcal{S}(\mathcal{C})$. In that sense, the chronological product is the right coregular action of $\mathcal{S}(\mathcal{C})^*$ on $\mathcal{S}(\mathcal{C})$ by $T(u) = u \triangleleft t = \sum t(u_{(1)})u_{(2)}$ and $\mathcal{S}(\mathcal{C})$ is a right $\mathcal{S}(\mathcal{C})^*$-module if $\mathcal{S}(\mathcal{C})^*$ is endowed with the convolution product (see [7] p. 21). The Hopf algebraic properties of $T$ were discussed in detail in [27].

The map $T$ enables us to write the S-matrix as $S = T\big(\exp(\lambda a)\big)$, where $\exp(\lambda a) = 1 + \sum_{n=1}^{\infty}(\lambda^n/n!)a^n$, where the product $a^n$ is the product in $\mathcal{S}(\mathcal{C})$. Moreover, the map $t$ is useful to define the Green functions of a theory:

**Definition 2.18** The $n$-point Green function of the scalar field theory with Lagrangian $a$ is the function

$$G(x_1, \ldots, x_k) \quad = \quad t\big(\varphi(x_1) \ldots \varphi(x_k) \exp(\lambda a)\big).$$

The usefulness of the Green functions stems from the fact that most important physical quantities can be expressed in terms of Green functions with a small number of arguments. For instance, the charge density is proportional to $G(x, x)$ [28], the optical spectrum is a linear function of $G(x_1, x_2, x_3, x_4)$ [29], etc.

Let us consider the example of the $\varphi^4$ theory with Lagrangian $a = g\int_{\mathbb{R}^4}\varphi^4(x)\mathrm{d}x$. Strictly speaking, we are not allowed to consider an infinite number of points as in the integral over $\mathbb{R}^4$, but this drawback can be cured in the perturbative regime, where $\exp(\lambda a)$ is expanded as a series in $\lambda$ and each term $a^n$ is rewritten

$$a^n \quad = \quad g^n \int_{\mathbb{R}^{4n}} \mathrm{d}y_1 \ldots \mathrm{d}y_n \varphi^4(y_1) \ldots \varphi^4(y_n).$$

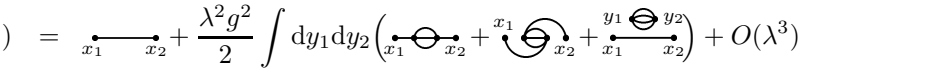
Thus, the perturbative expansion of the Green function is

$$G(x_1, \ldots, x_k) \quad = \quad \sum_{n=0}^{\infty} \frac{(\lambda g)^n}{n!} \int_{\mathbb{R}^{4n}} \mathrm{d}y_1 \ldots \mathrm{d}y_n t\big(\varphi(x_1) \ldots \varphi(x_k)\varphi^4(y_1) \ldots \varphi^4(y_n)\big).$$

Then, in each term of the expansion the coregular action $t$ acts on a finite number of points and the present formulation is valid. For instance, the first nonzero terms of $G(x_1, x_2)$ are

$$\begin{aligned} G(x_1, x_2) \quad &= \quad t\big(\varphi(x_1)\varphi(x_2)\big) + \frac{\lambda^2 g^2}{2} \int_{\mathbb{R}^8} \mathrm{d}y_1 \mathrm{d}y_2 t\big(\varphi(x_1)\varphi(x_2)\varphi^4(y_1)\varphi^4(y_2)\big) + O(\lambda^3), \\ &= \quad g(x_1, x_2) + \lambda^2 g^2 \int_{\mathbb{R}^8} \mathrm{d}y_1 \mathrm{d}y_2 \Big(48 g(x_1, y_1) g(y_1, y_2)^3 g(y_2, x_2) \\ &\quad + 48 g(x_1, y_2) g(y_1, y_2)^3 g(y_1, x_2) + 12 g(x_1, x_2) g(y_1, y_2)^4\Big) + O(\lambda^3) \end{aligned}$$

In terms of Feynman diagrams, this gives us

$G(x_1, x_2) \quad = \quad$  $+ O(\lambda^3)$



For the clarity of the figure, the labels $y_1$ and $y_2$ were given explicitly only for the last diagram. The three diagrams under the integral sign correspond to the three matrices

$$M_1 \quad = \quad \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 3 \\ 0 & 1 & 3 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 3 \\ 1 & 0 & 3 & 0 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 4 & 0 \end{pmatrix}.$$

Note that the first and second integrals have the same value because $g(y_1, y_2) = g(y_2, y_1)$. Their sum would be counted as a single diagram in the standard theory. The advantage of the present procedure is that we do not have to determine the number of elements of the symmetry group of the diagrams. Note also that the last term is not connected. To define connected Green functions, just replace $t$ by $t_c$ in definition 2.18.

We considered only Feynman diagrams without tadpoles (a tadpole is an edge going from a vertex to itself). For some applications, it is useful to consider diagrams that can possibly contain tadpoles. Burge defined a similar correspondence for this more general case [22]. The semi-standard tableaux are now required to have an even number of boxes in each row (and no condition on the columns). In the corresponding adjacency matrix, the tadpoles attached to vertex $x_k$ are described by assigning to the diagonal matrix element $m_{kk}$ the value of twice the number of tadpoles attached to $x_k$.

### 2.9 The noncocommutative case

When the coalgebra $\mathcal{C}$ is not cocommutative, the above construction is not valid because the twisted product is not associative. However, it is still possible to define a bialgebra $\mathcal{T}(\mathcal{C})$ from any coalgebra $\mathcal{C}$ over $\mathbb{C}$ as follows. As an algebra, $\mathcal{T}(\mathcal{C})$ is the tensor algebra of $\mathcal{C}$: $\mathcal{T}(\mathcal{C}) = \bigoplus_{n=0}^{\infty} \mathcal{T}^n(\mathcal{C})$. The coproduct $\Delta$ of $\mathcal{T}(\mathcal{C})$ is defined by extending the coproduct of $\mathcal{C}$ by algebra morphism and the counit $\varepsilon$ of $\mathcal{T}(\mathcal{C})$ is defined by extending the counit of $\mathcal{C}$ by algebra morphism. With this coproduct and counit, the algebra $\mathcal{T}(\mathcal{C})$ becomes a bialgebra. This bialgebra was used by Ritter [30] in the framework of noncommutative QFT.

For any element $t$ of $\mathcal{T}(\mathcal{C})^*$, the dual of $\mathcal{T}(\mathcal{C})$, we can define the right coregular action of $\mathcal{T}(\mathcal{C})^*$ on $\mathcal{T}(\mathcal{C})$ by $T(u) = u \triangleleft t = \sum t(u_{(1)})u_{(2)}$. If $\mathcal{T}(\mathcal{C})^*$ is endowed with the convolution product $(t * t')(u) = \sum t(u_{(1)})t'(u_{(2)})$, then $\mathcal{T}(\mathcal{C})^*$ is a unital associative algebra with unit $\varepsilon$. Moreover,

**Lemma 2.19** *$\mathcal{T}(\mathcal{C})$ is a right $\mathcal{T}(\mathcal{C})^*$-module, i.e. $u \triangleleft \varepsilon = u$ and $(u \triangleleft t) \triangleleft t' = u \triangleleft (t * t')$.*

Proof. By definition of the action and of the counit, we have $u \triangleleft \varepsilon = \sum \varepsilon(u_{(1)})u_{(2)} = u$. By definition of the action,

$$\begin{aligned}(u \triangleleft t) \triangleleft t' &= \sum \big(t(u_{(1)})u_{(2)}\big) \triangleleft t' = \sum t(u_{(1)})(u_{(2)} \triangleleft t') = \sum t(u_{(1)})t'(u_{(2)})u_{(3)} \\ &= \sum (t * t')(u_{(1)})u_{(2)} = u \triangleleft (t * t').\end{aligned}$$

□

## 3 Connected chronological products

In example 2.16, we saw that some of the Feyman diagrams are disconnected. According to a basic result of QFT (the Gell-Mann and Low theorem [31]), these diagrams must be eliminated. In this section, we show that this elimination can be carried out by a purely algebraic method, using the fact that $\mathcal{S}(\mathcal{C})$ can be equipped with a second coproduct.

### 3.1 A second coproduct on $\mathcal{S}(\mathcal{C})$

For the calculation of connected diagrams, it is convenient to define a second coproduct $\delta$ on the symmetric algebra $\mathcal{S}(\mathcal{C})$ by $\delta a = 1 \otimes a + a \otimes 1$ for $a$ in $\mathcal{C}$, extended to $\mathcal{S}(\mathcal{C})$ by algebra morphism. Besides, the counit $\varepsilon_\delta$ is defined by $\varepsilon_\delta(1) = 1$, $\varepsilon_\delta(a) = 0$ for $a$ in $\mathcal{C}$ and extended by algebra morphism. We denote by $\mathcal{S}_\delta(\mathcal{C})$ the resulting bialgebra. This is the standard symmetric algebra on $\mathcal{C}$ considered as a vector space. The Sweedler notation for the action of $\delta$ on an element $u$ of $\mathcal{S}(\mathcal{C})$ is $\delta u = \sum u_{\{1\}} \otimes u_{\{2\}}$.

We prove now the following crucial

**Proposition 3.1** *$\mathcal{S}_\delta(\mathcal{C})$ is a right $\mathcal{S}(\mathcal{C})$-comodule coalgebra for the right coaction $\beta = \Delta$.*

Proof. The map $\beta : \mathcal{S}_\delta(\mathcal{C}) \to \mathcal{S}_\delta(\mathcal{C}) \otimes \mathcal{S}(\mathcal{C})$ defined by $\beta = \Delta$ is a right coaction of $\mathcal{S}(\mathcal{C})$ on $\mathcal{S}_\delta(\mathcal{C})$ because we obviously have (see ref.[7] p.22 or ref.[32] p.29) $(\beta \otimes \mathrm{Id})\beta = (\mathrm{Id} \otimes \Delta)\beta$ and $(\mathrm{Id} \otimes \varepsilon)\beta = \mathrm{Id}$. Thus, $\mathcal{S}_\delta(\mathcal{C})$

is a right $\mathcal{S}(\mathcal{C})$-comodule coalgebra if we can prove the two properties (see ref.[7] p.23 or ref.[32] p.352)

$$\begin{aligned}(\varepsilon_\delta \otimes \mathrm{Id})\beta &= 1\varepsilon_\delta,\\ (\delta \otimes \mathrm{Id})\beta &= (\mathrm{Id}\otimes\mathrm{Id}\otimes\mu)(\mathrm{Id}\otimes\tau\otimes\mathrm{Id})(\beta\otimes\beta)\delta,\end{aligned}$$

where $\mu$ is the algebra product of $\mathcal{S}(\mathcal{C})$ and $\tau$ is the flip. In Sweedler's notation this becomes

$$\sum \varepsilon_\delta(u_{(1)})u_{(2)} = \varepsilon_\delta(u)1, \tag{3.1}$$

$$\sum u_{(1)\{1\}}\otimes u_{(1)\{2\}}\otimes u_{(2)} = \sum u_{\{1\}(1)}\otimes u_{\{2\}(1)}\otimes u_{\{1\}(2)}u_{\{2\}(2)}. \tag{3.2}$$

We first show equation (3.1). If $u\in\mathcal{S}^n(\mathcal{C})$, then $\varepsilon_\delta(u)=0$ by definition of $\varepsilon_\delta$. Moreover, $\beta u = \Delta u = \sum u_{(1)}\otimes u_{(2)}$, where all $u_{(1)}$ and $u_{(2)}$ belong to $\mathcal{S}^n(\mathcal{C})$ by definition of the coproduct $\Delta$. Thus, $\varepsilon_\delta(u_{(1)})=0$ and $\sum\varepsilon_\delta(u_{(1)})u_{(2)}=0=\varepsilon_\delta(u)1$. The remaining case is $u\in\mathcal{S}^0(\mathcal{C})$, so that $u=\lambda 1$ for some complex number $\lambda$. Then $\sum\varepsilon_\delta(u_{(1)})u_{(2)}=\lambda\varepsilon_\delta(1)1=\varepsilon_\delta(u)1$ and equation (3.1) is proved.

To show equation (3.2) we use a recursive proof. It is obviously true for $u=1$, assume that it is true for elements of degree up to $n$. Take $u$ an element of degree $n$ and $a\in\mathcal{C}$. From $\delta(au)=(\delta a)(\delta u)=\sum au_{\{1\}}\otimes u_{\{2\}}+u_{\{1\}}\otimes au_{\{2\}}$ we can calculate

$$\begin{aligned}(\delta\otimes\mathrm{Id})\Delta(au) &= \sum\delta(a_{(1)}u_{(1)})\otimes a_{(2)}u_{(2)}\\ &= \sum(a_{(1)}u_{(1)\{1\}}\otimes u_{(1)\{2\}}+u_{(1)\{1\}}\otimes a_{(1)}u_{(1)\{2\}})\otimes a_{(2)}u_{(2)}\\ &= \sum(a_{(1)}u_{\{1\}(1)}\otimes u_{\{2\}(1)}+u_{\{1\}(1)}\otimes a_{(1)}u_{\{2\}(1)})\otimes a_{(2)}u_{\{1\}(2)}u_{\{2\}(2)}\\ &= (\mathrm{Id}\otimes\mathrm{Id}\otimes\mu)(\mathrm{Id}\otimes\tau\otimes\mathrm{Id})(\Delta\otimes\Delta)\delta(au),\end{aligned}$$

where we go from the first line to the second with the expression for $\delta(a_{(1)}u_{(1)})$, from second line to the third with the recursion hypothesis and from the third to the fourth with the expression for $\delta(au)$. □

For the coalgebra of the scalar field, one might be tempted to replace all $\varphi^0(x_i)$ by 1, the unit of $\mathcal{S}(\mathcal{C})$. However, if we do this the coproduct of an element $u$ will contain the terms $u\otimes 1$ and $1\otimes u$ that spoils the validity of equations (3.1) and (3.2).

### 3.2 The connected chronological product

We denote the reduced coproduct $\underline{\delta}u=\delta u-1\otimes u-u\otimes 1$ with Sweedler's notation $\underline{\delta}u=\sum u_{\{\underline{1}\}}\otimes u_{\{\underline{2}\}}$.

**Definition 3.2** For $u\in\mathcal{S}(\mathcal{C})$ with $\varepsilon_\delta(u)=0$, we define the connected chronological product $T_c(u)$ as

$$T_c(u) = -\sum_{n=1}^{\infty}\frac{(-1)^n}{n}T(u_{\{\underline{1}\}})\ldots T(u_{\{\underline{n}\}}).$$

Note that the connected chronological product is related to the first Eulerian idempotent [33] for $u\in\ker\varepsilon_\delta$

$$e(u) = -\sum_{n=1}^{\infty}\frac{(-1)^n}{n}u_{\{\underline{1}\}}\ \ldots\ u_{\{\underline{n}\}},$$

in the sense that the operator $T$ is applied on each term $u_{\{\underline{i}\}}$. Recall that the first Eulerian idempotent projects onto the primitive elements of a connected cocommutative bialgebra. Reciprocally, we can express $T$ in terms of $T_c$ by

**Lemma 3.3** *For* $u\in\ker\varepsilon_\delta$

$$T(u) = \sum_{n=1}^{\infty}\frac{1}{n!}T_c(u_{\{\underline{1}\}})\ldots T_c(u_{\{\underline{n}\}}).$$

P r o o f. From the definition of $T_c$, we have

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n!}T_c(u_{\{\underline{1}\}})\dots T_c(u_{\{\underline{n}\}}) &= \sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\sum_{i_1,\dots,i_n}\frac{(-1)^{i_1+\dots+i_n}}{i_1\dots i_n}T(u_{\{\underline{i_1}\}})\dots T(u_{\{\underline{i_1+\dots+i_n}\}}),\\ &= \sum_{k=1}^{\infty}T(u_{\{\underline{1}\}})\dots T(u_{\{\underline{k}\}})\sum_{n=1}^{k}\frac{(-1)^{k+n}}{n!}\sum_{i_1+\dots+i_n=k}\frac{1}{i_1\dots i_n}.\end{aligned}$$

The sum over $n$ and $i_1,\dots,i_n$ is the coefficient of $x^k$ in the series expansion of $\mathrm{e}^{\log(1+x)}$. From $\mathrm{e}^{\log(1+x)}=1+x$ we deduce that the sum is $\delta_{k,1}$. Therefore

$$\sum_{n=1}^{\infty}\frac{1}{n!}T_c(u_{\{\underline{1}\}})\dots T_c(u_{\{\underline{n}\}}) = T(u_{\{\underline{1}\}}) = T(u).$$

□

For example,

$$\begin{aligned}T(a) &= T_c(a),\\ T(ab) &= T_c(ab)+T_c(a)T_c(b),\\ T(abc) &= T_c(abc)+T_c(ab)T_c(c)+T_c(a)T_c(bc)+T_c(ac)T_c(b)+T_c(a)T_c(b)T_c(c),\end{aligned}$$

where $a$, $b$ and $c$ are elements of $\mathcal{C}$. The relations between $T$ and $T_c$ were given in one form of another by Haag [34], Epstein and Glaser [35], Brunetti and Fredenhagen [36], Dütsch and Fredenhagen [37] and Mestre and Oeckl [38].

The chronological product $T(u)$ is a coregular action: $T(u)=\sum t(u_{(1)})u_{(2)}$, where the coproduct is $\Delta$. The connected chronological product $T_c(u)$ is defined in terms of $T(u)$ through the coproduct $\delta$. Therefore, it is rather surprising that $T_c$ is also a coregular action: there is an element $t_c$ of $\mathcal{S}(\mathcal{C})^*$ such that $T_c(u)=\sum t_c(u_{(1)})u_{(2)}$. As we shall see, this is a consequence of the fact that $\mathcal{S}_\delta(\mathcal{C})$ is a comodule coalgebra over $\mathcal{S}(\mathcal{C})$.

**Proposition 3.4** *The connected chronological product is a coregular action:* $T_c(u)=\sum t_c(u_{(1)})u_{(2)}$ *for* $u\in\ker\varepsilon$, *with*

$$t_c(u) = -\sum_{n=1}^{\infty}\frac{(-1)^n}{n}t(u_{\{\underline{1}\}})\dots t(u_{\{\underline{n}\}}).$$

P r o o f. From $\delta u=\underline{\delta}u+1\otimes u+u\otimes 1$, it is straightforward to show that

$$(\mathrm{Id}\otimes\mathrm{Id}\otimes\mu)(\mathrm{Id}\otimes\tau\otimes\mathrm{Id})(\beta\otimes\beta)\delta = (\delta\otimes\mathrm{Id})\beta,$$

implies

$$(\mathrm{Id}\otimes\mathrm{Id}\otimes\mu)(\mathrm{Id}\otimes\tau\otimes\mathrm{Id})(\beta\otimes\beta)\underline{\delta} = (\underline{\delta}\otimes\mathrm{Id})\beta.$$

In Sweedler's notation,

$$\sum u_{\{\underline{1}\}(1)}\otimes u_{\{\underline{2}\}(1)}\otimes u_{\{\underline{1}\}(2)}u_{\{\underline{2}\}(2)} = \sum u_{(1)\{\underline{1}\}}\otimes u_{(1)\{\underline{2}\}}\otimes u_{(2)}. \tag{3.3}$$

Take now two coregular actions $A(u)=\sum a(u_{(1)})u_{(2)}$ and $B(u)=\sum b(u_{(1)})u_{(2)}$. We have, using equation (3.3) for $u\in\ker\varepsilon$,

$$\begin{aligned}\sum A(u_{\{\underline{1}\}})B(u_{\{\underline{2}\}}) &= \sum a(u_{\{\underline{1}\}(1)})b(u_{\{\underline{2}\}(1)})u_{\{\underline{1}\}(2)}u_{\{\underline{2}\}(2)}\\ &= \sum a(u_{(1)\{\underline{1}\}})b(u_{(1)\{\underline{2}\}})u_{(2)} = \sum c(u_{(1)})u_{(2)},\end{aligned}$$

with $c(u)=\sum a(u_{\{\underline{1}\}})b(u_{\{\underline{2}\}})$. Therefore, $\sum A(u_{\{\underline{1}\}})B(u_{\{\underline{2}\}})$ is a coregular action. Using this argument recursively, we obtain that, if $A_1(u)=\sum a_1(u_{(1)})u_{(2)},\dots,A_k(u)=\sum a_k(u_{(1)})u_{(2)}$ are $k$ coregular actions, then $\sum A_1(u_{\{\underline{1}\}})\dots A_k(u_{\{\underline{k}\}})$ is a coregular action $\sum c(u_{(1)})u_{(2)}$, with $c(u)=\sum a_1(u_{\{\underline{1}\}})\dots a_k(u_{\{\underline{k}\}})$. This proves that all terms of $T_c$ are coregular actions, so that their sum is also a coregular action. □

### 3.3 The linked-cluster theorem

The name of the connected chronological product comes from the fact that, for the coalgebra of the scalar field, $t_c(u)$ is made of exactly the connected diagrams of $t(u)$. This was proved, for example by Mestre and Oeckl [38]. We do not offer a complete proof of this but, for a S-matrix $S = T\big(\exp(\lambda a)\big)$, we can calculate an expression relating $t(u)$ and $t_c(u)$. First, we have for $k > 0$ and $n \geq 0$,

$$\underline{\delta}^{k-1} a^n = \sum_{i_1+\cdots+i_k=n} \frac{n!}{i_1! \dots i_k!} a^{i_1} \otimes \cdots \otimes a^{i_k}, \tag{3.4}$$

where all $i_j$ are strictly positive integers. Therefore,

$$t(a^n) = \sum_{k=1}^{n} \frac{1}{k!} \sum_{i_1+\cdots+i_k=n} \frac{n!}{i_1! \dots i_k!} t_c(a^{i_1}) \dots t_c(a^{i_k}).$$

If we write $E = \exp(\lambda a)$, this gives us

$$\begin{aligned} t(E) &= 1 + \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \sum_{k=1}^{n} \frac{1}{k!} \sum_{i_1+\cdots+i_k=n} \frac{n!}{i_1! \dots i_k!} t_c(a^{i_1}) \dots t_c(a^{i_k}) \\ &= 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{n=k}^{\infty} \sum_{i_1+\cdots+i_k=n} \frac{1}{i_1! \dots i_k!} t_c((\lambda a)^{i_1}) \dots t_c((\lambda a)^{i_k}) \\ &= 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \big(t_c(E-1)\big)^k = \mathrm{e}^{t_c(E-1)}. \end{aligned}$$

The last line was obtained because all $i_j > 0$. The fact that $t_c(E-1)$ contains only connected Feynman diagrams follows from the fact that the logarithm of $t(E)$ is the sum of all connected vacuum diagrams [14].

The same proof holds for the $T$-products, so that

$$S = T(E) = \exp\big(T_c(E-1)\big).$$

If we define a connected $S$-matrix by $S_c = T_c(E-1)$, we obtain the linked-cluster theorem [39] $S = \mathrm{e}^{S_c}$.

### 3.4 A noncommutative analogue

If the tensor bialgebra $\mathcal{T}(\mathcal{C})$ is used instead of the symmetric bialgebra $\mathcal{S}(\mathcal{C})$, the construction is similar. We start from the bialgebra $\mathcal{T}(\mathcal{C})$ and we define the coalgebra $\mathcal{T}_\delta(\mathcal{C})$ to be the vector space $\mathcal{T}(\mathcal{C})$ endowed with the deconcatenation coproduct: $\delta 1 = 1 \otimes 1$, $\delta a = a \otimes 1 + 1 \otimes a$ for $a \in \mathcal{C}$ and

$$\delta u = u \otimes 1 + 1 \otimes u + \sum_{k=1}^{n-1} a_1 \dots a_k \otimes a_{k+1} \dots a_n,$$

for $u = a_1 \dots a_n \in \mathcal{T}^n(\mathcal{C})$ and $n > 1$. The counit $\varepsilon_\delta$ of $\mathcal{T}_\delta(\mathcal{C})$ is defined by $\varepsilon_\delta(1) = 1$ and $\varepsilon_\delta(u) = 0$ if $u \in \mathcal{T}^n(\mathcal{C})$ with $n > 0$. If $\mathcal{T}_\delta(\mathcal{C})$ is equipped with the concatenation product, then $\mathcal{T}_\delta(\mathcal{C})$ is not a bialgebra because the coproduct is not an algebra morphism. Loday and Ronco [4] showed that the deconcatenation coproduct and the concatenation product satisfy the compatibility rule $\delta(uv) = (u \otimes 1)\delta v + \delta(u)(1 \otimes v) - u \otimes v$, which makes $\mathcal{T}_\delta(\mathcal{C})$ a unital infinitesimal bialgebra. Note that, if $u = a$, the compatibility rule becomes

$$\delta(av) = (a \otimes 1)\delta v + 1 \otimes av, \tag{3.5}$$

for $a \in \mathcal{C}$ and $u \in \mathcal{T}_\delta(\mathcal{C})$.

We have the following

**Proposition 3.5** *$\mathcal{T}_\delta(\mathcal{C})$ is a right $\mathcal{T}(\mathcal{C})$-comodule coalgebra for the right coaction $\beta = \Delta$.*

P r o o f. The proof of condition (3.1) on the counit is exactly the same as for the symmetric case. We prove (3.2) recursively. It is obviously true for $u = 1$, assume that this is true for elements of degree up to $n$. Take $u$ an element of degree $n$ and $a \in \mathcal{C}$. We rewrite equation (3.5) as $\delta(au) = \sum au_{\{1\}} \otimes u_{\{2\}} + 1 \otimes au$. Thus,

$$(\Delta \otimes \Delta)\delta au = \sum a_{(1)}u_{\{1\}(1)} \otimes a_{(2)}u_{\{1\}(2)} \otimes u_{\{2\}(1)} \otimes u_{\{2\}(2)} + \sum 1 \otimes 1 \otimes a_{(1)}u_{(1)} \otimes a_{(2)}u_{(2)}.$$

$$\begin{aligned}(\mathrm{Id} \otimes \tau \otimes \mathrm{Id})(\Delta \otimes \Delta)\delta au &= \sum a_{(1)}u_{\{1\}(1)} \otimes u_{\{2\}(1)} \otimes a_{(2)}u_{\{1\}(2)} \otimes u_{\{2\}(2)} \\ &\quad + \sum 1 \otimes a_{(1)}u_{(1)} \otimes 1 \otimes a_{(2)}u_{(2)}. \end{aligned} \tag{3.6}$$

Thus,

$$\begin{aligned}(\delta \otimes \mathrm{Id})\Delta(au) &= \sum \delta(a_{(1)}u_{(1)}) \otimes a_{(2)}u_{(2)} \\ &= \sum a_{(1)}u_{(1)\{1\}} \otimes u_{(1)\{2\}} \otimes a_{(2)}u_{(2)} + \sum 1 \otimes a_{(1)}u_{(1)} \otimes a_{(2)}u_{(2)} \\ &= \sum a_{(1)}u_{\{1\}(1)} \otimes u_{\{2\}(1)} \otimes a_{(2)}u_{\{1\}(2)}u_{\{2\}(2)} + \sum 1 \otimes a_{(1)}u_{(1)} \otimes a_{(2)}u_{(2)} \\ &= (\mathrm{Id} \otimes \mathrm{Id} \otimes \mu)(\mathrm{Id} \otimes \tau \otimes \mathrm{Id})(\Delta \otimes \Delta)\delta(au),\end{aligned}$$

where we go from the first line to the second using equation (3.5), from the second to the third with the recursion hypothesis and from the third to the fourth using equation (3.6). This completes the proof. □

Inspired by the first Eulerian idempotent (for $u \in \ker \varepsilon_\delta$)

$$e = -\sum_{n=1}^{\infty} (-1)^n u_{\{\underline{1}\}} \ldots u_{\{\underline{n}\}},$$

defined by Loday and Ronco [4] for connected unital infinitesimal bialgebras, we define the connected chronological product $T_c$ by

$$T_c(u) = -\sum_{n=1}^{\infty} (-1)^n T(u_{\{\underline{1}\}}) \ldots T(u_{\{\underline{n}\}}),$$

or, reciprocally,

$$T(u) = \sum_{n=1}^{\infty} T_c(u_{\{\underline{1}\}}) \ldots T_c(u_{\{\underline{n}\}}),$$

still for $u \in \ker \varepsilon_\delta$. Again, $T_c$ is a coregular action if $T$ is a coregular action.

## 4 Renormalization

Renormalization is a fundamental aspect of quantum field theory. It was discovered because many Feynman diagrams containing loops are divergent when evaluated straightforwardly. After several attempts, notably by Dyson [40], the problem was essentially solved by Bogoliubov [41]. The renormalization theory found in most textbooks [14] is a development of the Bogoliubov approach called the BPHZ renormalization. However, it appeared recently that the original Bogoliubov approach has decisive advantage over the BPHZ renormalization. In particular, it can be used for the renormalization of quantum field theory in curved spacetime [42, 43].

We first present Bogoliubov's solution in Hopf algebraic terms, then we consider in more detail a simplified model.

### 4.1 The Bogoliubov formula

Bogoliubov ([41], section 26.2) and Epstein-Glaser [35] showed that the relation between the bare (i.e. divergent) chronological product $T$ and the renormalized chronological product $T'$ is

$$T'(u) \;=\; \sum_{n=1}^{\infty} \frac{1}{n!} T\big(\Lambda(u_{\{\underline{1}\}}) \dots \Lambda(u_{\{\underline{n}\}})\big), \tag{4.1}$$

for $u \in \ker \varepsilon$ and $T'(1) = T(1) = 1$. In equation (4.1), $\Lambda$ is a linear operator $\ker \varepsilon_\delta \to \mathcal{C}$ called a generalized vertex [41]. Epstein and Glaser proved that the standard BPHZ renormalization is a consequence of this formula [35]. Note that the renormalized chronological product $T'$ is not in general a coregular action.

To see the effect of the operator $\Lambda$ we calculate $T'(E)$ for $E = \mathrm{e}^{\lambda a}$. We first use equation (3.4) to write

$$T'(a^n) \;=\; \sum_{k=1}^{n} \frac{1}{k!} \sum_{i_1+\dots+i_k=n} \frac{n!}{i_1! \dots i_k!} T\big(\Lambda(a^{i_1}) \dots \Lambda(a^{i_k})\big),$$

where all $i_j > 0$. This gives us

$$\begin{aligned} T'(E) \;&=\; 1 + \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} T'(a^n) = 1 + \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{1}{k!} \sum_{i_1+\dots+i_k=n} \frac{1}{i_1! \dots i_k!} T\big(\Lambda((\lambda a)^{i_1}) \dots \Lambda((\lambda a)^{i_k})\big) \\ &=\; 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{i_1,\dots,i_k} \frac{1}{i_1! \dots i_k!} T\big(\Lambda((\lambda a)^{i_1}) \dots \Lambda((\lambda a)^{i_k})\big) \\ &=\; 1 + \sum_{k=1}^{n} \frac{1}{k!} T(\Lambda(\mathrm{e}^{\lambda a} - 1) \dots \Lambda(\mathrm{e}^{\lambda a} - 1)) = T\big(\exp(\Lambda(\mathrm{e}^{\lambda a} - 1))\big). \end{aligned}$$

If we define $a' \in \mathcal{C}$ by

$$a' \;=\; \frac{1}{\lambda} \Lambda(\mathrm{e}^{\lambda a} - 1) = \Lambda(a) + \sum_{n=2}^{\infty} \frac{\lambda^{n-1}}{n!} \Lambda(a^n), \tag{4.2}$$

the previous equality can be rewritten $T'(\mathrm{e}^{\lambda a}) = T(\mathrm{e}^{\lambda a'})$ In other words, the change of chronological product from $T$ to $T'$ amounts to a change of Lagrangian from $a$ to $a'$. This result was obtained by Hollands and Wald [43] who showed that it holds also in curved spacetime.

In flat spacetime, the chronological product satisfies $T(a) = T'(a) = a$ for $a \in \mathcal{C}$. Therefore, $a = T'(a) = T(\Lambda(a)) = \Lambda(a)$ because $\Lambda(a) \in \mathcal{C}$. This implies $\Lambda(a) = a$ and the renormalized Lagrangian starts with the unrenormalized one. The terms with $n > 1$ in equation (4.2) are called the renormalization *counterterms*. In curved spacetime the situation is more complicated and Hollands and Wald [44] showed that we have in general $T'(a) = \sum t'(a_{(1)}) T(a_{(2)})$, where $t'$ is a linear map from $\mathcal{C}$ to $\mathbb{C}$. In that case $\Lambda(a) = \sum t'(a_{(1)}) a_{(2)}$.

### 4.2 The renormalization group: preparation

In this section, we define a product on linear maps $\Lambda : \mathcal{S}(V)^+ \to V$ for any vector space $V$ on the complex numbers, where $\mathcal{S}(V) = \mathbb{C}1 \oplus \mathcal{S}(V)^+$ is the symmetric Hopf algebra on $V$, with coproduct $\delta$ and counit $\varepsilon_\delta$. The Sweedler notatin for the coproduct $\delta$ and the reduced coproduct $\underline{\delta}$ ($\underline{\delta} u = \delta u - u \otimes 1 - 1 \otimes u$) are $\delta u = \sum u_{\{1\}} \otimes u_{\{2\}}$ and $\underline{\delta} u = \sum u_{\{\underline{1}\}} \otimes u_{\{\underline{2}\}}$.

We first define linear maps $\Lambda^{(n)}$ from $\mathcal{S}(V)$ to $\mathcal{S}^n(V)$ by

**Definition 4.1** Let $u \in \mathcal{S}(V)$, we define $\Lambda^{(0)} : \mathcal{S}(V) \to \mathcal{S}^0(V)$ by $\Lambda^{(0)}(u) = \varepsilon_\delta(u) 1$. If $u = 1$, we define $\Lambda^{(n)}(1) = 0$ for any integer $n > 0$, if $u \in \mathcal{S}(V)^+$, we define recursively $\Lambda^{(n)} : \mathcal{S}(V)^+ \to \mathcal{S}^n(V)$ by $\Lambda^{(1)}(u) = \Lambda(u)$ and $\Lambda^{(n)}(u) = \sum \Lambda^{(1)}(u_{\{1\}}) \Lambda^{(n-1)}(u_{\{2\}}) = \sum \Lambda(u_{\{\underline{1}\}}) \Lambda^{(n-1)}(u_{\{\underline{2}\}})$ for any integer $n > 1$. Moreover, we define the linear map $\mathrm{e}^\Lambda : \mathcal{S}(V) \to \mathcal{S}(V)$ by

$$\mathrm{e}^\Lambda(u) \;=\; \sum_{n=0}^{\infty} \frac{1}{n!} \Lambda^{(n)}(u).$$

Note that the exponential is well defined because, for $u \in \mathcal{S}^k(V)$, $\Lambda^{(n)}(u) = 0$ for $n > k$. This is true for $k = 0$ by definition. For $k > 0$, $\Lambda^{(n)}(u)$ can be rewritten

$$\Lambda^{(n)}(u) \quad = \quad \sum \Lambda(u_{\{\underline{1}\}}) \dots \Lambda(u_{\{\underline{n}\}}).$$

The right hand side is zero for $n > k$ because $\underline{\delta}^{n-1} u = 0$ for $n > k$. The following special cases are illustrative: $\mathrm{e}^\Lambda(1) = 1$, $\mathrm{e}^\Lambda(a) = \Lambda(a)$, $\mathrm{e}^\Lambda(ab) = \Lambda(ab) + \Lambda(a)\Lambda(b)$ and

$$\mathrm{e}^\Lambda(abc) \quad = \quad \Lambda(abc) + \Lambda(a)\Lambda(bc) + \Lambda(b)\Lambda(ac) + \Lambda(c)\Lambda(ab) + \Lambda(a)\Lambda(b)\Lambda(c),$$

for $a$, $b$ and $c$ in $V$. Note also that $\mathrm{e}^\Lambda$ maps $\mathcal{S}(V)^+$ to $\mathcal{S}(V)^+$.

We first prove the useful lemma

**Lemma 4.2** *For $u \in \mathcal{S}(V)$, we have $\delta \mathrm{e}^\Lambda(u) = \big(\mathrm{e}^\Lambda \otimes \mathrm{e}^\Lambda\big)\delta u$.*

P r o o f. We first give a recursive proof of the identity

$$\delta\Lambda^{(n)}(u) \quad = \quad \sum_{k=0}^{n} \binom{n}{k} \sum \Lambda^{(k)}(u_{\{1\}}) \otimes \Lambda^{(n-k)}(u_{\{2\}}). \tag{4.3}$$

For $n = 0$, equation (4.3) is satisfied because

$$\delta\Lambda^{(0)}(u) \quad = \quad \varepsilon_\delta(u) 1 \otimes 1 = \sum \varepsilon_\delta(u_{\{1\}}) 1 \otimes \varepsilon_\delta(u_{\{2\}}) 1 = \sum \Lambda^{(0)}(u_{\{1\}}) \otimes \Lambda^{(0)}(u_{\{2\}}).$$

Equation (4.3) is obviously true for $u = 1$ and all $n > 0$. Thus, we assume from now on that $u \in \mathcal{S}(V)^+$. For $n = 1$, equation (4.3) holds because

$$\begin{aligned} \delta\Lambda^{(1)}(u) \quad &= \quad \Lambda(u) \otimes 1 + 1 \otimes \Lambda(u) = \sum \Lambda^{(0)}(u_{\{1\}}) 1 \otimes \varepsilon_\delta(u_{\{2\}}) 1 + \sum \varepsilon_\delta(u_{\{1\}}) 1 \otimes \Lambda^{(0)}(u_{\{2\}}) \\ &= \quad \sum_{k=0}^{1} \binom{1}{k} \sum \Lambda^{(k)}(u_{\{1\}}) \otimes \Lambda^{(1-k)}(u_{\{2\}}). \end{aligned}$$

Assume that equation (4.3) is true up to $n$, then the definition of $\Lambda^{(n+1)}(u)$ and the recursion hypothesis imply

$$\begin{aligned} \delta\Lambda^{(n+1)}(u) \quad &= \quad \sum \big(\delta\Lambda^{(1)}(u_{\{1\}})\big)\big(\delta\Lambda^{(n)}(u_{\{2\}})\big) \\ &= \quad \sum_{k=0}^{n} \binom{n}{k} \sum \Lambda^{(1)}\Lambda^{(k)}(u_{\{2\}}) \otimes \Lambda^{(n-k)}(u_{\{3\}}) \\ &\quad + \sum_{k=0}^{n} \binom{n}{k} \sum \Lambda^{(k)}(u_{\{2\}}) \otimes \Lambda^{(1)}(u_{\{1\}})\Lambda^{(n-k)}(u_{\{3\}}). \end{aligned}$$

The cocommutativity of the coproduct and the definition of $\Lambda^{(k+1)}$ and $\Lambda^{(n-k+1)}$ yield

$$\begin{aligned} \delta\Lambda^{(n+1)}(u) \quad &= \quad \sum_{k=0}^{n} \binom{n}{k} \sum \Big(\Lambda^{(k+1)}(u_{\{1\}}) \otimes \Lambda^{(n-k)}(u_{\{2\}}) + \Lambda^{(k)}(u_{\{1\}}) \otimes \Lambda^{(n-k+1)}(u_{\{2\}})\Big) \\ &= \quad \sum_{k=0}^{n+1} \binom{n+1}{k} \sum \Lambda^{(k)}(u_{\{1\}}) \otimes \Lambda^{(n+1-k)}(u_{\{2\}}), \end{aligned}$$

by Pascal's formula. The lemma follows immediately because, using equation (4.3),

$$\begin{aligned} \delta\mathrm{e}^\Lambda(u) \quad &= \quad \sum_{n=0}^{\infty} \frac{1}{n!} \delta\Lambda^{(n)}(u) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} \sum \Lambda^{(k)}(u_{\{1\}}) \otimes \Lambda^{(n-k)}(u_{\{2\}}) \\ &= \quad \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{l=0}^{\infty} \frac{1}{l!} \sum \Lambda^{(k)}(u_{\{1\}}) \otimes \Lambda^{(l)}(u_{\{2\}}) = \sum \mathrm{e}^\Lambda(u_{\{1\}}) \otimes \mathrm{e}^\Lambda(u_{\{2\}}). \end{aligned}$$

□

A second lemma will be useful to derive recursive proofs.

**Lemma 4.3** *For $a \in V$ and $u \in \mathcal{S}(V)$, we have* $\mathrm{e}^{\Lambda}(au) = \sum \Lambda(au_{\{1\}})\mathrm{e}^{\Lambda}(u_{\{2\}})$.

P r o o f. We first show recursively that, for $a \in V$, $u \in \mathcal{S}(V)^+$ and $n > 1$, then

$$\Lambda^{(n)}(au) \quad = \quad n\Lambda(a)\Lambda^{(n-1)}(u) + n\sum \Lambda(au_{\{1\}})\Lambda^{(n-1)}(u_{\{2\}}). \tag{4.4}$$

When $u \in \mathcal{S}(V)^+$, we take the definition $\Lambda^{(n)}(u) = \sum \Lambda(u_{\{1\}})\Lambda^{(n-1)}(u_{\{2\}})$ and we start from the identity

$$\underline{\delta}(au) \quad = \quad a \otimes u + u \otimes a + \sum au_{\{1\}} \otimes u_{\{2\}} + \sum u_{\{1\}} \otimes au_{\{2\}},$$

which is obtained by expanding $\delta(au) = (a\otimes 1+1\otimes a)(u\otimes 1+1\otimes u+\underline{\delta}u)$ in $\underline{\delta}(au) = \delta(au)-au\otimes 1-1\otimes au$, to derive

$$\Lambda^{(2)}(au) \quad = \quad \sum \Lambda((au)_{\{1\}})\Lambda((au)_{\{2\}}) = 2\Lambda(a)\Lambda(u) + 2\sum \Lambda(au_{\{1\}})\Lambda(u_{\{2\}}),$$

where we used the commutativity of the algebra $\mathcal{S}(V)$ and the cocommutativity of the coproduct. Therefore, equation (4.4) is satisfied for $n = 2$. Assume that it is true up to $n$, then, using again the expression for $\underline{\delta}(au)$, we find

$$\begin{aligned}\Lambda^{(n+1)}(au) \quad &= \quad \sum \Lambda((au)_{\{1\}})\Lambda^{(n)}((au)_{\{2\}}) \\ &= \quad \Lambda(a)\Lambda^{(n)}(u) + \Lambda(u)\Lambda^{(n)}(a) + \sum \Lambda(au_{\{1\}})\Lambda^{(n)}(u_{\{2\}}) + \sum \Lambda(u_{\{1\}})\Lambda^{(n)}(au_{\{2\}}).\end{aligned}$$

For $n > 1$, $\Lambda^{(n)}(a) = 0$ and the recursion hypothesis gives us

$$\begin{aligned}\Lambda^{(n+1)}(au) \quad &= \quad \Lambda(a)\Lambda^{(n)}(u) + \sum \Lambda(au_{\{1\}})\Lambda^{(n)}(u_{\{2\}}) + n\sum \Lambda(u_{\{1\}})\Lambda(a)\Lambda^{(n-1)}(u_{\{2\}}) \\ &\quad +n\sum \Lambda(u_{\{1\}})\Lambda(au_{\{2\}})\Lambda^{(n-1)}(u_{\{3\}}).\end{aligned}$$

Using the definition of $\Lambda^{(n)}(u)$ for $u \in \mathcal{S}(V)^+$, the commutativity of the product and the cocommutativity of the coproduct, we find

$$\begin{aligned}\Lambda^{(n+1)}(au) \quad &= \quad \Lambda(a)\Lambda^{(n)}(u) + \sum \Lambda(au_{\{1\}})\Lambda^{(n)}(u_{\{2\}}) + n\sum \Lambda(a)\Lambda^{(n)}(u) \\ &\quad +n\sum \Lambda(au_{\{1\}})\Lambda^{(n)}(u_{\{2\}}),\end{aligned}$$

and equation (4.4) is proved for $n + 1$.

To prove the lemma, we first consider the case $u = 1$, which holds because the left hand side is $\mathrm{e}^{\Lambda}(a) = \Lambda(a)$ and the right hand side is $\Lambda(a1)\mathrm{e}^{\Lambda}(1) = \Lambda(a)1 = \Lambda(a)$. Assume now that $u \in \mathcal{S}(V)^+$. Then

$$\mathrm{e}^{\Lambda}(au) \quad = \quad \Lambda(au) + \sum_{n=2}^{\infty} \frac{1}{n!}\Lambda^{(n)}(au).$$

Using equation (4.4), we obtain

$$\begin{aligned}\mathrm{e}^{\Lambda}(au) \quad &= \quad \Lambda(au) + \Lambda(a)\sum_{n=2}^{\infty} \frac{1}{(n-1)!}\Lambda^{(n-1)}(u) + \sum \Lambda(au_{\{1\}})\sum_{n=2}^{\infty} \frac{1}{(n-1)!}\Lambda^{(n-1)}(u_{\{2\}}) \\ &= \quad \Lambda(au) + \Lambda(a)\mathrm{e}^{\Lambda}(u) + \sum \Lambda(au_{\{1\}})\mathrm{e}^{\Lambda}(u_{\{2\}}).\end{aligned}$$

The lemma follows from the fact that $\delta u = u \otimes 1 + 1 \otimes u + \underline{\delta}u$ and $\mathrm{e}^{\Lambda}(1) = 1$. □

We are now ready to define a product on $\Lambda$ by

**Definition 4.4** If $\Lambda'$ and $\Lambda$ are linear maps from $\mathcal{S}(V)^+$ to $V$, the product of $\Lambda'$ and $\Lambda$ to be the linear map $\Lambda' * \Lambda$ defined by

$$(\Lambda' * \Lambda)(u) \quad = \quad \Lambda'\big(\mathrm{e}^{\Lambda}(u)\big)$$

This definition enables us to write the last lemma of this section. Let us denoted by $\mathcal{L}(\mathcal{S}(V)^+, V)$ the set of linear maps from $\mathcal{S}(V)^+$ to $V$.

**Lemma 4.5** *For $\Lambda'$ and $\Lambda$ in $\mathcal{L}(\mathcal{S}(V)^+, V)$ and $u \in \mathcal{S}(V)$, we have* $\mathrm{e}^{\Lambda'}\big(\mathrm{e}^{\Lambda}(u)\big) = \mathrm{e}^{\Lambda' * \Lambda}(u)$.

Proof. The lemma is true for $u = 1$ and $u = a \in V$ because $\mathrm{e}^{\Lambda'}\big(\mathrm{e}^{\Lambda}(1)\big) = \mathrm{e}^{\Lambda'}(1) = 1 = \mathrm{e}^{\Lambda' * \Lambda}(1)$ and $\mathrm{e}^{\Lambda'}\big(\mathrm{e}^{\Lambda}(a)\big) = \mathrm{e}^{\Lambda'}(\Lambda(a)) = \Lambda'(\Lambda(a)) = (\Lambda' * \Lambda)(a) = \mathrm{e}^{\Lambda' * \Lambda}(a)$. Assume that the lemma is true for all elements of $\mathcal{S}^k(V)$ up to $k = n$. Take $a \in V$, $u \in \mathcal{S}^n(V)$ and use lemma 4.3 to calculate

$$\mathrm{e}^{\Lambda'}\big(\mathrm{e}^{\Lambda}(au)\big) \;=\; \sum \mathrm{e}^{\Lambda'}\big(\Lambda(au_{\{1\}})\mathrm{e}^{\Lambda}(u_{\{2\}})\big)$$

If we denote $\Lambda(au_{\{1\}})$ by $a'$ and $\mathrm{e}^{\Lambda}(u_{\{2\}})$ by $u'$, we can use lemma 4.3 again

$$\mathrm{e}^{\Lambda'}(a'u') \;=\; \sum \Lambda'(a'u'_{\{1\}})\mathrm{e}^{\Lambda'}(u'_{\{2\}}).$$

Lemma 4.2 enables us to calculate $\sum u'_{\{1\}} \otimes u'_{\{2\}} = \delta u' = \delta \mathrm{e}^{\Lambda}(u_{\{2\}}) = \sum \mathrm{e}^{\Lambda}(u_{\{2\}}) \otimes \mathrm{e}^{\Lambda}(u_{\{3\}})$. Therefore,

$$\begin{aligned}\mathrm{e}^{\Lambda'}\big(\mathrm{e}^{\Lambda}(au)\big) &= \sum \Lambda'\big(\Lambda(au_{\{1\}})\mathrm{e}^{\Lambda}(u_{\{2\}})\big)\mathrm{e}^{\Lambda'}(\mathrm{e}^{\Lambda}(u_{\{3\}})) \\ &= \sum \Lambda'\big(\Lambda(au_{\{1\}})\mathrm{e}^{\Lambda}(u_{\{2\}})\big)\mathrm{e}^{\Lambda' * \Lambda}(u_{\{3\}}),\end{aligned}$$

where we used the recursion hypothesis to evaluate $\mathrm{e}^{\Lambda'}(\mathrm{e}^{\Lambda}(u_{\{3\}}))$. Lemma 4.3 and the definition of $\Lambda' * \Lambda$ yield

$$\mathrm{e}^{\Lambda'}\big(\mathrm{e}^{\Lambda}(au)\big) \;=\; \sum \Lambda'\big(\mathrm{e}^{\Lambda}(au_{\{1\}})\big)\mathrm{e}^{\Lambda' * \Lambda}(u_{\{2\}}) = \sum (\Lambda' * \Lambda)(au_{\{1\}})\mathrm{e}^{\Lambda' * \Lambda}(u_{\{2\}}) = \mathrm{e}^{\Lambda' * \Lambda}(au),$$

where we used lemma 4.3 again to conclude. Thus, the lemma is true for $au \in \mathcal{S}^{n+1}(V)$. □

This list of lemma leads us to the main result of this section,

**Proposition 4.6** *$\mathcal{L}(\mathcal{S}(V)^+, V)$ endowed with the product $*$ is a unital associative algebra. The unit of this algebra is the map $\Lambda_0$ such that $\Lambda_0(a) = a$ for $a \in V$ and $\Lambda_0(u) = 0$ for $u \in \mathcal{S}^n(V)$ with $n > 1$. The invertible elements of this algebra are exactly the $\Lambda$ such that the restriction of $\Lambda$ to $V$ is invertible as a linear map from $V$ to $V$.*

Proof. Associativity follows essentially from lemma 4.5 because, for $\Lambda_1$, $\Lambda_2$ and $\Lambda_3$ in $\mathcal{L}(\mathcal{S}(V)^+, V)$ we have

$$\begin{aligned}\big((\Lambda_1 * \Lambda_2) * \Lambda_3\big)(u) &= (\Lambda_1 * \Lambda_2)\big(\mathrm{e}^{\Lambda_3}(u)\big) = \Lambda_1\big(\mathrm{e}^{\Lambda_2}(\mathrm{e}^{\Lambda_3}(u))\big) = \Lambda_1\big(\mathrm{e}^{\Lambda_2 * \Lambda_3}(u)\big) \\ &= \big(\Lambda_1 * (\Lambda_2 * \Lambda_3)\big)(u).\end{aligned}$$

$\Lambda_0$ is the unit of the algebra because $\mathrm{e}^{\Lambda_0}(u) = u$ for any $u \in \mathcal{S}(V)$. This is true for $u = 1$ by definition of $\mathrm{e}^{\Lambda}$ and for $u = a \in V$ by definition of $\Lambda_0$. Assume that this is true for $u$ of degree up to $n$. Take $u$ of degree $n$ and use lemma 4.3: $\mathrm{e}^{\Lambda_0}(au) = \sum \Lambda_0(au_{\{1\}})\mathrm{e}^{\Lambda_0}(u_{\{2\}})$. $\Lambda_0(au_{\{1\}}) = 0$ if the degree of $u_{\{1\}}$ is larger than 0. Thus $\mathrm{e}^{\Lambda_0}(au) = \sum \Lambda_0(a)\mathrm{e}^{\Lambda_0}(u) = au$ by the recursion hypothesis. Thus $(\Lambda * \Lambda_0)(u) = \Lambda\big(\mathrm{e}^{\Lambda_0}(u)\big) = \Lambda(u)$. Similarly, $(\Lambda_0 * \Lambda)(u) = \Lambda_0\big(\mathrm{e}^{\Lambda}(u)\big)$. $\Lambda_0$ is the identity on the elements of degree 1 and 0 on the elements of degree different from 1. The element of degree 1 of $\mathrm{e}^{\Lambda}(u)$ is $\Lambda(u)$, thus $(\Lambda_0 * \Lambda)(u) = \Lambda(u)$.

To prove the invertibility property, consider an invertible element $\Lambda$ with inverse $\Lambda^{-1}$. Then $\Lambda^{-1} * \Lambda = \Lambda_0$ and, on any element $a$ of $V$, $(\Lambda^{-1} * \Lambda)(a) = \Lambda_0(a) = a$. Thus, $\Lambda^{-1}\big(\Lambda(a)\big) = a$ and $\Lambda$ is invertible as a map from $V$ to $V$. Reciprocally, take a $\Lambda$ invertible as a map from $V$ to $V$ with inverse $\Lambda'$. We shall construct recursively the inverse $\Lambda^{-1}$ of $\Lambda$ in the algebra $\mathcal{L}(\mathcal{S}(V)^+, V)$. For $a \in V$ we have $\Lambda^{-1}(a) = \Lambda'(a)$. To see how the recurrence works, we calculate the next term. For $a$ and $b$ in $V$ we are looking for a $\Lambda'$ such that $(\Lambda^{-1} * \Lambda)(ab) = \Lambda_0(ab) = 0$. Thus, $0 = \Lambda^{-1}(\mathrm{e}^{\Lambda}(ab)) = \Lambda^{-1}(\Lambda(ab) + \Lambda(a)\Lambda(b))$. This defines $\Lambda^{-1}(\Lambda(a)\Lambda(b)) = -\Lambda'(\Lambda(ab))$, because $\Lambda^{-1}$ is $\Lambda'$ on $V$. The map $\Lambda$ being bijective on $V$, this defines $\Lambda^{-1}$ on $\mathcal{S}^2(V)$. Assume now that $\Lambda^{-1}$ is defined on $\mathcal{S}^k(V)$ for all $k < n$, take $a_1, \ldots, a_n$ in $V$ and put $u = a_1, \ldots, a_n$ in $\mathcal{S}^n(V)$. We want to solve $(\Lambda^{-1} * \Lambda)(u) = 0$, with $(\Lambda^{-1} * \Lambda)(u) = \Lambda^{-1}(\mathrm{e}^{\Lambda}(u))$. The term of highest degree in $\mathrm{e}^{\Lambda}(u)$ is $\Lambda(a_1) \ldots \Lambda(a_n)$. The inverse $\Lambda^{-1}$ is defined on all the other terms, thus the equation $(\Lambda^{-1} * \Lambda)(u) = 0$ defines $\Lambda^{-1}$ on $\Lambda(a_1) \ldots \Lambda(a_n)$. In other words, $\Lambda^{-1}$ is now uniquely defined on $\mathcal{S}^n(V)$. Therefore, $\Lambda^{-1}$ is uniquely defined on $\mathcal{S}(V)$. □

### 4.3 Renormalization group: QFT

If $\Lambda$ is a linear map from $\mathcal{S}(\mathcal{C})^+$ to $\mathcal{C}$, we saw in equation (4.2) that the renormalization encoded in $\Lambda$ can be considered as a change of Lagrangian from $a$ to $a'$ with

$$a' \quad = \quad \Lambda(a) + \sum_{n=2}^{\infty} \frac{\lambda^{n-1}}{n!}\Lambda(a^n).$$

Thus, it is possible to consider $a'$ as the result of the action of $\Lambda$ on $a$: $a' = \Lambda \triangleright a$. If we renormalize the Lagrangian $a'$ with renormalization encoded in a map $\Lambda'$, we obtain a new Lagrangian $a'' = \Lambda' \triangleright a' = \Lambda' \triangleright (\Lambda \triangleright a)$. The first terms of $a''$ are

$$\begin{aligned} a'' \quad &= \quad \Lambda'(\Lambda(a)) + \frac{\lambda}{2}\big(\Lambda'(\Lambda(a^2)) + \Lambda'(\Lambda(a)\Lambda(a))\big) \\ &\quad + \frac{\lambda^2}{6}\big(\Lambda'(\Lambda(a^3)) + 3\Lambda'(\Lambda(a)\Lambda(a^2)) + \Lambda'(\Lambda(a)\Lambda(a)\Lambda(a))\big) + O(\lambda^3) \end{aligned}$$

The main result of this section is

**Proposition 4.7** *If $\Lambda$ and $\Lambda'$ are linear maps from $\mathcal{S}(\mathcal{C})^+$ to $\mathcal{C}$ and $a \in C$, then $\Lambda' \triangleright (\Lambda \triangleright a) = (\Lambda' * \Lambda) \triangleright a$.*

P r o o f. Equation (3.4) for $\underline{\delta}^{k-1}a^n$ gives us

$$\mathrm{e}^{\Lambda}(a^n) \quad = \quad \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{i_1+\cdots+i_k=n} \frac{n!}{i_1!\dots i_k!}\Lambda(a^{i_1})\dots\Lambda(a^{i_k}),$$

where all the $i_j$ are strictly positive integers. Thus, for $E = \mathrm{e}^{\lambda a}$,

$$\begin{aligned} \mathrm{e}^{\Lambda}(\frac{E-1}{\lambda}) \quad &= \quad \sum_{n=1}^{\infty} \lambda^{n-1} \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{i_1+\cdots+i_k=n} \frac{\Lambda(a^{i_1})}{i_1!} \cdots \frac{\Lambda(a^{i_k})}{i_k!} \\ &= \quad \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{i_1,\dots,i_k} \lambda^{i_1+\cdots+i_k-1} \frac{\Lambda(a^{i_1})}{i_1!} \cdots \frac{\Lambda(a^{i_k})}{i_k!}. \end{aligned}$$

On the other hand,

$$\Lambda' \triangleright a' \quad = \quad \Lambda'\Big(\sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{k!}(a')^k\Big),$$

with $a' = \sum_{i=1}^{\infty} \lambda^{i-1}\Lambda(a^i)/i!$. Therefore,

$$\Lambda' \triangleright a' \quad = \quad \Lambda'\Big(\sum_{k=1}^{\infty} \frac{1}{k!} \sum_{i_1,\dots,i_k} \lambda^{i_1+\cdots+i_k-1} \frac{\Lambda(a^{i_1})}{i_1!} \cdots \frac{\Lambda(a^{i_k})}{i_k!}\Big).$$

Thus,

$$\Lambda' \triangleright a' \quad = \quad \Lambda'\Big(\mathrm{e}^{\Lambda}\big(\frac{E-1}{\lambda}\big)\Big) = (\Lambda' * \Lambda)\big(\frac{E-1}{\lambda}\big) = (\Lambda' * \Lambda) \triangleright a,$$

where we used lemma 4.5 and the fact that, for any $\Lambda$, $\Lambda \triangleright a = (1/\lambda)\Lambda(E-1)$. □

In standard QFT, the linear maps $\Lambda$ satisfy $\Lambda(a) = a$. Thus, they are invertible for the product $*$ and they form a group, which is one of the many faces of the *renormalization group*.

### 4.4 Connected renormalization

We argued that, in QFT, the connected chronological product is physically more useful than the standard chronological product. Thus, it is important to investigate how connected chronological products are renormalized.

**Proposition 4.8** *The relation between the connected renormalized chronological product $T'_c$ and the connected chronological product $T_c$ is, for $u \in \ker \varepsilon_\delta$,*

$$T'_c(u) = \sum_{n=1}^{\infty} \frac{1}{n!} T_c\big(\Lambda(u_{\{\underline{1}\}}) \dots \Lambda(u_{\{\underline{n}\}})\big),$$

Proof. We first note that we can use definition 4.1 to rewrite equation (4.1) under the form $T'(u) = T(\mathrm{e}^\Lambda(u))$. Lemma 3.3 expresses $T(\mathrm{e}^\Lambda(u))$ in terms of connected chronological products. To evaluate this expression we need $\underline{\delta}^{n-1}\mathrm{e}^\Lambda(u)$. The identity $\delta v = v\otimes 1 + 1\otimes v + \underline{\delta} v$ transforms lemma 4.2 into $\underline{\delta}\mathrm{e}^\Lambda(u) = \big(\mathrm{e}^\Lambda \otimes \mathrm{e}^\Lambda\big)\underline{\delta} u$. By iterating and using the coassociativity of $\underline{\delta}$ we find

$$\underline{\delta}^{n-1}\mathrm{e}^\Lambda(u) = \sum \mathrm{e}^\Lambda(u_{\{\underline{1}\}}) \otimes \dots \otimes \mathrm{e}^\Lambda(u_{\{\underline{n}\}}), \tag{4.5}$$

and we can rewrite the renormalized chronological product in terms of the bare connected chronological products as

$$T'(u) = \sum_{n=1}^{\infty} \frac{1}{n!} T_c(\mathrm{e}^\Lambda(u_{\{\underline{1}\}})) \dots T_c(\mathrm{e}^\Lambda(u_{\{\underline{n}\}})).$$

To conclude, we use definition 3.2 to express $T'_c(u)$ in terms of $T'$. Then, we expand each $T'(u_{\{\underline{i}\}})$ using the last equation

$$\begin{aligned} T'_c(u) &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} T'(u_{\{\underline{1}\}}) \dots T'(u_{\{\underline{n}\}}) \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sum_{i_1,\dots,i_n} \frac{1}{i_1! \dots i_n!} T_c(\mathrm{e}^\Lambda(u_{\{\underline{1}\}})) \dots T_c(\mathrm{e}^\Lambda(u_{\{\underline{i_1+\dots+i_n}\}})) \\ &= \sum_{k=1}^{\infty} T_c(\mathrm{e}^\Lambda(u_{\{\underline{1}\}})) \dots T_c(\mathrm{e}^\Lambda(u_{\{\underline{k}\}})) \sum_{n=1}^{k} \frac{(-1)^{n+1}}{n} \sum_{i_1+\dots+i_n=k} \frac{1}{i_1! \dots i_n!}. \end{aligned}$$

The sum over $n$ and $i_1, \dots, i_n$ is the coefficient of $x^k$ in the series expansion of $\log(\mathrm{e}^x) = \log(1 + (\mathrm{e}^x - 1))$. From $\log(\mathrm{e}^x) = x$ we deduce that the sum is $\delta_{k,1}$. Therefore $T'_c(u) = T_c(\mathrm{e}^\Lambda(u))$ and the lemma is proved. □

In other words, the connected chronological product is renormalized with the same formula and the same generalized vertices $\Lambda$ as the standard chronological product. Such an expression for the renormalization of the connected chronological product was used, for instance, by Hollands [45].

### 4.5 A simplified model

In QFT, the linear maps $\Lambda$ have a very particular form. In the example of the $\varphi^4(x)$ theory, the Lagrangian is $a = g\int_{\mathbb{R}^4} \varphi^4(x)\mathrm{d}x$ and

$$\begin{aligned} \Lambda(a^n) = {} & g^n C_1^{(n)} \int_{\mathbb{R}^4} \varphi^4(x)\mathrm{d}x + g^n C_3^{(n)} \int_{\mathbb{R}^4} \varphi^2(x)\mathrm{d}x \\ & + g^n C_2^{(n)} \int_{\mathbb{R}^4} \varphi(x)\big(\partial\cdot\partial - m^2\big)\varphi(x)\mathrm{d}x \end{aligned}$$

where $C_1^{(n)}$, $C_2^{(n)}$ and $C_2^{(n)}$ are real numbers related to the charge, wavefunction and mass renormalization [41]. Such a Lagrangian cannot be manipulated directly with our approach because the integral over $\mathbb{R}^4$ involves an infinite number of points. However, as explained in section 2.8, it can be given a meaning in the perturbative approach. A more serious problem is the presence of derivatives in $\int_{\mathbb{R}^4} \varphi(x)\big(\partial\cdot\partial - m^2\big)\varphi(x)\mathrm{d}x$. To deal with

such terms, we must include derivatives of fields into our algebra $\mathcal{C}$. This poses several problems that are debated by Stora, Boas, Dütsch and Fredenhagen [46, 47, 48, 49], concerning the status of the Action Ward Identity or whether the fields should be taken on-shell or off-shell. Before the situation is fully clarified, we can propose a model without derivatives (i.e. where the divergences are only logarithmic, in the QFT parlance [14]). In that case, it was shown in a recent paper with Bill Schmitt [50], that the coalgebra $\mathcal{C}$ has to be replaced by a bialgebra $\mathcal{B}$ and that renormalization becomes a functor on bialgebras. In the case of the scalar field, the product is defined by $\varphi^n(x_i)\cdot\varphi^m(x_j)=\delta_{ij}\varphi^{n+m}(x_i)$. It can be checked that, with this product, the coalgebra of the scalar field becomes indeed a commutative bialgebra $\mathcal{B}$.

This simplified model can be extended to any commutative and cocommutative bialgebra $\mathcal{B}$ by defining the maps $\Lambda$ as $\Lambda(u)=\sum\lambda(u_{(1)})\prod u_{(2)}$, where $\lambda$ is a linear map from $\mathcal{S}(\mathcal{B})^+$ to $\mathbb{C}$ and $\prod u$ is defined as follows: if $u=a\in\mathcal{B}$, then $\prod u=a$, if $u=a_1\dots a_n\in\mathcal{S}^n(\mathcal{B})$, then $\prod u=a_1\cdot\ldots\cdot a_n$ where the product $\cdot$ is in $\mathcal{B}$. With this definition, it is clear that $\Lambda$ is a linear map from $\ker\varepsilon_\delta$ to $\mathcal{B}$. The fact that such $\Lambda$ form a subgroup of the renormalization group defined in the previous paragraph follows from the existence of a bialgebraic structure on $\mathcal{S}(\mathcal{S}(\mathcal{B})^+)$, studied in detail in [50]. If the product of $n$ elements $u_1,\dots,u_n$ of $\mathcal{S}(\mathcal{B})^+$ in $\mathcal{S}^n(\mathcal{S}(\mathcal{B})^+)$ is denoted by $u_1\vee\cdots\vee u_n$, the renormalization coproduct $\Delta_R$ defined in [50] can be written, for $u\in\mathcal{S}(\mathcal{B})^+$,

$$\Delta_R u \;=\; \sum_{n=1}^{\infty}\frac{1}{n!}u_{\{\underline{1}\}}\vee\cdots\vee u_{\{\underline{n}\}}\otimes\Big(\prod u_{\{\underline{2}\}}\Big)\dots\Big(\prod u_{\{\underline{n}\}}\Big).$$

This construction is functorial and has a noncommutative analogue [50].

### 4.6 The renormalization group: the noncommutative case

In this section, we want to describe the renormalization group in the noncommutative case. We first define a product on linear maps $\Lambda:\mathcal{T}(V)^+\to V$ for any vector space $V$ on the complex numbers, where $\mathcal{T}(V)=\mathbb{C}1\oplus\mathcal{T}(V)^+$ is the tensor algebra on $V$, with deconcatenation coproduct $\delta$ and counit $\varepsilon_\delta$. We first define linear maps $\Lambda^{(n)}$ from $\mathcal{T}(V)$ to $\mathcal{T}^n(V)$ by

**Definition 4.9** Let $u\in\mathcal{T}(V)$, we define $\Lambda^{(0)}:\mathcal{T}(V)\to\mathcal{T}^0(V)$ by $\Lambda^{(0)}(u)=\varepsilon_\delta(u)1$. If $u=1$, we define $\Lambda^{(n)}(1)=0$ for any integer $n>0$, if $u\in\mathcal{T}(V)^+$, we define recursively $\Lambda^{(n)}:\mathcal{T}(V)^+\to\mathcal{T}^n(V)$ by $\Lambda^{(1)}(u)=\Lambda(u)$ and $\Lambda^{(n)}(u)=\sum\Lambda^{(1)}(u_{\{1\}})\Lambda^{(n-1)}(u_{\{2\}})=\sum\Lambda(u_{\{\underline{1}\}})\Lambda^{(n-1)}(u_{\{\underline{2}\}})$ for any integer $n>1$. Moreover, we define the linear map $I_\Lambda:\mathcal{T}(V)\to\mathcal{T}(V)$ by

$$I_\Lambda(u) \;=\; \sum_{n=0}^{\infty}\Lambda^{(n)}(u).$$

Note that $I_\Lambda$ is well defined because, for $u\in\mathcal{T}^k(V)$, $\Lambda^{(n)}(u)=0$ if $n>k$. The following special cases are illustrative: $I_\Lambda(1)=1$, $I_\Lambda(a)=\Lambda(a)$, $I_\Lambda(ab)=\Lambda(ab)+\Lambda(a)\Lambda(b)$ and

$$I_\Lambda(abc) \;=\; \Lambda(abc)+\Lambda(a)\Lambda(bc)+\Lambda(ab)\Lambda(c)+\Lambda(a)\Lambda(b)\Lambda(c)$$

for $a$, $b$ and $c$ in $V$. Note also that $I_\Lambda$ maps $\mathcal{T}(V)^+$ to $\mathcal{T}(V)^+$. As for the commutative case, we have the useful lemma

**Lemma 4.10** *For $u\in\mathcal{T}(V)$, we have $\delta I_\Lambda(u)=\big(I_\Lambda\otimes I_\Lambda\big)\delta u$.*

P r o o f. We first give a recursive proof of the identity

$$\delta\Lambda^{(n)}(u) \;=\; \sum_{k=0}^{n}\sum\Lambda^{(k)}(u_{\{1\}})\otimes\Lambda^{(n-k)}(u_{\{2\}}). \tag{4.6}$$

For $n=0$, equation (4.6) is satisfied because

$$\delta\Lambda^{(0)}(u) \;=\; \varepsilon_\delta(u)1\otimes 1=\sum\varepsilon_\delta(u_{\{1\}})1\otimes\varepsilon_\delta(u_{\{2\}})1=\Lambda^{(0)}(u_{\{1\}})\otimes\Lambda^{(0)}(u_{\{2\}}).$$

Equation (4.6) is obviously true for $u = 1$ and all $n > 0$. Thus, we take from now on $u \in \mathcal{T}(V)^+$. Assume that equation (4.6) is true up to $n$, then the definition of $\Lambda^{(n+1)}(u)$, the relation $\delta(au) = (a \otimes 1)\delta u + 1 \otimes au$, and the recursion hypothesis imply

$$\begin{aligned}
\delta\Lambda^{(n+1)}(u) &= \sum_{k=0}^{n} \sum \Lambda^{(1)}(u_{\{1\}})\Lambda^{(k)}(u_{\{2\}}) \otimes \Lambda^{(n-k)}(u_{\{3\}}) + \sum 1 \otimes \Lambda^{(1)}(u_{\{1\}})\Lambda^{(n)}(u_{\{2\}}) \\
&= \sum_{k=0}^{n} \sum \Lambda^{(k+1)}(u_{\{1\}}) \otimes \Lambda^{(n-k)}(u_{\{2\}}) + 1 \otimes \Lambda^{(n+1)}(u) \\
&= \sum_{k=0}^{n+1} \sum \Lambda^{(k)}(u_{\{1\}}) \otimes \Lambda^{(n+1-k)}(u_{\{2\}}).
\end{aligned}$$

The lemma follows immediately. □

A second lemma is very close to its commutative analogue.

**Lemma 4.11** *For $a \in V$ and $u \in \mathcal{S}(V)$ we have $I_\Lambda(au) = \sum \Lambda(au_{\{1\}})I_\Lambda(u_{\{2\}})$.*

Proof. We first show that, for $a \in V$, $u \in \mathcal{S}(V)^+$ and $n > 1$, then

$$\Lambda^{(n)}(au) = \Lambda(a)\Lambda^{(n-1)}(u) + \sum \Lambda(au_{\{\underline{1}\}})\Lambda^{(n-1)}(u_{\{\underline{2}\}}). \tag{4.7}$$

When $u \in \mathcal{S}(V)^+$, we take the definition $\Lambda^{(n)}(u) = \sum \Lambda(u_{\{\underline{1}\}})\Lambda^{(n-1)}(u_{\{\underline{2}\}})$. We start now from the identity $\underline{\delta}(au) = a \otimes u + \sum au_{\{\underline{1}\}} \otimes u_{\{\underline{2}\}}$, which is obtained by expanding $\delta(au) = (a \otimes 1)(1 \otimes u + u \otimes 1 + \underline{\delta}u) + 1 \otimes au$, in $\underline{\delta}(au) = \delta(au) - au \otimes 1 - 1 \otimes au$, to derive

$$\Lambda^{(n)}(au) = \sum \Lambda((au)_{\{\underline{1}\}})\Lambda^{(n-1)}((au)_{\{\underline{2}\}}) = \Lambda(a)\Lambda^{(n)}(u) + \sum \Lambda(au_{\{\underline{1}\}})\Lambda^{(n-1)}(u_{\{\underline{2}\}}).$$

To prove the lemma, we first see that the case $u = 1$ obviously holds. Assume now that $u \in \mathcal{S}(V)^+$. Then

$$\begin{aligned}
I_\Lambda(au) &= \Lambda(au) + \sum_{n=2}^{\infty} \Lambda^{(n)}(au) \\
&= \Lambda(au) + \sum_{n=2}^{\infty} \Lambda(a)\Lambda^{(n-1)}(u) + \sum_{n=2}^{\infty} \sum \Lambda(au_{\{\underline{1}\}})\Lambda^{(n-1)}(u_{\{\underline{2}\}}) \\
&= \sum_{n=2}^{\infty} \sum \Lambda(au_{\{1\}})\Lambda^{(n-1)}(u_{\{2\}}) = \sum \Lambda(au_{\{1\}})I_\Lambda(u_{\{2\}}),
\end{aligned}$$

which follows from $\delta u = u \otimes 1 + 1 \otimes u + \underline{\delta}u$ and $\Lambda^{(0)}(u) = 0$. □

We are now ready to define a product on $\Lambda$ by

**Definition 4.12** If $\Lambda'$ and $\Lambda$ are linear maps from $\mathcal{T}(V)^+$ to $V$, the product of $\Lambda'$ and $\Lambda$ to be the linear map $\Lambda' * \Lambda$ defined by

$$(\Lambda' * \Lambda)(u) = \Lambda'\big(I_\Lambda(u)\big)$$

This definition enables us to write the last lemma of this section. Let us denoted by $\mathcal{L}(\mathcal{T}(V)^+, V)$ the set of linear maps from $\mathcal{T}(V)^+$ to $V$.

**Lemma 4.13** *For $\Lambda'$ and $\Lambda$ in $\mathcal{L}(\mathcal{T}(V)^+, V)$ and $u \in \mathcal{T}(V)$, we have $I_{\Lambda'}\big(I_\Lambda(u)\big) = I_{\Lambda' * \Lambda}(u)$.*

Proof. The proof is the same as for the commutative case. □

We can now state the main result of this section,

**Proposition 4.14** *$\mathcal{L}(\mathcal{T}(V)^+, V)$ endowed with the product $*$ is a unital associative algebra. The unit of this algebra is the map $\Lambda_0$ such that $\Lambda_0(a) = a$ for $a \in V$ and $\Lambda_0(u) = 0$ for $u \in \mathcal{T}^n(V)$ with $n > 1$. The invertible elements of this algebra are exactly the $\Lambda$ such that the restriction of $\Lambda$ to $V$ is invertible as a linear map from $V$ to $V$. In particular, the subset of $\mathcal{L}(\mathcal{T}(V)^+, V)$ characterized by $\Lambda(a) = a$ for $a \in V$ is a group.*

P r o o f. The proof is the same as for the commutative case. □

## 5 Conclusion

This paper described the first steps of a complete description of QFT in Hopf algebraic terms. Although these steps look encouraging, many open problems still have to be solved. The main one is analytical: the use of a finite number of points is not really satisfactory and we should allow for coalgebras containing elements such as $\int \varphi^n(x)g(x)\mathrm{d}x$ for some test functions $g$.

Other open problems are easier. We list now three of them: (i) The renormalization approach presented here is equivalent to the Connes-Kreimer approach because both are equivalent to the standard BPHZ renormalization [50]. However, it would be quite interesting to describe this equivalence explicitly. (ii) We proved that a QFT is renormalized once its connected chronological product is renormalized. In fact, a deeper result is true: a QFT is renormalized once its one-particle irreducible diagrams are renormalized [14]. To cast this result into our framework, we would need to write the connected chronological product $T_c$ in terms of a one-particle irreducible chronological product. Although such a connection was announced by Epstein and Glaser [35], it was described as complicated and was apparently never published. Similarly, it would be worthwhile to determine a Hopf algebraic expression of Green functions in terms of $n$-particle irreducible functions, which is usually done by Legendre transformation techniques [51]. (iii) It would be important to develop the analogue of the constructions presented in this paper to the case of gauge theories. Along that line, van Suijlekom obtained recently the remarkable result that the Ward and Slavnov-Taylor identities generate a Hopf ideal of the Hopf algebra of renormalization [52]. It would be nice to see how this result can be adapted to our framework.

The most original aspect of this work is the determination of noncommutative analogues of some QFT concepts (i.e. the replacement of $\mathcal{S}(\mathcal{C})$ by $\mathcal{T}(\mathcal{C})$, or of $\mathcal{S}(V)$ by $\mathcal{T}(V)$). Such a noncommutative analogue was first determined for quantum electrodynamics [53] and it lead to the definition of a noncommutative Faà di Bruno algebra [54], generalized to many variables by Anshelevich et al. [55]. These algebras provide an effective way to manipulate series in noncommutative variables. The noncommutative constructions defined in the present paper can also be useful to manipulate noncommutative objects.

The present approach enables us to recover the Feynman diagram formulation of QFT, but its most interesting aspect is that it is defined at the operator level. For example, in our notation, the relation between connected and standard chronological product is given not only at the level of the coregular actions $t$ and $t_c$, but at the level of the maps from $\mathcal{S}(\mathcal{C})$ to $\mathcal{S}(\mathcal{C})$ (i.e. the relation between $T$ and $T_c$). As a consequence, we can calculate Green functions such as

$$G_\rho(x_1, \ldots, x_n) \quad = \quad \rho\Big(T\big(\varphi(x_1)\ldots\varphi(x_n)\exp(\lambda a)\big)\Big),$$

where $\rho$ is a map from $\mathcal{S}(\mathcal{C})$ to $\mathbb{C}$. Such more general Green functions are the basic objects of the quantum field theory with initial correlations (or QFT of degenerate systems) which is well suited to the calculation of highly-correlated systems [56].

**Acknowledgements** I am very indebted to Raymond Stora for sending me some of his fascinating manuscripts and for pointing out Ruelle's work [1]. I am grateful to Jean-Yves Thibon for pointing out Burge's article [22].

## References


[1] D. Ruelle, Statistical Mechanics, Rigorous Results (W.A. Benjamin Inc., New York, 1969).
[2] A. Connes and D. Kreimer, Hopf algebras, renormalization and noncommutative geometry, Commun. Math. Phys. **199**, 203–42 (1998).
[3] H. Figueroa and J. Gracia-Bondia, Combinatorial Hopf algebras in quantum field theory I, Rev. Math. Phys. **17**, 881–976 (2005).
[4] J.-L. Loday and M. Ronco, On the structure of cofree Hopf algebras, J. f. reine u. angew. Math. **592**, 123–55 (2006).
[5] S. A. Joni and G.-C. Rota, Coalgebras and bialgebras in combinatorics, Stud. Appl. Math. **61**, 93–139 (1979).
[6] A. S. Wightman, Orientation (renormalisation theory), in G. Velo and A. S. Wightman (eds.), Proceedings of the NATO Advanced Study Institute on Renormalization Theory (Reidel, Dordrecht, 1976), 1–24.
[7] S. Majid, Foundations of Quantum Group Theory (Cambridge University Press, Cambridge, 1995).
[8] C. Brouder, Quantum groups and interacting quantum fields, in J.-P. Gazeau, R. Kerner, J.-P. Antoine, S. Métens, and J.-Y. Thibon (eds.), Group 24: Physical and Mathematical Aspects of Symmetries (Institute of Physics Publishing, Bristol, 2003), *Institute of Physics Conference Series*, volume 173, 787–90.
[9] G. 't Hooft and M. Veltman, Diagrammar, CERN report **73-9**, 1–119 (1973).
[10] P. J. Mohr, G. Plunien, and G. Soff, QED corrections in heavy atoms, Phys. Repts **293**, 227–372 (1998).
[11] M. I. Eides, H. Grotch, and V. A. Shelyuto, Theory of light hydrogenlike atoms, Phys. Repts. **342**, 63–261 (2001).
[12] P. Doubilet, G.-C. Rota, and J. A. Stein, On the foundations of combinatorial theories IX: Combinatorial methods in invariant theory, Stud. Appl. Math. **53**, 185–216 (1974).
[13] F. D. Grosshans, G.-C. Rota, and J. A. Stein, Invariant Theory and Superalgebras (Amer. Math. Soc., Providence, 1987).
[14] C. Itzykson and J.-B. Zuber, Quantum Field Theory (McGraw-Hill, New York, 1980).
[15] M. Reed and B. Simon, Methods of Modern Mathematical Physics. II Fourier Analysis, Self-adjointness (Academic Press, New York, 1975).
[16] M. E. Sweedler, Cohomology of algebras over Hopf algebras, Trans. Am. Math. Soc. **133**, 205–39 (1968).
[17] G.-C. Rota and J. A. Stein, Plethystic Hopf algebras, Proc. Natl. Acad. Sci USA **91**, 13057–61 (1994).
[18] M. E. Peskin and D. V. Schroeder, An Introduction to Quantum Field Theory (Addison-Wesley, Reading, 1995).
[19] R. C. Read, The enumeration of locally restricted graphs (I), J. London Math. Soc. **34**, 417–36 (1959).
[20] R. C. Read, The enumeration of locally restricted graphs (II), J. London Math. Soc. **35**, 344–51 (1960).
[21] R. C. Read, The use of S-functions in combinatorial analysis, Canad. J. Math. **20**, 808–41 (1968).
[22] W. H. Burge, Four corresponences between graphs and generalized Young tableaux, J. Combin. Theo. A **17**, 12–30 (1974).
[23] I. G. MacDonald, Symmetric Functions and Hall Polynomials (Clarendon Press, Oxford, 1996), second edition.
[24] W. Fulton, Young Tableaux (Cambridge University Press, Cambridge, 1997).
[25] V. Fack and J. van der Jeugt, Algorithms for generating labelled graphs with given degree, J. Comput. Appl. Math. **37**, 187–94 (1991).
[26] D. Knuth, Permutations, matrices and generalized Young tableaux, Pacific J. Math. **34**, 709–27 (1970).
[27] C. Brouder, B. Fauser, A. Frabetti, and R. Oeckl, Quantum field theory and Hopf algebra cohomology, J. Phys. A: Math. Gen. **37**, 5895–927 (2004).
[28] A. L. Fetter and J. D. Walecka, Quantum Theory of Many-Particle Systems (McGraw-Hill, Boston, 1971).
[29] G. Strinati, Effects of dynamical screening on resonances at inner-shell thresholds in semiconductors, Phys. Rev. B **29**, 5718–26 (1984).
[30] W. G. Ritter, Description of noncommutative theories and matrix models by Wightman functions, J. Math. Phys. **45**, 4980–5002 (2004).
[31] M. Gell-Mann and F. Low, Bound states in quantum field theory, Phys. Rev. **84**, 350–4 (1951).
[32] A. Klimyk and K. Schmüdgen, Quantum Groups and Their Representations (Springer Verlag, Berlin, 1997).
[33] J.-L. Loday, Série de Hausdorff, idempotents eulériens et algèbres de Hopf, Exposition. Math. **12**, 165–178 (1994).
[34] R. Haag, Quantum field theories with composite particles and asymptotic conditions, Phys. Rev. **112**, 669–73 (1958).
[35] H. Epstein and V. Glaser, The role of locality in perturbation theory, Ann. Inst. Henri Poincaré **19**, 211–95 (1973).
[36] R. Brunetti and K. Fredenhagen, Interacting quantum fields in curved space: Renormalization of $\phi^4$, in S. Doplicher, R. Longo, J. Roberts, and L. Zsido (eds.), Operator Algebras and Quantum Field Theory (International Press of Boston, Cambridge, 1997), 546–63.
[37] M. Dütsch and K. Fredenhagen, Algebraic quantum field theory, perturbation theory and the loop expansion, Commun. Math. Phys. **219**, 5–30 (2001).
[38] A. Mestre and R. Oeckl, Combinatorics of $n$-point functions via Hopf algebra in quantum field theory, J. Math. Phys. **47**, 052301 (2006).
[39] I. Lindgren and J. Morrison, Atomic Many-Body Theory (Springer, Berlin, 1986), second edition.
[40] F. J. Dyson, The S matrix in quantum electrodynamics, Phys. Rev. **75**, 1736–55 (1949).
[41] N. N. Bogoliubov and D. V. Shirkov, Introduction to the Theory of Quantized Fields (Interscience Pub. Inc., New York, 1959).

[42] R. Brunetti and K. Fredenhagen, Microlocal analysis and interacting quantum field theories: renormalization on physical backgrounds, Commun. Math. Phys. **208**, 623–61 (2000).
[43] S. Hollands and R. M. Wald, On the renormalisation group in curved spacetime, Commun. Math. Phys. **237**, 123–60 (2003).
[44] S. Hollands and R. M. Wald, Local Wick polynomials and time ordered products of quantum fields in curved spacetime, Commun. Math. Phys. **223**, 289–326 (2001).
[45] S. Hollands, Algebraic approach to the 1/N expansion in quantum field theory, Rev. Math. Phys. **16**, 509–58 (2004).
[46] M. Dütsch and F.-M. Boas, The master Ward identity, Rev. Math. Phys. **14**, 977–1049 (2002).
[47] M. Dütsch and K. Fredenhagen, The master Ward identity and generalized Schwinger-Dyson equation in classical field theory, Commun. Math. Phys. **243**, 275–314 (2003).
[48] M. Dütsch and K. Fredenhagen, Causal perturbation theory in terms of retarded products, and a proof of the Action Ward Identity, Rev. Math. Phys. **16**, 1291–348 (2004).
[49] M. Dütsch and K. Fredenhagen, Action Ward Identity and the Stueckelberg-Petermann renormalization group, in A. B. de Monvel, D. Buchholz, D. Iagolnitzer, and U. Moschella (eds.), Rigorous Quantum Field Theory: A Festschrift for Jacques Bros (Birkhäuser, Basel, 2006), *Progress in Mathematics*, volume 251, 113–124.
[50] C. Brouder and B. Schmitt, Renormalization as a functor on bialgebras, J. Pure Appl. Alg. doi:10.1016/j.jpaa.2006.06.013.
[51] Y. Pis'mak, $n$-particle problem in quantum field theory and the functional Legendre transforms, Int. J. Mod. Phys. **7**, 2793–808 (1992).
[52] W. van Suijlekom, The Hopf algebra of Feynman graphs in QED, Lett. Math. Phys. **77**, 265–81 (2006).
[53] C. Brouder and A. Frabetti, Renormalization of QED with planar binary trees, Euro. Phys. J. C **19**, 715–41 (2001).
[54] C. Brouder, A. Frabetti, and C. Krattenthaler, Non-commutative Hopf algebra of formal diffeomorphisms, Adv. Math. **200**, 479–524 (2006).
[55] M. Anshelevich, E. G. Effros, and M. Popa, Zimmermann type cancellation in the free Faà di Bruno algebra, J. Funct. Anal. **237**, 76–104 (2006).
[56] C. Brouder, Green function hierarchy for open shells, Euro. Phys. Lett. **71**, 556–62 (2005).