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# The Lévy-Fokker-Planck equation: $\Phi$ -entropies and convergence to equilibrium

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ABSTRACT. In this paper, we study a Fokker-Planck equation of the form  $u_t = \mathcal{I}[u] + \text{div}(xu)$  where the operator  $\mathcal{I}$ , which is usually the Laplacian, is replaced here with a general Lévy operator. We prove by the entropy production method the exponential decay in time of the solution to the only steady state of the associated stationary equation.

**Keywords:** Fokker-Planck equation, Ornstein-Uhlenbeck equation, Lévy process,  $\Phi$ -entropy inequalities, entropy production method, logarithmic Sobolev inequalities, fractional Laplacian

**Mathematics Subject Classification:** 46N20, 47G20, 35K15

## 1 Introduction

In this paper, we are interested in the long time behaviour of the following generalized Fokker-Planck equation:

$$\partial_t u = \mathcal{I}[u] + \text{div}(u\nabla V) \quad x \in \mathbb{R}^d, t > 0, \quad (1)$$

submitted to the initial condition:

$$u(0, x) = u_0(x) \quad x \in \mathbb{R}^d \quad (2)$$

where  $u_0$  is nonnegative and in  $L^1(\mathbb{R}^d)$  and  $V$  is a given proper potential for which there exists a nonnegative steady state (see below). We will assume without loss of generality that  $\int u_0(x)dx = 1$ . We refer to (1) as the Lévy-Fokker-Planck equation since the operator  $\mathcal{I}$  is the infinitesimal generator of a Lévy process.

As pointed out in [4], Equation (1) is studied at least for two important reasons. First, it is deeply connected with stochastic differential equations driven

by Gaussian and Lévy processes, see e.g. [7]. Secondly, in the case of a quadratic potential, *i.e.*  $V(x) = \frac{1}{2}|x|^2$ , and the fractional Laplacian (even for general  $\alpha$ -stable Lévy process – see Section 2 for a definition), it permits to describe the long time behaviour of the solution of:

$$\partial_t U = \mathcal{I}[U] \tag{3}$$

with an initial condition  $U_0$  nonnegative in  $L^1(\mathbb{R}^d)$ . Solution of (3) is a semi-group and it is often noted  $P_t(U_0)(x)$ , this notation is used on section 4. Indeed, the reader can check that the function

$$u(t, x) = R^d(t)U(\tau(t), xR(t)) \tag{4}$$

satisfies (1) as soon as one chooses  $R(t) = e^t$  and  $\tau(t) = (e^{\alpha t} - 1)/\alpha$  and

$$U_0(x) = u_0(x).$$

On the one hand,  $U$  vanishes as  $t \rightarrow +\infty$  and on the other hand,  $u$  is expected to tend towards the steady state of (1), that is to say towards the nonnegative stationary solution  $u_\infty$  of (1) with the quadratic potential  $V(x) = \frac{1}{2}|x|^2$ :

$$\mathcal{I}[u_\infty] + \operatorname{div}(u_\infty x) = 0 \tag{5}$$

such that  $\int u_\infty = \int u_0 = 1$ . We see that dealing with the quadratic potential is of particular interest and we will be able to say more in this special case.

In full generality, we expect that the solution  $u$  of (1) converges towards the unique nonnegative solution of:

$$\mathcal{I}[u_\infty] + \operatorname{div}(u_\infty \nabla V) = 0$$

such that  $\int u_\infty = \int u_0 = 1$ . Conditions on  $V$  must ensure that such a steady state exists and is nonnegative.

In [4], the authors exhibit a class of functionals that decrease along trajectories of (1): consider a smooth convex function  $\Phi : \mathbb{R}^+ \mapsto \mathbb{R}$  and define for any nonnegative function  $f$ :

$$\operatorname{Ent}_{u_\infty}^\Phi(f) := \int \Phi(f) u_\infty dx - \Phi\left(\int f u_\infty dx\right).$$

We follow [5] by referring to this functional as a  $\Phi$ -entropy. Since the initial condition  $u_0$  is nonnegative, the maximum principle ensures that

$$E_\Phi(u_0)(t) = \operatorname{Ent}_{u_\infty}^\Phi\left(\frac{u(t)}{u_\infty}\right)$$

is well defined for  $t > 0$ . Biler and Karch prove that  $E_\Phi(u_0)$  decreases in time for general confinement potentials  $V$ ; they also prove that for  $\Phi(r) = r^2/2$  and

for the quadratic potential  $V(x) = \frac{1}{2}|x|^2$ , there exists  $C = C(u_0, \mathcal{I})$  and  $\varepsilon$  such that:

$$E_{|\cdot|^{2/2}}(u_0)(t) \leq Ce^{-\varepsilon t}. \quad (6)$$

To prove the latter result, they assume that the symbol  $\psi$  satisfies for some  $\alpha \in (0, 2]$ :

$$\begin{cases} 0 < \liminf_{\xi \rightarrow 0} \frac{\psi(\xi)}{|\xi|^\alpha} \leq \limsup_{\xi \rightarrow 0} \frac{\psi(\xi)}{|\xi|^\alpha} < +\infty, \\ 0 < \inf \frac{\psi(\xi)}{|\xi|^2}. \end{cases} \quad (7)$$

In particular, the second assumption says more or less that there is a nontrivial Gaussian part ( $\sigma \neq 0$ ). These results are the starting point of this paper. We tried to understand if one can generalize them and obtain an exponential decay for any convex function  $\Phi$  under a sharper form. Our two main contributions are:

1. to exhibit the so-called Fisher information associated with the entropies  $E_\Phi$
2. and to take advantage of this result to prove an exponential decay of the  $\Phi$ -entropies for a class of convex functions  $\Phi$  and for a larger class of operators. Moreover, the constant  $\varepsilon$  only depends on  $\mathcal{I}$  and  $C = E_\Phi(u_0)(0)$  in (6).

Let us describe our results more precisely. Using the notation of Chafaï in [5], let us consider the Bregman distance associated with  $\Phi$ :

$$\forall (a, b) \in \mathbb{R}^+, \quad D_\Phi(a, b) = \Phi(a) - \Phi(b) - \Phi'(b)(a - b) \geq 0. \quad (8)$$

Our first result states that:

$$\forall t \geq 0, \quad \frac{d}{dt} E_\Phi(u_0)(t) = - \int D_\Phi(v(t, x), v(t, x + z)) \nu(dz) u_\infty(x) dx \quad (9)$$

where  $v(t, x) = \frac{u(t, x)}{u_\infty(x)}$ , as soon as  $(a, b) \mapsto D_\Phi(a + b, b)$  and  $(x, y) \mapsto \Phi''(x) < \sigma y$ ;  $y >$  are convex. Biler and Karch were very close to this formula (see their proof of the entropy decreasing property). These conditions appear in [5] (see  $(\mathcal{H}1)$  and  $(\mathcal{H}2)$  in this paper). Basic examples of such convex functions are given by  $\Phi(x) = x^p - 1 - p(x - 1)$  with  $p \in (1, 2]$ ,  $\Phi(x) = x \log x - x + 1$ . When  $p = 2$ ,  $D_\Phi(a, b) = (a - b)^2$  and in the latter case,  $D_\Phi(a, b) = a \ln \frac{a}{b} + (b - a)$ . If  $\Phi$  is  $C^2$ , these conditions are satisfied if  $\Phi, \Phi''$  are convex and  $1/\Phi''$  is concave.

A nice observation made in [4] is that, under their assumptions, the steady state  $u_\infty$  is the density of an infinitely divisible law  $\mu$ . It turns out that we can get the same result under our weaker assumptions (see Proposition 1). In addition, such probability measures  $\mu$  satisfy a logarithmic Sobolev inequality involving  $\int D_\Phi(v(x), v(x + z)) \nu_\mu(dz) \mu(dx)$  where  $\nu_\mu$  is the associated Lévy measure (see Theorem 3). This functional inequality is proved in [8] for  $\sigma = 0$  and  $\Phi(x) = x^2$  or  $\Phi(x) = x \ln x$ , and for general  $\Phi$  with  $\sigma = 0$  in [5]. The

general case, *i.e.*  $\sigma \neq 0$ , is announced in [5] but is not given explicitly, we thus make it precise. Combining it with (9) yields exponential decay as soon as:

$$\nu \leq C\nu_\infty \tag{10}$$

where  $\nu_\infty$  is the Lévy measure associated with  $u_\infty$ . This condition is fulfilled by any  $\alpha$ -stable Lévy process with (eventually) no Gaussian part. It is also satisfied by multifractal operators. See the next section for definitions. Furthermore, it should be compared with the first part of (7). Indeed, freely speaking, it says that the symbol lies between two  $\alpha$ -stable ones for small  $\xi$ .

The paper is organized as follows. In Section 2, we recall basic facts about Lévy operators and associated Lévy measures. Section 3 is devoted to derivating with respect to time the  $\Phi$ -entropies for general potentials  $V$ . In the last section, we suppose that the potential is quadratic and we prove exponential decay (Theorem 2) by combining a modified logarithmic Sobolev inequality (Theorem 3) and the study of the steady state (Proposition 1). We also discuss Condition (10) in Proposition 2.

**Notation.**  $\langle x ; y \rangle$  or  $x \cdot y$  denotes the usual scalar product in  $\mathbb{R}^d$ . The unit ball of  $\mathbb{R}^d$  is denoted by  $B$ .

## 2 Preliminaries

Let us next recall basic definitions about Lévy operators and introduce notations. See [3] for further details.

**Characteristic exponents and Lévy measures.** The Lévy-Khinchine formula implies that there exists a symmetric semi-definite  $d \times d$  matrix  $\sigma = (\sigma_{i,j})$ , a vector  $b = (b_i) \in \mathbb{R}^d$  and a nonnegative measure  $\nu$  on  $\mathbb{R}^d$  such that the Lévy operator writes:

$$\mathcal{I}[u](x) = \langle \sigma \nabla u ; \nabla u \rangle + \langle b ; \nabla u \rangle + \int_{\mathbb{R}^d} (u(x+z) - u(x) - \nabla u(x) \cdot zh(z)) \nu(dz)$$

where  $h$  is a truncature function that we choose as follows:  $h(z) = \frac{1}{1+|z|^2}$ . The so-called Lévy measure  $\nu$  can be singular at the origin and satisfies:

$$\nu(\{0\}) = 0 \quad \text{and} \quad \int \min(1, |z|^2) \nu(dz) < +\infty \tag{11}$$

and The matrix  $\sigma$  characterizes the diffusion (or Gaussian) part of the operator (it can be null if  $\sigma = 0$ ), while  $b$  and  $\nu$  characterises the drift part and the pure jump part, respectively. The characteristic exponent of the Lévy process is denoted by  $\psi$ :

$$\psi(\xi) = \langle \sigma \xi ; \xi \rangle + \langle b ; \xi \rangle + a(\xi)$$

where  $a$  is associated with the pure jump part of the underlying process:

$$a(\xi) = \int (e^{iz \cdot \xi} - 1 - iz \cdot \xi) \nu(dz).$$

**The pseudodifferential point of view.** It is convenient to introduce  $\mathcal{I}_g$ , the operator associated with the Gaussian part, and  $\mathcal{I}_a$ , the operator associated with the pure jump part. The operator  $\mathcal{I}_a$  can be seen as a pseudodifferential operator of symbol  $a$ :

$$\mathcal{I}_a(u) = \mathcal{F}^{-1}(a\mathcal{F}u)$$

where  $\mathcal{F}$  stands for the Fourier transform. We choose the probabilistic convention in defining, for all function  $w \in L^1(\mathbb{R}^d)$ :

$$\forall \xi \in \mathbb{R}^d, \quad \hat{w}(\xi) = \mathcal{F}(w)(\xi) = \int e^{ix \cdot \xi} w(x) dx. \quad (12)$$

**Multifractal and  $\alpha$ -stable Lévy operators.** Lévy operators whose characteristic exponent is positively homogenous of index  $\alpha \in (0, 2]$  are called  $\alpha$ -stable. The fractional Laplacian corresponds to a particular  $\alpha$ -stable Lévy process whose characteristic exponent is  $\psi(\xi) = |\xi|^\alpha$ . Note that if  $\alpha = 2$  the Lévy operator has no jump part. They play a central role in this paper since they are the operators for which we have equality in Condition (10). See the discussion in the last section.

Lévy operators whose characteristic exponent can be written as  $\psi(\xi) = \sum_{i=1}^n \psi_i(\xi)$  where  $\psi_i$  is  $\alpha_i$ -homogenous with  $\alpha_i \in (0, 2]$ , are often referred to as multifractal Lévy operators. These operators are typically those who satisfy Condition (10).

### 3 $\Phi$ -Entropy and associated Fisher information

In this section, we are interested in the (time) derivative of  $E_\Phi(t)$  in the case where a steady state is given. Precisely, the following theorem is proved.

**Theorem 1.** *Assume that the initial condition  $u_0$  is non-negative and satisfies  $\text{Ent}_{u_\infty}^\Phi \left( \frac{u_0}{u_\infty} \right) < \infty$  with  $u_\infty$ , a solution of (5). Then for any convex smooth function  $\Phi : \mathbb{R}^+ \rightarrow \mathbb{R}$  and any  $t \geq 0$ , the solution  $u$  of (1)-(2) satisfies:*

$$\frac{d}{dt} E_\Phi(u_0)(t) = - \int \Phi''(v) \langle \sigma \nabla v; \nabla v \rangle u_\infty - \int \int D_\Phi(v(x), v(x+z)) \nu(dz) u_\infty(x) dx$$

where  $v(t, x) = \frac{u(t, x)}{u_\infty(x)}$  and  $\nu$  is the Lévy measure associated to  $\mathcal{L}$ .

- Remark 1.*
1. The quantity  $E_\Phi$  is often called the Fisher information or energy in the literature.
  2. The existence of a steady state  $u_\infty$  is studied in the case of a quadratic potential in the next section.

**Corollary 1** ([4]). *Assume that  $\sigma = 0$ ,  $b = 0$  and  $u$  a solution of (1) with  $u_0$  a non-negative initial solution in  $L^1(\mathbb{R}^d)$ . Then for any convex function  $\Phi : \mathbb{R}^+ \rightarrow \mathbb{R}$ , the function*

$$t \mapsto \text{Ent}_{u_\infty}^\Phi \left( \frac{u(t)}{u_\infty} \right)$$

*decreases with time.*

If one considers the special case  $\Phi(r) = r^2/2$ , then  $D_\Phi(a, b) = (a - b)^2$ . Now, let us define the ‘‘carré du champ’’ operator  $\Gamma^\mathcal{I}$ , associated to the operator  $\mathcal{I}$ . For all smooth functions  $u$  and  $v$ :

$$\begin{aligned} \Gamma^\mathcal{I}[u, v] &= \mathcal{I}[uv] - u\mathcal{I}[v] - v\mathcal{I}[u] \\ &= \langle \sigma \nabla u; \nabla v \rangle + \int (u(x+z) - u(x))(v(x+z) - v(x))\nu(dz). \end{aligned}$$

Hence, Theorem 1 can be reformulated as follows:

**Corollary 2.** *If  $\Phi(r) = r^2/2$ , then:*

$$E'_\Phi(u_0)(t) = - \int \Gamma^\mathcal{I}[v, v]u_\infty \leq 0.$$

*Remark 2.* Recall that in the case of the Laplacian, the energy depends only on the operator  $\Delta$  and not on the potential  $V$ . We recover a similar result in the general case.

To prove Theorem 1, since the  $\Phi$ -entropy involves the function  $v(t, x) = \frac{u(t, x)}{u_\infty(x)}$ , its derivative makes appear  $\partial_t v$  and it is natural to ask ourselves which partial differential equation  $v$  satisfies. A simple computation gives:

$$\begin{aligned} \partial_t v &= \frac{1}{u_\infty} \mathcal{I}[u_\infty v] + \text{div}(u_\infty v \nabla V) \\ &= \frac{1}{u_\infty} (\mathcal{I}[u_\infty v] - \mathcal{I}[u_\infty]v) + \nabla V \cdot \nabla v =: Lv. \end{aligned} \quad (13)$$

In the case where  $\mathcal{I}[u] = \Delta u$  (i.e.  $\sigma$  is the identity matrix,  $b = 0$  and  $a = 0$ ), Equation (13) writes:

$$\partial_t v = \Delta v - \nabla V \cdot \nabla v$$

and is known as the Ornstein-Uhlenbeck equation. This is the reason why we will refer to Equation (13) as the Lévy-Ornstein-Uhlenbeck equation. We next give a simpler formulation for the Lévy-Ornstein-Uhlenbeck operator appearing in (13).

**Lemma 1** (Lévy-Ornstein-Uhlenbeck equation). *If the integro-differential operator in the right-hand side of (13) is denoted  $L$ , we have for all smooth functions  $w_1$  and  $w_2$ :*

$$\int w_1 L w_2 u_\infty = \int (\mathcal{I}[w_1] - \nabla V \cdot \nabla w_1) w_2 u_\infty.$$

This can be expressed by the formula:  $L^* = \mathcal{I} - \nabla V \cdot \nabla$  where duality is understood with respect to the measure  $u_\infty dx$ .

*Proof.* The main tool is the integration by parts for the operator  $\mathcal{I}$ , for any smooth functions  $u, v$  one gets

$$\int v \mathcal{I}[u] dx = \int u \mathcal{I}[v] dx.$$

Let now  $w_1$  and  $w_2$  are two smooth functions on  $\mathbb{R}^d$  then:

$$\begin{aligned} \int w_1 L w_2 u_\infty &= \int w_1 (\mathcal{I}[u_\infty w_2] - \mathcal{I}[u_\infty] w_2 + u_\infty \nabla V \cdot \nabla w_2) \\ &= \int w_2 \mathcal{I}[w_1] u_\infty - \int \mathcal{I}[u_\infty] w_1 w_2 - \int \operatorname{div}(u_\infty w_1 \nabla V) w_2 \\ &= \int u_\infty (\mathcal{I}[w_1] - \nabla V \cdot \nabla w_1) w_2. \end{aligned}$$

□

*Proof of Theorem 1.* By using Lemma 1 with  $v = u/u_\infty$ , we get:

$$\begin{aligned} \frac{d}{dt} E_\Phi(u_0)(t) &= \int \Phi'(v) \partial_t v u_\infty = \int \Phi'(v) L v u_\infty \\ &= \int (\mathcal{I}[\Phi'(v)] - \nabla V \cdot \nabla(\Phi'(v))) v u_\infty. \end{aligned}$$

If now one remarks that  $r\Phi''(r) = (r\Phi'(r) - \Phi(r))'$ , we get:

$$\begin{aligned} E'_\Phi(u_0)(t) &= \int v \mathcal{I}[\Phi'(v)] u_\infty - \int \nabla V \cdot \nabla(v\Phi'(v) - \Phi(v)) u_\infty \\ &= \int v \mathcal{I}[\Phi'(v)] u_\infty + \int \operatorname{div}(u_\infty \nabla V)(v\Phi'(v) - \Phi(v)) \\ &= \int v \mathcal{I}[\Phi'(v)] u_\infty - \int \mathcal{I}[u_\infty](v\Phi'(v) - \Phi(v)) \\ &= \int (v \mathcal{I}[\Phi'(v)] - \mathcal{I}[v\Phi'(v)] + \mathcal{I}[\Phi(v)]) u_\infty \end{aligned}$$



$$\begin{aligned}
E'_\Phi(u_0)(t) &= \int (v\mathcal{I}_g[\Phi'(v)] - \mathcal{I}_g[v\Phi'(v)] + \mathcal{I}_g[\Phi(v)])u_\infty \\
&\quad + \int (v\mathcal{I}_a[\Phi'(v)] - \mathcal{I}_a[v\Phi'(v)] + \mathcal{I}_a[\Phi(v)])u_\infty \\
&= - \int \Phi''(v)\langle\sigma\nabla v; \nabla v\rangle u_\infty \\
&\quad + \int \int \left( v(x)(\Phi'(v(x+z)) - \Phi'(v(x))) - v(x+z)\Phi'(v(x+z)) \right. \\
&\quad \left. + v(x)\Phi'(v(x)) + \Phi(v(x+z)) - \Phi(v(x)) \right) \nu(dz) u_\infty(x) dx \\
&= - \int \Phi''(v)\langle\sigma\nabla v; \nabla v\rangle u_\infty \\
&\quad - \int \int (\Phi(v(x)) - \Phi(v(x+z)) - \Phi'(v(x+z))(v(x) - v(x+z))) \\
&\quad \times \nu(dz) u_\infty(x) dx.
\end{aligned}$$

Then the definition of the Bregman distance gives

$$E'_\Phi(u_0)(t) = - \int \Phi''(v)\langle\sigma\nabla v; \nabla v\rangle u_\infty - \int \int D_\Phi(v(x), v(x+z))\nu(dz) u_\infty(x) dx.$$

□

## 4 Quadratic potential and exponential decay to equilibrium

By combining a functional inequality proved by Wu and Chafaï in [8, 5] and the observation made by Biler and Karch [4] that  $u_\infty$  is an infinitely divisible measure, we are able to prove exponential decay to 0 of any  $\Phi$ -entropy under some conditions on the convex function  $\Phi$  and on the characteristic exponent  $\psi$ .

**Theorem 2** (Exponential decay to equilibrium). *Assume that  $\Phi$  is such that  $(a, b) \mapsto D_\Phi(a + b, b)$  and  $\Phi''(x)\langle\sigma y \cdot y\rangle$  are convex on  $\{a + b \geq 0, b \geq 0\}$  and  $\mathbb{R}^+ \times (\mathbb{R}^d)^2$  respectively. Assume also that  $V(x) = \frac{1}{2}|x|^2$  and the operator  $\mathcal{I}$  is the infinitesimal generator of a Lévy process with a characteristic exponent  $\psi$  associated to a Lévy measure  $\nu$ . We assume that  $\nu$  has a density  $N$  with respect to the Lebesgue measure and that  $N$  satisfies*

$$\int_{\mathbb{R}^d \setminus B} \ln |z| N(z) dz < +\infty \quad (14)$$

and

$$\int_1^{+\infty} N(sz)s^{d-1} ds \leq CN(z) \quad (15)$$

for some constant  $C > 0$ .

Then the  $\Phi$ -entropy of the solution  $u$  of (1)-(2) goes to 0 exponentially; precisely, for any non-negative initial condition  $u_0$  such that

$$\text{Ent}_{u_\infty}^\Phi \left( \frac{u_0}{u_\infty} \right) < \infty,$$

one gets:

$$\forall t \geq 0, \quad \text{Ent}_{u_\infty}^\Phi \left( \frac{u(t)}{u_\infty} \right) \leq e^{-\frac{t}{C}} \text{Ent}_{u_\infty}^\Phi \left( \frac{u_0}{u_\infty} \right) \quad (16)$$

where  $v(t, x) = \frac{u(t, x)}{u_\infty}$ ,  $u_\infty$  is the unique nonnegative solution of (5) and  $C$  appears in (15).

- Remark 3.*
1. Both conditions we impose to  $\Phi$  appear in [5].
  2. For  $\alpha$ -stable Lévy processes,  $C = \frac{1}{\alpha}$ . See the last section for further discussion.
  3. As far as steady states are concerned, we prove that there exists one (see Proposition 1) and that it is unique.

#### 4.1 A modified logarithmic Sobolev inequality

The functional inequality we need is the following.

**Theorem 3** ([8, 5]). *Assume that  $\Phi$  is such that  $(a, b) \mapsto D_\Phi(a + b, b)$  and  $\Phi''(x)\langle \sigma y \cdot y \rangle$  are convex on  $\{a + b \geq 0, b \geq 0\}$  and  $\mathbb{R}^+ \times (\mathbb{R}^d)^2$  respectively. Consider an infinitely divisible law  $\mu$ . Then for all smooth and positive functions  $v$ ,*

$$\text{Ent}_\mu^\Phi(v) \leq \int \Phi''(v)\langle \sigma \nabla v; \nabla v \rangle \mu(dx) + \int \int D_\Phi(v(x), v(x+z)) \nu_\mu(dz) \mu(dx) \quad (17)$$

where  $\nu_\mu$  and  $\sigma$  denote respectively the Lévy measure and diffusion matrix associated with  $\mu$ .

*Remark 4.* In the particular case where  $\Phi(x) = x^2$  then the  $\Phi$ -entropy  $\text{Ent}_{u_\infty}^\Phi$  reduces to the variance. We shed some light on the fact that one can obtain Inequality (17) by using  $\Gamma_2$  associated with  $\mathcal{I} = \mathcal{I}_\mu$  defined as follows:

$$\Gamma_2^\mathcal{I}(u, v) = \mathcal{I}[\Gamma^\mathcal{I}[u, v]] - \Gamma^\mathcal{I}[u, \mathcal{I}[v]] - \Gamma^\mathcal{I}[v, \mathcal{I}[u]].$$

Inequality (17) can be obtained by remarking that (assume  $\sigma = 0$  for simplicity):

$$\Gamma_2^\mathcal{I}(u, u) = \int \int (u(x+z+z') - u(x+z) - u(x+z') + u(x))^2 \nu(dz) \nu(dz') \geq 0,$$

with  $\nu$  the Lévy measure associated to  $\mathcal{I}$ . We refer to [2] for a review about the  $\Gamma_2$  operator. This computation was already done by Chafaï and Malrieu in a nice but unpublished note [6].

*Remark 5.* Remark that the drift of the law plays no role in this functional inequality.

This inequality is proved in [8] for  $\Phi(x) = x^2$  or  $\Phi(x) = x \log x$  and in this general form in [5] but only for a pure jump Lévy process. The two articles are generalization of the logarithmic Sobolev for Poisson measure, given in [1]. Surprisingly, this theorem is not given in [5] in this general form, for the completeness of this article we next give a sketch of the proof.

*Proof (sketch).* Let  $v$  be a smooth and positive function on  $\mathbb{R}$ . Denote by  $(P_t)_{t \geq 0}$  the semi-group associated to the operator  $\mathcal{I}$ :

$$\frac{d}{dt} P_t(v)(x) = \mathcal{I}(P_t(v))(x), \quad (18)$$

with  $P_0(v) = v$ . Define for  $s \in [0, t]$ ,

$$\psi(s) = P_s(\Phi(g)),$$

where  $g = P_{t-s}(v)$ . One gets, for  $s \in [0, t]$ ,

$$\begin{aligned} \psi'(s) &= P_s(\mathcal{I}(\Phi(g)) - \Phi'(\mathcal{I}(g))) \\ &= P_s(\Phi''(g)\langle \sigma \nabla g; \nabla g \rangle) + P_s\left(\int D_\Phi(g(\cdot + z), g(\cdot)) \nu_\mu(dz)\right). \end{aligned}$$

Since  $\mathcal{I}$  is associated with  $\mu$ , one gets that a solution of (18) is given by

$$P_t(v)(x) = K_\psi(t, \cdot) * v(x),$$

where  $K_\psi(t, y) = \mathcal{F}^{-1}(e^{-t\psi(\cdot)})(y)$  and  $\psi$  is the characteristic exponent of the law  $\mu$ . In particular,  $\mu = K_\psi(1, \cdot)$ . Then one gets

$$\nabla P_t(g)(x) = \nabla g * K_\psi(t, \cdot) = P_t(\nabla g)(x).$$

Using the fact that  $(a, b) \mapsto D_\Phi(a + b, a)$  and  $(x, y) \mapsto \Phi''(x)\langle \sigma y; y \rangle$  are convex, one gets by Jensen's inequality,

$$\psi'(s) \leq P_t(\Phi''(v)\langle \sigma \nabla v; \nabla v \rangle) + P_t\left(\int D_\Phi(v(\cdot + z), v(\cdot)) \nu_\mu(dz)\right).$$

Integrate over  $s \in [0, t]$ , we obtain for  $t \geq 0$  and  $x \in \mathbb{R}^d$ ,

$$\begin{aligned} P_t(\Phi(v))(x) - \Phi(P_t(v))(x) &\leq t P_t(\Phi''(v)\langle \sigma \nabla v; \nabla v \rangle)(x) \\ &\quad + t P_t\left(\int D_\Phi(v(\cdot + z), v(\cdot)) \nu_\mu(dz)\right)(x). \end{aligned} \quad (19)$$

Then choose  $t = 1$  and  $x = 0$  in Inequality (19) and get:

$$\text{Ent}_{K_\psi}^\Phi(v) \leq \int \Phi''(v)\langle \sigma \nabla v; \nabla v \rangle K_\psi dx + \int \int D_\Phi(v(x), v(x + z)) \nu_\mu(dz) K_\psi(x) dx.$$

□

## 4.2 Study of the steady state

Let us now make precise the statement about the steady state  $u_\infty$ .

**Proposition 1** (The steady state as an infinitely divisible measure). *Assume that the Lévy measure  $\nu$  has a density  $N$  and that it satisfies:*

$$\int_{|z| \geq 1} \ln |z| N(z) dz < +\infty.$$

*There then exists a steady state  $u_\infty$ , i.e. a solution of (5). Moreover, it is an infinitely divisible measure whose characteristic exponent  $A$  is defined by:*

$$A(\xi) = \langle b; \xi \rangle + \frac{1}{2} \langle \sigma \xi; \xi \rangle + \int_0^1 a(s\xi) \frac{ds}{s}. \quad (20)$$

*Moreover, the Lévy measure  $\nu_\infty$  has: a density  $N_\infty$  with respect to the Lebesgue measure given by the following formula:*

$$N_\infty(z) = \int_1^\infty N(tz) t^{d-1} dt; \quad (21)$$

*a drift part  $b - b_A$  where:*

$$b_A = \int \int_0^1 z \frac{(1 - \tau^2)|z|^2}{(1 + \tau^2|z|^2)(1 + |z|^2)} d\tau N(z) dz; \quad (22)$$

*and a Gaussian part equal to  $\sigma/2$ .*

*Remark 6.* In the general case, Condition (15) precisely says that  $N_\infty \leq CN$ .

*Proof of Proposition 1.* Let us start as in [4]. At least formally, the Fourier transform  $\hat{u}_\infty$  of any steady state  $u_\infty$  satisfies

$$\psi(\xi) \hat{u}_\infty + \xi \cdot \nabla \hat{u}_\infty = 0$$

so that  $\hat{u}_\infty = \exp(-A)$  with  $A$  such that:

$$\nabla A(\xi) \cdot \xi = \psi(\xi).$$

The solution of this equation is precisely given by (20). It is not clear that  $A$  is well defined and is the characteristic exponent of an infinitely divisible measure; this is what we prove next. This implies in particular that  $\mathcal{F}^{-1}(\exp(-A))$  is a nonnegative function (see [3]).

Define the nonnegative  $N_\infty$  by Equation (21). This integral is well-defined since for any  $R > 0$ : if  $d\sigma$  denote the uniform measure on  $S_{d-1}$  we get,

$$\int_R^\infty \int_{|D|=1} N(\tau D) \tau^{d-1} d\tau d\sigma(D) = \int_{|y| \geq R} N(y) dy < +\infty$$

We conclude that for any  $r \geq R > 0$  and almost every  $D$  on the unit sphere (where the set of nul measure depends only on  $R$ ),

$$r^d N_\infty(rD) = \int_r^\infty N(\tau D) \tau^{d-1} d\tau < +\infty$$

so that  $N_\infty(z)$  is well-defined almost everywhere outside  $B_R$ . Choose now a sequence  $R_n \rightarrow 0$  and conclude.

Let us define  $I_B(r) = \int_{|D|=1} N_B(rD) d\sigma(D)$  for  $B = a$  or  $A$ . The previous equality implies that:  $r^d I_A(r) = \int_r^{+\infty} I_a(\tau) \tau^{d-1} d\tau$ . We conclude that:

$$\begin{aligned} \int_{|z| \leq 1} |z|^2 N_\infty(z) dz &= \int_0^1 I_A(r) r^{d+1} dr = \int_0^1 r \int_r^{+\infty} I_a(\tau) \tau^{d-1} d\tau dr \\ &= \frac{1}{2} \int_{|z| \geq 1} N(z) dz + \frac{1}{2} \int_{|z| \leq 1} |z|^2 N(z) dz < +\infty \\ \int_{|z| \geq 1} N_\infty(z) dz &= \int_1^{+\infty} I_A(r) r^{d-1} dr = \int_1^{+\infty} \frac{1}{r} \int_r^{+\infty} I_a(\tau) \tau^{d-1} d\tau dr \\ &= \int_{|z| \geq 1} \ln |z| N(z) dz. \end{aligned}$$

Hence we have  $\int \min(1, |z|^2) N_\infty(z) dz < +\infty$ . We conclude that it is a Lévy measure. Now consider the associated characteristic exponent:

$$\tilde{A}(\xi) = \langle b; \xi \rangle + \frac{1}{2} \langle \sigma \xi; \xi \rangle + \int (e^{iz \cdot \xi} - 1 - i(z \cdot \xi) h(z)) N_\infty(z) dz.$$

Now compute:

$$\begin{aligned} \tilde{A}(\xi) - \langle b; \xi \rangle - \frac{1}{2} \langle \sigma \xi; \xi \rangle &= \int \int_1^\infty (e^{iz \cdot \xi} - 1 - i(z \cdot \xi) h(z)) N(s z) s^{d-1} ds dz \\ &= \int_1^\infty \left\{ \int \left( e^{i \tilde{z} \cdot \frac{\xi}{s}} - 1 - i \left( \tilde{z} \cdot \frac{\xi}{s} \right) h \left( \frac{\tilde{z}}{s} \right) \right) N(\tilde{z}) d\tilde{z} \right\} \frac{ds}{s} \\ &= \int_1^\infty a \left( \frac{\xi}{s} \right) \frac{ds}{s} - i \xi \cdot \int_1^\infty \left\{ \int \frac{\tilde{z}}{s} \left( h \left( \frac{\tilde{z}}{s} \right) - h(\tilde{z}) \right) N(\tilde{z}) d\tilde{z} \right\} \frac{ds}{s} \\ &= A(\xi) - i \xi \cdot b_A \end{aligned}$$

where  $b_A$  is defined by (22). Properties (11) of the Lévy measure  $\nu$  implies that  $b_A$  is well defined. We conclude that  $A$  is the characteristic exponent of an infinitely divisible law  $u_\infty$  whose drift is  $b - b_A$ , Gaussian part  $\sigma/2$  and whose Lévy measure is  $N_\infty(z) dz$ .  $\square$

*Proof of Theorem 2.* Theorem 1 gives for  $t \geq 0$ ,

$$\begin{aligned} \frac{d}{dt} \text{Ent}_{u_\infty}^\Phi(v(t, \cdot)) &= - \int \Phi''(v(t, \cdot)) \langle \sigma \nabla v(t, \cdot); \nabla v \rangle u_\infty \\ &\quad - \int \int D_\Phi(v(t, x), v(t, x+z)) \nu(dz) u_\infty dx. \end{aligned}$$

It is enough to prove the following inequality

$$\begin{aligned} \text{Ent}_{u_\infty}^\Phi(v) &\leq C \int \Phi''(v(t, \cdot)) \langle \sigma \nabla v(t, \cdot); \nabla v \rangle u_\infty \\ &\quad + C \int \int D_\Phi(v(x), v(x+z)) \nu(dz) u_\infty(x) dx \end{aligned}$$

for some constant  $C$  independent of  $v$  and Gronwall's lemma permits to conclude. But this inequality is a direct consequence of (17) for the infinitely divisible law  $u_\infty$ . Remark also that if one considers another nonnegative steady state  $\tilde{u}_\infty$  such that  $\text{Ent}_{\tilde{u}_\infty}^\Phi(\tilde{u}_\infty) < +\infty$  for some  $\Phi$  strictly convex and satisfying the assumptions of the theorem, then  $\text{Ent}_{\tilde{u}_\infty}^\Phi(\tilde{u}_\infty) = 0$  and we conclude that  $\tilde{u}_\infty = u_\infty$ .  $\square$

### 4.3 Discussion of Condition (15) and examples

Let us discuss the condition we impose in order to get exponential decay. We point out that equality in this condition holds true only for  $\alpha$ -stable operators and we give a necessary condition on the behaviour of the Lévy measure at infinity if one knows that it decreases faster than  $|x|^{-d}$ .

**Proposition 2.** • *Equality  $N_\infty = N/\lambda$  if and only if  $\psi$  is positively homogenous of index  $\lambda \in (0, 2]$ , i.e.*

$$\psi(t\xi) = t^\lambda \psi(\xi) \text{ for any } t > 0, \xi \in \mathbb{R}^d.$$

*In this case, we get  $A = \psi/\lambda$  and  $b_A = 0$ . Note that in the extreme case, where  $\lambda = 2$ , then we get  $N_\infty = N/2 = 0$ .*

- *If  $|x|^d N(x) \rightarrow 0$  as  $|x| \rightarrow +\infty$ , then the densities  $N$  and  $N_\infty$  satisfy:*

$$N = -\text{div}(xN_\infty).$$

- *In this case, Condition (15) is equivalent to:*

$$\begin{cases} N_\infty(tx) \leq N_\infty(x)t^{-d-1/C} & \text{if } t \geq 1 \\ N_\infty(tx) \geq N_\infty(x)t^{-d-1/C} & \text{if } t \leq 1 \end{cases} \quad (23)$$

*Proof.* The first item simply follows from the definition of  $A$ .

Let us first prove the second item.

$$\begin{aligned} N(x) &= -\left(\lim_{t \rightarrow +\infty} t^d N(tx)\right) + 1^d N(1 \times x) = -\int_1^{+\infty} \frac{d}{dt} (t^d N(tx)) dt \\ &= -dN_\infty(x) - x \cdot \nabla N_\infty(x) = -\text{div}(xN_\infty). \end{aligned}$$

To prove the third item, use the first one to rewrite (15) as follows:

$$x \cdot \nabla N_\infty(x) + (d + 1/C)N_\infty(x) \leq 0.$$

Integrate and conclude.  $\square$

*Example 1.* The Lévy measure  $\frac{1}{|z|}e^{-|z|}$  does not satisfy Condition (15). Indeed, it is equivalent to:

$$\int_1^{+\infty} s^{d-2} e^{-|x|(s-1)} ds \leq C$$

and the monotone convergence theorem implies that the left hand side of this inequality goes to  $+\infty$  as  $|x| \rightarrow +\infty$ .

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