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Post Processing for the Vector Finite Element Method: from Edge to Nodal Values

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Purpose

Propose post processing methods for the edge finite element method on a tetrahedral mesh. They make it possible to deduce vector values on the vertices from scalar values defined on the edges of the tetrahedra.

Approach

The new proposed techniques are based on a least squares formulation leading to a sparse matrix system to be solved. They are compared in terms of accuracy and CPU time on a FE formulation for open boundary – frequency domain problems.

Findings

A significant improvement of vector values accuracy on the vertices of the tetrahedra is obtained compared to a classical approach with a very small additional computation time.

Originality

This work presents techniques allowing:

- To obtain the values at the initial nodes of the mesh and not inside the tetrahedra

- To take into account the discontinuity to the interface between two media of different electromagnetic properties.

Keywords

Edge elements, finite element method, post processing

Research paper

Introduction

Edge-based finite elements (FE) are useful in modeling electromagnetic phenomena because of their correct physical sense. Furthermore, it has been shown that better approximation of the solution may be obtained compared to nodal-based FE [Webb, 1993]. The incomplete first order edge finite element is commonly used in electromagnetic modeling. The degrees of freedom are line integrals along the edges in the mesh. However, knowledge of the nodal field values remains necessary for various reasons. Maximal values located at the interfaces can be required to predict possible electric breakdowns. Nodal values may be necessary to achieve some additional computation: induced currents in the conductors, dual field, source term for coupled problem (magneto thermal), … [Zhao et al, 2000, Sekkak et al, 1994]. Post processors for visualization are usually based on the nodal representation of the fields.

For vector finite elements, two techniques are commonly used to compute nodal values from degrees of freedom [Volakis et al, 2000]:

- Only nodal values inside an element are computed from its edge values. However, this method doesn't give a unique value on its boundary, namely on the vertices.

- An average value of the nodal field is evaluated on each node and for each region by taking into account the contribution of all the elements connected to the considered node [Dibben et al, 1997].

The objective of this paper is to propose a method to compute accurate nodal values at the vertices of a tetrahedral mesh. Our approach is validated on a FE formulation for open boundary frequency domain problems. Spatial discretization is achieved using incomplete first order tetrahedral edge elements [Yao Bi, 1996].

Reference method

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A first and simplest method consists in computing local node values on each element of the mesh from the edge values by means of the local edge basis. As explained by (1), a value at a node is then obtained as an arithmetic mean involving the elements containing the node. For a node at the interface between two regions, two values are obtained by averaging the field values separately in each region. It is taken as reference in the following (method 1).

$$E(m) = \frac{\sum\limits_{T \ni m} E_T(m)}{n_T} \quad \text{with } E_T(m) = \sum_{i=1}^{6} \alpha_i p_i(m) \tag{1}$$

m is a node which belongs to n_T elements. Associated to edge e_i of tetrahedron T, p_i is the local shape function with α_i as degree of freedom.

New approach

Let a complex vector field $\vec{E}$ be obtained from some computations with $H(\text{curl};\Omega)$-conforming finite elements on a tetrahedral mesh τ_h of a bounded domain Ω of $\mathbb{R}^3$. We are seeking a "good representation" of $\vec{E}$ by a continuous vector field E* on τ_h. This problem may be formalised by introducing the minimization problem:

$$\text{Find a vector field } E^* \in V_h \text{ which minimizes } \left\| \vec{E} - E \right\| \text{ for all E in } V_h \tag{2}$$

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V_h is the nodal conforming FE space of degree 1 defined on τ_h and $\|\ \|$ is a norm induced by

a scalar product in H(curl; Ω). Vector field $\overset{\Box}{E}$ being given from incomplete first order edge

elements, it can be decomposed in the edge basis denoted $(w_e)_{e \in \varepsilon_h}$, as:

$$\overset{\Box}{E} = \sum_{e \in \varepsilon_h} \overset{\Box}{\alpha}_e w_e \quad \text{with } \overset{\Box}{\alpha}_e = \int_e \overset{\Box}{E}.t : \text{circulation of } \overset{\Box}{E} \text{ along edge e} \tag{3}$$

ε_h denotes the set of the edges in τ_h and t is an oriented tangent unit vector on edge e. So it

seems appropriate to introduce a norm deduced by the following scalar product:

$$\langle E, E' \rangle_\theta = (1 - \theta) \int_\Omega E . \overline{E'} + \theta \sum_{e \in \varepsilon_h} \left(\int_e E . t \right) \left(\int_e \overline{E'}.t \right) \tag{4}$$

θ is a weight such that $0 \le \theta < 1$. The limit case $\theta = 1$, for which the bilinear form $\langle , \rangle_\theta$ does

not remain a scalar product on H(curl; Ω), is tackled in the later.

By definition of the norm deduced by (4), the gap between vector fields $\overset{\Box}{E}$ and E is

$$\left\| \overset{\Box}{E} - E \right\|_\theta = \left[(1 - \theta) \int_\Omega \left| \overset{\Box}{E} - E \right|^2 + \theta \sum_{e \in \varepsilon_h} \left| \int_e (\overset{\Box}{E} - E).t \right|^2 \right]^{1/2} \tag{5}$$

By minimizing $\left\| \overset{\Box}{E} - E \right\|_\theta$, according to the choice of the weight θ, good approximation E* of

$\overset{\Box}{E}$ as well as preserving of the "circulation" along the edges may be expected.

The linear system to solve

For the norm defined by (5) with $0 \le \theta < 1$, the minimization problem (2) is a least squares problem with a unique solution E*. Moreover vector field E* is the orthogonal projection of $\overset{\square}{E}$ into the space V_h for the scalar product (4). The unknown field E* can be decomposed as:

$$E^* = \sum_{i=1}^{N_h} \xi_i \, v_i \qquad (6)$$

where (v_i), i=1 to N_h (the nodal space dimension) is the nodal vector basis of the FE space V_h and the unknown vector $\xi = (\xi_i)$, i=1 to N_h contains the values of each component of the field E* on the vertices of the mesh τ_h. Writing that $E^* - \overset{\square}{E}$ is orthogonal to any test function v_i gives ξ as solution to a linear system: $A_\theta \xi = b_\theta$. The matrix A_θ, the Gram matrix associated to the scalar product $\langle , \rangle_\theta$, is given by:

$$(A_\theta)_{ij} = \left\langle v_i, v_j \right\rangle_\theta \quad \text{i,j=1 to } N_h \qquad (7)$$

Then, A_θ is real symmetric and positive definite for $0 \le \theta < 1$. The right-hand side b_θ is given by:

$$(b_\theta)_i = \left\langle \overset{\square}{E}, v_i \right\rangle_\theta, \text{ i=1 to } N_h \qquad (8)$$

Numbering the unknowns according to the 3 components on the axes, the nodal basis can be decomposed as

$$\begin{bmatrix} \varphi_i \\ 0 \\ 0 \end{bmatrix} i=1 \text{ to n}, \quad \begin{bmatrix} 0 \\ \varphi_i \\ 0 \end{bmatrix} i=1 \text{ to n}, \quad \begin{bmatrix} 0 \\ 0 \\ \varphi_i \end{bmatrix} i=1 \text{ to n} \tag{9}$$

where φ_i, i=1 to n is the scalar nodal basis and n the number of vertices. It induces a partition

of the matrix A_θ into a 3 by 3 block matrix with $9\,(n \times n)$ blocks:

$$A_\theta = \begin{bmatrix} A_\theta^{XX} & A_\theta^{XY} & A_\theta^{XZ} \\ A_\theta^{YX} & A_\theta^{YY} & A_\theta^{YZ} \\ A_\theta^{ZX} & A_\theta^{ZY} & A_\theta^{ZZ} \end{bmatrix} \tag{10}$$

and similarly for b_θ and ξ with $3\,(n \times 1)$ blocks:

$$b_\theta = \begin{bmatrix} b_\theta^X \\ b_\theta^Y \\ b_\theta^Z \end{bmatrix} \quad \xi = \begin{bmatrix} \xi^X \\ \xi^Y \\ \xi^Z \end{bmatrix} \tag{11}$$

Entries of the matrix

It is convenient to split the system as

$$A_\theta = (1-\theta)A_0 + \theta A_1 \text{ and } b_\theta = (1-\theta)b_0 + \theta b_1 \tag{12}$$

Then from (4) and (7), it comes

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$$(A_0)_{ij} = \int_\Omega v_i . v_j , \; i,j=1 \text{ to } N_h \tag{13}$$

$$(A_1)_{ij} = \sum_{e\in\varepsilon_h} (\int_e v_i . t)(\int_e v_j . t) , \; i,j=1 \text{ to } N_h \tag{14}$$

and from (4) and (8)

$$(b_0)_i = \int_\Omega \vec{E} . v_i , \; i=1 \text{ to } N_h \tag{15}$$

$$(b_1)_i = \sum_{e\in\varepsilon_h} (\int_e \vec{E} . t)(\int_e v_i . t) , \; i=1 \text{ to } N_h \tag{16}$$

From (9) and (13), it is easy to check that A_0 is given by:

$$A_0 = \begin{pmatrix} M & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & M \end{pmatrix} \tag{17}$$

where M is the usual $(n \times n)$ mass matrix defined by:

$$M_{ij} = \int_\Omega \varphi_i \varphi_j , \; i,j=1 \text{ to } n \tag{18}$$

A_1 is not a block diagonal matrix but it is easy to see from (19) that each block is as sparse as M. Associated to each vertex a_i of the mesh, we introduce the sets of edges

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$\epsilon_h^i = \{e \in \epsilon_h \, / \, \text{vertex } a_i \text{ is an extremity of edge } e\}$. Then denoting by e_x, e_y, e_z the 3 components on the axes of an oriented edge e, from

$$\int_e \varphi_i t_x = \frac{e_x}{2} \quad \text{if } e \in \epsilon_h^i \,, \tag{19}$$

one gets

$$(A_1^{XY})_{ii} = \sum_{e \in \epsilon_h^i} \frac{e_x e_y}{4} \tag{20}$$

and

$$\text{for } i \neq j \qquad (A_1^{XY})_{ij} = \frac{e_x e_y}{4} \qquad \text{if } e \in \epsilon_h^i \cap \epsilon_h^j \tag{21}$$

The entries of the other blocks of A_1 can be easily deduced.

Entries of the right hand side

For each vertex a_i of the mesh, let us introduce the associated set of edges: $\tilde{\epsilon}_h^i = \{e \in \epsilon_h \, / \, \text{edge } e \text{ and vertex } a_i \text{ belong to the same tetrahedron} \}$ and the set of elements: $\tau_h^i = \text{support of } v_i = \{T \in \tau_h \, / \, a_i \text{ is a vertex of tetrahedron } T\}$ and for each edge $e \in \epsilon_h$, $\tau_h^e = \text{support of } w_e = \{T \in \tau_h \, / e \text{ is an edge of tetrahedron } T\}$. Then from (3),(9) and (15), it comes

$$\left(b_0^X\right)_i = \sum_{e\in\tilde{\mathcal{E}}_h^i} \hat{\alpha}_e \left[\sum_{T\in\tau_h^e\cap\tau_h^i} \int_T \left(w_e\right)_x \varphi_i \right], \quad i=1 \text{ to } n \tag{22}$$

For $T\in\tau_h^e \cap \tau_h^i$, the value of $\int_T \left(w_e\right)_x \varphi_i$ can be computed as follows.

For $e = ab$ where a and b are vertices of tetrahedron T and denoting by λ_a,λ_b the respective associated barycentric co-ordinates, one gets:

$$\int_T \left(w_e\right)_x \varphi_i = \frac{\mathrm{mes}(T)}{20}\eta \quad \text{with } \eta = \begin{cases} (\nabla\lambda_b - 2\nabla\lambda_a)_x & \text{if } e\in\varepsilon_h^i \text{ and } a_i = a \\ (2\nabla\lambda_b - \nabla\lambda_a)_x & \text{if } e\in\varepsilon_h^i \text{ and } a_i = b \\ (\nabla\lambda_b - \nabla\lambda_a)_x & \text{if } e\notin\varepsilon_h^i \end{cases} \tag{23}$$

Because of equality $\int_e \vec{E}.t = \hat{\alpha}_e$, an explicit expression of the components of b_1^X is more easily obtained from (9) and (16) as:

$$\left(b_1^X\right)_i = \sum_{e\in\varepsilon_h^i} \hat{\alpha}_e \int_e \varphi_i.t_x = \sum_{e\in\varepsilon_h^i} \hat{\alpha}_e \frac{e_x}{2} \tag{24}$$

Limit cases $\theta = 0$ and $\theta = 1$

In the particular case where $\theta = 0$, the linear system reduces to $A_0\xi^* = b_0$. As pointed out before, matrix A_0 has a block diagonal structure with mass matrices on the diagonal. To save

some computation time an explicit but approximate solution to the least squares problem can be obtained. Mass-lumping technique can be implemented to obtain an explicit solution of the least squares problem. The mass lumping allows to approximate the mass matrix by a diagonal matrix [Zienkiewicz, 1970]:

$$\int_{\Omega} v_i^2 = \frac{\beta}{4} \quad \text{with } \beta = \sum_{T \in \tau_h^i} \text{mes}(T) \tag{25}$$

The components of the approximate solution are given by:

$$(\xi^X)_i = \frac{4}{\beta}(b_0^X)_i \quad i=1 \text{ to } n \tag{26}$$

For $\theta = 1$, the minimization problem defined by (2),(5) is a pure discrete least squares problem with one solution at least. The solutions satisfy the linear system $A_1\xi = S^t S\xi = b_1$ where S is the rectangular matrix defined by:

$$\text{for k=1 to cardinal } (\varepsilon_h) \quad S_{ki} = \int_e v_i.t \quad i=1 \text{ to } N_h \text{ e edge of number k} \tag{27}$$

Uniqueness of the solution is ensured if there exists no $\xi \in \text{IR}^{N_h} - \{0\}$ such that $S\xi = 0$. i.e. if the columns of S are linearly independent. It requires that the number of lines card(ε_h) is larger than (N_h) the number of columns.

From (27), the uniqueness condition of the solution can be also written as

$$\text{there exists no } E \in V_h - \{0\} \text{ such that } \int_e E.t = 0 \text{ for any } e \in \varepsilon_h$$

It can be translated into the following mesh characterization:

there is no vector field defined on the nodes such that $E(c) \neq 0$ for some node c and

$(E(a)+E(b)).e=0$ for any edge e=ab.

In the numerical experiments presented above such a property appears to be satisfied by the

meshes. In particular, the necessary condition $\text{card}(\varepsilon_h) \geq N_h$ is always satisfied.

Results and discussions

In the following, method 1 (local averaging) is the reference method, method 2 (energy

approximation) is obtained with $\theta=0$, method 3 (circulation approximation) corresponds to

$\theta=1$, method 4 (combination of circulation approximation with energy approximation) is

obtained for $\theta=1/2$, and method 5 corresponds to $\theta=0$ with the mass lumping approximation.

For methods 2, 3, 4, a symmetric quasi-minimal residual method with SSOR preconditioning

(PQMR) is used to solve the matrix system $A_\theta \xi = b_\theta$.

The accuracy and the time required for each post processing method are tested on 3 examples.

In the first two examples, numerical results are compared with analytical solutions of

scattering of a plane wave by a sphere. Analytical solutions can be found in [Harrington,

1968] and numerical formulations of scattering problems are given in [Yaobi, 1996]. A 50

mm radius - sphere is meshed with 100 nodes on its surface. The frequency of the incident

plane wave is 1 GHz. Absorbing Boundary Condition (ABC) (1st order Engquist Majda) is

located at a half wavelength from the sphere. In order to estimate the influence of the Finite

Element discretization, a relative error estimator on the sphere's surface (denoted by S) is computed as

$$\text{Error} = \frac{\sum\limits_{e \in S} \left| \alpha_e^{anal} - \alpha_e^{num} \right|}{\sum\limits_{e \in S} \left| \alpha_e^{anal} \right|} \tag{28}$$

The edge values α_e^{num} are those obtained from the FE-code. Denoting by H^{anal} the analytical solution to the magnetic formulation, the values α_e^{anal} are approximations of $\int\limits_e H^{anal}.t$ computed for an oriented edge e=ab with nodes a and b as:

$$\alpha_e^{anal} = \frac{H^{anal}(a) + H^{anal}(b)}{2} \cdot \frac{e}{|e|} \tag{29}$$

The efficiency of the different post processing strategies is also evaluated on the surface of the sphere by means of relative error estimators. For each of the five methods, 3 relative errors concerning the nodal vector field, its normal component and its tangential component are computed as

$$\text{Error} = \frac{\sum\limits_{a \in S} \left| \vec{H}^{anal}(a) - \vec{H}^{num}(a) \right|}{\sum\limits_{a \in S} \left| H^{anal}(a) \right|} \tag{30}$$

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Where $\overline{H}$ indicates either the vector (3 components), or its normal component or its tangential component. Nodal values $H^{num}(a)$ are given by the post processing methods from the F.E. edge values α_e^{num}.

Scattering by a perfect electric conductor sphere:

In the first example (figure 1), the sphere is modeled as a perfect electrical conductor (PEC). The problem is meshed with 4481 nodes and 22108 tetrahedral elements leading to 28161 edges (294 edges on the sphere's surface). The edge values error, computed with (28), is about 0.067. The total solving time (assembling and solving of the matrix system) with the FE code is about 300s on a HP J5000. Note that the same solver as in the post processing is used (217 iterations of PQMR).

Figure 1

Various relative errors are given in Table I which also contains the post processing costs.

Table I: relative errors (30), number of iterations of the PQMR and CPU time for the post processing for the PEC sphere.

Method	Relative error on 3 components	Relative error on tangential component	Relative error on normal component	Iterations	Time (in s)
1	0.238	0.120	0.200	/	2
2	0.190	0.080	0.173	7	11
3	0.130	0.072	0.100	22	21
4	0.133	0.069	0.113	19	20

5	0.255	0.161	0.200	/	1

On this first example, the methods with circulation approximation (mixed with energy approximation in method 4, pure in 3) give close results and appear to be the most accurate. From the reference method (method 1) the increase in precision on the nodal vector field is about 0.1. The increase in time (about 21 s) is reasonable as compared with the time required to solve the FE problem (436 s). With method 2 (pure energy approximation), the accuracy gain is roughly divided by 2 (0.048) and disappears when the mass lumping is introduced (method 5). The costs behave in similar ways.

Scattering by a magnetic sphere:

In the second example, the sphere is magnetic with $\mu_r = 3$. A discontinuity of the magnetic field at the surface of the sphere is therefore introduced and the interior (region 1) has to be meshed. The same mesh as for the PEC sphere (figure 1) is used outside of the sphere (region 2). For the whole mesh, one gets 4547 nodes and 22783 tetrahedral elements leading to 28805 edges (294 edges on the sphere's surface). The total time for FE solving is about 436 s (537 iterations of the PQMR). The edge values error, computed with (28), is about 0.149. The post processing is made in the only region 2 (air) according to the analytical solution (Harrington, 1968).

Similarly as in Table I, Table II gives the results we obtain for each five methods. Compared to the reference method, the least squares methods behave as in the PEC case.

Table II: relative errors (30), number of iterations of the PQMR and CPU time for the post processing for the magnetic sphere.

Method	Relative error on 3 components	Relative error on tangential component	Relative error on normal component	Iterations	Time (in s)
1	0.265	0.133	0.230	/	2
2	0.188	0.098	0.163	7	9
3	0.127	0.092	0.087	20	19
4	0.129	0.089	0.090	18	19
5	0.289	0.129	0.260	/	1

Academic example:

In order to eliminate the error due to FE discretization, the performances of the post processing methods are finally evaluated on an academic example on the same spherical geometry. The same geometry and the same mesh as for the problem of scattering by the dielectric sphere (example 2) are used. The vector field to post process is no longer the solution to some scattering problem but a radial field analytically defined as (figure 2):

$$20r \text{ in region 1}; 2000r \text{ in region 2} \tag{31}$$

with r the radial component in spherical coordinates (r, θ, φ).

Figure 2

Approximate edge values are then evaluated as in (29). From these data, nodal values are computed with the different post processing methods independently in both regions (table III).

Table III: nodal values, number of iterations of the PQMR of the post processing for the

academic problem.

Method	Normal component (region 1)	Normal component (region 2)	Tangential component (region 1)	Tangential component (region 2)	Iterations
1	1.2	71	0.019	0.081	
2	1.1	87	0.002	0.177	8
3	1.0	100	3.10^{-14}	8.10^{-12}	23
4	1.0	98	1.10^{-5}	0.004	20
5	1.2	70	0.003	0.108	

Taking into account the starting field (31), the awaited results on the sphere surface should

be :

    region 1: tangential component = 0; normal component = 1

    region 2: tangential component = 0; normal component = 100

The results confirm an important increase of accuracy obtained with pure or mixed circulation

approximation methods. In particular exact results are given by method 3.

Conclusion

As shown from tables I, II and III, the most accurate vector fields are obtained with the

methods where some circulation approximation is satisfied. Theses methods are the most CPU

consuming. However, the time required for the post processing remains reasonable compared

to the total FE solving time. A significant improvement of the accuracy is so obtained with a small additional computation time.

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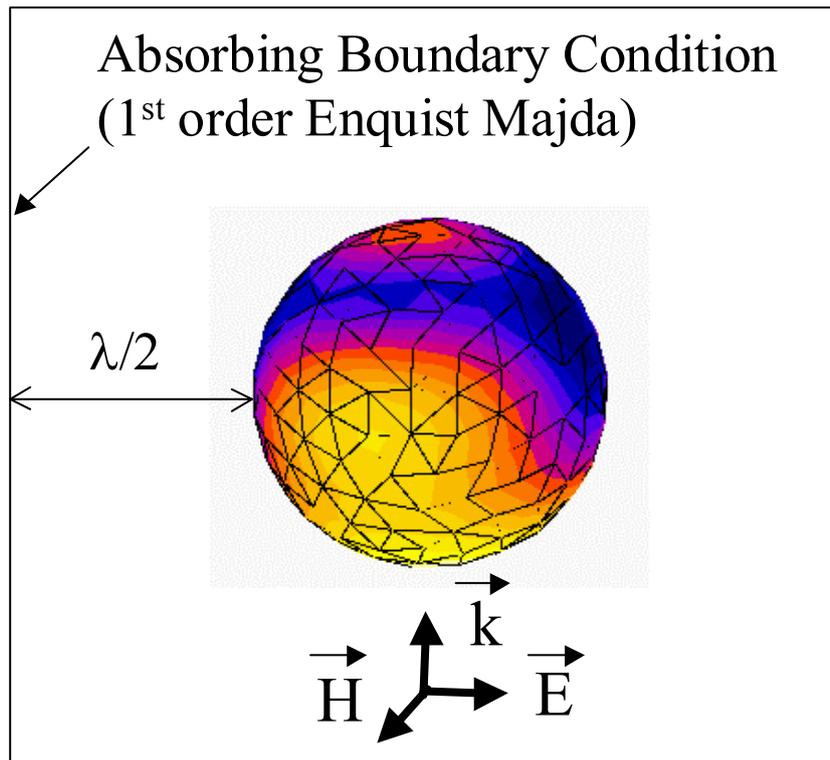Zienkiewicz, (1970), O. C., *The Finite Element Method*, Mc Graw-Hill Book Company.

Figures
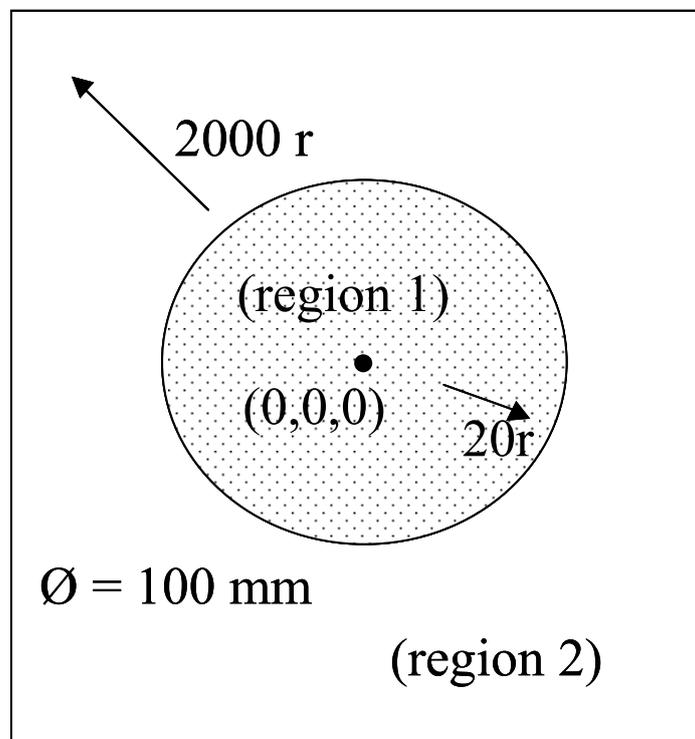
Figure 1 : H field modulus on a PEC sphere

Figure 2 : radial field of magnitude 20r in region 1 and 2000r in region 2