



Existence of subcritical regimes in the Poisson Boolean model of continuum percolation

Jean-Baptiste Gouéré

► To cite this version:

Jean-Baptiste Gouéré. Existence of subcritical regimes in the Poisson Boolean model of continuum percolation. The Annals of Probability, 2008, pp.36(4):1209-1220. <hal-00113319>

HAL Id: hal-00113319

<https://hal.science/hal-00113319v1>

Submitted on 13 Nov 2006

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



HAL Authorization

Existence of subcritical regimes in the Poisson Boolean model of continuum percolation

Jean-Baptiste Gou  r   *

Abstract

We consider the so-called Poisson Boolean model of continuum percolation. At each point of an homogeneous Poisson point process on the Euclidean space \mathbb{R}^d , we center a ball with random radius. We assume that the radii of the balls are independent, identically distributed and independent of the point process. We denote by Σ the union of the balls and by S the connected component of Σ that contains the origin.

We show that S is almost surely bounded for small enough density λ of the point process if and only if the mean volume of the balls is finite. Let us denote by D the diameter of S and by R one of the random radii. We also show that, for all positive real number s , D^s is integrable for small enough λ if and only if R^{d+s} is integrable.

1 Introduction

We consider the so-called Poisson Boolean model of continuum percolation. At each point of an homogeneous Poisson point process on the Euclidean space \mathbb{R}^d , we center a ball of random radius. We assume that the radii of the balls are independent, identically distributed and independent of the point process. We denote by Σ the union of the balls and by S the connected component of Σ that contains the origin. In this paper, we are interested in some properties of S when the density λ of the Poisson point process is small.

Let R be one of the random radii. In [2] (see also [3] and [5]), Hall proved that if $E(R^{2d-1})$ is finite, then the set S is almost surely bounded for small enough λ . If $E(R^d)$ is infinite, then such a behaviour does not happen. Actually, in that case, whatever the value of the density λ , the set Σ is almost surely the whole space. In this paper, we prove that the set S is almost surely bounded for small enough λ if and only if $E(R^d)$ is finite.

Let us denote by N the number of balls whose union is S . In [2] (see also [3] and [5]), Hall also proved that N is integrable for small enough λ if and only if $E(R^{2d})$ is finite. More generally, in [1],   laszczyszyn, Rau and Schmidt proved, among other things, that, for all integer $k \geq 1$, N^k is integrable for small enough λ if and only if the moment $E(R^{d(1+k)})$ is finite. In this paper, we prove a related result. Let us denote by D the

**Postal address:* Universit   d'Orl  ans MAPMO B.P. 6759 45067 Orl  ans Cedex 2 France *E-mail:* jbgouere@univ-orleans.fr

Euclidean diameter of S . We prove that, for all real $s > 0$, D^s is integrable for small enough λ if and only if the moment $E(R^{d+s})$ is finite.

In all these results, the main difficulty lies in proving that S is small for small enough λ when R is integrable enough. (Proving that this behaviour does not happen when R is not integrable enough is easy). The proofs of [1] and [2] both rely on the introduction of a multitype branching process that dominates the percolation process. Roughly speaking, the branching process is defined so that the size of the first generation dominates the number of balls that contain the origin, the size of the second generation dominates the number of balls that intersect the previous balls and so on. In this paper, the proof relies on an estimate that can be roughly described as follows (see Proposition 3.1 or Lemma 3.3 for a more precise statement): the probability of D being greater than a real α is bounded above by the square of the probability of D being greater than $\alpha/10$, up to error terms that are due to the existence of large balls.

2 Notations and statement of the main results

For all this paper, we fix an integer $d \geq 1$. Let $|\cdot|$ be the canonical Lebesgue measure on \mathbb{R}^d . We denote by $\|\cdot\|$ the canonical Euclidean norm on \mathbb{R}^d and by $B(x, r)$ the open Euclidean ball centered at $x \in \mathbb{R}^d$ with radius $r \geq 0$.

If ν is a probability measure on $]0, +\infty[$ and $\lambda > 0$ is a positive real, we define random variables on a probability space $(\Omega_{\lambda, \nu}, \mathcal{F}_{\lambda, \nu}, P_{\lambda, \nu})$ as follows. The measure ν will be the law of the radii of the balls and the real $\lambda > 0$ will be the density of the set of centers of the balls. Let ξ be a Poisson point process on $\mathbb{R}^d \times]0, +\infty[$ whose intensity measure is the product of $\lambda|\cdot|$ and ν . Almost surely, distinct points of ξ have distinct coordinates on \mathbb{R}^d . We work on this event. We denote by χ the projection of ξ on \mathbb{R}^d . Notice that χ is a Poisson point process on \mathbb{R}^d whose intensity is $\lambda|\cdot|$. If c belongs to χ , we denote by $r(c)$ the unique positive real such that $(c, r(c))$ belongs to ξ . We refer to [4, 6, 7] for background on point processes and to [3, 5] for Boolean processes.

We are interested in some properties of the following random set:

$$\Sigma = \bigcup_{c \in \chi} B(c, r(c)). \quad (1)$$

Let S denote the connected component of Σ which contains 0. (We let $S = \emptyset$ if 0 does not belong to Σ .) We say that percolation occurs if the set S is unbounded. We prove the following result:

Theorem 2.1 *For all probability measures ν on $]0, +\infty[$, the following assertions are equivalent:*

1. *The moment $\int r^d \nu(dr)$ is finite;*
2. *There exists a real $\lambda_0 > 0$ such that $P_{\lambda, \nu}(\text{Percolation}) = 0$ for all positive real $\lambda < \lambda_0$.*

Moreover, there exists a positive constant C that depends only on the dimension d such that, for all probability measures ν on $]0, +\infty[$, if the previous assertions hold, then:

$$P_{\lambda, \nu}(\text{Percolation}) = 0 \text{ for all positive real } \lambda < C \left(\int r^d \nu(dr) \right)^{-1}.$$

Let D denote the Euclidean diameter of S :

$$D = \sup_{x,y \in S} \|x - y\|. \quad (2)$$

(We let $D = 0$ if S is empty.) We prove the following result:

Theorem 2.2 *Let $s > 0$ be a positive real. For all probability measures ν on $]0, +\infty[$, the following assertions are equivalent:*

1. *The moment $\int r^{d+s} \nu(dr)$ is finite;*
2. *There exists a real $\lambda_0 > 0$ such that $E_{\lambda, \nu}(D^s)$ is finite for all positive real number $\lambda < \lambda_0$.*

Moreover, there exists a positive constant C that depends only on the dimension d such that, for all $s > 0$ and all probability measures ν on $]0, +\infty[$, if the previous assertions hold, then:

$$E_{\lambda, \nu}(D^s) \text{ is finite for all positive real number } \lambda < C \left(\int r^d \nu(dr) \right)^{-1}.$$

3 Proofs

In order to simplify some notations, in all this section, once a probability measure ν on $]0, +\infty[$ is given, we will denote by R a random variable whose law is ν .

3.1 Proof of some inequalities

In all this subsection, we fix a probability measure ν on $]0, +\infty[$ and a positive real λ . In order to simplify notations, we drop the subscript $\{\lambda, \nu\}$ of the probability measure P and of the expectation symbol E . For all Borel subsets $A \subset \mathbb{R}^d$, we define a set $\Sigma(A)$ as follows:

$$\Sigma(A) = \bigcup_{c \in \chi \cap A} B(c, r(c)). \quad (3)$$

We will study percolation through a family of events defined as follows. If $\alpha > 0$ is a real and x is a point of \mathbb{R}^d , we say that $G(x, \alpha)$ occurs if the connected component of

$$\Sigma(B(x, 10\alpha)) \cup B(x, \alpha)$$

containing x is not included in $B(x, 8\alpha)$. By stationarity, the probability of these event does not depend on x . We denote it by $\pi(\alpha)$:

$$\pi(\alpha) = P(G(0, \alpha)).$$

To deal with large radii, we introduce two other families of events as follows. For all real $\alpha > 0$, we define an event $H(\alpha)$ by:

$$H(\alpha) = \{\exists c \in \chi \setminus B(0, 10\alpha) : B(c, r(c)) \cap B(0, 9\alpha) \neq \emptyset\}$$

and an event $\tilde{H}(\alpha)$ by:

$$\tilde{H}(\alpha) = \{\exists c \in \chi \cap B(0, 100\alpha) : r(c) \geq \alpha\}.$$

Finally, we define a random variable M as follows:

$$M = \sup_{x \in S} \|x\|. \quad (4)$$

(We let $M = 0$ if S is empty.)

Our aim in this subsection is to prove the following proposition:

Proposition 3.1 *There exists a constant C , that depend only on the dimension d , such that the following assertions hold for all $\alpha > 0$:*

$$\pi(10\alpha) \leq C\pi(\alpha)^2 + \lambda C \int_{\alpha}^{+\infty} r^d \nu(dr), \quad (5)$$

$$P(M \geq 9\alpha) \leq \pi(\alpha) + \lambda C \int_{\alpha}^{+\infty} r^d \nu(dr) \quad (6)$$

and

$$\pi(\alpha) \leq C\lambda\alpha^d. \quad (7)$$

In the following lemma, we give a link between the percolation event and the families of events $G(0, \cdot)$ and $H(\cdot)$. This will partly give (6).

Lemma 3.2 *For all real number $\alpha > 0$, the following inclusion holds:*

$$\{M \geq 9\alpha\} \subset G(0, \alpha) \cup H(\alpha).$$

Proof. Let $\alpha > 0$ be a real. We assume that M is greater or equal to 9α and that $H(\alpha)$ does not occur. Recall Definitions (1) and (3). Let \tilde{S} be the connected component of $\Sigma(B(0, 10\alpha))$ that contains 0. (We let $\tilde{S} = \emptyset$ if $\Sigma(B(0, 10\alpha))$ does not contain 0.) As $H(\alpha)$ does not occurs, the set $\Sigma(B(0, 10\alpha)^c)$ is included in $B(0, 9\alpha)^c$. Therefore, \tilde{S} can not be included in $B(0, 8\alpha)$, because otherwise \tilde{S} would be the connected component of Σ containing 0 and one would have $M \leq 8\alpha$. Therefore, the connected component of $\Sigma(B(0, 10\alpha)) \cup B(0, \alpha)$ that contains 0 is not included in $B(0, 8\alpha)$. In other words, the event $G(0, \alpha)$ occurs. \square

In the following lemma, we give a way to control the probabilities $\pi(\alpha)$'s. This will partly give (5).

Lemma 3.3 *There exists a constant C_1 that only depends on the dimension d such that, for all real $\alpha > 0$, the following holds:*

$$\pi(10\alpha) \leq C_1\pi(\alpha)^2 + P(\tilde{H}(\alpha)).$$

Proof.

- Let S_{10} and S_{80} denote the Euclidean spheres centered at the origin with radii 10 and 80. Fix K and L , two subsets of \mathbb{R}^d such that the following properties hold:

1. The sets K and L are finite;
2. $K \subset S_{10}$ and $L \subset S_{80}$;
3. $S_{10} \subset K + B(0, 1)$ and $S_{80} \subset L + B(0, 1)$.

We define C_1 as the product of the cardinals of the sets K and L .

- Let $\alpha > 0$ be a real. In this step, we prove the following inclusion:

$$G(0, 10\alpha) \setminus \tilde{H}(\alpha) \subset \left(\bigcup_{k \in K} G(\alpha k, \alpha) \right) \cap \left(\bigcup_{l \in L} G(\alpha l, \alpha) \right). \quad (8)$$

We assume that the event $G(0, 10\alpha)$ occurs but that the event $\tilde{H}(\alpha)$ does not occur. Let $\phi : [0, 1] \rightarrow \Sigma(B(0, 100\alpha)) \cup B(0, 10\alpha)$ be a continuous function such that $\|\phi(0)\| = 10\alpha$, $\|\phi(1)\| = 80\alpha$ and, for all real $r \in]0, 1[$, $10\alpha < \|\phi(r)\| < 80\alpha$. Let $k \in K$ such that $\phi(0)$ belongs to the ball $B(\alpha k, \alpha)$. Let a be the smallest positive real such that $\phi(a)$ does not belong to $B(\alpha k, 8\alpha)$. Recall that $\phi([0, 1])$ is included in $\Sigma(B(0, 100\alpha)) \cup B(0, 10\alpha)$. As $\|\phi(r)\|$ is greater than 10α for all real $r \in]0, 1[$, we get that $\phi([0, a])$ is included in $\Sigma(B(0, 100\alpha))$. As $\phi(0)$ belongs to $B(\alpha k, \alpha)$, we then get that $\phi([0, a])$ is included in $\Sigma(B(0, 100\alpha)) \cup B(\alpha k, \alpha)$. As $\tilde{H}(\alpha)$ does not occur and as $\phi([0, a])$ is included in $B(\alpha k, 9\alpha)$ (by definition of a and k), we get that $\phi([0, a])$ is included in $\Sigma(B(\alpha k, 10\alpha)) \cup B(\alpha k, \alpha)$. As, moreover, $\phi(0)$ belongs to $B(\alpha k, \alpha)$, the set $\phi([0, a]) \cup B(\alpha k, \alpha)$ is a connected subset of $\Sigma(B(\alpha k, 10\alpha)) \cup B(\alpha k, \alpha)$ containing αk . As $\phi(a)$ does not belong to $B(\alpha k, 8\alpha)$, we finally deduce that the event $G(\alpha k, \alpha)$ occurs.

We have proved that the event $\cup_{k \in K} G(\alpha k, \alpha)$ occurs. We can prove in a similar way that the event $\cup_{l \in L} G(\alpha l, \alpha)$ occurs. Therefore the inclusion (8) is proved.

- The left-hand side of (8) is an intersection of two events. The first of them only depends on what happens in $B(0, 20\alpha)$. The other one only depends on what happens in $B(0, 70\alpha)^c$. These two events are therefore independent. We can then conclude from relation (8). \square

In the following two lemmas, we give a way to bound the probabilities of the events $P(H(\alpha))$'s and $P(\tilde{H}(\alpha))$'s. This will enable us to conclude the proof of (5) and (6).

Lemma 3.4 *There exists a constant C_2 , that only depends on the dimension d , such that for all real $\alpha > 0$, the following inequality holds:*

$$P(H(\alpha)) \leq \lambda C_2 \int_{\alpha}^{+\infty} r^d \nu(dr).$$

Proof. Let α be a positive real. We have:

$$\begin{aligned}
P(H(\alpha)) &\leq E(\text{card}(c \in \chi \setminus B(0, 10\alpha) : B(c, r(c)) \cap B(0, 9\alpha) \neq \emptyset)) \\
&\leq \lambda \int_{\mathbb{R}^d \setminus B(0, 10\alpha)} P(B(x, R) \cap B(0, 9\alpha) \neq \emptyset) dx \\
&\leq \lambda \int_0^{+\infty} |B(0, r + 9\alpha) \setminus B(0, 10\alpha)| \nu(dr) \\
&\leq \lambda \int_\alpha^{+\infty} |B(0, r + 9\alpha)| \nu(dr) \\
&\leq \lambda \int_\alpha^{+\infty} |B(0, 10r)| \nu(dr).
\end{aligned}$$

The inequality stated in the lemma is therefore fulfilled with $C_2 = |B(0, 10)|$. \square

Lemma 3.5 *There exists a constant C_3 , that only depends on the dimension d , such that for all real $\alpha > 0$, the following inequality holds:*

$$P(\tilde{H}(\alpha)) \leq \lambda C_3 \int_\alpha^{+\infty} r^d \nu(dr).$$

Proof. Let α be a positive real. We have:

$$\begin{aligned}
P(\tilde{H}(\alpha)) &\leq E(\text{card}(\{c \in \chi \cap B(0, 100\alpha) : r(c) \geq \alpha\})) \\
&\leq \lambda |B(0, 100\alpha)| \nu([\alpha, +\infty[) \\
&\leq \lambda |B(0, 100)| \int_\alpha^{+\infty} \alpha^d \nu(dr) \\
&\leq \lambda |B(0, 100)| \int_\alpha^{+\infty} r^d \nu(dr).
\end{aligned}$$

The inequality stated in the lemma is therefore fulfilled with $C_3 = |B(0, 100)|$. \square

The following lemma will enable us to make sure that π is small enough on a sufficiently large set. This will give (7).

Lemma 3.6 *There exists a constant C_4 , that only depends on the dimension d , such that for all real $\alpha > 0$, the following inequality holds:*

$$\pi(\alpha) \leq \lambda C_4 \alpha^d.$$

Proof. Let $\alpha > 0$. Notice that, if $B(0, 10\alpha) \cap \chi$ is empty, then $\Sigma(B(0, 10\alpha))$ is empty and therefore the event $G(0, \alpha)$ can not occur. As a consequence:

$$\begin{aligned}
P(G(0, \alpha)) &\leq P(B(0, 10\alpha) \cap \chi \neq \emptyset) \\
&\leq E(\text{card}(B(0, 10\alpha) \cap \chi)) \\
&\leq \lambda |B(0, 10\alpha)|.
\end{aligned}$$

The inequality stated in the lemma is therefore satisfied with $C_4 = |B(0, 10)|$. \square

Proof of Proposition 3.1. It is a consequence of Lemmas 3.2, 3.3, 3.4 and 3.5 and 3.6. \square

3.2 Proof of the existence of subcritical behaviours

We need the following lemma:

Lemma 3.7 *Let f and g be two measurable, bounded and nonnegative functions from $[1, +\infty]$ to \mathbb{R}^+ . We assume that f is bounded by $1/2$ on $[1, 10]$ and that g is bounded by $1/4$ on $[1, +\infty]$. We also assume that, for all real $\alpha \geq 10$, the following inequality holds:*

$$f(\alpha) \leq f(\alpha/10)^2 + g(\alpha). \quad (9)$$

Under those conditions, if $g(\alpha)$ converges to 0 as α tends to infinity, then $f(\alpha)$ converges to 0 as α tends to infinity. If, moreover, a real number s is such that the integral $\int_1^{+\infty} \alpha^s g(\alpha) d\alpha$ is finite, then the integral $\int_1^{+\infty} \alpha^s f(\alpha) d\alpha$ is also finite.

Proof. • We assume that g converges to 0. As f is bounded by $1/2$ on $[1, 10]$ and g is bounded by $1/4$ on $[1, +\infty]$, we get by (9) that f is bounded by $1/2$ on $[1, +\infty]$. Therefore, for all real $\alpha \geq 10$, we have:

$$f(\alpha) \leq f(\alpha/10)/2 + g(\alpha).$$

As a consequence, for all real $\alpha \in [1, 10]$ and all integer $n \geq 1$, the following inequality holds:

$$\begin{aligned} f(10^n \alpha) &\leq f(\alpha)/2^n + g(10\alpha)/2^{n-1} + \dots + g(10^n \alpha) \\ &\leq 1/2^{n+1} + g(10\alpha)/2^{n-1} + \dots + g(10^n \alpha). \end{aligned}$$

As g is bounded and converges to 0, we get that f converges to 0.

• Let s be a real number. We furthermore assume that the integral $\int_1^{+\infty} \alpha^s g(\alpha) d\alpha$ is finite. By the first step, we know that f converges to 0. There therefore exists a real $A \geq 10$, that we fix, such that $f(\alpha)$ is bounded by $10^{-s-1}/2$ on $[A/10, +\infty[$. For all real $r \geq A$, we get, by (9):

$$\begin{aligned} \int_A^r f(\alpha) \alpha^s d\alpha &\leq \int_A^r f(\alpha/10)^2 \alpha^s d\alpha + \int_A^r g(\alpha) \alpha^s d\alpha \\ &\leq 10^{s+1} \int_{A/10}^{r/10} f(\alpha)^2 \alpha^s d\alpha + \int_A^{+\infty} g(\alpha) \alpha^s d\alpha \\ &\leq 1/2 \int_{A/10}^{r/10} f(\alpha) \alpha^s d\alpha + \int_A^{+\infty} g(\alpha) \alpha^s d\alpha \\ &\leq 1/2 \int_A^r f(\alpha) \alpha^s d\alpha + 1/2 \int_{A/10}^A f(\alpha) \alpha^s d\alpha + \int_A^{+\infty} g(\alpha) \alpha^s d\alpha. \end{aligned}$$

We therefore get:

$$\int_A^r f(\alpha) \alpha^s d\alpha \leq \int_{A/10}^A f(\alpha) \alpha^s d\alpha + 2 \int_A^{+\infty} g(\alpha) \alpha^s d\alpha.$$

As a consequence, the integral $\int_A^{+\infty} f(\alpha) \alpha^s d\alpha$ is finite. □

The following result gives one direction in the equivalences stated in the theorems.

Lemma 3.8 *There exists a positive constant C , that depend only on the dimension d , such that the following assertions hold for all probability measure ν on $]0, +\infty[$:*

1. *If $E(R^d)$ is finite, then $P_{\lambda,\nu}(\text{Percolation}) = 0$ for all positive real $\lambda < C(E(R^d))^{-1}$.*
2. *For all positive real number $s > 0$, if $E(R^{d+s})$ is finite, then $E_{\lambda,\nu}(M^s)$ is finite for all positive real $\lambda < C(E(R^d))^{-1}$.*

Proof. Let \tilde{C} be the constant given by Proposition 3.1. We define a constant C by:

$$C = (4\tilde{C}^2)^{-1}.$$

• Let ν be a probability measure on $]0, +\infty[$. In all the proof we assume that $E(R^d)$ is finite. We set:

$$\lambda_0 = C(E(R^d))^{-1}.$$

Let λ be a positive real such that $\lambda < \lambda_0$.

Let us define a positive real number A by:

$$A = (E(R^d))^{1/d}/10.$$

Let $f : [1, +\infty[$ be the function defined by:

$$f(\alpha) = \tilde{C}\pi(A\alpha)$$

and $g : [1, +\infty[$ be the function defined by:

$$g(\alpha) = \lambda\tilde{C}^2 \int_{A\alpha/10}^{\infty} r^d \nu(dr).$$

As $\lambda < \lambda_0$, we get, by (7), that f is bounded by $1/2$ on $[1, 10]$. As $\lambda < \lambda_0$, we get that g is bounded by $1/4$ on $[1, +\infty[$. As $E(R^d)$ is finite, the function g converges to 0. By (5), we get that (9) holds for all $\alpha \geq 10$. By Lemma 3.7 we therefore get that f , and then π converges to 0. By (6), we then get that M is almost surely finite. Therefore, almost surely, percolation does not occur.

• Let $s > 0$. In this step, we assume furthermore that $E(R^{d+s})$ is finite. The integral $\int_1^{+\infty} \alpha^{s-1} g(\alpha) d\alpha$ is therefore finite. By Lemma 3.7, we get that the integral $\int_1^{+\infty} \alpha^{s-1} f(\alpha) d\alpha$ is also finite. By (6), we then get that the integral $\int_1^{+\infty} \alpha^{s-1} P_{\lambda,\nu}(M \geq 9A\alpha) d\alpha$ is also finite. As a consequence, the moment $E_{\lambda,\nu}(M^s)$ is finite. \square

3.3 Proof of non-existence of subcritical behaviours

In the following lemma, we give the other direction of the equivalences stated in the theorems. Recall that Σ is defined by (1) and that M is defined by (4).

Lemma 3.9 *Let ν be a probability measure on $]0, +\infty[$. If $E(R^d)$ is infinite then, for all $\lambda > 0$, we have $P_{\lambda,\nu}$ -almost surely $\Sigma = \mathbb{R}^d$. If $s > 0$ is a real such that $E(R^{d+s})$ is infinite then, for all $\lambda > 0$, $E_{\lambda,\nu}(M^s)$ is infinite.*

Proof. Let ν be a probability measure on $]0, +\infty[$ and λ a positive real.

- We first prove that, for all real $r > 0$, the following inequality holds:

$$P_{\lambda,\nu}(\exists c \in \chi : B(0, r) \subset B(c, r(c))) \geq 1 - \exp \left(-\lambda 2^{-d} |B(0, 1)| \int_{[2r, +\infty[} \alpha^d \nu(d\alpha) \right). \quad (10)$$

Let $r > 0$. We have:

$$\begin{aligned} P_{\lambda,\nu}(\exists c \in \chi : B(0, r) \subset B(c, r(c))) &= 1 - \exp \left(-\lambda \int_{\mathbb{R}^d} P(R \geq \|x\| + r) dx \right) \\ &= 1 - \exp \left(-\lambda E(|B(0, R - r)| 1_{R \geq r}) \right) \\ &\geq 1 - \exp \left(-\lambda E(|B(0, R - r)| 1_{R \geq 2r}) \right) \\ &\geq 1 - \exp \left(-\lambda E(|B(0, R/2)| 1_{R \geq 2r}) \right). \end{aligned}$$

The relation (10) is proved.

- If $E(R^d)$ is infinite then, by (10), we get, for all $r > 0$:

$$P_{\lambda,\nu}(\exists c \in \chi : B(0, r) \subset B(c, r(c))) = 1.$$

Therefore, almost surely, we have $\Sigma = \mathbb{R}^d$.

- Let $s > 0$. We assume now that $E(R^{d+s})$ is infinite. If $E(R^d)$ is infinite, the desired result is a consequence of what we have proved in the previous step. We assume henceforth that $E(R^d)$ is finite. Let C be defined by:

$$C = \lambda 2^{-d} |B(0, 1)| \int_{[0, +\infty[} \alpha^d \nu(d\alpha).$$

This constant is finite. By (10) we get, for all $r > 0$, the following inequality:

$$P_{\lambda,\nu}(\exists c \in \chi : B(0, r) \subset B(c, r(c))) \geq C^{-1} (1 - \exp(-C)) \lambda 2^{-d} |B(0, 1)| \int_{[2r, +\infty[} \alpha^d \nu(d\alpha).$$

As $E(R^{d+s})$ is infinite, the integral

$$\int_0^{+\infty} \left(r^{s-1} \int_{2r}^{+\infty} \alpha^d \nu(d\alpha) \right) dr$$

is infinite. Therefore the integral

$$\int_0^{+\infty} r^{s-1} P_{\lambda,\nu}(\exists c \in \chi : B(0, r) \subset B(c, r(c))) dr$$

is infinite. As a consequence, the integral $\int_0^{+\infty} r^{s-1} P_{\lambda,\nu}(M \geq r) dr$ is infinite. The moment $E_{\lambda,\nu}(M^s)$ is then infinite. \square

3.4 Proof of the theorems

Proof of Theorems 2.1 and 2.2. This is a consequence of Lemmas 3.8 and 3.9. \square

Remark. With Lemma 3.8 and with the proof of Lemma 3.9 we could also give characterizations of the integrability of the volume of S or of the radius of the largest ball centered at the origin and included in S .

References

- [1] Bartłomiej Błaszczyszyn, Christian Rau, and Volker Schmidt. Bounds for clump size characteristics in the Boolean model. *Adv. in Appl. Probab.*, 31(4):910–928, 1999.
- [2] Peter Hall. On continuum percolation. *Ann. Probab.*, 13(4):1250–1266, 1985.
- [3] Peter Hall. *Introduction to the theory of coverage processes*. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons Inc., New York, 1988.
- [4] Olav Kallenberg. *Random measures*. Akademie-Verlag, Berlin, fourth edition, 1986.
- [5] Ronald Meester and Rahul Roy. *Continuum percolation*, volume 119 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1996.
- [6] Jesper Møller. *Lectures on random Voronoï tessellations*. Springer-Verlag, New York, 1994.
- [7] J. Neveu. Processus ponctuels. In *École d’Été de Probabilités de Saint-Flour, VI—1976*, pages 249–445. Lecture Notes in Math., Vol. 598. Springer-Verlag, Berlin, 1977.