



**HAL**  
open science

# Regularities of the distribution of beta-adic van der Corput sequences

Wolfgang Steiner

► **To cite this version:**

Wolfgang Steiner. Regularities of the distribution of beta-adic van der Corput sequences. Monatshefte für Mathematik, 2006, 149, pp.67-81. 10.1007/s00605-005-0370-8 . hal-00113240

**HAL Id: hal-00113240**

**<https://hal.science/hal-00113240>**

Submitted on 11 Nov 2006

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# REGULARITIES OF THE DISTRIBUTION OF $\beta$ -ADIC VAN DER CORPUT SEQUENCES

WOLFGANG STEINER\*

ABSTRACT. For Pisot numbers  $\beta$  with irreducible  $\beta$ -polynomial, we prove that the discrepancy function  $D(N, [0, y))$  of the  $\beta$ -adic van der Corput sequence is bounded if and only if the  $\beta$ -expansion of  $y$  is finite or its tail is the same as that of the expansion of 1. If  $\beta$  is a Parry number, then we can show that the discrepancy function is unbounded for all intervals of length  $y \notin \mathbb{Q}(\beta)$ . We give explicit formulae for the discrepancy function in terms of lengths of iterates of a reverse  $\beta$ -substitution.

## 1. INTRODUCTION

Let  $(x_n)_{n \geq 0}$  be a sequence with  $x_n \in [0, 1)$  and

$$D(N, I) = \#\{0 \leq n < N : x_n \in I\} - N\lambda(I)$$

its *discrepancy function* on the interval  $I$ , where  $\lambda(I)$  denotes the length of the interval. Then  $(x_n)_{n \geq 0}$  is uniformly distributed if and

---

*Date:* November 11, 2006.

2000 *Mathematics Subject Classification.* 11K31, 11K16, 37B10.

*Key words and phrases.* discrepancy, van der Corput sequence,  $\beta$ -expansion, substitution.

This work was supported by the Austrian Science Foundation, grant S8302-MAT.

\*Institute of Discrete Mathematics and Geometry, Vienna University of Technology, Wiedner Hauptstraße 8-10/104, 1040 Vienna, Austria.

only if  $D(N, I) = o(N)$  for all intervals  $I \subseteq [0, 1)$ . Van Aardenne-Ehrenfest [25] proved that the discrepancy function cannot be bounded (in  $N$ ) for all intervals  $I \subseteq [0, 1)$ . W.M. Schmidt showed in [23] that the set of lengths of intervals with bounded discrepancy function,  $\{\lambda(I) : \sup_{N \geq 0} D(N, I) < \infty\}$ , is at most countable and in [22] that  $\sup_{I \subseteq [0, 1)} D(N, I) \geq C \log N$  for some constant  $C > 0$ . For more details on the discrepancy, see Drmota and Tichy [4].

For some special sequences, the intervals with bounded discrepancy function were determined. If  $x_n = \{n\alpha\}$ , then  $D(N, I)$  is bounded if and only if  $\lambda(I) = \{m\alpha\}$  for some  $m \geq 0$  (Hecke [10] and Kesten [13]). More generally, Rauzy [18] and Ferenczi [8] characterized bounded remainder sets for irrational rotations on the torus  $\mathbb{T}^s$ . Liardet [14] extended Hecke's and Kesten's result on these rotations and studied bounded remainder sets for  $x_n = \{p(n)\}$ , where  $p(n)$  is a real polynomial with irrational leading coefficient, as well as for  $q$ -multiplicative sequences.

If  $(x_n)_{n \geq 0}$  is the van der Corput sequence in base  $q$ , then  $D(N, I)$  is bounded if and only if  $\lambda(I)$  has finite  $q$ -ary expansion (W.M. Schmidt [23] and Shapiro [24] for  $q = 2$ , Hellekalek [11] for integers  $q \geq 2$ ). Faure extended this result in [6] on generalized van der Corput sequences and recently in [7] on digital  $(0, 1)$ -sequences over  $\mathbb{Z}_q$  generated by a nonsingular upper triangular matrix where  $q$  is a prime number (see also Drmota, Larcher and Pillichshammer [3]). Hellekalek [12] considered generalizations of the Halton sequences in higher dimensions.

The aim of this article is to determine the intervals with bounded discrepancy function for the  $\beta$ -adic van der Corput sequences, which were introduced by Ninomiya [15] who proved that these sequences are low discrepancy sequences, i.e.  $\sup_{I \subseteq [0,1)} D(N, I) = \mathcal{O}(\log N)$ , if  $\beta$  is a Pisot number with irreducible  $\beta$ -polynomial.

For a given real number  $\beta > 1$ , the *expansion of 1* with respect to  $\beta$  is the sequence of nonnegative integers  $(a_j)_{j \geq 1}$  satisfying

$$1 = .a_1 a_2 \dots = \frac{a_1}{\beta} + \frac{a_2}{\beta^2} + \dots \text{ with } a_j a_{j+1} \dots < a_1 a_2 \dots \text{ for all } j \geq 2$$

(Throughout this article, let  $<$  denote the lexicographical order for words.) For  $x \in [0, 1)$ , the  $\beta$ -*expansion* of  $x$ , introduced by Rényi [19] and characterized by Parry [16], is given by

$$x = .\epsilon_1 \epsilon_2 \dots = \frac{\epsilon_1}{\beta} + \frac{\epsilon_2}{\beta^2} + \dots \text{ with } \epsilon_j \epsilon_{j+1} \dots < a_1 a_2 \dots \text{ for all } j \geq 1.$$

The elements of the  $\beta$ -*adic van der Corput sequence*  $(x_n)_{n \geq 0}$  are the real numbers  $x \in [0, 1)$  with finite  $\beta$ -expansion,

$$\{x_n : n \geq 0\} = \{.\epsilon_1 \epsilon_2 \dots :$$

$$\epsilon_j \epsilon_{j+1} \dots < a_1 a_2 \dots \text{ for all } j \geq 1, \epsilon_\ell \epsilon_{\ell+1} \dots = 0^\infty \text{ for some } \ell \geq 1\},$$

ordered lexicographically with respect to the (inversed) word  $\dots \epsilon_2 \epsilon_1$ , i.e. for  $x_n = .\epsilon_1 \epsilon_2 \dots$  and  $x_{n'} = .\epsilon'_1 \epsilon'_2 \dots$ , we have  $n < n'$  if we have some  $k \geq 1$  such that  $\epsilon_k < \epsilon'_k$  and  $\epsilon_j = \epsilon'_j$  for all  $j > k$ .

If the expansion of 1 is finite,  $a_1 a_2 \dots = a_1 \dots a_d 0^\infty$ , or eventually periodic,  $a_1 a_2 \dots = a_1 \dots a_{d-p} (a_{d-p+1} \dots a_d)^\infty$ , then  $\beta$  is a *Parry number* and it is the dominant root of the  $\beta$ -*polynomial*  $x^d - a_1 x^{d-1} - \dots - a_d$

(with  $a_d > 0$ ) and  $(x^d - a_1x^{d-1} - \dots - a_d) - (x^{d-p} - a_1x^{d-p-1} - \dots - a_{d-p})$  (where  $p$  is assumed to be minimal) respectively. In this case, we obtain results for the discrepancy function.

**Theorem 1.** *If  $\beta$  is a Parry number and  $D(N, I)$  is bounded (in  $N$ ), then  $\lambda(I) \in \mathbb{Q}(\beta)$ .*

Bertrand [1] and K. Schmidt [21] proved that all *Pisot numbers* (algebraic integers for which all algebraic conjugates have modulus  $< 1$ ) are Parry numbers. If furthermore the  $\beta$ -polynomial is the minimal polynomial of  $\beta$ , then we can completely characterize the intervals  $[0, y)$  with bounded discrepancy function.

**Theorem 2.** *If  $\beta$  is a Pisot number with irreducible  $\beta$ -polynomial, then  $D(N, [0, y))$  is bounded (in  $N$ ) for  $y \in [0, 1)$  if and only if the  $\beta$ -expansion of  $y$  is finite or its tail is the same as that of the expansion of 1 with respect to  $\beta$ , i.e. if  $y = .y_1y_2\dots$  with  $y_ky_{k+1}\dots = 0^\infty$  or  $y_ky_{k+1}\dots = (a_{d-p+1}\dots a_d)^\infty$  for some  $k \geq 1$ .*

*Remark.* Another way to formulate the condition on  $y$  is: the infinite  $\beta$ -expansion of  $y$  has the same tail as the infinite expansion of 1 (which is  $1 = .(a_1\dots a_{d-1}(a_d - 1))^\infty$  if  $1 = .a_1\dots a_d$ ).

The classification for general intervals  $I$  seems to be more difficult. Of course,  $D(N, [y, y'))$  is bounded if  $D(N, [0, y))$  and  $D(N, [0, y'))$  are bounded because of  $D(N, [y, y')) = D(N, [0, y')) - D(N, [0, y))$ . From the proof of Theorem 2 we see that  $D(N, [y, y'))$  is bounded if  $y = .y_1y_2\dots$  and  $y' = .y'_1y'_2\dots$  with  $y_ky_{k+1}\dots = y'_ky'_{k+1}\dots$  for some  $k \geq 1$ .

The boundedness of  $D(N, I)$  is not necessarily invariant under translation of the interval. E.g. for  $1 = .31^\infty$ ,  $D(N, [0, .1^\infty))$  is bounded, but  $D(N, [.1^\infty, .2^\infty))$  is unbounded. It is also possible that  $D(N, [y, y'))$  is bounded and  $D(N, [0, y' - y))$  is unbounded:  $D(N, [.02, 1))$  is bounded and  $D(N, [0, 1 - .02)) = D(N, [0, .2^\infty))$  is unbounded.

This article is organised as follows. In Section 2 we recapitulate some facts about number systems defined by substitutions (due to Dumont and Thomas [5]) and define a reverse  $\beta$ -substitution which determines  $x_n$ . Theorem 1 is proved in Section 3 similarly to Shapiro [24]. The remaining parts of Theorem 2 are proved in Section 4, where explicit formulae for the discrepancy function in terms of lengths of iterates of the reverse  $\beta$ -substitution are given.

## 2. NUMBER SYSTEMS DEFINED BY SUBSTITUTIONS

**2.1. Generalities.** Let  $\sigma$  be a substitution on the alphabet  $\mathcal{A} = \{1, \dots, d\}$ , i.e. a mapping from  $\mathcal{A}$  into the set of nonempty finite words on  $\mathcal{A}$ , which is extended to a mapping on words by concatenation,  $\sigma(w w') = \sigma(w)\sigma(w')$ . A sequence of words  $m_k, \dots, m_1$  is called  $\sigma$ -*b-admissible* if we have a companion sequence of letters  $b_j$  with  $b_{k+1} = b$  such that  $m_j b_j \leq_p \sigma(b_{j+1})$  for all  $j \leq k$  (where  $w \leq_p w'$  means that  $w$  is a prefix of  $w'$ ). For a given sequence  $m_k, \dots, m_1$ , clearly the sequence  $b_k, \dots, b_1$  is unique.

If  $\sigma(1) = 1w$  for some word  $w$ , then the limit  $\sigma^\infty(1) = \lim_{k \rightarrow \infty} \sigma^k(1)$  exists because of  $\sigma^{k+1}(1) = \sigma^k(1w) = \sigma^k(1)\sigma^k(w)$  and we have

$$(1) \quad \sigma^{k-1}(m_k) \dots \sigma^0(m_1) \leq_p \sigma^k(1) \leq_p \sigma^\infty(1)$$

for all  $\sigma$ -1-admissible sequences  $m_k, \dots, m_1$ . Furthermore, every prefix  $u_1 \dots u_n \leq_p \sigma^\infty(1)$ ,  $n \geq 1$ , can be written as the left hand side of (1) with a unique  $\sigma$ -1-admissible sequence  $m_k, \dots, m_1$  with  $|m_k| > 0$  (where  $|m|$  denotes the length of  $m$ ). Denote these  $m_j$  by  $m_{j,\sigma}(n)$  and set  $m_{j,\sigma}(n) = \varepsilon$  (the empty word) for all  $j > k$ . For  $n = 0$ , set  $m_{j,\sigma}(0) = \varepsilon$  for all  $j \geq 1$ . Then

$$n = \sum_{j=1}^{\infty} |\sigma^{j-1}(m_{j,\sigma}(n))| = \sum_{j=1}^{\infty} \sum_{b=1}^d |m_{j,\sigma}(n)|_b |\sigma^{j-1}(b)|,$$

where  $|m|_b$  denotes the number of  $b$ 's in  $m$ . If  $m_{j,\sigma}(n') = m_{j,\sigma}(n)$  for all  $j > k$  and  $|m_{k,\sigma}(n')| > |m_{k,\sigma}(n)|$ , i.e.  $m_{k,\sigma}(n') = m_{k,\sigma}(n)b_jw$  for some word  $w$ , then  $\sigma^{k-2}(m_{k-1,\sigma}(n)) \dots \sigma^0(m_{1,\sigma}(n))$  is a strict prefix of  $\sigma^{k-1}(b_k)$ , hence  $\sum_{j=1}^{k-1} |\sigma^{j-1}m_{j,\sigma}(n)| < \sigma^{k-1}(b_k)$  and we have

$$n' \geq \sum_{j=k}^{\infty} |\sigma^{j-1}(m_{j,\sigma}(n'))| \geq \sum_{j=k}^{\infty} |\sigma^{j-1}(m_{j,\sigma}(n))| + |\sigma^{k-1}(b_k)| > n,$$

thus

$$(2) \quad n < n' \text{ if } \dots |m_{2,\sigma}(n)| |m_{1,\sigma}(n)| < \dots |m_{2,\sigma}(n')| |m_{1,\sigma}(n')|$$

2.2.  **$\beta$ -substitution.** If  $\beta$  is a Parry number, then the  $\beta$ -substitution  $\sigma$  is defined by

$$\sigma(b) = \begin{cases} 1^{a_b}(b+1) & \text{if } 1 \leq b < d \\ 1^{a_d} & \text{if } b = d, 1 = .a_1 \dots a_d \\ 1^{a_d}(d-p+1) & \text{if } b = d, 1 = .a_1 \dots a_{d-p}(a_{d-p+1} \dots a_d)^\infty \end{cases}$$

(where  $1^{a_j}$  denotes the concatenation of  $a_j$  letters 1).

If we set  $G_k = |\sigma^k(1)|$  for all  $k \geq 0$ , then

$$G_k = \sum_{j=1}^k a_j G_{k-j} + \begin{cases} 1 & \text{if } a_j = 0 \text{ for all } j > k \\ 0 & \text{else} \end{cases}$$

(in particular  $G_k = \sum_{j=1}^d a_j G_{k-j}$  if  $1 = .a_1 \dots a_d$  and  $k > d$ ) and

$$n = \sum_{j=1}^{\infty} |m_{j,\sigma}(n)| |\sigma^{j-1}(1)| = \sum_{j=1}^{\infty} |m_{j,\sigma}(n)| G_{j-1}$$

since the words  $m_{j,\sigma}(n)$  consist only of ones. Thus the  $|m_{j,\sigma}(n)|$  are the digits in the  $G$ -ary expansion of  $n$  with  $G = (G_j)_{j \geq 0}$  and the  $\sigma$ -1-admissible sequences  $m_k, \dots, m_1$  are exactly those sequences consisting only of ones with  $|m_j| \dots |m_1| 0^\infty < a_1 a_2 \dots$  for all  $j \leq k$ .

*Example.* If  $1 = .402$ , then

$$\sigma(1) = 11112, \quad \sigma(2) = 3, \quad \sigma(3) = 11.$$

An example of a  $\sigma$ -1-admissible sequence with  $k = 5$  is

$$(m_5, b_5), \dots, (m_1, b_1) = (11, 1), (1111, 2), (\varepsilon, 3), (\varepsilon, 1), (1, 1)$$

which corresponds to

$$n = |\sigma^4(11)\sigma^3(1111)\sigma^2(\varepsilon)\sigma(\varepsilon)1| = 2G_4 + 4G_3 + 1 = 1053.$$



**2.3. Reverse  $\beta$ -substitution.** For a Parry number  $\beta$ , set  $t_1 = 0^\infty$  and let  $\{t_2, \dots, t_{d+1}\}$  be the set of words  $\{a_j a_{j+1} \dots : j \geq 2\}$  with

$$0^\infty = t_1 < t_2 < \dots < t_d < t_{d+1} = a_1 a_2 \dots$$

For  $1 \leq b \leq d$  set

$$\tau(b) = \begin{cases} u_0(b) \dots u_{a_1}(b) & \text{if } a_1 t_b < a_1 a_2 \dots \\ u_0(b) \dots u_{a_1-1}(b) & \text{else} \end{cases}$$

with

$$u_j(b) = b' \text{ if } t_{b'} \leq j t_b < t_{b'+1}.$$

We clearly have  $u_0(1) = 1$ , thus  $\tau^\infty(1)$  exists and every  $n \geq 1$  corresponds to a unique  $\tau$ -1-admissible sequence  $m_k, \dots, m_1$  with  $|m_k| > 0$ .

The following example and proposition show (for  $b = 1$ ) that the possible sequences of “digits”  $|m_{j,\tau}(n)|$  are the same as for  $|m_{j,\sigma}(n)|$ , but in reversed order. Therefore we call  $\tau$  *reverse  $\beta$ -substitution*.

*Example.* For  $1 = .402$ , we have  $t_1 = 0^\infty$ ,  $t_2 = 020^\infty$ ,  $t_3 = 20^\infty$ ,  $t_4 = 4020^\infty$ , thus

$$\tau(1) = 12333, \quad \tau(2) = 1233, \quad \tau(3) = 2233.$$

We have a  $\tau$ -1-admissible sequence with  $|m_5| \dots |m_1| = 10042$ ,

$$(m_5, b_5), \dots, (m_1, b_1) = (1, 2), (\varepsilon, 1), (\varepsilon, 1), (1233, 3), (22, 3)$$

which corresponds to

$$n = |\tau^4(1)\tau^3(\varepsilon)\tau^2(\varepsilon)\tau(1233)22| = G_4 + 19 = 373.$$

**Proposition 1.** *Each  $\tau$ - $b$ -admissible sequence  $m_k, \dots, m_1$  satisfies*

$$(3) \quad |m_j| \dots |m_k| t_b < a_1 a_2 \dots \text{ for all } j \leq k.$$

*Conversely, for each sequence  $\epsilon_1 \dots \epsilon_k$  with  $\epsilon_j \dots \epsilon_k t_b < a_1 a_2 \dots$  for all  $j \geq 1$ , we have a (unique)  $\tau$ - $b$ -admissible sequence  $m_k, \dots, m_1$  with  $|m_1| \dots |m_k| = \epsilon_1 \dots \epsilon_k$ .*

*Proof.* Assume first that  $m_k, \dots, m_1$  is  $\tau$ - $b$ -admissible and let  $b_k, \dots, b_1$  be its companion sequence ( $m_j b_j \leq_p \tau(b_{j+1}), b_{k+1} = b$ ). Assume further

$$|m_j| \dots |m_{\ell-1}| = a_1 \dots a_{\ell-j} \text{ and } t_{b_\ell} < a_{\ell-j+1} a_{\ell-j+2} \dots$$

(which is trivially true for  $j = \ell$ ). We have  $b_\ell = u_{|m_\ell|}(b_{\ell+1})$ , hence

$$|m_\ell| t_{b_{\ell+1}} < t_{b_{\ell+1}} \leq a_{\ell-j+1} a_{\ell-j+2} \dots$$

This implies  $|m_j| \dots |m_\ell| < a_1 \dots a_{\ell-j+1}$  or

$$|m_j| \dots |m_\ell| = a_1 \dots a_{\ell-j+1} \text{ and } t_{b_{\ell+1}} < a_{\ell-j+2} a_{\ell-j+3} \dots$$

In the latter case, we proceed inductively and obtain

$$|m_j| \dots |m_k| t_{b_{k+1}} = |m_j| \dots |m_k| t_b < a_1 a_2 \dots$$

Hence, (3) is proved.

For the converse, assume  $\epsilon_j \dots \epsilon_k t_b < a_1 a_2 \dots$  for all  $j \geq 1$  and

$$t_{b_{\ell+1}} \leq \epsilon_{\ell+1} t_{b_{\ell+2}} \text{ for all } \ell \in \{j+1, \dots, k\}$$

(which is trivially true for  $j = k$ ). Then we have

$$\epsilon_j t_{b_{j+1}} \leq \epsilon_j \epsilon_{j+1} t_{b_{j+2}} \leq \dots \leq \epsilon_j \dots \epsilon_k t_{b_{k+1}} = \epsilon_j \dots \epsilon_k t_b < a_1 a_2 \dots,$$

thus  $b_j = u_{\epsilon_j}(b_{j+1})$  exists and  $m_j = u_0(b_{j+1}) \dots u_{\epsilon_{j-1}}(b_{j+1})$ . Furthermore, we have  $t_{b_j} \leq \epsilon_j t_{b_{j+1}}$  and obtain, by induction, a (unique)  $\tau$ - $b$ -admissible sequence  $m_k, \dots, m_1$  with  $|m_1| \dots |m_k| = \epsilon_1 \dots \epsilon_k$ .  $\square$

By Proposition 1 ( $b = 1$ ), every finite  $\beta$ -expansion  $\epsilon_1 \dots \epsilon_k 0^\infty$  corresponds to some  $n < |\tau^k(1)|$  such that  $\epsilon_1 \dots \epsilon_k = |m_{1,\tau}(n)| \dots |m_{k,\tau}(n)|$ . By (2), we have  $n < n'$  for  $n, n' < |\tau^k(1)|$  if

$$\epsilon_k \dots \epsilon_1 = |m_{k,\tau}(n)| \dots |m_{1,\tau}(n)| < |m_{k,\tau}(n')| \dots |m_{1,\tau}(n')| = \epsilon'_k \dots \epsilon'_1.$$

Therefore the  $\beta$ -adic van der Corput sequence is given by

$$x_n = \sum_{j=1}^{\infty} |m_{j,\tau}(n)| \beta^{-j}.$$

Note that we have  $|\tau^k(1)| = |\sigma^k(1)| = G_k$  for all  $k \geq 0$ .

### 3. PROOF OF THEOREM 1

Let  $\mathcal{D}$  be the set of all sequences  $(m_j, b_j)_{j \geq 1}$  of words  $m_j$  and letters  $b_j$  with  $m_j b_j \leq_p \tau(b_{j+1})$  for all  $j \geq 1$ . Set

$$\delta((m_j, b_j)_{j \geq 1}, (m'_j, b'_j)_{j \geq 1}) = 1/k$$

if  $(m_j, b_j) = (m'_j, b'_j)$  for all  $j < k$  and  $(m_j, b_j) \neq (m'_j, b'_j)$ . Then  $\mathcal{D}$  is a compact metric space with the metric  $\delta$ .

In order to extend the addition of 1 in the number system defined by  $\tau$ ,  $(m_{j,\tau}(n))_{j \geq 1} \mapsto (m_{j,\tau}(n+1))_{j \geq 1}$ , define the *successor function* (or

odometer or *adic transformation*) on  $\mathcal{D}$  by

$$S((m_j, b_j)_{j \geq 1}) = (m'_j, b'_j)_{j \geq 1} \text{ with } (m'_j, b'_j) = \begin{cases} (m_j, b_j) & \text{if } j > k \\ (m_k b_k, b'_k) & \text{if } j = k \\ (\varepsilon, u_0(b'_{j+1})) & \text{if } j < k \end{cases}$$

where  $k \geq 1$  is the smallest integer such that  $\tau(b_{k+1}) = m_k b_k b'_k w$  for some letter  $b'_k$  and some word  $w$ . If  $(m_j, b_j)_{j \geq 1}$  is a maximal sequence, i.e.  $m_k b_k = \tau(b_{k+1})$  for all  $k \geq 1$ , then let its successor be the (unique) minimal sequence  $(\varepsilon, 1), (\varepsilon, 1), \dots$

If the maximal sequence is unique, then  $S$  is a homeomorphism and  $(\mathcal{D}, S)$  is a transformation group, but in many cases the maximal sequence is not unique. In particular if  $a_2 a_3 \dots > (a_1 - 1)^\infty$ , then every maximal sequence satisfies  $|m_j| = a_1$ ,  $|m_{j'}| = a_1 - 1$  for some  $j, j' \geq 1$ , and we obtain a different maximal sequence by shifting this sequence. Hence  $(\mathcal{D}, S)$  is only a transformation semigroup.

Define a continuous function  $f : \mathcal{D} \rightarrow [0, 1]$  by

$$f((m_j, b_j)_{j \geq 1}) = \sum_{j=1}^{\infty} |m_j| \beta^{-j}.$$

Then we have  $x_n = f(S^n((\varepsilon, 1), (\varepsilon, 1), \dots))$ . If  $S$  is invertible, then  $(x_0, x_1, \dots)$  can be extended to a bisequence  $(x_n)_{n \in \mathbb{Z}}$  by this definition.

Let  $X$  denote the orbit closure of  $(x_0, x_1, \dots)$  under the shift  $T$ , and define  $\varphi : \mathcal{D} \rightarrow X$  by

$$(\varphi((m_j, b_j)_{j \geq 1}))_k = f(S^k((m_j, b_j)_{j \geq 1}))$$

Then  $\varphi$  is a homeomorphism and  $\varphi \circ S = T \circ \varphi$ . Hence the transformation (semi)group  $(X, T)$  is isomorphic to  $(\mathcal{D}, S)$ . If  $S$  is invertible,

then  $(X, T)$  is minimal by Theorem 2.2 of Shapiro [24] and we can apply Theorem 5.1 of this article, which states that  $\exp(2\pi i\lambda(I))$  is an eigenvalue of  $T$  and thus of  $S$  if  $D(N, I)$  is bounded. Lemma 1 shows that Shapiro's proof is valid for our transformation semigroup as well.

By Théorème 5.2 of Canterini and Siegel [2], we have a continuous and surjective “desubstitution map”  $\Gamma : \Omega \rightarrow \mathcal{D}$ , where  $\Omega$  is the set of biinfinite words which have the same language as  $\tau^\infty(1)$ . Let  $\Delta$  be the shift on  $\Omega$ . By Théorème 5.1 of this article and since the minimal sequence in  $\mathcal{D}$  is unique, we have  $S \circ \Gamma = \Gamma \circ \Delta$ . Therefore the eigenvalues of  $S$  are a subset of the eigenvalues of  $\Delta$  and, by Proposition 5 of Ferenczi, Mauduit and Nogueira [9], these eigenvalues are of the form  $\exp(2\pi iy)$  with  $y \in \mathbb{Q}(\beta)$ . This concludes the proof of Theorem 1.

*Remarks.* Ferenczi, Mauduit and Nogueira [9] gave a more precise description of the set of eigenvalues of  $\Delta$  in their Proposition 4, which is too complicated to be cited here.

For more details on the spectrum of these dynamical systems, see Chapter 7.3 in Pytheas Fogg [17], but note that the result of [9] is cited incorrectly: According to Theorem 7.3.28 of [17], the eigenvalues of  $\Delta$  associated with the trivial coboundary are in  $\exp(2\pi i\mathbb{Z}[\beta])$ , but  $\mathbb{Z}[\beta]$  should be  $\mathbb{Q}[\beta]$  and the condition on the coboundary is unnecessary. Nevertheless, the author considered the coboundary and showed that all reverse  $\beta$ -substitutions  $\tau$  have only the trivial coboundary, but the proof is rather lengthy and technical and therefore not given in this article.

**Lemma 1.** *If  $D(N, I)$  is bounded, then  $\exp(2\pi i\lambda(I))$  is an eigenvalue of  $S$ .*

*Proof.* Set

$$g((m_j, b_j)_{j \geq 1}) = \chi_I \left( \sum_{j=1}^{\infty} |m_j| \beta^{-j} \right) - \lambda(I)$$

where  $\chi_I$  denotes the indicator function of  $I$ . Let  $\omega = (m_j, b_j)_{j \geq 1}$  be a sequence with  $|m_1| |m_2| \dots = y_1 y_2 \dots$ , hence  $\sum_{j=0}^{N-1} g(S^j \omega) = D(N, I)$  is bounded. Set  $U(x, \eta) = (Sx, \eta + g(x))$  for  $x \in \mathcal{D}$ ,  $\eta \in \mathbb{R}$ . Then we have

$$U^k(x, \eta) = \left( S^k x, \eta + \sum_{j=0}^{k-1} g(S^j x) \right).$$

The positive semi-orbit  $\{U^k(\omega, 0) : k \geq 0\}$  is bounded and has therefore compact closure. Denote by  $M$  the set of limit points of this semi-orbit. Then  $M$  is nonempty, closed and invariant under  $U$  (NCI). It is easy to see that  $\{S^k x : k \geq 0\}$  is dense in  $\mathcal{D}$  for all  $x \in \mathcal{D}$ . Since  $M$  is NCI, we must therefore have some point  $(x, \eta) \in M$  for all  $x \in \mathcal{D}$ .

Below we show that, for a given  $x$ , this  $\eta$  is unique, i.e.  $\eta = \eta(x)$ . Then the graph  $(x, \eta(x))$  is the compact set  $M$ , therefore  $\eta$  is continuous. Since  $U(x, \eta(x)) = (Sx, \eta(x) + g(x))$ , we have

$$\eta(Sx) = \eta(x) + g(x),$$

$$\exp(-2\pi i\lambda(I)) = \exp(2\pi i g(x)) = \exp(2\pi i \eta(Sx)) / \exp(2\pi i \eta(x)).$$

Therefore  $K(x) = \exp(-2\pi i \eta(x))$  is a continuous function with

$$K(Sx) = \exp(2\pi i \lambda(I)) K(x)$$

and  $\exp(2\pi i\lambda(I))$  is an eigenvalue of  $S$ .

To prove that  $\eta(x)$  is unique, we show first  $\eta(\omega) = 0$ . Suppose  $(\omega, \eta) \in M$ . Since  $M$  consists of limit points of  $\{U^k(\omega, 0) : k \geq 0\}$ , we have a sequence  $k_j \rightarrow \infty$  with

$$\lim_{j \rightarrow \infty} U^{k_j}(\omega, 0) = (\omega, \eta).$$

This implies

$$\lim_{j \rightarrow \infty} S^{k_j} \omega = \omega \quad \text{and} \quad \lim_{j \rightarrow \infty} \sum_{i=0}^{k_j-1} g(S^i \omega) = \eta,$$

hence

$$\lim_{j \rightarrow \infty} U^{k_j}(\omega, \eta) = \left( \lim_{j \rightarrow \infty} S^{k_j} \omega, \eta + \lim_{j \rightarrow \infty} \sum_{i=0}^{k_j-1} g(S^i \omega) \right) = (\omega, \eta + \eta).$$

Since  $M$  is invariant, we have  $U^{k_j}(\omega, \eta) \in M$  for all  $j$  and, since  $M$  is closed,  $(\omega, 2\eta) \in M$ . Inductively we obtain  $(\omega, k\eta) \in M$  for all  $M$ , which implies  $\eta = 0$  since  $M$  is bounded.

Next suppose  $(x, \eta) \in M$  and  $(x, \eta') \in M$ . Since  $\{S^k x : k \geq 0\}$  is dense, we have some  $k_j \rightarrow \infty$  such that

$$\lim_{j \rightarrow \infty} S^{k_j} x = \omega.$$

Since  $M$  is compact, we can refine the sequence  $k_j$  so that the sequences  $U^{k_j}(x, \eta)$  and  $U^{k_j}(x, \eta')$  converge (to points in  $M$ ). Since the first coordinate of the limit points is  $\omega$ , the second coordinate must be 0 for both points. Therefore

$$\lim_{j \rightarrow \infty} \left( \eta + \sum_{\ell=0}^{k_j-1} g(S^\ell x) \right) = \lim_{j \rightarrow \infty} \left( \eta' + \sum_{\ell=0}^{k_j-1} g(S^\ell x) \right),$$

hence  $\eta = \eta'$  and we have proved that  $\eta(x)$  is unique.  $\square$

#### 4. PROOF OF THEOREM 2

Because of Theorem 1, we just have to consider  $y \in \mathbb{Q}(\beta)$  for Theorem 2, but first we compute formulae for the discrepancy function of arbitrary intervals  $[0, y)$ . Let  $A(N, I) = \#\{x_n \in I : 0 \leq n < N\}$ . Then we have, for  $y = .y_1y_2\dots$ ,

$$D(N, [0, y)) = \sum_{k=1}^{\infty} (A(N, [.y_1\dots y_{k-1}, .y_1\dots y_k)) - Ny_k\beta^{-k}).$$

**Lemma 2.** *We have*

$$A(N, [.y_1\dots y_{k-1}, .y_1\dots y_k)) = y_k \sum_{\ell=k+1}^{\infty} \sum_{b=1}^d |m_{\ell, \tau}(N)|_b |\tau^{\ell-k-1}(b)| + \mu_k(N, y)$$

with

$$\mu_k(N, y) = \begin{cases} y_k & \text{if } |m_{k, \tau}(N)| \geq y_k \\ |m_{k, \tau}(N)| + 1 & \text{if } |m_{k, \tau}(N)| < y_k, \\ |m_{k-1, \tau}(N)| \dots |m_{1, \tau}(N)| > y_{k-1} \dots y_1 & \\ |m_{k, \tau}(N)| & \text{else.} \end{cases}$$

*Proof.* For  $G_L \leq N < G_{L+1}$ , we have

$$\begin{aligned} & \{(m_{1, \tau}(n), \dots, m_{L, \tau}(n)) : 0 \leq n < N\} \\ &= \bigcup_{\ell=1}^L \bigcup_{m: mb \leq_p m_{\ell, \tau}(N)} \{(m_1, \dots, m_{\ell-1}, m, m_{\ell+1, \tau}(N), \dots, m_{L, \tau}(N)) : \\ & \qquad \qquad \qquad m_{\ell-1}, \dots, m_1 \text{ is } \tau\text{-}b\text{-admissible}\} \end{aligned}$$

and  $x_n \in [.y_1\dots y_{k-1}, .y_1\dots y_k)$  if and only if

$$|m_{1, \tau}(n)| \dots |m_{k-1, \tau}(n)| = y_1 \dots y_{k-1}, \quad |m_{k, \tau}(n)| < y_k.$$



Thus, for  $\ell > k$ , we have to count the  $\tau$ - $b$ -admissible sequences  $m_{\ell-1}, \dots, m_1$  with  $|m_1| \dots |m_{k-1}| = y_1 \dots y_{k-1}$ ,  $|m_k| < y_k$ . By Proposition 1, every  $\tau$ - $b$ -admissible sequence  $m_{\ell-1}, \dots, m_{k+1}$  can be prolonged to such a sequence for all  $|m_k| < y_k$  because of

$$|m_j| \dots |m_{\ell-1}| t_b < y_j \dots y_k \leq a_1 a_2 \dots \text{ for } j \leq k.$$

Therefore we have  $y_k |\tau^{\ell-k-1}(b)|$  such sequences for every letter  $b$  in  $m_{\ell, \tau}(N)$ .

For  $\ell = k$ , we need  $|m| < |m_{k, \tau}(N)|$  and  $|m| < y_k$ . For each such  $|m|$  (and the corresponding  $b$ ), there is one  $\tau$ - $b$ -admissible sequence  $m_{k-1}, \dots, m_1$  with  $|m_1| \dots |m_{k-1}| = y_1 \dots y_{k-1}$ . Thus, the contribution is  $\max(|m_{k, \tau}(N)|, y_k)$ .

Finally, for  $\ell < k$ , we need  $|m| = y_\ell < |m_{\ell, \tau}(N)|$ ,  $|m_{k, \tau}(N)| < y_k$  and  $|m_{\ell+1, \tau}(N)| \dots |m_{k-1, \tau}(N)| = y_{\ell+1} \dots y_{k-1}$ . Thus the contribution is 1 if  $|m_{k, \tau}(N)| < y_k$ ,  $|m_{k-1, \tau}(N)| \dots |m_{1, \tau}(N)| > y_{k-1} \dots y_1$  and 0 else.  $\square$

The characteristic polynomial of the incidence matrix of the  $\beta$ -substitution  $\sigma$  is the  $\beta$ -polynomial. Hence  $\sigma$  is of Pisot type (one eigenvalue is  $> 1$  and all other eigenvalues have modulus  $< 1$ ) if and only if  $\beta$  is a Pisot number and the  $\beta$ -polynomial is irreducible. Since  $|\sigma^k(1)| = |\tau^k(1)|$  for all  $k \geq 0$ ,  $\beta$  is an eigenvalue of  $\tau$  as well. Furthermore,  $\tau$  is of Pisot type because the alphabet has the same size as the alphabet of  $\sigma$ . Hence we have some constants  $c_{b,j}$  and  $\rho < 1$  such that

$$|\tau^k(b)| = c_{b,1}\beta^k + c_{b,2}\beta_2^j + \dots + c_{b,d}\beta_d^k = c_{b,1}\beta^k + \mathcal{O}(\rho^k),$$

where the  $\beta_j$ ,  $2 \leq j \leq d$  are the conjugates of  $\beta$ . Thus

$$\begin{aligned}
D(N, [0, y)) &= \sum_{k=1}^{\infty} \left( y_k \sum_{\ell=k+1}^{\infty} \sum_{b=1}^d |m_{\ell, \tau}(N)|_b |\tau^{\ell-k-1}(b)| + \mu_k(N, y) \right. \\
&\quad \left. - y_k \sum_{\ell=1}^{\infty} \sum_{b=1}^d |m_{\ell, \tau}(N)|_b |\tau^{\ell-1}(b)| \beta^{-k} \right) \\
&= \sum_{k=1}^{\infty} \left( y_k \sum_{\ell=k+1}^{\infty} \sum_{b=1}^d |m_{\ell, \tau}(N)|_b \sum_{j=2}^d c_{b,j} (\beta_j^{\ell-k-1} - \beta_j^{\ell-1} \beta^{-k}) + \mu_k(N, y) \right. \\
&\quad \left. - y_k \sum_{\ell=1}^k \sum_{b=1}^d |m_{\ell, \tau}(N)|_b \left( c_{b,1} \beta^{\ell-1-k} + \sum_{j=2}^d \beta_j^{\ell-1} \beta^{-k} \right) \right) = \sum_{k=1}^{\infty} y_k \mathcal{O}(1)
\end{aligned}$$

and

$$\begin{aligned}
D(N, [0, y)) &= \sum_{\ell=1}^{\infty} \left( \sum_{b=1}^d |m_{\ell, \tau}(N)|_b \left( \sum_{k=1}^{\ell-1} y_k \sum_{j=2}^d c_{b,j} (\beta_j^{\ell-k-1} - \beta_j^{\ell-1} \beta^{-k}) \right) \right. \\
&\quad \left. + \mu_{\ell}(N, y) - \sum_{b=1}^d |m_{\ell, \tau}(N)|_b \sum_{k=\ell}^{\infty} y_k \left( c_{b,1} \beta^{\ell-k-1} + \sum_{j=2}^d c_{b,j} \beta_j^{\ell-1} \beta^{-k} \right) \right) \\
&= \sum_{\ell=1}^{\infty} \left( \mu_{\ell}(N, y) \right. \\
&\quad \left. - \sum_{b=1}^d |m_{\ell, \tau}(N)|_b \left( c_{b,1} \sum_{k=\ell}^{\infty} y_k \beta^{\ell-k-1} - \sum_{j=2}^d c_{b,j} \sum_{k=1}^{\ell-1} y_k \beta_j^{\ell-k-1} \right) \right) + \mathcal{O}(1)
\end{aligned}$$

By the above formulae, we easily see that  $D(N, [0, y))$  is bounded if  $y_k > 0$  for only finitely many  $k \geq 1$ . Now we consider  $y \in \mathbb{Q}(\beta)$ . Bertrand [1] and K. Schmidt [21] proved independently that the elements  $y \in \mathbb{Q}(\beta)$  are exactly those who have eventually periodic  $\beta$ -expansion. (See Rigo and Steiner [20] for an alternative proof including number systems defined by substitutions.) Furthermore, by the above formulae, a finite number of digits of the  $\beta$ -expansion of  $y$  as well as

a shift of digits has no influence on the boundedness of  $D(N, [0, y])$ .

Therefore we may assume that the  $\beta$ -expansion of  $y$  is purely periodic.

For  $y = .(y_1 \dots y_q)^\infty$ , we have

$$\sum_{k=\ell}^{\infty} y_k \beta^{\ell-k-1} = \frac{y_\ell \beta^{p-1} + \dots + y_{\ell+p-1}}{\beta^p - 1} = s_{\ell, d-1} \beta^{d-1} + \dots + s_{\ell, 0} \beta^0 = P_\ell(\beta)$$

for some  $s_{\ell, j} \in \mathbb{Q}$ . If we set  $y_k = y_{k+q}$  for  $k \leq 0$ , then we obtain

$$\sum_{k=-\infty}^{\ell-1} y_k \beta_i^{\ell-k-1} = \frac{y_{\ell-p} \beta_i^{p-1} + \dots + y_{\ell-1}}{1 - \beta_i^p} = -P_\ell(\beta_i),$$

$$\begin{aligned} \gamma_\ell(b) &= c_{b,1} \sum_{k=\ell}^{\infty} y_k \beta^{\ell-k-1} - \sum_{i=2}^d c_{b,i} \sum_{k=-\infty}^{\ell-1} y_k \beta_i^{\ell-k-1} \\ &= s_{\ell, d-1} |\tau^{d-1}(b)| + \dots + s_{\ell, 0} |\tau^0(b)| \end{aligned}$$

and

$$D(N, [0, y]) = \sum_{\ell=1}^{\infty} \left( \mu_\ell(N, y) - \gamma_\ell(m_{\ell, \tau}(N)) \right) + \mathcal{O}(1)$$

by extending  $\gamma_\ell$  naturally on words,  $\gamma_\ell(w) = \sum_{b=1}^d |w|_b \gamma_\ell(b)$ .

We split the remaining part of the proof into two lemmata.

**Lemma 3.** *If  $\beta$  is a Pisot number with irreducible  $\beta$ -polynomial, then  $D(N, [0, .(a_{d-p+1} \dots a_d)^\infty])$  is bounded.*

*Proof.* We have

$$.y_\ell y_{\ell+1} \dots = .a_{d-p+\ell} a_{d-p+\ell+1} \dots = \beta^{d-p+\ell-1} - a_1 \beta^{d-p+\ell-2} - \dots - a_{d-p+\ell-1}$$

and, by Proposition 1, we easily see

$$|\tau^k(b)| = a_1 |\tau^{k-1}(b)| + \dots + a_k |\tau^0(b)| + \begin{cases} 1 & \text{if } a_1 \dots a_k t_b < a_1 a_2 \dots \\ 0 & \text{else} \end{cases}$$

for all  $k > 0$ , hence

$$\gamma_\ell(b) = \begin{cases} 1 & \text{if } t_b < a_{d-p+\ell}a_{d-p+\ell+1}\dots \\ 0 & \text{else.} \end{cases}$$

By definition, we have  $t_{u_j(b_{\ell+1})} \leq jt_{b_{\ell+1}} < t_{u_j(b_{\ell+1})+1}$ , therefore

$$\gamma_\ell(u_j(b_{\ell+1})) = \begin{cases} 1 & \text{if } jt_{b_{\ell+1}} < a_{d-p+\ell}a_{d-p+\ell+1}\dots \\ 0 & \text{else.} \end{cases}$$

With  $m_{\ell,\tau}(N) = u_0(b_{\ell+1}) \dots u_{|m_{\ell,\tau}(N)|-1}(b_{\ell+1})$ , we obtain

$$\gamma_\ell(m_{\ell,\tau}(N)) = \begin{cases} |m_{\ell,\tau}(N)| & \text{if } |m_{\ell,\tau}(N)| \leq a_{d-p+\ell} \\ a_{d-p+\ell} & \text{if } |m_{\ell,\tau}(N)| > a_{d-p+\ell}, \\ & t_{b_{\ell+1}} \geq a_{d-p+\ell+1}a_{d-p+\ell+2}\dots \\ a_{d-p+\ell} + 1 & \text{else} \end{cases}$$

and

$$\begin{aligned} \Delta_\ell &= \mu_\ell(N, (a_{d-p+1} \dots a_d)^\infty) - \gamma_\ell(m_{\ell,\tau}(N)) \\ &= \begin{cases} -1 & \text{if } |m_{\ell,\tau}(N)| > a_{d-p+\ell}, t_{b_{\ell+1}} < a_{d-p+\ell+1}a_{d-p+\ell+2}\dots \\ 1 & \text{if } |m_{\ell,\tau}(N)| < a_{d-p+\ell}, \\ & |m_{\ell-1,\tau}(N)| \dots |m_{1,\tau}(N)| > a_{d-p+\ell-1} \dots a_{d-p+1} \\ 0 & \text{else.} \end{cases} \end{aligned}$$

If  $\Delta_\ell = -1$ , then  $t_{b_{\ell+1}} < a_{d-p+\ell+1}a_{d-p+\ell+2}\dots$  and

$$t_{b_{\ell+1}} \leq |m_{\ell+1,\tau}(N)|t_{b_{\ell+2}} < t_{b_{\ell+1}+1} \leq a_{d-p+\ell+1}a_{d-p+\ell+2}\dots$$

implies either  $|m_{\ell+1,\tau}(N)| < a_{d-p+\ell+1}$ , thus  $\Delta_{\ell+1} = 1$ , or

$$|m_{\ell+1,\tau}(N)| = a_{d-p+\ell+1}, t_{b_{\ell+2}} < a_{d-p+\ell+2}a_{d-p+\ell+3}\dots \text{ and } \Delta_{\ell+1} = 0.$$

Inductively, we obtain some  $k > \ell$  such that  $\Delta_{\ell+1} = \dots = \Delta_{k-1} = 0$  and  $\Delta_k = 1$ .

If  $\Delta_\ell = 1$ , then  $|m_{\ell-1,\tau}(N)| \dots |m_{1,\tau}(N)| > a_{d-p+\ell-1} \dots a_{d-p+1}$  implies either

$$|m_{\ell-1,\tau}(N)| > a_{d-p+\ell-1} \text{ and } t_{b_\ell} \leq |m_{\ell,\tau}(N)| t_{b_{\ell+1}} < a_{d-p+\ell},$$

thus  $\Delta_{\ell-1} = -1$ , or

$$|m_{\ell-1,\tau}(N)| = a_{d-p+\ell-1}, |m_{\ell-2,\tau}(N)| \dots |m_{1,\tau}(N)| > a_{d-p+\ell-2} \dots a_{d-p+1}$$

and  $\Delta_{\ell-1} = 0$ . Inductively, we obtain some  $k < \ell$  such that  $\Delta_k = -1$  and  $\Delta_{k+1} = \dots = \Delta_{\ell-1} = 0$ .

Therefore we have  $\sum_{\ell=1}^{\infty} \Delta_\ell = 0$  and the discrepancy function is bounded.  $\square$

$D(N, [0, (a_{d-p+j} \dots a_d a_{d-p+1} \dots a_{d-p+j-1})^\infty])$ ,  $1 < j \leq p$ , is bounded as well because a shift of digits does not change the boundedness.

**Lemma 4.** *If  $D(N, [0, y))$  is bounded and  $y \neq 0$  has purely periodic  $\beta$ -expansion, then the expansion of 1 is eventually periodic and  $y = .a_L a_{L+1} \dots$  for some  $L > d - p$ .*

*Proof.* Let the  $\beta$ -expansion of  $y$  be  $.y_1 y_2 \dots = .(y_1 \dots y_q)^\infty$ . Consider sequences of integers  $N_K$  given by

$$(m_{1,\tau}(N_K), m_{2,\tau}(N_K), \dots) = ((m_1, \dots, m_{J_q})^K, \varepsilon, \varepsilon, \dots)$$

with  $m_{\ell+1} = \dots = m_{Jq} = \varepsilon$  for some  $\ell \geq 1$ ,  $J \geq 1$  such that  $b_{\ell+1} = 1$  and  $y_{\ell+1} \dots y_{Jq} > 0 \dots 0$ . For these sequences, we have

$$\mu_{j+kJq}(N_K, y) = \mu_j(N_K, y), \quad \gamma_{j+kJq}(m_{j+kJq, \tau}(N_K)) = \gamma_j(m_j)$$

for all  $j \leq Jq$ ,  $k < K$ . Thus  $D(N_K, [0, y))$  is bounded if and only if

$$\sum_{j=1}^{Jq} (\mu_j(N_1, y) - \gamma_j(m_j)) = 0$$

Let furthermore  $m_1 = \dots = m_{k-1} = \varepsilon$  for some  $k \in \{1, \dots, \ell\}$ , hence  $\mu_j(N_1, y) = \gamma_j(m_j)$  for all  $j < k$ . Consider simultaneously integers  $N'_K$  with  $m'_k = \varepsilon$  and  $m'_j = m_j$  for all  $j \neq k$ . Then we have  $\mu_j(N'_1, y) = \gamma_j(m'_j) = 0$  for all  $j < k$ ,  $\gamma_j(m'_j) = \gamma_j(m_j)$  for all  $j > k$  and

$$\sum_{j=k+1}^{Jq} \mu_j(N_1, y) = \sum_{j=k+1}^{Jq} \mu_j(N'_1, y) + \begin{cases} 1 & \text{if } |m_k| > y_k, \\ & |m_{k+1}| \dots |m_{Jq}| < y_{k+1} \dots y_{Jq} \\ 0 & \text{else,} \end{cases}$$

thus

$$\begin{aligned} \gamma_k(m_k) - \mu_k(N_1, y) &= \sum_{j=k+1}^{Jq} (\mu_j(N_1, y) - \gamma_j(m_j)) \\ &= \begin{cases} 1 & \text{if } |m_k| > y_k, |m_{k+1}| \dots |m_\ell| \leq y_{k+1} \dots y_\ell \\ 0 & \text{else} \end{cases} \end{aligned}$$

and

$$\gamma_k(m_k) = \begin{cases} |m_k| & \text{if } |m_k| \leq y_k \\ y_k & \text{if } |m_k| > y_k, |m_{k+1}| \dots |m_\ell| > y_{k+1} \dots y_\ell \\ y_k + 1 & \text{else.} \end{cases}$$

If  $m_k b_k <_p \tau(b_{k+1})$ , then  $m_\ell, \dots, m_{k+1}, m_k b_k$  is a  $\tau$ -1-admissible sequence and we obtain

$$(4) \quad \gamma_k(b_k) = \gamma_k(m_k b_k) - \gamma_k(m_k) = \begin{cases} 1 & \text{if } |m_k| \dots |m_\ell| \leq y_k \dots y_\ell \\ 0 & \text{else,} \end{cases}$$

in particular  $\gamma_k(1) = 1$  for all  $k \geq 1$  (with  $k = \ell, m_k = \varepsilon$ ).

If  $m_k b_k = \tau(b_{k+1})$ , consider

$$\cdot y_{k+1} y_{k+2} \dots = \beta \times \cdot y_k y_{k+1} \dots - y_k = s_{k,d-1} \beta^d + \dots + s_{k,0} \beta - y_k,$$

hence

$$\begin{aligned} \gamma_{k+1}(b_{k+1}) &= s_{k,d-1} |\tau^d(b_{k+1})| + \dots + s_{k,0} |\tau(b_{k+1})| - y_k \\ &= s_{k,d-1} |\tau^{d-1}(m_k b_k)| + \dots + s_{1,0} |m_k b_k| - y_k = \gamma_k(m_k) + \gamma_k(b_k) - y_k \\ &= \gamma_k(b_k) + \begin{cases} -1 & \text{if } |m_k| < y_k \text{ (i.e. } |m_k| = a_1 - 1, y_k = a_1) \\ 0 & \text{if } |m_k| = y_k \text{ or } |m_k| > y_k, |m_{k+1}| \dots |m_\ell| > y_{k+1} \dots y_\ell \\ 1 & \text{else.} \end{cases} \end{aligned}$$

In case  $|m_k| = |\tau(b_{k+1})| - 1 = a_1 - 1, y_k = a_1$ , we have  $a_1 t_{b_{k+1}} \geq a_1 a_2 \dots, y_{k+1} y_{k+2} \dots < a_2 a_3 \dots$  and  $t_{b_{k+1}} \leq |m_{k+1}| t_{b_{k+2}} \leq \dots \leq |m_{k+1}| \dots |m_\ell| 0^\infty$ , hence  $|m_{k+1}| \dots |m_\ell| \geq a_2 \dots a_{\ell-k+1} \geq y_{k+1} \dots y_\ell$ . One of these inequalities is strict because  $t_{b_{k+1}} = |m_{k+1}| \dots |m_\ell| 0^\infty = a_2 \dots a_{\ell-k+1} 0^\infty$  implies  $|m_{k+1}| \dots |m_\ell| = a_2 \dots a_d 0^{\ell-k-d+1} > y_{k+1} \dots y_\ell$ . Therefore we

have, for all  $b_k, b_{k+1}$ ,

$$\begin{aligned} & \gamma_k(b_k) - \gamma_{k+1}(b_{k+1}) \\ &= \begin{cases} 1 & \text{if } |m_k| \dots |m_\ell| \leq y_k \dots y_\ell, |m_{k+1}| \dots |m_\ell| > y_{k+1} \dots y_\ell \\ -1 & \text{if } |m_k| \dots |m_\ell| > y_k \dots y_\ell, |m_{k+1}| \dots |m_\ell| \leq y_{k+1} \dots y_\ell \\ 0 & \text{else.} \end{cases} \end{aligned}$$

and, with  $\gamma_{\ell+1}(b_{\ell+1}) = \gamma_{\ell+1}(1) = 1$ , (4) holds for all  $m_k, b_k$ .

Now, let  $k = 1$  and  $m_\ell, \dots, m_1$  and  $m'_\ell, \dots, m'_1$  be  $\tau$ -1-admissible sequences with companion sequences  $b_\ell, \dots, b_1$  and  $b'_\ell, \dots, b'_1$ . If  $b_1 < b'_1$ , then we have  $|m_1|t_{b_2} < t_{b_1+1} \leq t_{b'_1} \leq |m'_1|t_{b'_2}$ , thus either  $|m_1| < |m'_1|$  or  $|m_1| = |m'_1|, b_2 < b'_2$ . Inductively, we obtain  $|m_1| \dots |m_\ell| < |m'_1| \dots |m'_\ell|$  and  $\gamma_1(b_1) \geq \gamma_1(b'_1)$ . Therefore we have some  $b' \geq 2$  such that

$$\gamma_1(b) = \begin{cases} 1 & \text{if } b < b' \\ 0 & \text{else.} \end{cases}$$

Finally, consider the system of linear equations

$$s_{1,d-1}|\tau^{d-1}(b)| + \dots + s_{1,0}|\tau^0(b)| = \begin{cases} 1 & \text{if } b < b' \\ 0 & \text{else} \end{cases}$$

for  $1 \leq b \leq d$ . We have  $t_{b'} = a_L a_{L+1} \dots$  for some  $L \geq 2$ . Then, by the proof of Lemma 3,  $(s_{1,d-1}, \dots, s_{1,0}) = (0, \dots, 0, 1, -a_1, \dots, -a_{L-1})$  is a solution of this system, i.e.  $y = .a_L a_{L+1} \dots$ . To show that these solutions are unique, consider linear combinations of the column vectors



$(|\tau^\ell(1)|, \dots, |\tau^\ell(d)|)^T$  (over  $\mathbb{Q}$ ). We have, with  $\beta_1 = \beta$ ,

$$\sum_{\ell=0}^{d-1} r_\ell \begin{pmatrix} |\tau^\ell(1)| \\ \vdots \\ |\tau^\ell(d)| \end{pmatrix} = \sum_{\ell=0}^{d-1} r_\ell M^\ell \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \sum_{\ell=0}^{d-1} r_\ell \sum_{j=1}^d v_j \beta_j^\ell \mathbf{e}_j = \sum_{j=1}^d v_j \mathbf{e}_j \sum_{\ell=0}^{d-1} r_\ell \beta_j^\ell,$$

where  $M$  is the incidence matrix of  $\tau$ ,  $M = (|\tau(b)|_c)_{1 \leq b, c \leq d}$ , and the  $\mathbf{e}_j$ ,  $1 \leq j \leq d$ , are right eigenvectors of  $M$  to the eigenvalues  $\beta_j$ . If  $r_\ell \in \mathbb{Q}$ , then all  $r_\ell$  must be zero, hence the vectors  $(|\tau^\ell(1)|, \dots, |\tau^\ell(d)|)$ ,  $0 \leq \ell < d$ , are linearly independent and the system of linear equations has a unique solution.

To conclude the proof of the lemma, note that  $a_L a_{L+1} \dots$  is purely periodic if and only if  $L > d - p$ .  $\square$

*Acknowledgements.* I want to thank Klaus Scheicher for bringing my attention to this subject and for many fruitful discussions.

#### REFERENCES

- [1] Bertrand A (1977) Développements en base de Pisot et répartition modulo 1. C R Acad Sci, Paris, Sér A **285**: 419–421
- [2] Canterini V, Siegel A (2001) Automate des préfixes-suffixes associé à une substitution primitive. J Théor Nombres Bordx **13**: 353–369
- [3] Drmota M, Larcher G, Pillichshammer F (to appear) Precise distribution properties of the van der Corput sequence and related sequences. Manusc Math
- [4] Drmota D, Tichy R F (1998) Sequences, Discrepancies and Applications. Lecture Notes in Mathematics, vol 1651. Berlin: Springer
- [5] Dumont J-M, Thomas A (1989) Systèmes de numération et fonctions fractales relatifs aux substitutions. Theoret Comput Sci **65**: 153–169

- [6] Faure H (1983) Étude des restes pour les suites de van der Corput généralisées. *J Number Theory* **16**: 376–394
- [7] Faure H (2005) Discrepancy and diaphony of digital (0,1)-sequences in prime base. *Acta Arith* **117**: 125–148
- [8] Ferenczi S (1992) Bounded remainder sets. *Acta Arith* **61**: 319–326.
- [9] Ferenczi S, Mauduit S, Nogueira A (1996) Substitution dynamical systems: Algebraic characterization of eigenvalues. *Ann Sci Éc Norm Supér, IV Sér* **29**: 519–533
- [10] Hecke E (1921) Analytische Funktionen und die Verteilung von Zahlen mod. eins. *Abh Math Sem Univ Hamburg* **1**: 54–76.
- [11] Hellekalek P (1980) On regularities of the distribution of special sequences. *Monatsh Math* **90**: 291–295
- [12] Hellekalek P (1984) Regularities in the distribution of special sequences. *J Number Theory* **18**: 41–55
- [13] Kesten H (1966) On a conjecture of Erdős and Szüsz related to uniform distribution mod 1. *Acta Arith* **12**: 193–212
- [14] Liardet P (1987) Regularities of distribution. *Compos Math* **61**: 267–293
- [15] Ninomiya S (1998) Constructing a new class of low-discrepancy sequences by using the  $\beta$ -adic transformation. *Math Comput Simulation* **47**: 403–418
- [16] Parry P (1960) On the  $\beta$ -expansions of real numbers. *Acta Math Acad Sci Hung* **11**: 401–416
- [17] Pytheas Fogg N (2002) Substitutions in dynamics, arithmetics and combinatorics. Berthé V, Ferenczi S, Mauduit S, Siegel A (eds). *Lecture Notes in Mathematics*, vol 1794. Berlin: Springer
- [18] Rauzy G, (1984) Ensembles à restes bornés, *Sém Théor Nombres, Univ Bordeaux I 1983-1984*, Exp. No. 24: 12 p.
- [19] Rényi A (1957) Representations for real numbers and their ergodic properties. *Acta Math Acad Sci Hung* **8**: 477-493

- [20] Rigo M, Steiner W (2005) Abstract  $\beta$ -expansions and ultimately periodic representations. *J Théor Nombres Bordx* **17**: 283–299
- [21] Schmidt K (1980) On periodic expansions of Pisot numbers and Salem numbers. *Bull Lond Math Soc* **12**: 269–278
- [22] Schmidt W M (1972) Irregularities of distribution VI. *Comput Math* **24**: 63–74
- [23] Schmidt W M (1974) Irregularities of distribution VIII. *Trans Am Math Soc* **198**: 1–22
- [24] Shapiro L (1978) Regularities of distribution. In: *Studies in probability and ergodic theory*, *Adv in Math Suppl Stud*, vol 2, pp 135–154. New York-London: Academic Press
- [25] Van Aardenne-Ehrenfest T (1949) On the impossibility of a just distribution. *Indag Math* **11**: 264–269