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On the existence of some ARCH(∞) processes

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Abstract

A new sufficient condition for the existence of a stationary causal solution of an ARCH(∞) equation is provided. This condition allows to consider polynomially decaying coefficients, so that it can be applied to the so-called FIGARCH processes, whose existence is thus proved.

Key words: ARCH processes, Long memory

1 Introduction

It can arguably be said that autoregressive conditionnally heteroskedastic (ARCH) and long memory processes are two success stories of the nineties, so that there were bound to meet. Their tentative offspring was the FIGARCH process, introduced by Baillie et al. (1996) without proving its existence, which has remained controversial up to now. More precisely, the

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FIGARCH(p, d, q) process is the solution of the equations

$$X_n = \sigma_n z_n, \quad (1)$$

$$\sigma_n^2 = a_0 + \left\{ I - (I - L)^d \frac{\theta(L)}{\phi(L)} \right\} X_n^2, \quad (2)$$

where $\{z_n\}$ is an i.i.d. sequence with zero mean and unit variance, $a_0 > 0$, $d \in (0, 1)$, L is the backshift operator and $(I - L)^d$ is the fractional differencing operator:

$$(I - L)^d = I + \sum_{j=1}^{\infty} \frac{(-d)(1-d) \cdots (j-1-d)}{j!} L^j,$$

and θ and ϕ are polynomials such that $\theta(0) = \phi(0) = 1$, $\phi(z) \neq 0$ for all complex number z in the closed unit disk and the coefficients of the series expansion of $\theta(z)/\phi(z)$ are nonnegative. Then the coefficients $\{a_j\}_{j \geq 1}$ defined by $\sum_{j=1}^{\infty} a_j L^j = I - (I - L)^d \theta(L)/\phi(L)$ satisfy $a_j \sim c j^{-d-1}$ for some constant $c > 0$ and $\sum_{j=1}^{\infty} a_j = 1$.

These processes are subcases of what can be called IARCH(∞), defined as solutions of the equations (1) and

$$\sigma_n^2 = a_0 + \sum_{j=1}^{\infty} a_j X_{n-j}^2, \quad (3)$$

for some sequence $\{a_j\}$ such that $a_0 > 0$ and $\sum_{j=1}^{\infty} a_j = 1$. The letter I stands for integrated, by analogy to ARIMA processes. An important property of such processes is that a stationary solution necessarily has infinite variance. If the condition $\sum_{j=1}^{\infty} a_j = 1$ is not imposed, a solution to equations (1) and (3) is simply called an ARCH(∞) process.

A solution of an ARCH(∞) equation is said to be causal with respect to the i.i.d. sequence $\{z_n\}$ if for all n , σ_n is \mathcal{F}_{n-1}^z measurable, where \mathcal{F}_n^z is the sigma-field generated by $\{z_n, z_{n-1}, \dots\}$. Note that to avoid trivialities, here

and in the sequel, σ_n is the positive square root of σ_n^2 . There exist an important literature on ARCH(∞), IARCH(∞) and FIGARCH processes. For a recent review of this literature, see for instance Giraitis et al. (2007). The known conditions for the existence of stationary causal conditions to ARCH equations are always a compromise between conditions on the distribution of the innovation sequence $\{z_n\}$ and summability conditions on the coefficients $\{a_j, j \geq 1\}$. Giraitis and Surgailis (2002) provides a necessary and sufficient condition for the solution to have finite fourth moment. The only rigorous result in the IARCH(∞) case was obtained by Kazakevičius and Leipus (2003). They prove the existence of a causal stationary solution under the condition that the coefficients a_j decay geometrically fast, which rules out FIGARCH processes, and on a mild condition on the distribution of z_0 .

The purpose of this note is to provide a new sufficient condition for the existence of a stationary solution to an ARCH(∞) equation, which allows polynomial decay of the coefficients a_j s, even in the IARCH(∞) case. This condition is stated in Section 2. It is applied to the IARCH(∞) case in Section 3 and the existence of a stationary solution to the FIGARCH equation is proved. Further research directions are given in Section 4. In particular, the memory properties of FIGARCH processes are still to be investigated. This is an important issue, since the original motivation of these processes was the modelling of long memory in volatility.

2 A sufficient condition for the existence of ARCH(∞) processes

Theorem 1 *Let $\{a_j\}_{j \geq 0}$ be a sequence of nonnegative real numbers and $\{z_k\}_{k \in \mathbb{Z}}$ a sequence of i.i.d. random variables. For $p > 0$, define*

$$A_p = \sum_{j=1}^{\infty} a_j^p \quad \text{and} \quad \mu_p = \mathbb{E}[z_0^{2p}] .$$

If there exists $p \in (0, 1]$ such that

$$A_p \mu_p < 1, \tag{4}$$

then there exists a strictly stationary solution of the ARCH(∞) equation:

$$X_n = \sigma_n z_n , \tag{5}$$

$$\sigma_n^2 = a_0 + \sum_{j=1}^{\infty} a_j X_{n-j}^2 , \tag{6}$$

given by (5) and

$$\sigma_n^2 = a_0 + a_0 \sum_{k=1}^{\infty} \sum_{j_1, \dots, j_k \geq 1} a_{j_1} \dots a_{j_k} z_{n-j_1}^2 \dots z_{n-j_1-\dots-j_k}^2 . \tag{7}$$

The process $\{X_n\}$ so defined is the unique causal stationary solution to equations (5) and (6) such that $\mathbb{E}[|X_n|^{2p}] < \infty$.

Proof. Denote $\xi_k = z_k^2$, so that $\mathbb{E}[\xi_k^p] = \mu_p$, and define the $[0, \infty]$ -valued r.v.

$$\aleph_0 = a_0 + a_0 \sum_{k=1}^{\infty} \sum_{j_1, \dots, j_k \geq 1} a_{j_1} \dots a_{j_k} \xi_{-j_1} \dots \xi_{-j_1-\dots-j_k} \tag{8}$$

Since $p \in (0, 1]$, we apply the inequality $(a+b)^p \leq a^p + b^p$ valid for all $a, b \geq 0$, and we obtain, by independence of the ξ_j 's,

$$\begin{aligned} \mathbb{E}[\aleph_0^p] &\leq a_0^p + a_0^p \sum_{k=1}^{\infty} \sum_{j_1, \dots, j_k \geq 1} a_{j_1}^p \dots a_{j_k}^p \mathbb{E}[\xi_{-j_1}^p \dots \xi_{-j_1-\dots-j_k}^p] \\ &= a_0^p \left[1 + \sum_{k=1}^{\infty} (\mu_p A_p)^k \right] = \frac{a_0^p}{1 - A_p \mu_p} , \end{aligned} \tag{9}$$

where we used (4). This bound shows that $\aleph_0 < \infty$ a.s. and the sequence

$$\aleph_n = a_0 + a_0 \sum_{k=1}^{\infty} \sum_{j_1, \dots, j_k \geq 1} a_{j_1} \cdots a_{j_k} \xi_{n-j_1} \cdots \xi_{n-j_1-\dots-j_k}, \quad n \in \mathbb{Z},$$

is a sequence of a.s. finite r.v.'s. Since only nonnegative numbers are involved in the summation, we may write

$$\begin{aligned} \sum_{j=1}^{\infty} a_j \aleph_{n-j} \xi_{n-j} &= a_0 \sum_{j_0=1}^{\infty} a_{j_0} \xi_{n-j_0} \\ &+ a_0 \sum_{j_0=1}^{\infty} a_{j_0} \xi_{n-j_0} \sum_{k=1}^{\infty} \sum_{j_1, \dots, j_k \geq 1} a_{j_1} \cdots a_{j_k} \xi_{n-j_0-j_1} \cdots \xi_{n-j_0-j_1-\dots-j_k} \\ &= a_0 \sum_{k=0}^{\infty} \sum_{j_0, j_1, \dots, j_k \geq 1} a_{j_0} \cdots a_{j_k} \xi_{n-j_0} \cdots \xi_{n-j_0-j_1-\dots-j_k}. \end{aligned}$$

Hence $\{\aleph_n, n \in \mathbb{Z}\}$ satisfies the recurrence equation

$$\aleph_n = a_0 + \sum_{j=1}^{\infty} a_j \aleph_{n-j} \xi_{n-j}.$$

This proves the existence of a strictly stationary solution for (5) and (6) by setting $\sigma_n^2 = \aleph_n$ and $X_n = \sigma_n z_n$. Using (9), we moreover have $\mathbb{E}[|X_n|^{2p}] \leq \mu_p a_0^p / (1 - A_p \mu_p)$.

Suppose now that $\{X_n\}$ is a strictly stationary causal solutions of the ARCH(∞) equations (5) and (6). Then, for any $q \geq 1$, the following expansion holds:

$$\sigma_n^2 = a_0 + a_0 \sum_{k=0}^q \sum_{j_1, \dots, j_k \geq 1} a_{j_1} \cdots a_{j_k} \xi_{n-j_1} \cdots \xi_{n-j_1-\dots-j_k} \quad (10)$$

$$+ \sum_{j_1, \dots, j_{q+1} \geq 1} a_{j_1} \cdots a_{j_{q+1}} \xi_{n-j_1} \cdots \xi_{n-j_1-\dots-j_q} X_{n-j_1-\dots-j_{q+1}}^2. \quad (11)$$

This implies that the series on the right-hand side of (10) converges to \aleph_n as q tends to infinity. Denote by $R_{n,q}$ the remainder term in (11). Since $\{X_n\}$ is a causal solution, $X_{n-j_1-\dots-j_{q+1}}$ is independent of $\xi_{n-j_1} \cdots \xi_{n-j_1-\dots-j_q}$ for all $j_1, \dots, j_{q+1} \geq 1$. Hence, for any $p \leq 1$,

$$\mathbb{E}[R_{n,q}^p] \leq (A_p \mu_p)^q \mathbb{E}[X_0^{2p}].$$

If Assumption (4) holds and $\mathbb{E}[X_0^{2p}] < \infty$, then $\mathbb{E}[R_{n,q}^p]$ tends to zero and this proves that $\sigma_n^2 = \aleph_n$. \square

3 IARCH(∞) processes

IARCH (Integrated ARCH) processes are particular ARCH(∞) processes for which $A_1\mu_1 = 1$, or, equivalently up to a scale factor,

$$A_1 = 1 \quad \text{and} \quad \mu_1 = 1 \tag{12}$$

To the best of our knowledge, the only rigorous result on IARCH(∞) processes was obtained by Kazakevičius and Leipus (2003). See Giraitis et al. (2007) for a recent discussion of the literature on this problem. In Theorem 2.1 of Kazakevičius and Leipus (2003), it is proved that if

$$\mathbb{E}[|\log(z_0)|^2] < \infty, \tag{13}$$

$$\sum_i a_i q^i < \infty \quad \text{for some } q > 1, \tag{14}$$

hold, then there exists a unique stationary causal solution to the ARCH(∞) equations (5)-(6). Condition (13) on the distribution of z_0 is mild, but the condition (14) rules out polynomial decay of the coefficients $\{a_j\}$.

Theorem (13) yields the following sufficient condition for the existence of a IARCH(∞) process.

Corollary 2 *If $A_1 = 1$ and $\mu_1 = 1$, (4) holds for some $p \in (0, 1]$ if and only if there exists $p^* < 1$ such that $A_{p^*} < \infty$ and*

$$\sum_{j=1}^{\infty} a_j \log(a_j) + \mathbb{E}[\xi_0 \log(\xi_0)] \in (0, \infty]. \tag{15}$$

Then, the process defined by (5) and (7) is a solution of the ARCH(∞) equation and $\mathbb{E}[|X_n|^q] < \infty$ for all $q \in [0, 2)$ and $\mathbb{E}[X_n^2] = \infty$.

Condition (15) is not easily comparable to conditions (13) and (14) of Kazakevičius and Leipus (2003). The moment $\mathbb{E}[\xi_0 \log(\xi_0)]$ can be arbitrarily large (possibly infinite) if the distribution of ξ_0 has a sufficiently heavy tail. It is infinite for instance if the distribution of ξ_0 is absolutely continuous with a density bounded from below by $1/(x^2 \log^2(x))$ for x large enough. In that case, condition (15) holds for any sequence $\{a_j\}$ such that $A_1 = 1$ and $\sum_{j=1}^{\infty} a_j |\log(a_j)| < \infty$. This conditions allows for a polynomial decay of the coefficients a_j , for instance $a_j \sim cj^{-\delta}$, for some $\delta > 1$.

This proves that under a suitable condition on the distribution of ξ_0 , there exists a causal strictly stationary version of the FIGARCH(p, d, q) process.

Proof of Corollary 2.

First assume that there exists $p \in (0, 1]$ such that (4) holds. Since $A_1 = \mu_1 = 1$, then $p < 1$ and for all $q \in [p, 1]$, $A_q < \infty$. Thus we can define the function $\phi : [p, 1] \rightarrow \mathbb{R}$ by

$$\phi(q) = \log(A_q \mu_q) = \log \sum_{j=1}^{\infty} a_j^q + \log \mathbb{E}[z_0^{2q}].$$

Hölder inequality implies that the functions $q \mapsto \log \sum_{j=1}^{\infty} a_j^q$ and $q \mapsto \log \mathbb{E}[z_0^{2q}]$ are both convex on $[p, 1]$. Thus ϕ is also convex on $[p, 1]$ and, since $\phi(p) < 0$ and $\phi(1) = 0$, the left derivative of ϕ at 1, which is given by the left-hand side of (15), is positive (possibly infinite).

Conversely suppose that there exists $p^* < 1$ such that $A_{p^*} < \infty$ and that (15) holds. Then ϕ is a convex function on $[p^*, 1]$ and (15) implies that $\phi(q) < 1$

for $q < 1$ sufficiently close to 1.

By convexity of ϕ and since $\phi(1) = 0$, we also get that $A_p\mu_p < 1$ implies $A_q\mu_q < 1$ for all $q \in [p, 1]$. Then, by Theorem 1, the process $\{X_n, n \in \mathbb{Z}\}$ defined by (7) and (6) is a solution to the ARCH(∞) equation and satisfies $\mathbb{E}[|X_0|^q] < \infty$ for all positive $q < 2$. \square .

4 Open problems

Now that a proof of existence of FIGARCH and related processes is obtained under certain conditions, there still remain some open questions. We state a few of them here.

- Condition (15) is probably not necessary for the existence of a stationary causal solution. If a solution exists, say $\{X_n\}$, then, as seen in the proof of Theorem 1, the sequence $\{\aleph_n\}$ defined in (8) is well defined and $Y_n = \aleph_n^{1/2}z_n$ is also a stationary causal solution which satisfies moreover $Y_n^2 \leq X_n^2$. But we cannot prove without more assumptions that these solutions are equal.
- Tail behaviour of the marginal distribution of *GARCH* processes have investigated by Basrak et al. (2002), following Nelson (1990), but there are no such results in the ARCH(∞) case. Under suitable conditions, we have shown that the squares of the FIGARCH process X_n^2 have finite moments of all order $p < 1$, but necessarily, $\mathbb{E}[X_n^2] = \infty$. Thus, it is natural to conjecture that perhaps under additional conditions on the distribution of z_0 , the function $x \rightarrow \mathbb{P}(X_n^2 > x)$ is regularly varying with index -1.
- The memory properties of the FIGARCH process are of course of great interest. The sequence $\{X_n\}$ is a strictly stationary martingale increment

sequence, but $\mathbb{E}[X_n^2] = \infty$. So does it hold that the partial sum process $n^{-1/2} \sum_{k=1}^{\lfloor nt \rfloor} X_k$ converges weakly to the Brownian motion? For $p \in [1, 2)$, do the sequences $\{|X_n|^p\}$ have distributional long memory in the sense that $n^{-H} \sum_{k=1}^{\lfloor nt \rfloor} \{|X_k|^p - \mathbb{E}[|X_k|^p]\}$ converge to the fractional Brownian motion with Hurst index H for a suitable $H > 1/2$?

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