The Kato smoothing effect for Schrödinger equations with unbounded potentials in exterior domains

Luc Robbiano, Claude Zuily

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by

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1. Introduction

The Kato $\frac{1}{2}-$ smoothing effect for Schrödinger equations has received much attention during the last years. See Constantin-Saut [C-S], Sjölin [Sj], Vega [V], Yajima [Y] for the case of the flat Laplacian in $\mathbb{R}^d$. It has been successively extended to variable coefficients operators by Doi (see [D1], [D2]) and to perturbations of such operators by potentials growing at most quadratically at infinity (see Doi [D3]). The aim of this paper is to consider exterior boundary value problems for variable coefficients operators with unbounded potentials. The case of potentials decaying at infinity has been considered by Burq [B1] using resolvent estimates.

Our main smoothing estimate is proved by contradiction. The idea of proving estimates by contradiction (with the appropriate technology) goes back to Lebeau [L] and it has been subsequently used with success by several authors, (see e.g. Burq [B2]).

In this paper, some ideas of Gérard-Leichtnam [G-L], Burq [B3] and Miller [Mi] will be also used.

Let us briefly outline how this method applies here. Assuming that our estimate is false gives rise, after renormalization, to a sequence $\big(u_k\big)$ which is bounded in $L^2_{loc}\big([0,T]\times\mathbb{R}^d\big)$. To a subsequence we associate a microlocal semi-classical defect measure μ in the sense of Gérard [G]. Then, roughly speaking, there are three main steps in the proof. First μ does not vanish identically. Moreover μ vanishes somewhere (in the incoming region). Finally the support of μ is invariant by the generalized bicharacteristic flow (in the sense of Melrose-Sjöstrand [M-S]). Since one of our assumptions (the non trapping condition) ensures that the backward generalized flow always meet the incoming region (where μ vanishes) we obtain a contradiction thus proving the desired estimate.

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Let us now describe more precisely the content of each section.

In the next one we describe the assumptions and state the main result of this paper. In the third section we begin our contradiction argument and we show in the next one how to obtain a bounded sequence in $L^2\big([0,T],L^2_{loc}(\mathbb{R}^d)\big)$. Then in the fifth section we introduce the semi classical defect measure μ and we state without proof the invariance of its support by the generalized Melrose-Sjöstrand bicharacteristic flow. In the next section we show that μ does not vanish identically while in section seven we show that μ vanishes in the incoming region. In the section eight we end the proof of our main result by achieving a contradiction. Finally in the appendix (section nine) we recall the geometrical framework we prove the invariance of the support of μ and we end by proving some technical Lemmas used in the preceeding sections.

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2. Statement of the result

Let K be a compact obstacle in $\mathbb{R}^d$ whose complement Ω is a connected open set with $\mathcal{C}^\infty$ boundary $\partial\Omega$.

Let P be a second order differential operator of the form

$$P=\sum_{j,k=1}^{d} D_j\big(a^{jk}(x)D_k\big)+V(x) \quad , \quad D_j=\frac{1}{i}\frac{\partial}{\partial x_j} \tag{2.1}$$

whose coefficients a^{jk} and V are assumed (for simplicity) to be in $\mathcal{C}^\infty(\overline{\Omega})$, real valued and $a^{jk}=a^{kj}$, $1\leq j,k\leq d$

We shall set

$$p(x,\xi)=\sum_{j,k=1}^{d} a^{jk}(x)\xi_j\xi_k \tag{2.2}$$

and we shall assume that

$$\exists\, c>0 \ : \ p(x,\xi)\geq c\,|\xi|^2\,, \text{ for } x \text{ in } \overline{\Omega} \text{ and } \xi \text{ in } \mathbb{R}^d. \tag{2.3}$$

To express the remaining assumptions on the coefficients we introduce the metric

$$g=\frac{\mathrm{d}x^2}{\langle\, x\,\rangle^2}+\frac{\mathrm{d}\xi^2}{\langle\,\xi\,\rangle^2} \tag{2.4}$$

where $\langle\,\cdot\,\rangle=\big(1+|\cdot|^2\,\big)^{1/2}$ and we shall denote by $S_\Omega(M,g)$ the Hörmander's class of symbols if M is a weight. Then $a\in S_\Omega(M,g)$ iff $a\in\mathcal{C}^\infty\big(\overline{\Omega}\times\mathbb{R}^d\big)$ and for all α, β in $\mathbb{N}^d$ one can find $C_{\alpha,\beta}>0$ such that

$$\big|D_x^\beta D_\xi^\alpha a(x,\xi)\big|\leq C_{\alpha\beta}M(x,\xi)\,\langle\, x\,\rangle^{-|\beta|}\,\langle\,\xi\,\rangle^{-|\alpha|}$$

for all x in $\overline{\Omega}$ and ξ in $\mathbb{R}^d$.

Next we assume

$$\text{(2.5)}\qquad \begin{cases} (i)\ \ a^{jk} \in S_\Omega(1,g),\ \ \nabla_x a^{jk}(x) = o\Big(\dfrac{1}{|x|}\Big),\ \ |x| \to +\infty,\ \ 1 \le j,k \le d. \\ (ii)\ \ V \in S_\Omega\big(\langle x \rangle^2, g\big),\ \ V \ge -C_0 \text{ for some positive constant } C_0. \end{cases}$$

Under the assumptions (2.3), (2.5) the operator P is essentially self adjoint on $\big\{u \in \mathcal{C}_0^\infty(\overline{\Omega})\ : u_{|\partial\Omega} = 0\big\}$. We shall denote by P_D (D means Dirichlet) its self adjoint extension.

Let us describe now our geometrical assumptions. We shall assume

$$\text{(2.6)}\qquad \begin{cases} \text{the generalized bicharacteristic flow (in the sense of} \\ \text{Melrose-Sjöstrand) is not backward trapped.} \end{cases}$$

This assumption needs some explanations. Let $M = \Omega \times \mathbb{R}_t$. Let us set $T_b^*M = T^*M \setminus \{0\} \cup T^*\partial M \setminus \{0\}$. We have a natural restriction map $\pi\ :\ T^*\mathbb{R}^{d+1}_{|\overline{M}} \setminus \{0\} \to T_b^*M$ which is the identity on $T^*\mathbb{R}^{d+1}_{|M} \setminus \{0\}$.

Let $\Sigma = \big\{(x,t,\xi,\tau) \in T^*\mathbb{R}^{d+1} \setminus \{0\} : x \in \overline{\Omega},\, t \in \big[0,T\big],\, \tau + p(x,\xi) = 0\big\}$ and $\Sigma_b = \pi(\Sigma)$. For $a \in \Sigma_b$ the generalized bicharacteristic $\Gamma(t,a)$ lives in Σ_b (see section 9.1 for details). Then (2.6) means the following.

For any a in Σ_b there exists s_0 such that for all $s \le s_0$ we have $\Gamma(s,a) \subset T^*M \setminus \{0\}$, then $\Gamma(s,a) = \big(x(s),t,\xi(s),\tau\big)$ where $\big(x(s),\xi(s)\big)$ is the usual flow of p and $\lim\limits_{s\to-\infty} |x(s)| = +\infty$.

We shall need another assumption on the flow whose precise meaning will be given in the appendix, section 9.1, Definition 9.3.

$$\text{(2.7)}\qquad \begin{cases} \text{The bicharacteristics have no contact} \\ \text{of infinite order with the boundary } \partial\Omega \end{cases}$$

Now we set

$$\text{(2.8)}\qquad \Lambda_D = \Big(\big(1+C_0\big)\mathrm{Id} + P_D\Big)^{1/2}$$

which is well defined by the functionnal calculus of self adjoint positive operators.

We shall consider the problem

$$\text{(2.9)}\qquad \begin{cases} i\,\dfrac{\partial u}{\partial t} + P_D u = 0, \\ u_{|t=0} = u_0, \\ u_{|\partial\Omega\times\mathbb{R}_t} = 0, \end{cases}$$

where $u_0 \in L^2(\Omega)$.

Then we can state our main result.

Theorem 2.1. *Let $T > 0$, $\chi \in \mathcal{C}_0^\infty(\overline{\Omega})$, $s \in \big[-1, 1\big]$. Let P be defined by (2.1) satisfying the assumptions (2.3), (2.5), (2.6) and (2.7). Then one can find a positive constant $C(T, \chi, s) = C$ such that*

$$\int_0^T \Big\| \chi \Lambda_D^{s+\frac{1}{2}} u(t) \Big\|^2_{L^2(\Omega)} \mathrm{d}t \leq C \left\| \Lambda_D^s u_0 \right\|^2_{L^2(\Omega)} \tag{2.10}$$

for all u_0 in $\mathcal{C}_0^\infty(\Omega)$, where u denotes the solution of (2.9).

Here are some remarks

Remarks 2.2.
(i) Theorem 2.1 can be extended to operators of the form

$$P = \sum_{j,k=1}^{d} \left(D_j - b_j(x)\right) a^{jk}(x) \left(D_k - b_k(x)\right) + V(x)$$

where $b_j \in S_\Omega\big(\langle x \rangle, g\big)$.
(ii) In the case $\Omega = \mathbb{R}^d$ the above result has been proved by Doi [D3].
(iii) Without lack of generality one may assume $s = 0$ in the theorem.
Moreover working with $\widetilde{u}(t) = \mathrm{e}^{-i(1+C_0)t} u(t)$ one may assume $V \geq 1$ in (2.5)(ii) and $\Lambda_D = P_D^{1/2}$ which we will assume in that follows.

3. The contradiction argument

Our goal is to begin the proof by contradiction of Theorem 2.1. We shall first consider a version of the estimate which is localized in frequency.

Let $T > 0$ and $I = \big]0, T\big[$. Let $\theta \in \mathcal{C}_0^\infty(\mathbb{R})$ be such that $\operatorname{supp} \theta \subset \Big\{ t \ : \ \frac{1}{2} \leq |t| \leq 2 \Big\}$.

Theorem 3.1. *Let $\chi_0 \in \mathcal{C}_0^\infty\big(\mathbb{R}^d\big)$ be fixed. There exists $C > 0$, $h_0 > 0$ such that for all h in $\big]0, h_0\big[$ we have*

$$\int_0^T \Big\| \chi_0 \theta\big(h^2 P_D\big) P_D^{1/4} u(t) \Big\|^2_{L^2(\Omega)} \leq C \left\| u_0 \right\|^2_{L^2(\Omega)} \tag{3.1}$$

for all $u_0 \in L^2(\Omega)$.

Here $\theta\big(h^2 P_D\big)$ is defined by the functionnal calculus of selfadjoint operators.

Recall that K is our compact obstacle. We take $R_0 \geq 1$ so large that

$$K \subset \big\{ x \in \mathbb{R}^n \ : \ |x| < R_0 \big\}. \tag{3.2}$$

Let $R_1 > R_0$ be such that $\operatorname{supp}\chi_0 \subset \{x \in \mathbb{R}^d \ : \ |x| < R_1\}$. Let $\chi_1 \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ be such that $0 \leq \chi_1 \leq 1$ and

$$\begin{cases} \chi_1(x) = 1 \text{ if } |x| \leq R_1 + 2 \\ \operatorname{supp}\chi_1 \subset \{x \ : \ |x| \leq R_1 + 3\}. \end{cases} \tag{3.3}$$

Then $\chi_0\chi_1 = \chi_0$. Moreover let us set

$$\theta_1(t) = t^{\frac{1}{4}}\theta(t), \ \ \theta_2(t) = t^{-\frac{1}{4}}\theta(t). \tag{3.4}$$

It is easy to see that (3.1) will be implied by the following estimate.

$$\begin{cases} \exists C > 0, \exists h_0 > 0 \ : \ \forall h \in \left]0, h_0\right[, \forall u_0 \in \mathcal{C}_0^\infty(\Omega), \\ \displaystyle\int_0^T \left\| \chi_1 h^{-\frac{1}{2}} \theta_1\big(h^2 P_D\big) u(t) \right\|^2_{L^2(\Omega)} \mathrm{d}t \leq C \left\|u_0\right\|^2_{L^2}. \end{cases} \tag{3.5}$$

We shall prove (3.5) by contradiction. Assuming it is false, taking $h_0 = \dfrac{1}{k}$, $C = k$, $k \in \mathbb{N}^*$, we deduce sequences $\big(h_k\big) \to 0$, $u_k^0 \in \mathcal{C}_0^\infty(\Omega)$, such that

$$\int_0^T \left\| \chi_1 h_k^{-\frac{1}{2}} \theta_1\big(h_k^2 P_D\big) u_k(t) \right\|^2_{L^2(\Omega)} \mathrm{d}t > k \left\|u_k^0\right\|^2_{L^2}.$$

It follows that the left hand side does not vanish. Therefore if we set

$$\begin{cases} \alpha_k^2 = \displaystyle\int_0^T \left\| \chi h_k^{-\frac{1}{2}} \theta_1\big(h_k^2 P_D\big) u_k(t) \right\|^2_{L^2(\Omega)} \mathrm{d}t > 0, \\ \widetilde{u}_k^{\,0} = \dfrac{1}{\alpha_k} u_k^0, \ \ \widetilde{u}_k = \dfrac{1}{\alpha_k} u_k, \\ w_k = h_k^{-\frac{1}{2}} \theta_1\big(h_k^2 P_D\big) \widetilde{u}_k, \end{cases} \tag{3.6}$$

we see that

$$\begin{cases} (i) \quad \displaystyle\int_0^T \left\|\chi_1 w_k(t)\right\|^2_{L^2(\Omega)} \mathrm{d}t = 1, \\ (ii) \quad \left\|\widetilde{u}_k^{\,0}\right\|_{L^2(\Omega)} < \dfrac{1}{k}. \end{cases} \tag{3.7}$$

4. The sequence (w_k) is bounded in $L^2\big(\mathbb{R}, L^2_{loc}(\mathbb{R}^d)\big)$

We shall prove in this section the following result.

Proposition 4.1. *For any $\chi \in \mathcal{C}_0^\infty\big(\mathbb{R}^d\big)$ one can find a positive constant C such that*

$$\int_0^T \|\chi w_k(t)\|^2_{L^2(\Omega)}\,\mathrm{d}t \,\leq\, C \tag{4.1}$$

for all $k \geq 1$.

Proof

We begin by extending to the whole $\mathbb{R}^d$ the operator P given in (2.1).
Let $\chi_2 \in \mathcal{C}_0^\infty\big(\mathbb{R}^d\big)$ be such that $0 \leq \chi_2 \leq 1$ and

$$\chi_2(x) = 1 \text{ if } |x| \leq R_0,\;\; \chi_2(x) = 0 \text{ if } |x| \geq R_0 + 1. \tag{4.2}$$

Then we set for $x \in \mathbb{R}^d$,

$$\widetilde{P} = \sum_{j,k=1}^{d} D_j\big(\chi_2 \delta_{jk} D_k\big) + \sum_{j,k=1}^{d} D_j\,\big(\big(1-\chi_2\big)a^{jk}(x)D_k\big) + \chi_2 + \big(1-\chi_2\big)V \tag{4.3}$$

where δ_{jk} denotes the Kronecker symbol.

The principal symbol of $\widetilde{P}$ is

$$\begin{cases} \widetilde{p}(x,\xi) = \displaystyle\sum_{j,k=1}^{d} \widetilde{a}^{jk}(x)\xi_j\xi_k \\ \widetilde{a}^{jk}(x) = \chi_2(x)\delta_{jk} + \big(1-\chi_2(x)\big)a^{jk}(x) \end{cases} \tag{4.4}$$

According to conditions (2.2), (2.5), (2.6) we have the following,

$$\begin{cases} (i) & \widetilde{P} = P \text{ if } |x| \geq R_0 + 1, \\ (ii) & \widetilde{p}(x,\xi) \geq \widetilde{c}\,|\xi|^2\,, x \in \mathbb{R}^d, \xi \in \mathbb{R}^d, \widetilde{c} > 0, \\ (iii) & \widetilde{a}^{jk} \in S_{\mathbb{R}^d}(1,g) = S(1,g), \nabla_x \widetilde{a}^{jk}(x) = o\big(\,|x|^{-1}\,\big), |x| \to +\infty, \\ (iv) & \widetilde{V} = \chi_2 + \big(1-\chi_2\big)V \in S\big(\,\langle\, x\,\rangle^2\,, g\big) \text{ and } \widetilde{V} \geq 1, \\ (v) & \text{The flow of } \widetilde{p} \text{ in non trapping.} \end{cases} \tag{4.5}$$

Let $\chi_3 \in \mathcal{C}_0^\infty\big(\mathbb{R}^d\big)$ be such that $0 \leq \chi_3 \leq 1$ and with R_1 defined in (3.3),

$$\chi_3(x) = 1 \text{ if } |x| \leq R_1 + 1,\;\; \chi_3(x) = 0 \text{ if } |x| \geq R_1 + 2. \tag{4.6}$$

Then, according to (3.3), we have for $\alpha \neq 0$,

$$\operatorname{supp} \partial^\alpha \chi_3 \subset \big\{x \;:\; R_1 + 1 \leq |x| \leq R_1 + 2\big\} \subset \big\{x \;:\; \chi_1(x) = 1\big\}. \tag{4.7}$$

Moreover since $R_1 > R_0$ (see (3.2)) we have

$$\text{supp}\,\big(1-\chi_3\big) \subset \big\{x \ : \ |x| > R_0+1\big\}. \tag{4.8}$$

It follows from (4.5) (i) that

$$P = \widetilde{P} \ \text{ on } \ \text{supp}\,\big(1-\chi_3\big). \tag{4.9}$$

Now with w_k defined in (3.6) we set,

$$U_k = \big(1-\chi_3\big)w_k. \tag{4.10}$$

Then we have

$$\begin{cases} \big(D_t - \widetilde{P}\big)U_k = G_k \\ G_k = \big[\widetilde{P}, \chi_3\big]w_k \\ \\ U_k(0) = \big(1-\chi_3\big)h_k^{-\frac{1}{2}}\theta_1\big(h_k^2 P_D\big)\widetilde{u}_k^0. \end{cases} \tag{4.11}$$

According to conditions (ii) to (v) in (4.5) we may apply Theorem 2.8 in [D3] with $s = -\dfrac{1}{2}$. It follows that for any $\chi \in \mathcal{C}_0^\infty\big(\mathbb{R}^d\big)$ and any $\nu > 0$ we have,

$$\int_0^T \|\chi U_k(t)\|_{L^2}^2 \,\mathrm{d}t \ \le \ C_\nu\Big(\Big\|E_{-\frac{1}{2}}U_k(0)\Big\|_{L^2}^2 + \int_0^T \Big\| \langle\, x\,\rangle^{\frac{1+\nu}{2}} E_{-1}G_k(t)\Big\|_{L^2}^2 \mathrm{d}t\Big) \tag{4.12}$$

where E_s is the pseudo-differential operator with symbol $e_s(x,\xi) = \big(1+\widetilde{p}(x,\xi)+|x|^2\,\big)^{s/2}$ which belongs to $S\big(\big(\,|\xi|+\langle\, x\,\rangle\,\big)^s, g\big)$.

To handle the first term in the right hand side of (4.12) we shall need the following Lemma.

Lemme 4.2. *Let $Q = P_D^{\frac{1}{2}}\big(1-\chi_3\big)A_{-1}$ where $A_{-1} \in \mathcal{O}pS\,\Big(\big(\,|\xi|+\langle\, x\,\rangle\,\big)^{-1}, g\Big)$. Then Q is bounded from $L^2\big(\mathbb{R}^d\big)$ to $L^2(\Omega)$.*

Proof

Let $\mathcal{V}(\Omega) = \big\{u \in H_0^1(\Omega) \ : V^{\frac{1}{2}}u \in L^2(\Omega)\big\}$ endowed with the norm $\|\,u\,\|_{\mathcal{V}(\Omega)}^2 = \|\,u\,\|_{H^1}^2 + \|\,V^{\frac{1}{2}}u\,\|_{L^2}^2$. It is well known that $\mathcal{V}(\Omega)$ is the domain of $P_D^{\frac{1}{2}}$ and that $\|\,u\,\|_{\mathcal{V}(\Omega)}$ is equivalent to $\|\,P_D^{\frac{1}{2}}u\,\|_{L^2(\Omega)}$. Moreover since $|V|^{\frac{1}{2}} \le C\,\langle\, x\,\rangle$ we have,

$$\big\|\big(1-\chi_3\big)f\big\|_{\mathcal{V}(\Omega)} \le C\,\Big(\|\,f\,\|_{H_1(\mathbb{R}^d)} + \|\langle\, x\,\rangle\, f\|_{L^2(\mathbb{R}^d)}\Big)$$

whenever the right hand side is finite.

It follows that we can write

$$\begin{aligned}\| Qu \|_{L^2(\Omega)} \le C_1 \left\| (1-\chi_3) A_{-1} u \right\|_{\mathcal{V}(\Omega)} &\le C_2 \left(\| A_{-1} u \|_{H^1(\mathbb{R}^d)} + \| \langle x \rangle A_{-1} u \|_{L^2(\mathbb{R}^d)} \right) \\ &\le C_3 \| u \|_{L^2(\mathbb{R}^d)}\end{aligned}$$

□

Now let us set $① = \left\| E_{-\frac{1}{2}} U_k(0) \right\|^2_{L^2(\mathbb{R}^d)}$. According to (4.11) we have,

$$① \le C \left\| E_{-\frac{1}{2}} (1-\chi_3) h_k^{-\frac{1}{2}} \theta_1(h_k^2 P_D) \widetilde{u}_k^0 \right\|^2_{L^2} = C \left\| E_{-\frac{1}{2}} (1-\chi_3) P_D^{\frac{1}{4}} \theta(h_k^2 P_D) \widetilde{u}_k^0 \right\|^2_{L^2}$$

Introducing $S = E_{-\frac{1}{2}} (1-\chi_3) P_D^{\frac{1}{4}}$ we can write

$$\begin{aligned}① &\le C \left(S\theta(h_k^2 P_D) \widetilde{u}_k^0, S\theta(h_k^2 P_D) \widetilde{u}_k^0 \right) = \left(\theta(h_k^2 P_D) S^* S \theta(h_k^2 P_D) \widetilde{u}_k^0, \widetilde{u}_k^0 \right) \\ ① &\le C \left\| \theta(h_k^2 P_D) S^* S \theta(h_k^2 P_D) \widetilde{u}_k^0 \right\|_{L^2} \left\| \widetilde{u}_k^0 \right\|_{L^2}\end{aligned}$$

Now

$$S^* S = P_D^{-\frac{1}{4}} P_D^{\frac{1}{2}} (1-\chi_3) A_{-1} (1-\chi_3) P_D^{\frac{1}{4}} = P_D^{-\frac{1}{4}} Q (1-\chi_3) P_D^{\frac{1}{4}}$$

where Q has been defined in Lemme 4.2 and $A_{-1} = E^*_{-\frac{1}{2}} E_{-\frac{1}{2}}$.

Using (3.4) we obtain

$$① \le C \left\| h_k^{\frac{1}{2}} \theta_2(h_k^2 P_D) Q h_k^{-\frac{1}{2}} (1-\chi_3) \theta_1(h_k^2 P_D) \widetilde{u}_k^0 \right\|_{L^2} \| \widetilde{u}_k^0 \|_{L^2}$$

Since the operators $\theta_j(h_k^2 P_D)$, $j = 1, 2$, are uniformly bounded in $L^2(\Omega)$, using Lemma 4.2 and (3.7) we obtain

$$\left\| E_{-\frac{1}{2}} U_k(0) \right\|^2_{L^2(\mathbb{R}^d)} \le C \| \widetilde{u}_k^0 \|^2_{L^2(\Omega)} \le C \tag{4.13}$$

with a uniform constant $C > 0$.

We claim now that we have (see (4.11)), uniformly in $k \ge 1$,

$$\int_0^T \left\| \langle x \rangle^{\frac{1+\nu}{2}} E_{-1} G_k(t) \right\|^2_{L^2(\mathbb{R}^d)} \mathrm{d}t = \mathcal{O}(1). \tag{4.14}$$

By (4.7) we can write $G_k = \left[\widetilde{P}, \chi_3 \right] \chi_1 w_k$. Moreover the symbolic calculus shows that the symbol of $\left[\widetilde{P}, \chi_3 \right]$ belongs to $S\left(\frac{\langle \xi \rangle^2}{\langle x \rangle \langle \xi \rangle}, g \right)$.

It follows that the symbol of $\langle x \rangle^{\frac{1+\nu}{2}} E_{-1}\big[\widetilde{P}, \chi_3\big]$ belongs to $S(M, g)$ where

$$M(x,\xi) = \frac{\langle x \rangle^{\frac{1+\nu}{2}}}{\langle x \rangle + \langle \xi \rangle} \cdot \frac{\langle \xi \rangle^2}{\langle x \rangle \langle \xi \rangle} \leq C.$$

This operator is therefore L^2 bounded, so using (3.7) we obtain

$$\int_0^T \Big\| \langle x \rangle^{\frac{1+\nu}{2}} E_{-1} G_k(t) \Big\|^2_{L^2(\mathbb{R}^d)} \mathrm{d}t \leq C' \int_0^T \|\chi_1 w_k(t)\|^2_{L^2(\Omega)} \, \mathrm{d}t \leq C'$$

which proves (4.14).

Using (4.10), (4.12), (4.13) and (4.14) we conclude that

$$\int_0^T \big\|\chi\big(1-\chi_3\big) w_k(t)\big\|^2_{L^2(\Omega)} \, \mathrm{d}t = \mathcal{O}(1). \tag{4.15}$$

Since by (3.7) we have $\displaystyle\int_0^T \|\chi_1 w_k(t)\|^2_{L^2(\Omega)} \, \mathrm{d}t = 1$ and since by (3.3) and (4.6) we have $\chi_1 + \big(1-\chi_3\big) \geq 1$ we obtain (4.1). The proof of Proposition 4.1 is complete. □

5. The measure μ and its properties

We shall set

$$\begin{cases} \underline{w}_k(t) = \mathbb{1}_\Omega w_k(t), \\ W_k = \mathbb{1}_{[0,T]} \, \underline{w}_k. \end{cases} \tag{5.1}$$

It follows from Proposition 4.1 that the sequence $\big(W_k\big)$ is bounded in $L^2\left(\mathbb{R}_t, L^2_{loc}\big(\mathbb{R}^d\big)\right)$.

Now to a symbol $a = a(x,t,\xi,\tau) \in \mathcal{C}_0^\infty\big(T^*\mathbb{R}^{d+1}\big)$ we associate the semi-classical pseudo-differential operator (pdo) by the formula

$$\begin{aligned} &\mathcal{O}p(a)\big(x,t,hD_x,h^2D_t\big)v(x,t) = \\ &\qquad (2\pi h)^{-(d+1)} \iint \mathrm{e}^{i\left(\frac{x-y}{h}\xi + \frac{t-s}{h^2}\tau\right)} \varphi(y) a(x,t,\xi,\tau) v(y,s) \mathrm{d}y \mathrm{d}s \mathrm{d}\xi \mathrm{d}\tau \end{aligned} \tag{5.2}$$

where $\varphi \in \mathcal{C}_0^\infty\big(\mathbb{R}^d\big)$ is equal to one on a neighborhood of the x-projection of the support of a.

We note that by the symbolic calculus the operator $\mathcal{O}p(a)$ is, modulo operators bounded in L^2 by $\mathcal{O}\big(h^\infty\big)$, independant of the function φ. The following result is classical and introduces the notion of semi-classical defect measure.

Proposition 5.1. *There exists a subsequence* $\big(W_{\sigma(k)}\big)$ *and a Radon measure* μ *on* $T^*\mathbb{R}^{d+1}$ *such that for every* $a \in \mathcal{C}_0^\infty\big(T^*\mathbb{R}^{d+1}\big)$ *one has*

$$\lim_{k\to+\infty} \Big(\mathcal{O}p(a)\big(x,t,h_{\sigma(k)}D_x,h_{\sigma(k)}^2D_t\big)W_{\sigma(k)},W_{\sigma(k)}\Big)_{L^2(\mathbb{R}^{d+1})} = \langle\,\mu,a\,\rangle\,.$$

Here are the two main properties of the measure μ which will be used later on.

Theorem 5.2. *The support of* μ *is contained in the set*

$$\Sigma = \big\{(x,t,\xi,\tau)\in T^*\mathbb{R}^{d+1}\setminus\{0\}\ :\ x\in\overline{\Omega},t\in\big[0,T\big]\ \textit{and}\ \tau+p(x,\xi)=0\big\}.$$

Proof

See section 9.2 in the appendix. □

To state the propagation result let us recall some notations. Let $M=\Omega\times\mathbb{R}_t$. We set

$$T_b^*M = T^*M\setminus\{0\}\cup T^*\partial M\setminus\{0\}.$$

We have a natural application of restriction

$$\pi\ :\ T^*\mathbb{R}^{d+1}_{|\overline{M}}\setminus\{0\}\to T_b^*M$$

which is the identity on $T^*\mathbb{R}^{d+1}_{|M}\setminus\{0\}$ (see section 9.1 for details).

With Σ defined in Theorem 5.2 we set $\Sigma_b=\pi(\Sigma)$. The measure μ has its support in $\Sigma\subset T^*\mathbb{R}^{d-1}_{|\overline{M}}\setminus\{0\}$ while for $\zeta\in\Sigma_b$ the generalized bicharacteristic $\Gamma(t,\zeta)$ lives in Σ_b.

Then we can state an important result of this paper.

Theorem 5.3. *Let* $\zeta\in\Sigma_b$ *and* $s_1,s_2\in\mathbb{R}$. *Then we have*

$$\pi^{-1}\big(\Gamma(s_1,\zeta)\big)\cap\operatorname{supp}\mu=\emptyset\iff\pi^{-1}\big(\Gamma(s_2,\zeta)\big)\cap\operatorname{supp}\mu=\emptyset$$

For the proof, see the appendix, section 9.2.

6. The measure μ does not vanish identically

The purpose of this section is to prove the following results.

Let $A\geq1$, $R\geq1$, $\psi_A\in\mathcal{C}_0^\infty(\mathbb{R})$, $\Phi_R\in\mathcal{C}_0^\infty(\mathbb{R})$ be such that $0\leq\psi_A,\Phi_R\leq1$ and

$$\psi_A(\tau)=1\ \text{if}\ |\tau|\leq A,\ \ \Phi_R(t)=1\ \text{if}\ |t|\leq R. \tag{6.1}$$

Proposition 6.1. *There exist positive constants* A_0, R_0, k_0 *such that*

$$\int_{\mathbb{R}}\big\|\psi_A\big(h_k^2D_t\big)\Phi_R\big(h_k^2\Delta\big)\mathbb{1}_{[0,T]}\chi_1\underline{w}_k(t)\big\|^2_{L^2(\mathbb{R}^d)}\,\mathrm{d}t\geq\frac{1}{2}, \tag{6.2}$$

when $A \geq A_0$, $R \geq R_0$, $k \geq k_0$. Here χ_1, $\underline{w}_k$ have been defined in (3.3), (5.1) and Δ is the usual Laplacien.

Corollary 6.2. *The measure μ defined in Proposition 5.1 does not vanish identically.*

Proof

Let $\widetilde{\chi}_1 \in \mathcal{C}_0^\infty\big(\mathbb{R}^d\big)$ be such that $\widetilde{\chi}_1 = 1$ on $\operatorname{supp}\chi_1$. Let $\varphi = \varphi(t) \in \mathcal{C}_0^\infty(\mathbb{R})$ and $a(x,t,\xi,\tau) = \varphi(t)\chi_1(x)\psi_A^2(\tau)\Phi_R^2\big(\,|\xi|^2\,\big)\chi_1$. It follows from (6.2) that

$$\big(a\big(x,t,hD_x,h^2D_t\big)\widetilde{\chi}_1 \mathbb{1}_{[0,T]}\underline{w}_k(t),\, \mathbb{1}_{[0,T]}\underline{w}_k\big)_{L^2(\mathbb{R}^{d+1})} \geq \frac{1}{3}.$$

Since the left hand side with the subsequence $\sigma(k)$ tends to $\langle\, \mu, a\,\rangle$ when $k \to +\infty$ the Corollary follows.

□

Proof of Proposition 6.1

We shall need the following Lemma.

Lemma 6.3. *Let $\theta \in \mathcal{C}_0^\infty(\mathbb{R})$, $\chi \in \mathcal{C}_0^\infty\big(\mathbb{R}^d\big)$. Then there exists $C > 0$ such that*

$$\begin{aligned}
&(i) \quad \big\|\big[\theta\big(h^2P_D\big),\chi\big]u\big\|^2_{L^2(\Omega)} \leq C\|\,hu\,\|^2_{L^2(\Omega)},\\
&(ii) \quad \big\|\partial_j\theta\big(h^2P_D\big)u\big\|^2_{L^2(\Omega)} \leq C\,\big\|h^{-1}u\big\|_{L^2(\Omega)}\,,\\
&(iii) \quad \big\|\partial_j\big[\theta\big(h^2P_D\big),\chi\big]u\big\|_{L^2(\Omega)} \leq C\|\,u\,\|_{L^2(\Omega)},
\end{aligned}$$

for all $j = 1,\cdots,d$, $h > 0$ and $u \in L^2(\Omega)$.

Proof

See the Appendix, section 9.3.

□

Let us set

$$\text{(6.3)} \qquad \begin{cases} I = \big(\mathrm{Id} - \psi_A\big(h^2D_t\big)\big)\,\mathbb{1}_{[0,T]}\chi_1 w_k \\ \widetilde{\psi}(\tau) = \dfrac{1-\psi(\tau)}{\tau}. \end{cases}$$

Then $\widetilde{\psi} \in L^\infty(\mathbb{R})$ and $\Big|\widetilde{\psi}(\tau)\Big| \leq \dfrac{1}{A}$ for all $\tau \in \mathbb{R}$.

Now we can write $I = \widetilde{\psi}\big(h_k^2D_t\big)h_k^2D_t\big(\mathbb{1}_{[0,T]}\chi_1 w_k\big)$. Using (3.6) and the fact that $D_t\widetilde{u}_k = P_D\widetilde{u}_k$ we deduce that

$$\text{(6.4)} \qquad \begin{cases} I = \ ① + \ ② \\ ① = \dfrac{1}{i}\widetilde{\psi}\big(h_k^2D_t\big)\chi_1 h_k^2\big(w_k(0)\delta_{t=0} - w_k(T)\delta_{t=T}\big) \\ ② = -\widetilde{\psi}\big(h_k^2D_t\big)\,\mathbb{1}_{[0,T]}\chi_1 h_k^2P_D h_k^{-\frac{1}{2}}\theta_1\big(h_k^2P_D\big)\widetilde{u}_k. \end{cases}$$

Estimate of ①

If $a \in \mathbb{R}$, we have $\widetilde{\psi}(h_k^2 D_t)\delta_{t=a} = \overline{\mathcal{F}}\left(\widetilde{\psi}(h_k^2\tau)\mathrm{e}^{-ia\tau}\right)$, so by Parseval formula we have

$$\left\|\widetilde{\psi}(h_k^2 D_t)\delta_{t=a}\right\|^2_{L^2(\mathbb{R})} = c_n \int \left|\widetilde{\psi}(h_k^2\tau)\right|^2 \mathrm{d}\tau = c_n h_k^{-2} \int \frac{|\psi_A(\tau)-1|^2}{|\tau|^2}\mathrm{d}\tau$$

It follows from (3.6) that

$$\int_{\mathbb{R}} \| \text{①} \|^2_{L^2(\Omega)}\mathrm{d}t \leq C\, h_k^4\, h_k^{-2}\, h_k^{-1}\left(\|\widetilde{u}_k(0)\|^2_{L^2} + \|\widetilde{u}_k(T)\|^2_{L^2}\right)$$

Applying the energy estimate and (3.7) we obtain

$$\int_{\mathbb{R}} \| \text{①} \|^2_{L^2}\mathrm{d}t \leq C\, h_k \left\|\widetilde{u}_k^0\right\|^2_{L^2} = o(1). \tag{6.5}$$

Estimate of ②

Let $\widetilde{\theta} \in \mathcal{C}_0^\infty(\mathbb{R})$ be such that $\widetilde{\theta} = 1$ on the support of θ_1. Then we can write with $\widetilde{\theta}_1(t) = t\,\widetilde{\theta}(t)$

$$\text{②} = -\widetilde{\psi}(h_k^2 D_t)\left[\chi_1, \widetilde{\theta}_1(h_k^2 P_D)\right] \mathbb{1}_{]0,T]} h_k^{-\frac{1}{2}} \theta_1(h_k^2 P_D)\widetilde{u}_k - \widetilde{\psi}(h_k^2 D_t)\widetilde{\theta}_1(h_k^2 P_D)\mathbb{1}_{[0,T]}\chi_1 w_k(t)$$

Using Lemma 6.3 (i) and the fact that,

$$\left\|\widetilde{\psi}(h_k^2 D_t)\right\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} = \mathcal{O}\left(\frac{1}{A}\right), \ \left\|\widetilde{\theta}_1(h_k^2 P_D)\right\|_{L^2(\Omega)\to L^2(\Omega)} = \mathcal{O}(1)$$

uniformly in k we obtain,

$$\int_{\mathbb{R}} \| \text{②} \|^2_{L^2(\Omega)}\mathrm{d}t \leq \frac{C}{A}\left(\int_0^T \left\|h_k^{\frac{1}{2}}\widetilde{u}_k(t)\right\|^2_{L^2(\Omega)}\mathrm{d}t + \int_0^T \|\chi_1 w_k(t)\|^2_{L^2(\Omega)}\,\mathrm{d}t\right)$$

Using the energy estimate and (3.7) we deduce that

$$\int_{\mathbb{R}} \| \text{②} \|^2_{L^2(\Omega)}\mathrm{d}t = o(1) + \mathcal{O}\left(\frac{1}{A}\right). \tag{6.6}$$

Taking k and A sufficiently large and using (3.7), (6.3), (6.4), (6.5), (6.6) we obtain

$$\int_{\mathbb{R}} \left\|\psi_A(h_k^2 D_t)\mathbb{1}_{[0,T]}\chi_1 w_k(t)\right\|^2_{L^2(\Omega)} \geq \frac{1}{2}. \tag{6.7}$$

Now with Φ_R defined in (6.1) we set

$$\mathrm{II} = h_k^{-\frac{1}{2}} \big(\mathrm{Id} - \Phi_R\big(h_k^2\Delta\big)\big)\, \psi_A\big(h_k^2 D_t\big)\mathbb{1}_{[0,T]}\chi_1 \underline{w}_k(t). \tag{6.8}$$

Since $\mathrm{supp}\,\big(1-\Phi_R(t)\big) \subset \big\{t\in\mathbb{R}\ :\ |t|\geq R\big\}$ we have by Fourier transform

$$\big\|\big(\mathrm{Id}-\Phi_R\big(h^2\Delta\big)\big)h_k^{-1}v\big\|^2_{L^2(\mathbb{R}^d)} \leq \frac{C}{R}\sum_{j=1}^d \|\partial_j v\|^2_{L^2(\mathbb{R}^d)}\ \ ,\ \ v\in H^1\big(\mathbb{R}^d\big). \tag{6.9}$$

Now by (3.6) we have $h_k^{-\frac{1}{2}}\underline{w}_k = h_k^{-1}\underline{v}_k$, $v_k = \theta_1\big(h_k^2 P_D\big)\widetilde{u}_k$.

Thus applying (6.9) we obtain

$$\int_{\mathbb{R}} \|\,\mathrm{II}\,\|^2_{L^2(\mathbb{R}^d)}\mathrm{d}t \leq \frac{C}{R}\sum_{j=1}^d \int_{\mathbb{R}} \big\|\partial_j\psi_A\big(h^2D_t\big)\mathbb{1}_{[0,T]}\chi_1\underline{v}_k\big\|^2_{L^2(\mathbb{R}^d)}\,.$$

Since $v_k\in H^1_0(\Omega)$ we have

$$\partial_j\big(\chi_1\underline{v}_k\big) = \partial_j\big(\mathbb{1}_\Omega\chi_1 v_k\big) = \mathbb{1}_\Omega\partial_j\big(\chi_1 v_k\big)$$

It follows that

$$\int_{\mathbb{R}} \|\,\mathrm{II}\,\|^2_{L^2(\mathbb{R}^d)}\mathrm{d}t \ \leq\ \frac{C}{R}\sum_{j=1}^d \int_{\mathbb{R}} \big\|\partial_j\psi_A\big(h_k^2D_t\big)\mathbb{1}_{[0,T]}\chi_1 v_k\big\|^2_{L^2(\Omega)}\,\mathrm{d}t$$

Let $\widetilde{\theta}\in\mathcal{C}_0^\infty(\mathbb{R})$ be such that $\widetilde{\theta}=1$ near the support of θ_1. Then $\big(1-\widetilde{\theta}(t)\big)\theta_1(t)=0$. We first consider

$$①\ =\ \int_{\mathbb{R}} \Big\|\partial_j\widetilde{\theta}\big(h_k^2P_D\big)\psi_A\big(h_k^2D_t\big)\mathbb{1}_{[0,T]}\chi_1 v_k\Big\|^2_{L^2(\Omega)}\,\mathrm{d}t. \tag{6.10}$$

Using Lemma 6.3 (ii) we obtain

$$①\ \leq C\int_{\mathbb{R}} \big\|\psi_A\big(h_k^2D_t\big)h_k^{-1}\mathbb{1}_{[0,T]}\chi_1 v_k(t)\big\|^2_{L^2(\Omega)}\,\mathrm{d}t \leq C'\int_0^T \Big\|h_k^{-\frac{1}{2}}\chi_1 w_k(t)\Big\|^2_{L^2(\Omega)}\mathrm{d}t$$

so by (3.6) we obtain

$$①\ \leq C' h_k^{-1}. \tag{6.11}$$

It remains to consider

$$②\ =\ \int_{\mathbb{R}} \Big\|\partial_j\Big(\mathrm{Id}-\widetilde{\theta}\big(h_k^2P_D\big)\Big)\,\psi_A\big(h_k^2D_t\big)\mathbb{1}_{[0,T]}\chi_1 v_k(t)\Big\|^2_{L^2(\Omega)}\,\mathrm{d}t. \tag{6.12}$$

Since $v_k(t) = \theta_1\big(h_k^2 P_D\big)\widetilde{u}_k$ and $\Big(\mathrm{Id} - \widetilde{\theta}\big(h_k^2 P_D\big)\Big)\,\theta_1\big(h_k^2 P_D\big) = 0$ we obtain

$$② \leq \int_{\mathbb{R}} \Big\| \partial_j \Big[\widetilde{\theta}\big(h_k^2 P_D\big), \chi_1\Big]\, \psi_A\big(h_k^2 D_t\big)\, \mathbb{1}_{[0,T]} \widetilde{\chi}_1 v_k(t) \Big\|^2_{L^2(\Omega)}\, \mathrm{d}t$$

where $\widetilde{\chi}_1 \in \mathcal{C}_0^\infty\big(\overline{\Omega}\big)$, $\widetilde{\chi}_1 = 1$ on $\operatorname{supp}\chi_1$.

By Lemma 6.3 (iii) we obtain

$$② \leq C \int_{\mathbb{R}} \big\| \psi_A\big(h_k^2 D_t\big)\, \mathbb{1}_{[0,T]} \widetilde{\chi}_1 \theta_1\big(h_k^2 P_D\big) \widetilde{u}_k(t) \big\|^2_{L^2(\Omega)}\, \mathrm{d}t.$$

Since the operator $\psi_A\big(h_k^2 D_t\big)$ is uniformly L^2 bounded we obtain by the energy estimate

$$② \leq C' \int_0^T \|\widetilde{u}_k(t)\|^2_{L^2}\, \mathrm{d}t = \mathcal{O}(1).$$

It follows from (6.10), (6.11), (6.12) that

$$\int_{\mathbb{R}} \|\mathrm{II}\|^2_{L^2}\, \mathrm{d}t \;\leq\; \frac{C}{R}\big(h_k^{-1} + \mathcal{O}(1)\big). \tag{6.13}$$

Using (6.8) we deduce that

$$\int_{\mathbb{R}} \big\| \big(\mathrm{Id} - \Phi_{\mathbb{R}}\big(h^2\Delta\big)\big)\, \psi_A\big(h^2 D_t\big)\, \mathbb{1}_{[0,T]} \chi_1 \underline{w}_k(t) \big\|^2_{L^2(\mathbb{R}^d)}\, \mathrm{d}t \;\leq\; \frac{C}{R}\left(1 + \mathcal{O}\big(h_k\big)\right). \tag{6.14}$$

Taking R sufficiently large and using (6.7) we obtain

$$\int_{\mathbb{R}} \big\| \Phi_{\mathbb{R}}\big(h^2\Delta\big) \psi_A\big(h^2 D_t\big)\, \mathbb{1}_{[0,T]} \chi_1 \underline{w}_k(t) \big\|^2_{L^2}\, \mathrm{d}t \;\geq\; \frac{1}{3},$$

which is (6.2). The proof of Proposition 6.1 is complete. □

7. The measure μ vanishes in the incoming set

We pursue here our reasoning by contradiction in proving that the measure μ vanishes in the incoming set. Let us state the main result of this section.

Let $\widetilde{P}$ be the operator defined by (4.3) satisfying the conditions (4.5).

Theorem 7.1. *Let* $m_0 = \big(x_0, t_0, \xi_0, \tau_0\big) \in T^*\mathbb{R}^{d+1}$ *be such* $\xi_0 \neq 0$, $\tau_0 + \widetilde{p}\big(x_0, \xi_0\big) = 0$, $|x_0| \geq 3R_0$, $\displaystyle\sum_{j,k=1}^{d} \widetilde{a}^{jk}\big(x_0\big) x_{0j}\xi_{0k} \leq -3\delta\,|x_0|\,|\xi_0|$ *(for some* $\delta > 0$ *small enough). Then* $m_0 \notin \operatorname{supp}\mu$

The rest of this section will be devoted to the proof of this result. Il will be a consequence of an estimate which will be proved in constructing an appropriate escape function and will require several Lemmas.

Lemma 7.2. *Let us set* $e_0(x,\xi) = \sum_{j,k=1}^{d} \widetilde{a}^{jk}(x) x_j \frac{\xi_k}{\langle\, \xi \,\rangle}$. *Then there exist positive constants* R, C_0, C_1 *such that*

$$H_{\tilde{p}} e_0(x,\xi) \geq C_0 \,|\xi| - C_1, \quad \forall (x,\xi) \in T^*\mathbb{R}^d, \quad |x| \geq R$$

where $H_{\tilde{p}}$ *denotes the Hamiltonian field of* $\widetilde{P}$.

Proof

It is an easy computation which uses the conditions (ii) and (iii) in (4.5). □

Lemma 7.3. *Under condition* (v) *in (4.5) there exist* $e \in S\big(\langle\, x \,\rangle, g\big)$ *and positive constants* C, C', R' *such that*

$$(a) \quad H_{\tilde{p}} e(x,\xi) \geq C\,|\xi| - C', \quad \forall (x,\xi) \in T^*\mathbb{R}^d,$$
$$(b) \quad e(x,\xi) = e_0(x,\xi) \ \text{ if } \ |x| \geq R'.$$

Proof See Doi [D3]. □

The symbol e is an escape function. However it is not adapted to our situation because its Poisson bracket with our potentiel $\widetilde{V}$ (see (4.5) (iv)) belongs to $S\Big(\frac{\langle\, x \,\rangle^2}{\langle\, \xi \,\rangle}, g\Big)$ so does not correspond to an operator bounded in L^2 which will be required later on. We shall describe below a construction by Doi [D3] which will take care of this problem.

Let $\psi \in \mathcal{C}^\infty(\mathbb{R})$ be such that $0 \leq \psi \leq 1$ and

$$\psi(t) = 1 \text{ if } t \geq 2\varepsilon, \quad \operatorname{supp}\psi \subset \big[\varepsilon, +\infty\big[, \quad \psi'(t) \geq 0 \ \ \forall t \in \mathbb{R}, \tag{7.1}$$

where $\varepsilon > 0$ is a small constant choosen later on.

We set

$$\begin{cases} \psi_0(t) = 1 - \psi(t) - \psi(-t) = 1 - \psi(|t|) \\ \psi_1(t) = \psi(-t) - \psi(t) = -(\operatorname{sgn} t)\psi\big(\,|t|\,\big). \end{cases} \tag{7.2}$$

Then $\psi_j \in \mathcal{C}^\infty(\mathbb{R})$, $j = 1, 2$, and we have

$$\begin{cases} \psi_0'(t) = -(\operatorname{sgn} t)\psi'\big(\,|t|\,\big) \\ \psi_1'(t) = -\psi'\big(\,|t|\,\big). \end{cases} \tag{7.3}$$

Let $\chi \in \mathcal{C}^\infty(\mathbb{R})$ be such that $0 \leq \chi \leq 1$ and

$$\chi(t) = 1 \text{ if } t \leq \frac{\rho}{2}, \quad \chi(t) = 0 \text{ if } t \geq \rho, \quad \rho > 0 \text{ small.} \tag{7.4}$$

With e defined in Lemma 7.3 we set

$$-\lambda = \Big(\frac{e}{\langle\, x \,\rangle}\psi_0\Big(\frac{e}{\langle\, x \,\rangle}\Big) - \Big(M_0 - \langle\, e \,\rangle^{-\nu}\Big)\,\psi_1\Big(\frac{e}{\langle\, x \,\rangle}\Big)\Big)\chi\Big(\frac{\langle\, x \,\rangle}{\sqrt{\widetilde{p}(x,\xi)}}\Big) \tag{7.5}$$

where $\nu > 0$ is an arbitrary small constant and M_0 is a sufficiently large constant.

Lemma 7.4 (Doi [D3], Lemma 8.3)**.**

$$\begin{aligned}&(i)\ \lambda \in S(1,g)\\ &(ii)\ \big[\widetilde{P}, \mathcal{O}p^w(\lambda)\big] - \frac{1}{i}\big(H_{\tilde{p}}\lambda\big)^w \in \mathcal{O}p^w S(1,g)\\ &(iii)\ \textit{There exists } M_0 > 0 \textit{ such that for any } \nu > 0 \textit{ there exist } C > 0,\ C' > 0\\ &\qquad \textit{such that } -H_{\tilde{p}}\lambda(x,\xi) \geq C\,\langle\, x\,\rangle^{-1-\nu}\,\big(\,|x| + |\xi|\,\big) - C', \quad \forall (x,\xi) \in T^*\mathbb{R}^d.\end{aligned} \tag{7.6}$$

We must now localize this escape function near the incoming set.
We shall need the following Lemma. Let us set

$$a(x,\xi) = \sum_{j,k=1}^{d} \widetilde{a}^{jk}(x) x_j \xi_k \tag{7.7}$$

Let $\xi_0 \neq 0$ be defined in Theorem 8.1.

Lemma 7.5. *There exists a symbol $\Phi \in S(1,g)$ such that $0 \leq \Phi \leq 1$ and*

(i) $\operatorname{supp}\Phi \subset \Big\{(x,\xi) \in T^*\mathbb{R}^d \ :\ |x| \geq 2R_0,\ \ a(x,\xi) \leq -\dfrac{\delta}{2}\,|x|\,|\xi|\,,\ \ |\xi| \geq \dfrac{|\xi_0|}{4}\Big\}$,

(ii) $\Big\{(x,\xi)\ :\ |x| \geq \dfrac{5}{2}R_0, \quad a(x,\xi) \leq -\delta\,|x|\,|\xi|\,,\ \ |\xi| \geq \dfrac{|\xi_0|}{2}\Big\} \subset \big\{(x,\xi)\ :\ \Phi(x,\xi) = 1\big\}$,

(iii) $\Phi(x,h\xi) = \Phi(x,\xi)$ *when* $|h\xi| \geq \dfrac{|\xi_0|}{2}$ *and* $0 < h \leq 1$,

(iv) $H_{\tilde{p}}\Phi(x,\xi) \leq 0$ *on the support of* λ,

(v) $\lambda(x,\xi) \geq 0$ *on the support of* Φ.

Proof

Let φ_j, $j = 1,2,3$, be such $\varphi_j \in \mathcal{C}^\infty(\mathbb{R})$, $0 \leq \varphi_j \leq 1$ and

$$\begin{cases} \varphi_1(s) = 0 \text{ if } s \leq R_0,\ \ \varphi_1(s) = 1 \text{ if } s \geq \dfrac{5}{2}R_0,\ \ \varphi_1 \text{ increasing},\\ \varphi_2(s) = 0 \text{ if } s \geq -\dfrac{1}{2}\delta,\ \ \varphi_2(s) = 1 \text{ if } s \leq -\delta,\ \ \varphi_2 \text{ decreasing},\\ \varphi_3(s) = 0 \text{ if } s \leq \dfrac{1}{4}\,|\xi_0|\,,\ \ \varphi_3(s) = 1 \text{ if } s \geq \dfrac{1}{2}\,|\xi_0|\,. \end{cases} \tag{7.8}$$

Let us set

$$\Phi(x,\xi) = \varphi_1\big(\,|x|\,\big)\varphi_2\left(\frac{a(x,\xi)}{|x|\,|\xi|}\right)\varphi_3\big(\,|\xi|\,\big). \tag{7.9}$$

Then (i) and (ii) follow immediatly. Now if $|h\xi| \geq \frac{|\xi_0|}{2}$ then $|\xi| \geq \frac{|\xi_0|}{2h} \geq \frac{|\xi_0|}{2}$ so $\varphi_3\big(h\,|\xi|\,\big) = \varphi_3\big(\xi\big) = 1$ and (iii) follows.

Let us prove (iv). We have

$$
\left\{
\begin{array}{l}
H_{\tilde{p}}\Phi(x,\xi) = \ \textcircled{1} + \ \textcircled{2} + \ \textcircled{3}, \\
\textcircled{1} = \varphi_1'\big(\,|x|\,\big) H_{\tilde{p}}\,|x|\,\varphi_2\left(\dfrac{a}{|x|\ |\xi|}\right)\varphi_3\big(\,|\xi|\,\big), \\
\textcircled{2} = \varphi_1\big(\,|x|\,\big)\varphi_2'\left(\dfrac{a}{|x|\ |\xi|}\right) H_{\tilde{p}}\left(\dfrac{a}{|x|\ |\xi|}\right)\varphi_3\big(\,|\xi|\,\big), \\
\textcircled{3} = \varphi_1\big(\,|x|\,\big)\varphi_2\left(\dfrac{a}{|x|\ |\xi|}\right)\varphi_3'\big(\,|\xi|\,\big) H_{\tilde{p}}\,|\xi|\,.
\end{array}
\right. \tag{7.10}
$$

According to (7.4) and (7.5) we have $\widetilde{p}(x,\xi) \geq \dfrac{1}{\rho^2}\,\langle\, x\,\rangle^2 \geq \dfrac{1}{\rho^2}$ on the support of λ. Therefore we can choose ρ so small that $|\xi| > \dfrac{1}{2}\,|\xi_0|$ on the support of λ. It follows that $\textcircled{3} = 0$ on this set. Now an easy computation shows that $H_{\tilde{p}}\,|x| = \dfrac{2a(x,\xi)}{|x|}$ when $|x| \geq R_0$ which implies that

$$
\textcircled{1} = 2\varphi_1'\big(\,|x|\,\big)\frac{a}{|x|}\varphi_2\left(\frac{a}{|x|\ |\xi|}\right)\varphi_3\big(\,|\xi|\,\big).
$$

On the support of φ_2 we have $a \leq -\dfrac{1}{2}\delta\,|x|\ |\xi|$. Since $\varphi_1' \geq 0$, $\varphi_2 \geq 0$, $\varphi_3 \geq 0$, we conclude that

$$
\textcircled{1} \leq 0. \tag{7.11}
$$

Let us look to $\textcircled{2}$. First of all we have on the support of Φ

$$
H_{\tilde{p}}\left(\frac{a}{|x|\ |\xi|}\right) = \frac{1}{|x|\ |\xi|}H_{\tilde{p}}a + a\ H_{\tilde{p}}\left(\frac{1}{|x|\ |\xi|}\right). \tag{7.12}
$$

Since we have (see (4.5)) $\big(\widetilde{a}^{jk}(x)\big) \geq C\ \mathrm{Id}$, $\big|\nabla_x\,\widetilde{a}^{jk}(x)\big| = o\big(\,|x|^{-1}\big)$ as $|x| \to +\infty$ and $|x| \geq R_0$ on the support of Φ, taking R_0 large enough we obtain by an easy computation

$$
H_{\tilde{p}}a(x,\xi) \geq C_0\,|\xi|^2 \ \text{ on } \ \mathrm{supp}\,\Phi. \tag{7.13}
$$

We also obtain

$$
H_{\tilde{p}}\left(\frac{1}{|x|\ |\xi|}\right) = -2\frac{a(x,\xi)}{|\xi|\ |x|^3} + o\left(\frac{1}{|x|^2}\right) \ \text{ as } \ |x| \to +\infty. \tag{7.14}
$$

It follows from (7.12), (7.13), (7.14) and $|a| \leq C\,|x|\,|\xi|$ that

$$H_{\tilde{p}}\left(\frac{a}{|x|\,|\xi|}\right) \geq C_0\frac{|\xi|}{|x|} - 2\frac{a^2(x,\xi)}{|\xi|\,|x|^3} + o(1)\frac{|\xi|}{|x|}.$$

On the support of $\varphi_2'\left(\frac{a}{|x|\,|\xi|}\right)$ we have, by (8.8), $-\delta \leq \frac{a}{|x|\,|\xi|} \leq -\frac{1}{2}\delta$. It follows that $|a| \leq \delta\,|x|\,|\xi|$ so

$$-2\frac{a^2}{|\xi|\,|x|^3} \geq -\delta^2\frac{|\xi|}{|x|}.$$

Moreover on the support of $\varphi_1\big(|x|\big)$ we have $|x| \geq R_0$. So taking R_0 large enough and δ small, we obtain

$$H_{\tilde{p}}\left(\frac{a}{|x|\,|\xi|}\right) \geq \frac{C_0}{2}\frac{|\xi|}{|x|}.$$

Since, by (7.5), we have $\varphi_2'\left(\frac{a}{|x|\,|\xi|}\right) \leq 0$, we conclude that

$$\textcircled{2} \leq 0. \tag{7.15}$$

The claim (iv) in Lemma 7.5 follows then from (7.11), (7.15) and (7.10) since $\textcircled{3} = 0$ on $\operatorname{supp}\lambda$.

Finally let us look to the claim (v).

On the support of Φ we have $|x| \geq R_0$, $|\xi| \geq \frac{1}{4}\,|\xi_0|$ and $a(x,\xi) \leq -\frac{1}{2}\delta\,|x|\,|\xi|$. It follows that $\langle\, x\,\rangle \leq \sqrt{2}\,|x|$, $\langle\,\xi\,\rangle \leq C\,|\xi|$ and $a(x,\xi) \leq -C'\delta\,\langle\, x\,\rangle\,\langle\,\xi\,\rangle$. Moreover since $|x| \geq R_0$, taking R_0 large enough, we deduce from Lemma 7.3 that $e(x,\xi) = e_0(x,\xi) = \frac{a(x,\xi)}{\langle\,\xi\,\rangle}$ by (7.7). It follows that $\frac{e}{\langle\, x\,\rangle} \leq -C'\delta$ which implies that $\frac{|e|}{\langle\, x\,\rangle} \geq C'\delta$. Using (7.1), (7.2) and taking $\varepsilon \ll \delta$ we see that $\psi_0\left(\frac{e}{\langle\, x\,\rangle}\right) = 0$ and $\psi_1\left(\frac{e}{\langle\, x\,\rangle}\right) \geq 0$. It follows from (7.5) that $-\lambda = -\big(M_0 - \langle\, e\,\rangle^{-\nu}\big)\psi_1\left(\frac{e}{\langle\, x\,\rangle}\right)\chi \leq 0$.

The proof of Lemma 7.5 is complete. □

Corollary 7.6. *Let $\lambda_1 = \Phi^2\lambda$ where λ has been defined in Lemma 7.4. Then*

$$\begin{aligned}
&(i)\ \lambda_1 \in S(1,g),\\
&(ii)\ \big[\widetilde{P}, \lambda_1^w\big] - \frac{1}{i}\mathcal{O}p^w\big(H_{\tilde{p}}\lambda_1\big) \in \mathcal{O}p^w S(1,g),\\
&(iii)\ \textit{There exist two positive constants } C,\ C' \textit{ such that}\\
&\qquad -H_{\tilde{p}}\lambda_1 \geq C\,\langle\, x\,\rangle^{-1-\nu}\,\Phi^2(x,\xi)\big(\,|x| + |\xi|\,\big) - C'\Phi^2(x,\xi).
\end{aligned} \tag{7.16}$$

Proof

(i), (ii) follow from Lemma 7.4 and the fact that $\Phi^2 \in S(1,g)$. Let us look to (iii). We have

$$-H_{\tilde{p}}\lambda_1 = \big(-H_{\tilde{p}}\lambda\big)\Phi^2 - 2\lambda\Phi H_{\tilde{p}}\Phi.$$

By Lemma 7.5 we have $H_{\tilde{p}}\Phi \leq 0$ on $\operatorname{supp}\lambda$ and $\lambda \geq 0$ on $\operatorname{supp}\Phi$. It follows that $-2\lambda\Phi H_{\tilde{p}}\Phi \geq 0$. Thus (iii) follows from (7.6). □

Let now $\big(x_0\,,\xi_0\big)$ be given as in Theorem 7.1. We set

$$V_{(x_0,\xi_0)} = \big\{(x,\xi) \in T^*\mathbb{R}^d \ : \ |x-x_0| + |\xi-\xi_0| \leq \varepsilon_0\big\}.$$

Since $|x_0| \geq 3R_0$, $a\big(x_0,\xi_0\big) \leq -3\delta\,|x_0|\,|\xi_0|$ we can take ε_0 so small that we will have

$$V_{(x_0,\xi_0)} \subset \left\{(x,\xi)\ : |x| \geq \frac{5}{2}R_0, \quad a(x,\xi) \leq -\delta\,|x|\,|\xi|\,, \quad |\xi| \geq \frac{|\xi_0|}{2}\right\}.$$

It follows from Lemma 7.5 (ii) that

$$V_{(x_0,\xi_0)} \subset \big\{(x,\xi) \in T^*\mathbb{R}^d \ : \ \Phi(x,\xi) = 1\big\}. \tag{7.17}$$

Let $b \in \mathcal{C}_0^\infty\big(V_{(x_0,\xi_0)}\big)$ be such that $b\big(x_0,\xi_0\big) = 1$. It follows from (7.17) that one can find $C > 0$ such that

$$|b(x,\xi)| \leq C\Phi(x,\xi), \quad \forall (x,\xi) \in T^*\mathbb{R}^d. \tag{7.18}$$

Therefore we will have $|b(x,h\xi)| \leq C\Phi(x,h\xi)$ for all (x,ξ) in $T^*\mathbb{R}^d$ and all $h \in \big]0,1\big]$. Now on the support of $b(x,h\xi)$ we have $h\,|\xi| \geq \dfrac{|\xi_0|}{2}$ so it follows from Lemma 7.5(iii) that $\Phi(x,h\xi) = \Phi(x,\xi)$. Therefore

$$\begin{cases} \text{There exists } C > 0 \text{ such that} \\ |b(x,h\xi)| \leq C\Phi(x,\xi), \quad \forall (x,\xi) \in T^*\mathbb{R}^d, \quad \forall h \in \big]0,1\big]. \end{cases} \tag{7.19}$$

We deduce from Corollary 7.6 (iii) that

$$-H_{\tilde{p}}\lambda_1 \geq C\,\langle\, x\,\rangle^{1-\nu}\,|b(x,h\xi)|^2\,|\xi| - C', \quad \forall (x,\xi) \in T^*\mathbb{R}^d, \quad \forall h \in \big]0,1\big] \tag{7.20}$$

Let now $m_0 = \big(x_0,t_0,\xi_0,\tau_0\big)$ be as in Theorem 7.1.

Let $\varphi_0 \in \mathcal{C}_0^\infty(\mathbb{R})$, $\psi \in \mathcal{C}_0^\infty(\mathbb{R})$, $\varphi_1 \in \mathcal{C}_0^\infty\big(\mathbb{R}^d\big)$ be such that

$$\begin{cases} \varphi_0\big(t_0\big) \neq 0, \quad \psi\big(\tau_0\big) \neq 0, \\ \varphi_1(x) = 1 \text{ if } |x| \leq \dfrac{4}{3}R_0, \quad \operatorname{supp}\varphi_1 \subset \left\{x \ : \ |x| \leq \dfrac{3}{2}R_0\right\}. \end{cases} \tag{7.21}$$

Let

$$b_1(x,\xi) = b(x,\xi)\,|\xi|^{\frac{1}{2}}\,. \tag{7.22}$$

Then $b_1 \in \mathcal{C}_0^\infty\left(B\big((x_0,\xi_0),\varepsilon_0\big)\right)$ and $b_1\big(x_0,\xi_0\big) \neq 0$.

Finally let us recall for convenience that in (3.6) and (5.1) we have set

$$\begin{cases} W_k(t) = \mathbb{1}_{[0,T]}\,\underline{w}_k(t), \quad \underline{w}_k(t) = \mathbb{1}_\Omega w_k(t), \\ \\ w_k(t) = h_k^{-\frac{1}{2}} v_k(t), \quad v_k(t) = \theta_1\big(h_k^2 P_D\big)\widetilde{u}_k(t). \end{cases} \tag{7.23}$$

Lemma 7.7. *We have*

$$\int_{\mathbb{R}} \left\|\varphi_0(t)\psi\big(h_k^2 D_t\big) b_1\big(x, h_k D_x\big)\big(1-\varphi_1(x)\big) W_k(t)\right\|^2_{L^2\left(\mathbb{R}^d\right)} \mathrm{d}t = o(1) \text{ as } k \to +\infty$$

Proof

With λ_1 defined in Corollary 7.6 we set

$$N(t) = \Big(\big(M - \big(1-\varphi_1\big)\lambda_1^w\big(1-\varphi_1\big)\big)v_k(t), v_k(t)\Big)_{L^2(\Omega)}\,,$$

where M is a large constant and λ_1^w the Weyl quantization of the symbol $\lambda_1 \in S(1,g)$. Then there exists $C>0$ such that for $k \geq 1$,

$$\|v_k(t)\|^2_{L^2(\Omega)} \leq C\, N(t). \tag{7.24}$$

Setting $\Lambda = M - \big(1-\varphi_1\big)\lambda_1^w\big(1-\varphi_1\big)$ and $(.,.) = (.,.)_{L^2(\Omega)}$ we can write

$$\frac{\mathrm{d}}{\mathrm{d}t}N(t) = \left(\Lambda\frac{\mathrm{d}}{\mathrm{d}t}v_k(t), v_k(t)\right) + \left(\Lambda v_k(t), \frac{\mathrm{d}}{\mathrm{d}t}v_k(t)\right)$$

Since $\dfrac{\mathrm{d}v_k}{\mathrm{d}t} = iP_D v_k$ and P_D is self adjoint in $L^2(\Omega)$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}N(t) = -i\left(\big[P_D,\Lambda\big]v_k(t), v_k(t)\right)$$

Now on the support of $1-\varphi_1$ we have $|x| \geq \dfrac{4}{3}R_0$. It follows from (4.5)(i) that $\big[P_D,\Lambda\big] = \big[\widetilde{P},\Lambda\big]$. Therefore we have,

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}N(t) = \ ① + \ ② + \ ③ \quad \text{where,} \\ \qquad ① = -i\left(\big[\widetilde{P},\varphi_1\big]\lambda_1^w\big(1-\varphi_1\big)v_k(t), v_k(t)\right), \\ \qquad ② = i\left(\big[\widetilde{P},\lambda_1^w\big]\big(1-\varphi_1\big)v_k(t), \big(1-\varphi_1\big)v_k(t)\right), \\ \qquad ③ = -\left(\big(1-\varphi_1\big)\lambda_1^w\big[\widetilde{P},\varphi_1\big]v_k(t), v_k(t)\right). \end{cases} \tag{7.25}$$

Since $\lambda_1 = \lambda\Phi^2$, Lemma 7.5 shows that the support of λ_1 is contained in $\{x \,:\, |x| \geq 2R_0\}$. By (7.21) the Poisson bracket $\{\widetilde{p}, \varphi_1\}$ has its support in $\left\{x \,:\, \frac{4}{3}R_0 \leq |x| \leq \frac{3}{2}R_0\right\}$. It follows that $\{\widetilde{p}, \varphi_1\}\lambda_1 \equiv 0$ from which we deduce that $\big[\widetilde{P}, \varphi_1\big]\lambda_1^w$, is a zero$^{\text{th}}$ order operator. It follows that

$$\big|\ ①\big| + \big|\ ③\big| \leq C\,\|v_k(t)\|^2_{L^2(\Omega)}\,. \tag{7.26}$$

Using the sharp Garding inequality, (ii) in Corollary 7.6 and (7.20) we see that

$$② = -\big(-\big(H_{\tilde{p}}\lambda_1\big)^w\big(1-\varphi_1\big)v_k(t), \big(1-\varphi_1\big)v_k(t)\big) + \mathcal{O}\left(\|v_k(t)\|^2_{L^2(\Omega)}\right)$$

$$② \leq -\Big\|\,\langle\, x\,\rangle^{-\frac{1+\nu}{2}}\, b\big(x, h_kD_x\big)(-\Delta)^{\frac{1}{4}}\big(1-\varphi_1\big)v_k(t)\Big\|^2_{L^2(\mathbb{R}^d)} + C\,\|v_k(t)\|^2_{L^2(\Omega)}\,. \tag{7.27}$$

It follows from (7.24), (7.25), (7.26), (7.27) and (7.22) that

$$N(t) + \int_0^t \Big\|\,\langle\, x\,\rangle^{-\frac{1+\nu}{2}}\, b_1\big(x, h_kD_x\big)\big(1-\varphi_1\big)\underline{w}_k(t)\Big\|^2_{L^2(\mathbb{R}^d)}\mathrm{d}t \;\leq\; C\int_0^t N(s)\mathrm{d}s + N(0). \tag{7.28}$$

Using the Gronwall inequality we see that $N(t) \leq N(0)e^{CT}$.

Now $N(0) \leq C\,\|v_k(0)\|^2_{L^2(\Omega)} \leq C'\,\big\|\widetilde{u}_k^0\big\|^2_{L^2(\Omega)}$. Thus using again (7.28) we obtain

$$\int_0^T \big\|b_1\big(x, h_kD_x\big)\big(1-\varphi_1\big)\underline{w}_k(t)\big\|^2_{L^2(\Omega)}\,\mathrm{d}t = 0(1) \tag{7.29}$$

by (3.7), since $\langle\, x\,\rangle^{-\frac{1+\nu}{2}} \approx 1$ on the support of $b_1\big(x, h_kD_x\big)$.

Now since $\varphi_0(t)\psi\big(h_k^2D_t\big)$ is bounded in $L^2(\mathbb{R})$ Lemma 7.7 follows from (7.29) and (7.23).

□

End of the proof of Theorem 7.1

Applying Lemma 7.7 to the subsequence $\big(W_{\sigma(k)}\big)$ and using Proposition 5.1 we see that $\langle\, \mu, a\,\rangle = 0$ with

$$a(x,t,\xi,\tau) = \big[\big(1-\varphi_1(x)\big)\varphi_0(t)\psi(\tau)b_1(x,\xi)\big]^2$$

Since by (7.21), (7.22) we have $a\big(x_0,t_0,\xi_0,\tau_0\big) \neq 0$ we conclude that $m_0 \notin \operatorname{supp}\mu$.

The proof of Theorem 7.1 is thus complete.

□

8. End of the proofs of Theorem 3.1 and 2.1

8.1 End of the proof of Theorem 3.1

According to Corollary 6.2 we will reach to a contradiction if we show that the measure μ vanishes identically. Recall that

$$\operatorname{supp}\mu \subset \Sigma = \big\{(x,t,\xi,\tau) \in T^*\mathbb{R}^{d+1},\ \ x \in \overline{\Omega},\ \ t \in \big[0,T\big],\ \ \tau + p(x,\xi) = 0\big\}.$$

Let $m = (x,t,\xi,\tau) \in \Sigma$ and $a = \pi(m) \in \Sigma_b$. The assumption (2.6) tell us that we can find $s_0 \in \mathbb{R}$ such that for all $s \leq s_0$ we have $\Gamma(s,a) \subset T^*M \setminus \{0\}$, $\Gamma(s,a) = \big(x(s),t,\xi(s),\tau\big)$ where $\big(x(s),\xi(s)\big)$ is the usual flow of p and $\lim\limits_{s\to-\infty} |x(s)| = +\infty$.

Then we have the following Lemma.

Lemma 8.1. *One can find $s_1 \leq s_0$ such that with the notations of Theorem 7.1*

$$(i)\ \big|x\big(s_1\big)\big| \geq 3R_0,$$

$$(ii)\ \sum_{j,k=1}^{d} \widetilde{a}^{jk}\big(x\big(s_1\big)\big)x_j\big(s_1\big)\xi_k\big(s_1\big) \leq -3\delta\,\big|x\big(s_1\big)\big|\,\big|\xi\big(s_1\big)\big|\,.$$

Let us assume this Lemma for a moment.

Since $\tau + \widetilde{p}\big(x\big(s_1\big),\xi\big(s_1\big)\big) = \tau + p\big(x\big(s_1\big),\xi\big(s_1\big)\big) = 0$ (because $\Gamma(s,a) \subset \Sigma_b$) we deduce from Theorem 7.1 that $\big(x\big(s_1\big),t,\xi\big(s_1\big),\tau\big) = \Gamma\big(s_1,a\big) = \pi^{-1}\big(\Gamma\big(s_1,a\big)\big) \notin \operatorname{supp}\mu$ (π is the identity on $T^*\mathbb{R}^{d+1}_M$). By Theorem 5.3 we have $\pi^{-1}\big(\Gamma(0,a)\big) = \pi^{-1}(a) \cap \operatorname{supp}\mu = \emptyset$. Since $m \in \pi^{-1}(a)$ it follows that $m \notin \operatorname{supp}\mu$. Therefore $\operatorname{supp}\mu = \emptyset$ which contradicts Corollary 6.2 and proves Theorem 3.1.

Proof of Lemma 8.1

Since $\lim\limits_{s\to-\infty} |x(s)| = +\infty$ we can find $\widetilde{s}_0$ such that

$$|x(s)| \geq 3R_0 \ \text{ for } \ s \leq \widetilde{s}_0. \tag{8.1}$$

Let us set for $s \in \big]-\infty,\widetilde{s}_0\big]$

$$\begin{cases} F(s) = F_1(s) + F_2(s), \\ F_1(s) = \displaystyle\sum_{j,k=1}^{d} a^{jk}\big(x(s)\big)x_j(s)\xi_k(s), \\ F_2(s) = 3\delta\,|x(s)|\ |\xi(s)|\,. \end{cases} \tag{8.2}$$

Let us remark that since $|x(s)| \geq 3R_0$ we have $a^{jk}\big(x(s)\big) = \widetilde{a}^{jk}\big(x(s)\big)$.

We have

$$\begin{cases} \dot{x}_j(s) = 2\displaystyle\sum_{k=1}^{d} a^{jk}\big(x(s)\big)\xi_k(s) \\ \dot{\xi}_j(s) = -\displaystyle\sum_{p,q=1}^{d} \frac{\partial a^{pq}}{\partial x_j}\big(x(s)\big)\xi_p(s)\xi_q(s) \end{cases} \tag{8.3}$$

and by assumption (2.5), $\left|\nabla_x a^j(x)\right| = o\left(\dfrac{1}{|x|}\right)$ as $|x| \to +\infty$.

Using (8.1), (8.3), (2.5), the ellipticity condition (2.3) and taking R_0 large enough we find by an easy computation that

$$\frac{\mathrm{d}}{\mathrm{d}s}F_1(s) \geq C\,|\xi(s)|^2\,, \quad s \in \left]-\infty, \widetilde{s}_0\right] \tag{8.4}$$

for some fixed constant $C > 0$.

Using again (8.3) and the same arguments we see easily that

$$\frac{\mathrm{d}}{\mathrm{d}s}F_2(s) \leq C'\delta\,|\xi(s)|^2\,. \tag{8.5}$$

It follows from (8.4), (8.5) and (8.2), taking δ small enough, that for $s \in \left]-\infty, \widetilde{s}_0\right]$ we have

$$\frac{\mathrm{d}}{\mathrm{d}s}F(s) \geq C_0\,|\xi(s)|^2 \geq C_1 p\big(x(s),\xi(s)\big) = C_1 p\left(x\big(\widetilde{s}_0\big),\xi\big(\widetilde{s}_0\big)\right) \geq C_2\left|\xi\big(\widetilde{s}_0\big)\right|^2.$$

Integrating this inequality between s and $\widetilde{s}_0$ we obtain

$$F(s) \leq F\big(\widetilde{s}_0\big) + C_2\left|\xi\big(\widetilde{s}_0\big)\right|^2\big(s - \widetilde{s}_0\big).$$

Since the right hand side tends to $-\infty$ when s goes to $-\infty$ we can find $s_1 \leq \widetilde{s}_0$ such that $F(s) \leq 0$ when $s \leq s_1$.

□

8.2 End of the proof of Theorem 2.1

We shall need the following Lemma.

Lemma 8.2. *Let $\theta \in \mathcal{C}_0^\infty(\mathbb{R})$, $\chi_0 \in \mathcal{C}_0^\infty\big(\overline{\Omega}\big)$. There exists $C > 0$ such that*

$$\left\|\left[\theta\big(h^2 P_D\big), \chi_0 P_D^{\frac{1}{4}}\right] v\right\|_{L^2(\Omega)} \leq C\,h^{\frac{1}{2}}\,\|\,v\,\|_{L^2(\Omega)}$$

for every $h \in \left]0,1\right]$ and $v \in L^2(\Omega)$.

Proof

See section 9.3

□

Now it is classical that one can find ψ, θ in $\mathcal{C}_0^\infty(\mathbb{R})$ such that

$$\begin{cases} \operatorname{supp}\psi \subset \big\{t\ :\ |t| \leq 1\big\}, \quad \operatorname{supp}\theta \subset \left\{t\ :\ \dfrac{1}{2} \leq |t| \leq 2\right\} \quad \text{and} \\ \\ \psi(t) + \displaystyle\sum_{p=0}^{+\infty} \theta\big(2^{-p}t\big) = 1 \text{ for all } t \in \mathbb{R}. \end{cases}$$

By the functionnel calculus we see easily that

$$\text{(8.6)}\qquad \begin{cases} \psi\big(P_D\big) + \displaystyle\sum_{p=0}^{+\infty} \theta\big(2^{-p}P_D\big) = \mathrm{Id} \quad \text{and} \\ \| v \|^2_{L^2(\Omega)} \leq C \left(\big\| \psi\big(P_D\big) v \big\|^2_{L^2(\Omega)} + \displaystyle\sum_{p=0}^{+\infty} \big\| \theta\big(2^{-p}P_D\big) v \big\|^2_{L^2(\Omega)} \right) \end{cases}$$

Let $u(t) = e^{itP_D} u_0$. Using (8.6) we see that

$$\text{(8.7)}\qquad \begin{cases} \Big\| \chi_0 P_D^{\frac{1}{4}} u(t) \Big\|^2_{L^2(\Omega)} \leq C\big(\text{①} + \text{②} \big) \quad \text{where} \\ \text{①} = \Big\| \psi\big(P_D\big) \chi_0 P_D^{\frac{1}{4}} u(t) \Big\|^2_{L^2(\Omega)} \\ \text{②} = \displaystyle\sum_{p=0}^{+\infty} \Big\| \theta\big(2^{-p}P_D\big) \chi_0 P_D^{\frac{1}{4}} u(t) \Big\|^2_{L^2(\Omega)} \end{cases}$$

We have

$$\text{①} \leq 2\Big(\Big\| \Big[\psi\big(P_D\big), \chi_0 P_D^{\frac{1}{4}} \Big] u(t) \Big\|^2_{L^2} + \Big\| \chi_0 \psi\big(P_D\big) P_D^{\frac{1}{4}} u(t) \Big\|^2_{L^2} \Big)$$

Using Lemma 8.2 with $h = 1$, the fact that the operator $\psi\big(P_D\big) P_D^{\frac{1}{4}}$ is $L^2(\Omega)$ bounded and the energy estimate we deduce that

$$\text{(8.8)}\qquad \text{①} \leq C \| u_0 \|^2_{L^2(\Omega)}.$$

On the other hand we have

$$\text{(8.9)}\qquad \begin{cases} \text{②} \leq C\big(\text{③} + \text{④} \big) \\ \text{③} = \displaystyle\sum_{p=0}^{+\infty} \Big\| \Big[\theta\big(2^{-p}P_D\big), \chi_0 P_D^{\frac{1}{4}} \Big] u(t) \Big\|^2_{L^2(\Omega)} \\ \text{④} = \displaystyle\sum_{p=0}^{+\infty} \Big\| \chi_0 P_D^{\frac{1}{4}} \theta\big(2^{-p}P_D\big) u(t) \Big\|^2_{L^2(\Omega)} \end{cases}$$

Using Lemma 8.2 we can write

$$\text{③} \leq C \left(\sum_{p=0}^{+\infty} 2^{-p} \right) \| u(t) \|^2_{L^2(\Omega)},$$

so by the energy estimate

$$\text{③} \leq C \left\| u_0 \right\|^2_{L^2(\Omega)} . \tag{8.10}$$

To handle the term ④ we use the Theorem 3.1. Let $\widetilde{\theta} \in \mathcal{C}_0^\infty(\mathbb{R})$ be such that $\operatorname{supp} \widetilde{\theta} \subset \left\{ t \, : \, \frac{1}{3} \leq |t| \leq 3 \right\}$, $\widetilde{\theta}(t) = 1$ on the support of θ and $0 \leq \widetilde{\theta} \leq 1$. Then $\theta\big(2^{-p} P_D\big) u(t) = \theta\big(2^{-p} P_D\big) \mathrm{e}^{itP_D} \widetilde{\theta}\big(2^{-p} P_D\big) u_0$.

It follows from Theorem 3.1 that

$$\int_0^T \text{④} \, \mathrm{d}t \, \leq \, C \sum_{p=0}^{+\infty} \left\| \widetilde{\theta}\big(2^{-p} P_D\big) u_0 \right\|^2_{L^2(\Omega)}$$

Now we have

$$\sum_{p=0}^{+\infty} \Big[\widetilde{\theta}\big(2^{-p} t\big) \Big]^2 \leq \left(\sum_{p=0}^{+\infty} \widetilde{\theta}\big(2^{-p} t\big) \right)^2 \leq M_0, \quad \text{for all } t \in \mathbb{R}$$

It follows that the operator $\displaystyle\sum_{p=0}^{+\infty} \Big[\widetilde{\theta}\big(2^{-p} P_D\big) \Big]^2$ is $L^2(\Omega)$ bounded, therefore

$$\int_0^T \text{④} \, \mathrm{d}t \, \leq \, C \left\| u_0 \right\|^2_{L^2(\Omega)} . \tag{8.11}$$

It follows from (8.7), (8.8), (8.9), (8.10) and (8.11) that

$$\int_0^T \left\| \chi_0 P_D^{\frac{1}{4}} \mathrm{e}^{itP_D} u_0 \right\|^2_{L^2(\Omega)} \mathrm{d}t \, \leq \, C \left\| u_0 \right\|^2_{L^2(\Omega)}$$

which is the claim in Theorem 2.1. The proof is complete. □

9. Appendix

9.1 The geometrical framework

We recall here the definition of the generalized bicharacteristic flow in the sense of Melrose and Sjöstrand. For this purpose we follow Hörmander [Hö].

Let $M = \Omega \times \mathbb{R}_t$. We set $T_b^* M = T^*M \setminus \{0\} \cup T^* \partial M \setminus \{0\}$. We have a natural restriction map $\pi \, : \, T^* \mathbb{R}^{d+1}_{|\overline{M}} \to T_b^* M$ (which will be describe more precisely in local coordinates below) which is the identity on $T^* \mathbb{R}^{d+1}_{|M} \setminus \{0\}$.

With p defined in (2.2) we introduce the characteristic set

$$\Sigma = \big\{ (x,t,\xi,\tau) \in T^* \mathbb{R}^{d+1}, \;\; x \in \overline{\Omega}, \;\; t \in \big[0,T\big], \;\; \tau + p(x,\xi) = 0 \big\} ,$$

and we set $\Sigma_b = \pi(\Sigma)$.

Definition 9.1. *Let $\zeta \in T^*\partial M \setminus \{0\}$. We shall say that*

(i) ζ is elliptic (or $\zeta \in \mathcal{E}$) iff $\zeta \notin \Sigma_b$

(ii) ζ is hyperbolic (or $\zeta \in \mathcal{H}$) iff $\#\left\{\pi^{-1}(\zeta) \cap \Sigma\right\} = 2$

(iii) ζ is glancing (or $\zeta \in \mathcal{G}$ iff $\#\left\{\pi^{-1}(\zeta) \cap \Sigma\right\} = 1$.

Let us describe π and these sets in local coordinates. As we said before, π is the identity map on $T^*\mathbb{R}^{d+1}_{|M} \setminus \{0\}$.

Near any point of ∂M we can use the geodesical coordinates where M is given by

$$\left\{\left(x_1, x', t\right) \in \mathbb{R} \times \mathbb{R}^{d-1} \times \mathbb{R} \ : \ x_1 > 0\right\},$$

∂M is given by $\left\{\left(x_1, x', t\right) \ : \ x_1 = 0\right\}$ and $\tau + p(x,\xi)$ is transformed to $\xi_1^2 + r\left(x_1, x', \xi'\right) + \tau$.

In these coordinates if $\rho \in T^*\mathbb{R}^{d+1}_{|\partial M} \setminus \{0\}$ then $\rho = \left(0, x', t, \xi_1, \xi', \tau\right)$ and $\pi(\rho) = \left(x', t, \xi', \tau\right) \in T^*\partial M \setminus \{0\}$.

Now let $\zeta = \left(x', t, \xi', \tau\right) \in T^*\partial M \setminus \{0\}$. Then

$$\left\{\begin{array}{l} \zeta \in \mathcal{E} \iff r\left(0, x', \xi'\right) + \tau > 0, \\ \zeta \in \mathcal{H} \iff r\left(0, x', \xi'\right) + \tau < 0, \\ \zeta \in \mathcal{G} \iff r\left(0, x', \xi'\right) + \tau = 0. \end{array}\right. \tag{9.1}$$

When $\zeta \in \mathcal{H}$ then $\pi^{-1}(\zeta) \cap \Sigma = \left\{\left(0, x', t, \xi_1^{\pm}, \xi', \tau\right)\right\}$ where

$$\xi_1^{\pm} = \pm\Big(- \left(r\left(0, x', \xi'\right) + \tau\right)\Big)^{\frac{1}{2}}. \tag{9.2}$$

When $\zeta \in \mathcal{G}$ then $\pi^{-1}(\zeta) \cap \Sigma = \left\{\left(0, x', t, 0, \xi', \tau\right)\right\}$.

For the purpose of the proofs it is important to decompose the set $\mathcal{G}$ of glancing points into several subsets. The following definition is given in local coordinates but could be written in an intrinsic way (see [H]). We shall set

$$r_0\left(x', \xi'\right) = r\left(0, x', \xi'\right) \tag{9.3}$$

and H_{r_0} will denote the Hamilton field of r_0 namely $H_{r_0} = \dfrac{\partial r_0}{\partial \xi'}\dfrac{\partial}{\partial x'} - \dfrac{\partial r_0}{\partial x'}\dfrac{\partial}{\partial \xi'}$.

Definition 9.2. *Let $\zeta = \left(x', t, \xi', \tau\right) \in \mathcal{G}$. We shall say that*

(i) ζ is diffractive (or $\zeta \in \mathcal{G}_d$) iff $\dfrac{\partial r}{\partial x_1}\left(0, x', \xi'\right) < 0$,

(ii) ζ is gliding (or $\zeta \in \mathcal{G}_g$) iff $\dfrac{\partial r}{\partial x_1}\left(0, x', \xi'\right) > 0$, and we set $\mathcal{G}^2 = \mathcal{G}_d \cup \mathcal{G}_g$,

(iii) ζ belongs to $\mathcal{G}^k$, $k \geq 3$, iff

$$H_{r_0}^j\left(\frac{\partial r}{\partial x_1}_{|x_1=0}\right)(\zeta) = 0, \ 0 \leq j < k-2, \ H_{r_0}^{k-2}\left(\frac{\partial r}{\partial x_1}_{|x_1=0}\right)(\zeta) \neq 0.$$

We can now give the meaning of the assumption made in (2.7).

Definition 9.3. *We shall say that the bicharacteristics have no contact of infinite order with the boundary if*

$$\mathcal{G} = \bigcup_{k=2}^{+\infty} \mathcal{G}^k.$$

We are going now to make a brief description of the generalized bicharacteristic flow and we refer to [M-S] or [Hö] for more details.

First of all we introduce some notations.

We shall denote by $\gamma(s) = \big(x(s), \xi(s)\big)$ the usual bicharacteristic of p in $T^*\Omega$ defined by

$$\big(\dot{x}(s), \dot{\xi}(s)\big) = \Big(\frac{\partial p}{\partial \xi}\big(\gamma(s)\big), -\frac{\partial p}{\partial x}\big(\gamma(s)\big)\Big).$$

We shall denote by $\gamma_g(s) = \big(x'_g(x), \xi'_g(s)\big)$ the gliding ray in $T^*\partial\Omega$ defined in the geodesic coordinates by the equations

$$\Big(\dot{x}\,'_g(s), \dot{\xi}\,'_g(s)\Big) = \Big(\frac{\partial r_0}{\partial \xi'}\big(\gamma_g(s)\big), -\frac{\partial r_0}{\partial x'}\big(\gamma_g(s)\big)\Big)$$

where r_0 has been introduced in (9.3).

The generalized flow lives in $\Sigma_b \subset T^*_b M$ and for $a \in \Sigma_b$ is denoted by $\Gamma(s, a)$. Since Σ_b is the disjoint union of $\Sigma_b \cap T^*M$, $\Sigma_b \cap \mathcal{H}$, $\Sigma_b \cap \mathcal{G}_d$, $\Sigma_b \cap \mathcal{G}_g$ and $\Sigma_b \cap \Big(\bigcup_{k\geq 3} \mathcal{G}^k\Big)$ we shall consider separatly the case where a belongs to each set. Moreover each description of $\Gamma(s, a)$ holds for small $|s|$.

Case 1 : $a \in \Sigma_b \cap T^*M$

Here $a = (x, t, \xi, \tau)$ where $x \in \Omega$, $t \in \big[0, T\big]$, $\tau + p(x, \xi) = 0$. Then for small $|s|$ we have

$$\Gamma(s, a) = \big(x(s), t, \xi(s), \tau\big) \subset T^*M$$

where $\big(x(s), \xi(s)\big)$ is the bicharacteristic of p starting from the point (x, ξ).

Case 2 : $a \in \Sigma_b \cap \mathcal{H}$

In the geodesic coordinates we have $a = \big(x', t, \xi', \tau\big)$ and the equation $\xi_1^2 + r\big(0, x', \xi'\big) + \tau = 0$ has two distinct roots ξ_1^+, ξ_1^- described in (9.2). For $s > 0$ (resp. $s < 0$) let $\gamma^+(s) = \big(x^+(s), \xi^+(s)\big)$ (resp. $\gamma^-(s) = \big(x^-(s), \xi^-(s)\big)$ be the bicharacteristic of p starting for $s = 0$ at the point $\big(0, x', \xi_1^+, \xi'\big)$ (resp. $\big(0, x', \xi_1^-, \xi'\big)$). They are contained in $T^*\Omega$ for small $|s| \neq 0$. Then $\Gamma(0, a) = a$ and

$$\Gamma(s, a) = \begin{cases} \big(x^+(s), t, \xi^+(s), \tau\big), & 0 < s < \varepsilon, \\ \big(x^-(s), t, \xi^-(s), \tau\big), & -\varepsilon < s < 0. \end{cases}$$

Here $\Gamma(s,a) \subset T^*M$ for $s \neq 0$.

Case 3 : $a \in \Sigma_b \cap \mathcal{G}_d$

Here $a = \big(x', t, \xi', \tau\big)$ and the equation $\xi_1^2 + r\big(0, x', \xi'\big) + \tau = 0$ has a double root $\xi_1 = 0$. Let $\gamma(s) = \big(x(s), \xi(s)\big)$ be the flow of p starting when $s = 0$ at the point $\big(0, x', \xi_1 = 0, \xi'\big)$. Then we have

$$\Gamma(s,a) = \big(x(s), t, \xi(s), \tau\big) \subset T^*M, \quad 0 < |s| < \varepsilon.$$

Case 4 : $a \in \Sigma_b \cap \mathcal{G}_g$

As above $a = \big(x', t, \xi', \tau\big)$ and $\xi_1 = 0$ is a double root. Let $\gamma_g(s) = \big(x'_g(s), \xi'_g(s)\big)$ be the gliding ray starting when $s = 0$ at the point $\big(x', \xi'\big)$. Then we have

$$\Gamma(s,a) = \big(x'_g(s), t, \xi'_g(s), \tau\big) \subset T^*\partial M, \quad |s| < \varepsilon.$$

Case 5 : $a \in \Sigma_b \cap \left(\bigcup_{k=3}^{+\infty} \mathcal{G}^k \right)$

Let $a = \big(x', t, \xi', \tau\big)$. Let $\gamma_g(s) = \big(x'_g(s), \xi'_g(s)\big)$ be the gliding ray starting when $s = 0$ at the point $\big(x', \xi'\big)$. Then (see Theorem 24.3.9 in [Hö]) one can find $\varepsilon > 0$ such that with $I = \big]0, \varepsilon\big[$ we have either $\gamma_g(s) \in \mathcal{G}_g$, $\forall s \in I$ and then $\Gamma(s,a) = \big(x'_g(s), t, \xi'_g(s), \tau\big) \subset T^*\partial M$, $\forall s \in I$, or $\gamma_g(s) \in \mathcal{G}_d$, $\forall s \in I$ and then $\Gamma(s,a) = \big(x(s), t, \xi(s), \tau\big) \subset T^*M$, $\forall s \in I$, where $\big(x(s), \xi(s)\big)$ is the bicharacteristic of p starting when $s = 0$ at the point $\big(0, x', \xi_1 = 0, \xi'\big)$.

The same discussion is independently valid for $-\varepsilon < s < 0$.

Remark 9.4. *Let $a \in \Sigma_b$ and $\Gamma(t,a)$ be the generalized bicharacteristing starting for $t = 0$ at the point a. Then the above discussion shows that one can find $\varepsilon > 0$ such that for $0 < |t| \leq \varepsilon$ we have $\Gamma(t,a) \subset T^*M \cup \mathcal{G}_g$. Let us note (see [M-S]) that the maps $s \mapsto \Gamma(s,a)$ and $a \mapsto \Gamma(s,a)$ are continuous, the later when T_b^*M is endowed with the topologie induced by the projection π. Moreover we have the usual relation $\Gamma(t+s,a) = \Gamma\big(t, \Gamma(s,a)\big)$ for s, t in $\mathbb{R}$.*

9.2 Proofs of Theorem 5.2 and Theorem 5.3

a) Proof of Theorem 5.2

According to (5.1), (5.2) it is obvious that

$$\operatorname{supp} \mu \subset \big\{(x,t,\xi,\tau) \in T^*\mathbb{R}^{d+1} \ : \ x \in \overline{\Omega} \text{ and } t \in \big[0,T\big]\big\}$$

Therefore it remains to show that if $m_0 = \big(x_0, t_0, \xi_0, \tau_0\big)$ with $x_0 \in \overline{\Omega}$, $t_0 \in \big[0,T\big]$, but $\tau_0 + p\big(x_0, \xi_0\big) \neq 0$ then $m_0 \notin \operatorname{supp} \mu$.

Case 1 : assume $x_0 \in \Omega$

Let $\varepsilon > 0$ be such that $B\big(x_0, \varepsilon\big) \subset \Omega$. Let $\varphi \in \mathcal{C}_0^\infty\big(B\big(x_0, \varepsilon\big)\big)$, $\varphi = 1$ on $B\left(x_0, \dfrac{\varepsilon}{2}\right)$ and $\widetilde{\varphi} \in \mathcal{C}_0^\infty(\Omega)$, $\widetilde{\varphi} = 1$ on $\operatorname{supp}\varphi$. Let $a \in \mathcal{C}_0^\infty\left(\mathbb{R}_x^d \times \mathbb{R}_\xi^d\right)$ such that $\pi_x \operatorname{supp} a \subset B\left(x_0, \dfrac{\varepsilon}{2}\right)$ and $\chi \in \mathcal{C}_0^\infty\left(\mathbb{R}_t \times \mathbb{R}_\tau\right)$. Recall that we have set $W_k = \mathbb{1}_{[0,T]}\mathbb{1}_\Omega w_k$ with $w_k = h_k^{-\frac{1}{2}}\theta\big(h_k^2 P_D\big)\widetilde{u}_k$ and that $\big(w_k\big)$ is a bounded sequence in $L^2\left(\big[0,\tau\big], L^2_{\mathrm{loc}}\big(\mathbb{R}^d\big)\right)$ (see Proposition 4.1). Now we set

$$I_k = \big(a\big(x, h_k D_x\big)\chi\big(t, h_k^2 D_t\big)\varphi h_k^2\big(D_t + P\big(x, D_x\big)\big)W_k, \widetilde{\varphi}W_k\big)_{L^2(\mathbb{R}^{d+1})} \tag{9.4}$$

We have $h_k^2\left(D_t + P\big(x, D_x\big)\right) = h_k^2 D_t + P_2\big(x, h_k D_x\big) + h_k P_1\big(x, h_k D_x\big) + h_k^2 P_0(X)$ where $P_j(x,\xi)$ are homogeneous in ξ of order j.

Using the semi classical symbolic calculus and the fact that $\big(\widetilde{\varphi}W_k\big)$ is bounded in $L^2\big(\mathbb{R}^{d+1}\big)$ we see easily that the terms in I_k corresponding to $h_k P_1\big(x, h_k D_x\big)$ and $h_k^2 P_0(x)$ tend to zero when $k \to +\infty$. It remains to consider the term $P_2\big(x, h_k D_x\big)$. But by the semi classical calculus

$$a\big(x, h_k D_x\big)\chi\big(t, h_k^2 D_t\big)\varphi\big(h_k^2 D_t + P_2\big(x, h_k D_x\big)\big) = \mathcal{O}p\big(a\chi(\tau + p)\big) + h_k R_k$$

where R_k is a uniformly bounded semi classical pseudo-differential operator in $L^2\big(\mathbb{R}^{d+1}\big)$. Therefore the term in I_k corresponding to $h_k R_k$ tends to zero.

It follows from Proposition 5.1 that

$$\lim_{k \to +\infty} I_{\sigma(k)} = \langle\, \mu, (\tau + p) a\chi \,\rangle \tag{9.5}$$

On the other hand we have since $\big[D_t + P\big(x, D_x\big)\big]\widetilde{u}_k = 0$ in $\Omega \times \mathbb{R}_t$ and $\varphi \in \mathcal{C}_0^\infty(\Omega)$,

$$\varphi\big(D_t + P\big(x, D_x\big)\big)W_k = \varphi\big(w_k(0)\delta_{t=0} - w_k(T)\delta_{t=T}\big) \tag{9.6}$$

Lemma 9.5. *Let $1 \leq p \leq +\infty$, $\chi \in \mathcal{C}_0^\infty(\mathbb{R} \times \mathbb{R})$ and $\ell \geq 1$. Then there exists $C > 0$ such that*

$$\big\|\chi\big(t, h^\ell D_t\big)\delta_{t=a}\big\|_{L^p(\mathbb{R})} \leq C\, h^{\frac{\ell}{p} - \ell}$$

for every $0 < h \leq 1$.

Proof

Let $\psi \in \mathcal{C}_0^\infty(\mathbb{R})$, $\psi(t) = 1$ on a neighborhood of $\pi_t \operatorname{supp}\chi$. Then $\psi\chi = \chi$. Now

$$① = \chi\big(t, h^\ell D_t\big)\delta_{t=a} = \frac{1}{2\pi}\int \mathrm{e}^{i(t-a)\tau}\chi\big(t, h^2\tau\big)\mathrm{d}\tau$$

On the other hand $\chi(t,\tau)=\psi(t)\chi(a,\tau)+\psi(t)(t-a)\widetilde{\chi}(t,\tau,a)$ where $\widetilde{\chi}\in\mathcal{C}^\infty$ has compact support in τ. It follows that

$$\text{(9.7)}\qquad \begin{cases} ① = \psi(t)h^{-\ell}\left(\overline{\mathcal{F}}_x\chi\right)\left(a,\dfrac{t-a}{h^\ell}\right)+\ ② \quad \text{with} \\ ② = \dfrac{1}{2\pi}\psi(t)\displaystyle\int (t-a)\mathrm{e}^{i(t-a)\tau}\widetilde{\chi}\big(t,h^\ell\tau,a\big)\mathrm{d}\tau \end{cases}$$

Noting that $(t-a)\mathrm{e}^{i(t-a)\tau}=\dfrac{1}{i}\dfrac{\partial}{\partial\tau}\mathrm{e}^{i(t-a)\tau}$ and making an integration by part, we see easily that

$$\text{(9.8)}\qquad \big|\ ②\big| \le C\,|\psi(t)|\,h^\ell\int\left|\frac{\partial\widetilde{\chi}}{\partial\tau}\big(t,h^\ell\tau,a\big)\right|\mathrm{d}\tau = C\,|\psi(t)|\int\left|\frac{\partial\widetilde{\chi}}{\partial\tau}(t,\tau,a)\right|\mathrm{d}\tau$$

Then the Lemma follows easily from (9.7) and (9.8). □

Now we see from (9.6) that I_k is a sum of two terms of the form

$$J_k=\big(a\big(x,h_kD_x\big)\varphi w_k(a)h_k^2\chi\big(t,h_k^2D_t\big)\delta_{t=a},\widetilde{\varphi}W_k\big)\,,\quad a=0 \text{ or } T$$

Since $\big(\widetilde{\varphi}W_k\big)$ is bounded in $L^2\big(\mathbb{R}^{d+1}\big)$ we see that

$$|J_k|^2\le C\left\|a\big(x,h_kD_x\big)\varphi w_k(a)\right\|^2_{L^2(\mathbb{R}^d)}\left\|h_k^2\chi\big(t,h_k^2D_C\big)\delta_{t=a}\right\|^2_{L^2(\mathbb{R})}$$

so using Lemma 9.5 with $p=2$ and $\ell=2$ we deduce that

$$|J_k|^2\le C\,h_k^{-1}\left\|\widetilde{u}_k(a)\right\|^2_{L^2(\Omega)}h_k^2\le C\,h_k\left\|\widetilde{u}_k^0\right\|^2_{L^2(\Omega)}$$

by the energy estimate. It follows from (3.7) that

$$\text{(9.9)}\qquad \lim_{k\to+\infty}I_k=0$$

Using (9.5) and (9.9) we see that $\langle\,\mu,(\tau+p)a\chi\,\rangle=0$. Since $\tau_0+p\big(x_0,\xi_0\big)\neq0$ and

$$\mathcal{C}_0^\infty\big(\mathbb{R}^d_x\times\mathbb{R}^d_\xi\big)\otimes\mathcal{C}_0^\infty\big(\mathbb{R}_t\times\mathbb{R}_\tau\big)$$

is dense in $\mathcal{C}_0^\infty\big(\mathbb{R}^{d+1}\times\mathbb{R}^{d+1}\big)$ we deduce that $m_0=\big(x_0,t_0,\xi_0,\tau_0\big)\notin\operatorname{supp}\mu$.

Case 2 : assume $x_0\in\partial\Omega$

We would like to show that one can find a neighborhood U_{x_0} of x_0 in $\mathbb{R}^d$ such that for any $a\in\mathcal{C}_0^\infty\big(U_{x_0}\times\mathbb{R}_t\times\mathbb{R}^d_\xi\times\mathbb{R}_\tau\big)$ we have

$$\text{(9.10)}\qquad \langle\,\mu,(\tau+p)a\,\rangle=0$$

Indeed this will imply that the point $m_0 = \big(x_0, t_0, \xi_0, \tau_0\big)$ (with $\tau_0 + \big(x_0, \xi_0\big) \neq 0$) does not belong to the support of μ as claimed.

Now (9.10) will be implied, according to Proposition 5.1 and (4.1) by

$$\text{(9.11)} \qquad \begin{cases} \lim\limits_{k\to+\infty} I_{\sigma(k)} = 0 \quad \text{where} \\ I_k = \big(a\big(x,t,h_kD_x,h_k^2D_t\big)\varphi h_k^2\big(D_t+P\big)W_k, W_k\big)_{L^2(\mathbb{R}^{d+1})} \end{cases}$$

where $\varphi \in \mathcal{C}_0^\infty\big(V_{x_0}\big)$, $\varphi = 1$ on $\pi_x \operatorname{supp} a$.

Now we may choose U_{x_0} so small that one can find a $\mathcal{C}^\infty$ diffeomorphism F from U_{x_0} to a neighborhood U_0 of the origin in $\mathbb{R}^d$ such that

$$\text{(9.12)} \qquad \begin{cases} F\big(U_{x_0} \cap \Omega\big) = \big\{y \in U_0 \ : \ y_1 > 0\big\} \\ F\big(U_{x_0} \cap \partial\Omega\big) = \big\{y \in U_0 \ : \ y_1 = 0\big\} \\ \big(P(x,D)W_k\big) \circ F^{-1} = \big(D_1^2 + R\big(y,D'\big)\big)\big(W_k \circ F^{-1}\big) \end{cases}$$

where R is a second order differential operator and $D' = \big(D_2, \cdots, D_d\big)$.

Let us set

$$\text{(9.13)} \qquad v_k = w_k \circ F^{-1}, \quad V_k = \mathbb{1}_{[0,T]}\mathbb{1}_{y_1>0}v_k$$

then we will have

$$\text{(9.14)} \qquad \begin{cases} \big(D_t + D_1^2 + R\big(y,D'\big)\big)\, v_k = 0 \text{ in } U_0 \times \mathbb{R}_t, \quad y_1 > 0, \\ v_k|_{y_1=0} = 0 \end{cases}$$

Making the change of variable $x = F^{-1}(y)$ in the right hand side of the second line of (9.11) we see that

$$I_k = \big(b\big(y,t,h_tD_y,h_k^2D_t\big)\psi h_k^2\big(D_t + D_1^2 + R\big(y,D'\big)\big)V_k, V_k\big)_{L^2(\mathbb{R}^{d+1})}$$

where $b \in \mathcal{C}_0^\infty\big(U_0 \times \mathbb{R}_t \times \mathbb{R}_\eta \times \mathbb{R}_\tau\big)$ and $\psi \in \mathcal{C}_0^\infty\big(U_0\big)$, $\psi = 1$ near $\pi_y \operatorname{supp} b$.

To prove (9.11) it is sufficient to prove that

$$\text{(9.15)} \qquad \lim J_k = \lim \big(T\psi_0\big(y_1\big)\psi_1\big(y'\big)h_k^2\big(D_t + D_1^2 + R\big(y,D'\big)\big)V_k, V_k\big)_{L^2(\mathbb{R}^{d+1})} = 0$$

where $T = \theta\big(y_1, h_kD_1\big)\Phi\big(y', h_kD'\big)\chi\big(t, h_k^2D_t\big)$, $\theta\Phi\chi \in \mathcal{C}_0^\infty\big(U_0 \times \mathbb{R}_t \times \mathbb{R}_y^d \times \mathbb{R}_\tau\big)$, $\psi_0\psi_1 \in \mathcal{C}_0^\infty\big(U_0\big)$ and $\psi_0\psi_1 = 1$ on $\pi_y \operatorname{supp} \theta\Phi\xi$.

Now according to (9.14) we have

$$\text{(9.16)} \qquad \begin{aligned} &\big(D_t + D_1^2 + R\big(y,D'\big)\big)V_k \\ &\qquad = -i\mathbb{1}_{x_1>0}v_k(0,\cdot)\delta_{t=0} + i\mathbb{1}_{X_1>0}v_k(T,\cdot)\delta_{t=T} - i\mathbb{1}_{[0,T]}\big(D_1v_k|_{x_1=0}\big) \otimes \delta_{x_1=0} \end{aligned}$$

Therefore (9.11) will be proved in we can prove that

$$(9.17)\quad \begin{cases} \lim\limits_{k\to+\infty} A_k^j = 0, \quad j=1,2, \quad \text{where} \\ A_k^1 = \big(\theta\big(y,h_kD_1\big)\Phi\big(y',h_kD'\big)\chi\big(t,h_k^2D_t\big)\psi_0\psi_1 h_k^2 \mathbb{1}_{y_1>0} v_k(a,\cdot)\delta_{t=a}, V_k\big)\,, a=0,T \\ A_k^2 = \big(\theta\big(y_1,h_kD_1\big)\Phi\big(y',h_kD'\big)\chi\big(t,h_k^2D_t\big)\psi_0\psi_1 h_k^2 \mathbb{1}_{[0,T]}\big(D_1 v_k|_{y_1=0}\big)\otimes\delta_{y_1=0}, V_k\big) \end{cases}$$

Since the operator $\theta\big(y_1,h_kD_1\big)\Phi\big(y',h_kD'\big)$ is uniformly bounded in $L^2\big(\mathbb{R}^d\big)$ we can write with $\psi_2\in\mathcal{C}_0^\infty\big(U_0\big)$, $\psi_2=1$ near $\pi_y\operatorname{supp}\theta\Phi$

$$\big|A_k^1\big|^2 \le C\,\big\|h_k^2\chi\big(t,h_k^2D_t\big)\delta_{t=a}\big\|^2_{L^2(\mathbb{R})}\,\|\psi_0\psi_1\mathbb{1}_{y_1>0}v_k(a,\cdot)\|^2_{L^2(\mathbb{R}^d)}\,\|\psi_2V_k\|^2_{L^2(\mathbb{R}^{d+1})}$$

By (9.13), the energy estimate and Proposition 4.1 we have

$$\begin{cases} \|\psi_0\psi_1\mathbb{1}_{y_1>0}v_k(a,\cdot)\|^2_{L^2} \le C\,\|w_k(a)\|^2_{L^2(\Omega)} \le C\,h_k^{-1}\,\|\widetilde{u}_k(a)\|^2_{L^2} \le C\,h_k^{-1}\,\big\|\widetilde{u}_k^{\,0}\big\|^2_{L^2} \\ \|\psi_2V_k\|^2_{L^2} \le C\displaystyle\int_0^T \big\|\big(\psi_2\circ F\big)w_k(t,\cdot)\big\|^2_{L^2(\Omega)}\,\mathrm{d}t = \mathcal{O}(1) \end{cases}$$

Using Lemma 9.5 with $\ell=2$, $p=2$, we obtain

$$(9.18)\qquad \big|A_k^1\big|^2 \le C\,h_k\,\big\|\widetilde{u}_k^0\big\|^2_{L^2(\Omega)}$$

To estimate the term A_k^2 we need a Lemma.

With U_0 introduced in (9.12) we set $U_0^+ = \big\{y\in U_0 \ : \ y_1>0\big\}$. We shall consider smooth solution of the problem

$$(9.19)\qquad \begin{cases} \big(D_t + D_1^2 + R(y,D')\big)\,u = 0 \quad \text{in } U_0^+\times\mathbb{R}_t \\ u|_{y_1=0} = 0 \end{cases}$$

Lemma 9.6. *Let $\chi\in\mathcal{C}_0^\infty\big(U_0\big)$ and $\chi_1\in\mathcal{C}_0^\infty\big(U_0\big)$ on $\operatorname{supp}\chi$. There exists $C>0$ such that for any solution u of (9.19) and all h in $\left]0,1\right]$ we have*

$$\int_0^T \Big\|\big(\chi h\partial_1 u\big)_{|y_1=0}(t)\Big\|^2_{L^2}\,\mathrm{d}t \le C\Bigg(\int_0^T\sum_{|\alpha|\le1}\|\chi_1(hD)^\alpha u(t)\|^2_{L^2(U_0^+)} + \Big\|h^{\frac12}\chi u(0)\Big\|_{L^2(U_0^+)} \Big\|h^{\frac12}\big(h\partial_1u\big)(0)\Big\|_{L^2(U_0^+)} + \Big\|h^{\frac12}u(T)\Big\|_{L^2(U_0^+)}\Big\|h^{\frac12}\big(h\partial_1u\big)(T)\Big\|_{L^2(U_0^0)}\Bigg)$$

Corollary 9.7. *One can find a constant $C>0$ such that*

$$\int_0^T \Big\|\big(\chi h_k\partial_1 v_k\big)_{|y_1=0}(t)\Big\|^2_{L^2}\,\mathrm{d}t \le C\left(\int_0^T\|\widetilde{\chi}w_k(t)\|^2_{L^2(\Omega)}\,\mathrm{d}t + \big\|\widetilde{u}_k^0\big\|^2_{L^2(\Omega)}\right) = \mathcal{O}(1)$$

where v_k has been defined in (9.13) and $\widetilde{\chi} \in \mathcal{C}_0^\infty\big(\mathbb{R}^d\big)$.

Proof of the Corollary

We use Lemma 9.6, (9.13), (9.14) the fact that $w_k = h_k^{-\frac{1}{2}}\theta\big(h_k^2 P_D\big)\widetilde{u}_k$ Lemma 6.3(ii) and the energy estimate for $\widetilde{u}_k$.

□

Proof of Lemma 9.6

Let us set with $L^2 = L^2\left(\mathbb{R}^+_{y_1} \times \mathbb{R}^{d-1}_{y'}\right)$

$$\text{(9.20)} \qquad \begin{cases} \mathrm{I} = \displaystyle\sum_{|\alpha|\leq 1} \|\chi_1 (hD)^\alpha u(t)\|^2_{L^2} \\ \mathrm{II} = \displaystyle\sum_{j=0}^{1} \left\| h^{\frac{1}{2}} \chi u\big(a_j\big) \right\|_{L^2} \cdot \left\| h^{\frac{1}{2}} \chi\big(h\partial_1 u\big)\big(a_j\big) \right\|_{L^2}, \quad a_0 = 0, a_1 = T \end{cases}$$

By (9.19) we have

$$\text{(9.21)} \qquad \begin{aligned} 2\,\mathrm{Re}\int_0^T \big(\chi h\left(D_1^2 u(t) + R\big(y,D'\big)u(t)\right), \chi h\partial_1 u(t)\big)_{L^2}\mathrm{d}t \\ = -2\,\mathrm{Im}\int_0^T \big(\chi h\partial_t u(t), \chi h\partial_1 u(t)\big)_{L^2}\mathrm{d}t \end{aligned}$$

By integration by part we have

$$\begin{aligned} \int_0^T \big(\chi h\partial_t u(t), \chi h\partial_1 u(t)\big)_{L^2}\mathrm{d}t - \int_0^T \big(\chi h\partial_1 u(t), \chi h\partial_t u(t)\big)_{L^2} \\ = \int_0^T \big(\big(\partial_1\chi^2\big)h\, u(t), h\partial_t u(t)\big)\,\mathrm{d}t + \mathcal{O}(\mathrm{II}) \end{aligned}$$

Since $h\partial_t u(t) = -ihD_1^2 u(t) - ihR\big(y,D'\big)u(t)$ integrating by part and using the fact that $u_{|y_1=0} = 0$ we find that

$$\int_0^T \big(\big(\partial_1\chi^2\big)h\, u(t), h\partial_t u(t)\big)_{L^2}\,\mathrm{d}t = \mathcal{O}(\mathrm{I})$$

It follows that

$$\text{(9.22)} \qquad \mathrm{Im}\int_0^T \big(\chi h\partial_t u(t), \chi h\partial_1 u(t)\big)_{L^2}\mathrm{d}t = \mathcal{O}(\mathrm{I}+\mathrm{II})$$

Now

$$\begin{aligned} -\int_0^T \big(\chi h\partial_1^2 u(t), \chi h\partial_1 u(t)\big)\,\mathrm{d}t \\ = \int_0^T \left\| \big(\chi h\partial_1 u(t)\big)_{|y_1=0} \right\|^2_{L^2(\mathbb{R}^{d-1})} + \mathcal{O}(\mathrm{I}) - \int_0^T \big(\chi h\partial_1 u(t), \chi h\partial_1^2 u(t)\big)_{L^2}\,\mathrm{d}t \end{aligned}$$

from which we deduce that

$$(9.23)\quad -2\,\mathrm{Re}\int_0^T \big(\chi h\partial_1^2 u(t),\chi h\partial_1 u(t)\big)_{L^2}\,\mathrm{d}t = \int_0^T \Big\|\big(\chi h\partial_1 u(t)\big)_{|y_1=0}\Big\|^2_{L^2(\mathbb{R}^{d-1})}\,\mathrm{d}t + \mathcal{O}(\mathrm{I})$$

Finally using again integration by parts, the fact that R is symmetric and $D'u_{|y_1=0}=0$ we find that

$$(9.24)\qquad 2\,\mathrm{Re}\int_0^T \big(\chi hR\big(y,D'\big)u(t),\chi h\partial_1 u(t)\big)_{L^2}\,\mathrm{d}t = \mathcal{O}(\mathrm{I})$$

Then the Lemma follows from (9.20) to (9.24). □

Let us go back to the estimate of A_k^2 defined in (9.17). We have

$$\big|A_k^2\big|^2 \le C\,h_k^2\,\big\|\theta\big(y_1,h_xD_1\big)\delta_{y_1=0}\big\|^2_{L^2(\mathbb{R})}\int_0^T \Big\|\psi_1\big(hD_1v_k(t)\big)_{|y_1=0}\Big\|^2_{L^2(\mathbb{R}^{d-1})}\,\|\psi_2V_k\|_{L^2(\mathbb{R}^{d+1})}$$

Applying Lemma 9.5 with $p=2$, $\ell=1$, Corollary 9.7 and Proposition 4.1 we obtain

$$(9.25)\qquad \big|A_k^2\big| \le C\,h_k$$

Using (9.18) and (9.25) we deduce (9.17) which implies (9.11) thus (9.10). The proof of Theorem 5.2 is complete. □

The measure on the boundary

Let us denote by $\dfrac{\partial}{\partial n}$ the normal derivative at the boundary $\partial\Omega$. By Corollary 9.6 we see that the sequence $\left(\mathbb{1}_{[0,T]}h_k\big(\dfrac{\partial w_k}{\partial n}\big)_{|\partial\Omega}\right)$ is bounded in $L^2\big(\mathbb{R}_t\times L^2(\partial\Omega)\big)$. Therefore with the notations in (5.1) and Proposition 5.1 we have the following Lemma.

Lemma 9.8. *There exist a subsequence $\big(W_{\sigma_1(k)}\big)$ of $\big(W_{\sigma(k)}\big)$ and a measure ν on $T^*\big(\partial\Omega\times\mathbb{R}_t\big)$ such that for every $a\in\mathcal{C}_0^\infty\big(T^*\big(\partial\Omega\times\mathbb{R}_t\big)\big)$ we have with*

$$J_k = \left(a\left(x,t,h_kD_x,h_k^2D_t\right)h_k\frac{1}{i}\frac{\partial W_k}{\partial n},h_k\frac{1}{i}\frac{\partial W_t}{\partial n}\right)_{L^2(\partial\Omega\times\mathbb{R}_t)}$$

$$(9.26)\qquad \lim_{k\to+\infty} J_{\sigma_1(k)} = \langle\,\nu,a\,\rangle$$

Proof of Theorem 5.3

We begin this proof by considering the case of points inside T^*M.

Proposition 9.9. *Let $m_0 = (x_0, \xi_0, t_0, \tau_0) \in T^*M$ and U_{m_0} a neighborhood of this point in T^*M. Then for every $a \in \mathcal{C}_0^\infty(U_{m_0})$ we have*

$$\langle \mu, H_p a \rangle = 0 \tag{9.27}$$

Proof

It is enough to prove (9.27) when $a(x,t,\xi,\tau) = \Phi(x,\xi)\chi(t,\tau)$ with $\pi_x \operatorname{supp}\Phi \subset V_{x_0} \subset \Omega$. Let $\varphi \in \mathcal{C}_0^\infty(\Omega)$ be such that $\varphi = 1$ on V_{x_0}. We introduce

$$\begin{aligned} A_k = \frac{i}{h_k}\Big[&\big(\Phi(x,h_kD_x)\chi(t,h_k^2D_t)\varphi h_k^2(D_t+P_D)\mathbb{1}_{[0,T]}\underline{w}_k, \mathbb{1}_{[0,T]}\underline{w}_k\big)_{L^2(\Omega\times\mathbb{R})} \\ &- \big(\Phi(x,h_kD_x)\chi(t,h_k^2D_t)\varphi\mathbb{1}_{[0,T]}\underline{w}_k, h_k^2(D_t+P_D)\mathbb{1}_{[0,T]}\underline{w}_k\big)_{L^2(\Omega\times\mathbb{R})}\Big]. \end{aligned} \tag{9.28}$$

We claim that we have

$$\lim_{k\to+\infty} A_k = 0 \tag{9.29).}$$

The two terms in A_k are of the same type and will tend both to zero. Moreover since $\pi_x \operatorname{supp}\Phi$ and φ have compact supports contained inside Ω we have in the scalar product $(D_t+P_D)\underline{w}_k = (D_t+P_D)w_k = 0$. Since $D_t\big(\mathbb{1}_{[0,T]}w_k\big) = \frac{1}{i}w_k(0)\otimes\delta_{t=0} - \frac{1}{i}w_k(T)\otimes\delta_{t=T}$, (9.29) will be proved if we show that

$$\begin{cases} \lim B_k = 0 \quad \text{where} \\ \quad B_k = \big(\chi(t,h_k^2D_t)\delta_{t=a}h_k\Phi(x,h_kD_x)\varphi(x)w_k(a), \mathbb{1}_{[0,T]}w_k\big)_{L^2(\Omega\times\mathbb{R})}, a = 0, T \end{cases} \tag{9.30}$$

Now we have

$$|B_k| \le \int_0^T \big|\chi(t,h_k^2D_t)\delta_{t=a}\big| \left(\int h_k \big|\Phi(x,h_kD_x)\varphi(x)w_k(a)\big|\ |w_k(t)|\,\mathrm{d}x\right)\mathrm{d}t$$

Using the Cauchy-Schwarz inequality in the second integral and the fact that $w_k(t) = h_k^{-\frac{1}{2}}\theta(h_k^2P_D)\widetilde{u}_k(t)$ we obtain

$$|B_k| \le C\int_0^T \big|\chi(t,h_k^2D_t)\delta_{t=a}\big|\,\|\widetilde{u}_k(a)\|_{L^2(\Omega)}\,\|\widetilde{u}_k(t)\|_{L^2(\Omega)}\,\mathrm{d}t$$

It follows then from the energy estimate on $[0,T]$ and Lemma 9.4 with $\ell = 2$, $p = 1$ that

$$|B_k| \le C\,\big\|\widetilde{u}_k^0\big\|_{L^2(\Omega)}^2 \le \frac{C'}{k} \quad \text{by (3.7)}$$

Thus (9.29) is proved.

Let us now compute A_k in another manner. We write

$$
\begin{cases}
A_k = A_k^1 + A_k^2 \\
A_k^j = \dfrac{i}{h_k}\Big[\big(\Phi\big(x,h_kD_x\big)\chi\big(t,h_k^2D_t\big)\varphi h_k^2Q_j\mathbb{1}_{[0,T]}\underline{w}_k,\mathbb{1}_{[0,T]}\underline{w}_k\big)_{L^2(\Omega\times\mathbb{R})} \\
\qquad\qquad -\big(\Phi\big(x,h_kD_x\big)\chi\big(t,h_k^2D_t\big)\varphi\mathbb{1}_{[0,T]}\underline{w}_k,h_k^2Q_k\mathbb{1}_{[0,T]}\underline{w}_k\big)_{L^2(\Omega\times\mathbb{R})}\Big] \\
\text{with}\quad Q_1 = D_t,\quad Q_2 = P_D\big(x,D_x\big)
\end{cases}
\tag{9.31}
$$

We claim that we have

$$
\lim_{k\to+\infty} A_k^1 = 0 \tag{9.32}
$$

Indeed we have

$$
A_k^1 = -h_k\left(\Phi\big(x,h_kD_x\big)\frac{\partial\chi}{\partial t}\big(t,h_k^2D_t\big)\varphi\mathbb{1}_{[0,T]}\underline{w}_k,\mathbb{1}_{[0,T]}\underline{w}_k\right)_{L^2(\Omega\times\mathbb{R})}
$$

Therefore we have

$$
\big|A_k^1\big| \le C\,h_k\int_0^T\Big\|\widetilde{\varphi}w_k(t)\Big\|^2_{L^2(\Omega)}\mathrm{d}t
$$

where $\widetilde{\varphi}\in\mathcal{C}_0^\infty(\Omega)$, $\widetilde{\varphi}=1$ on $\operatorname{supp}\varphi$ and (9.32) follows from (4.1).

Now since P_D is self adjoint on $L^2(\Omega)$ we can write

$$
A_k^2 = \frac{i}{h_k}\big(\big[\Phi\chi\varphi,h_k^2P_D\big]\widetilde{\varphi}\mathbb{1}_{[0,T]}\underline{w}_k,\widetilde{\varphi}\mathbb{1}_{[0,T]}\underline{w}_k\big)_{L^2(\Omega\times\mathbb{R})} \tag{9.33}
$$

It is easy to see that $h_k^2P_D = \displaystyle\sum_{j=0}^{2}h_k^{2-j}P_j\big(x,h_kD_x\big)$ where $P_j(x,\xi)$ is homogeneous in ξ of order j. Moreover in the semi classical pseudo-differential calculus we have $\big[P,Q\big] = \dfrac{h_k}{i}\mathcal{O}p\big(\{p,q\}\big)+h_k^2R$ where R is L^2 bounded. Using the fact that the sequence $\big(\widetilde{\varphi}\mathbb{1}_{[0,T]}\underline{w}_k\big)$ is uniformly bounded in $L^2\big(\mathbb{R},L^2\big(\mathbb{R}^d\big)\big)$ we see easily that the terms in A_k^2 corresponding to $j=0,1$ tend to zero when $k\to+\infty$. It follows that

$$
A_k^2 = \big(\mathcal{O}p\big(\{\Phi\chi,p\}\big)\widetilde{\varphi}\mathbb{1}_{[0,T]}\underline{w}_k,\mathbb{1}_{[0,T]}\underline{w}_k\big)+o(1)
$$

Using (5.1) and Proposition 5.1 we deduce that

$$
\lim_{k\to+\infty}A^2_{\sigma(k)} = -\big\langle\,\mu\,,\,H_p\big(\Phi\chi\big)\,\big\rangle \tag{9.34}
$$

It follows from (9.29), (9.31), (9.32) and (9.34) that $\langle\, \mu\,,\, H_p a\,\rangle = 0$ if $a = \Phi\chi$ which implies our Proposition.

□

We consider now the case of points $m_0 = \big(x_0, t_0, \xi_0, \tau_0\big)$ with $x_0 \in \partial\Omega$.

We take a neighborhood U_{x_0} so small that we can perform the diffeomorphism F described in (9.12). Let μ and ν the measures on $T^*\mathbb{R}^{d+1}$ and $T^*\big(\partial\Omega \times \mathbb{R}_t\big)$ defined in Proposition 5.1 and Lemma 9.7. We shall denote by $\widetilde{\mu}$ and $\widetilde{\nu}$ the measures on $T^*\big(U_0 \times \mathbb{R}_t\big)$ and $T^*\big(U_0 \cap \big\{y_1 = 0\big\} \times \mathbb{R}_t\big)$ which are the pull back of μ and ν by the diffeomorphism $\widetilde{F} : (x,t) \mapsto \big(F(x), t\big)$.

We first start a Lemma.

Lemma 9.10. *Let $a \in \mathcal{C}_0^\infty\left(T^*\big(U_0 \times \mathbb{R}_t\big)\right)$. We can find $a_j \in \mathcal{C}_0^\infty\left(U_0 \times \mathbb{R}_t \times \mathbb{R}^{d-1}_{\eta'} \times \mathbb{R}_\tau\right)$, $j = 0, 1$ and $a_2 \in \mathcal{C}^\infty\left(T^*\big(U_0 \times \mathbb{R}_t\big)\right)$ with compact support in $\big(y, t, \eta', \tau\big)$ such that with the notations of (9.12)*

$$a\big(y,t,\eta,\tau\big) = a_0\big(y,t,\eta',\tau\big) + a_1\big(y,t,\eta',\tau\big)\eta_1 + a_2(y,t,\eta,\tau)\big(\tau + \eta_1^2 + r\big(y,\eta'\big)\big)$$

where r is the principal symbol of $R\big(y, D'\big)$.

Proof

We apply a version of the Malgrange preparation theorem given by Theorem 7.5.4 in Hörmander [Hö]. With the notations there, for fixed $m' = \big(y, t, \eta', \tau\big)$ we shall take $t = \eta_1$, $g_{m'}\big(\eta_1\big) = a(y,t,\eta,\tau)$, $k = 2$, $b_1 = 0$, $b_0 = \tau + r\big(y,\eta'\big)$. According to this theorem we can write

$$a(y,t,\eta,\tau) = q\big(\eta_1, b_0, 0, g_{m'}\big)\left(\eta_1^2 + r\big(y,\eta'\big) + \tau\right) + \widetilde{a}_0\big(b_0, 0, g_{m'}\big) + \eta_1\widetilde{a}_1\big(b_0, 0, g_{m'}\big)$$

If we multiply both sides by a function $\varphi = \varphi\big(y,t,\eta',\tau\big) \in \mathcal{C}_0^\infty$ which is equal to one on the support in $\big(y,t,\eta',\tau\big)$ of a we obtain the claim of the Lemma.

□

In the following Remark we note that we can extend $\widetilde{\mu}$ to symbols which are not with compact support in η_1.

Remark 9.11. *Let $q(y,t,\eta,\tau) = \displaystyle\sum_{j=0}^{N} q_j\big(y,t,\eta',\tau\big)\eta_1^j$ where $q_j \in \mathcal{C}_0^\infty\big(\mathbb{R}^{2d+1}\big)$. Let $\phi \in \mathcal{C}_0^\infty(\mathbb{R})$, $\phi\big(\eta_1\big) = 1$ if $|\eta_1| \leq 1$. Then $\left\langle\, \widetilde{\mu}\,,\, q\phi\left(\dfrac{\eta_1}{R}\right)\right\rangle$ does not depend on R for large R. Indeed let $R_2 > R_1 \gg 1$. Then the symbol $q\left(\phi\left(\dfrac{\eta_1}{R_1}\right) - \phi\left(\dfrac{\eta_1}{R_2}\right)\right)$ has a support contained in the set $\big\{\, |\tau| + |\eta'| \leq C, |\eta_1| \geq R_1\big\}$. Therefore*

$$\operatorname{supp}\widetilde{\mu} \cap \operatorname{supp}\left(q\left(\phi\left(\frac{\eta_1}{R_1}\right) - \phi\left(\frac{\eta_1}{R_2}\right)\right)\right)$$
$$\subset \big\{\tau + \eta_1^2 + r\big(y,\eta'\big) = 0, |\tau| + |\eta'| \leq C, |\eta_1| \geq R\big\}$$

and the set in the right hand side is empty if R_1 is large enough.

We shall set $\langle\, \widetilde{\mu}\,,\, q\,\rangle = \lim_{R\to+\infty} \left\langle\, \widetilde{\mu}\,,\, q\phi\left(\frac{\eta_1}{R}\right)\right\rangle$.

We can now state the analogue of Proposition 9.9 in the case of boundary points.

Proposition 9.12. *With the notations of Lemma 9.10 for any* $a \in \mathcal{C}_0^\infty\big(T^*\big(U_0\times\mathbb{R}_t\big)\big)$ *we have*

$$\langle\, \widetilde{\mu}\,,\, H_p a\,\rangle = -\left\langle\, \widetilde{\nu}\,,\, a_{1\,|y_1=0}\,\right\rangle$$

Proof

Let us recall that v_k has been defined in (9.13) which satisfies

$$\begin{aligned}\big(D_t+P_D\big)v_k =& 0 \text{ in } U_0^+ = \big\{y\in U_0\ :\ y_1>0\big\}\\ v_{k\,|y_1=0} =& 0\\ v_k =& h_k^{-\frac{1}{2}}\left(\theta\big(h_k^2P_D\big)w_k\right)\circ F^{-1}\end{aligned}$$

For sake of shortness whe shall set

$$\Lambda_k = \big(y,t,h_kD_y,h_k^2D_t\big)\,,\quad L^2_+ = L^2\big(U_0^+\times\mathbb{R}_t\big) \tag{9.35}$$

The Proposition will be a consequence of the following Lemmas.

Lemma 9.13. *Let for* $j=0,1$, $a_j = a_j\big(y,t,\eta',\tau\big)\in\mathcal{C}_0^\infty\big(U_0\times\mathbb{R}^{d+1}\big)$ *and* $\varphi\in\mathcal{C}_0^\infty\big(U_0\big)$, $\varphi=1$ *on* $\pi_y\operatorname{supp} a_j$. *Then*

(9.36)

$$\begin{aligned}&\frac{i}{h_k}\Big[\Big(\big(a_0\big(\Lambda_k\big)+a_1\big(\Lambda_k\big)h_kD_1\big)\,\varphi h_k^2\big(D_t+P_D\big)\mathbb{1}_{[0,T]}v_k,\mathbb{1}_{[0,T]}v_k\Big)_{L^2_+}\\ &\qquad-\int_{U_0^+}\Big\langle\,\big(a_0\big(\Lambda_k\big)+a_1\big(\Lambda_k\big)h_kD_1\big)\,\varphi\mathbb{1}_{[0,T]}v_k\,,\,h_k^2\big(D_t+P_D\big)\overline{\mathbb{1}_{[0,T]}v_k}\,\Big\rangle\,\mathrm{d}y\Big]\\ &=-\frac{i}{h_k}\Big(\big[h_k^2\big(D_t+P_D\big),\big(a_0\big(\Lambda_k\big)+a_1\big(\Lambda_k\big)h_kD_1\big)\,\varphi\big]\,\mathbb{1}_{[0,T]}v_k,\mathbb{1}_{[0,T]}v_k\Big)_{L^2_+}\\ &-\Big(a_1\big(0,y',t,h_kD_{y'},h_k^2D_t\big)\varphi_{|y_1=0}\mathbb{1}_{[0,T]}\left(h_kD_1v_{k\,|y_1=0}\right),\mathbb{1}_{[0,T]}\left(h_kD_1v_{k\,|y_1=0}\right)\Big)_{L^2(\mathbb{R}^{d-1}\times\mathbb{R})}\end{aligned}$$

Here $\langle\ ,\ \rangle$ *denotes the bracket in* $\mathcal{D}'\big(\mathbb{R}_t\big)$.

Lemma 9.14. *Let* $b=b\big(y,t,\eta',\tau\big)\in\mathcal{C}_0^\infty\big(U_0\times\mathbb{R}^{d+1}\big)$ *and* $\varphi\in\mathcal{C}_0^\infty\big(U_0\big)$, $\varphi=1$ *on* $\pi_y\operatorname{supp} b$ *For* $j=0,1$ *we set, with the same notations as in Lemma 9.13*

$$\begin{aligned}I_k^j &= \Big(h_k^{-1}b\big(\Lambda_k\big)\varphi\big(h_kD_1\big)^jh_k^2\big(D_t+P_D\big)\mathbb{1}_{[0,T]}v_k,\mathbb{1}_{[0,T]}v_k\Big)_{L^2_+}\\ J_k^j &= \int_{U_0^+}\Big\langle\,h_k^{-1}\,b\big(\Lambda_k\big)\varphi\big(h_kD_1\big)^j\mathbb{1}_{[0,T]}v_k\,,\,h_k^2\big(D_t+P_D\big)\mathbb{1}_{[0,T]}v_k\,\Big\rangle\,\mathrm{d}y\end{aligned}$$

Then $\lim_{k\to+\infty} I_k^j = \lim_{k\to+\infty} J_k^j = 0$

Lemma 9.15. *Let for* $j = 0, 1, 2$, $b_j = b_j\big(y, t, \eta', \tau\big) \in \mathcal{C}_0^\infty\big(U_0 \times \mathbb{R}^d\big)$ *and* $\varphi \in \mathcal{C}_0^\infty\big(U_0\big)$, $\varphi = 1$ *on* $\pi_y \operatorname{supp} b_j$. *Let us set*

$$L_k^j = \Big(b_j\big(\Lambda_k\big)\varphi\big(h_k D_1\big)^j \mathbb{1}_{[0,T]} v_k, \mathbb{1}_{[0,T]} v_k\Big)_{L^2_+}$$

Then with $\sigma(k)$ *defined in Proposition 5.1 we have for* $j = 0, 1, 2$

$$\lim_{k\to+\infty} L^j_{\sigma(k)} = \Big\langle\, \widetilde{\mu}\,,\, b_j \eta_1^j \,\Big\rangle$$

Proof of Proposition 9.12

Let ϕ be as in Remark 9.11. If R is large enough we have $H_p a = H_p a \phi\left(\dfrac{\eta_1}{R}\right)$ so by Lemma 9.10

$$\langle\, \widetilde{\mu}\,,\, H_p a \,\rangle = \left\langle\, \widetilde{\mu}\,,\, H_p\big(a_0 + a_1\eta_1\big)\phi\left(\frac{\eta_1}{R}\right) \right\rangle + \left\langle\, \widetilde{\mu}\,,\, (\tau + p) H_p a_2 \phi\left(\frac{\eta_1}{R}\right) \right\rangle$$

Since Theorem 5.2 implies that $\left\langle\, \widetilde{\mu}\,,\, (\tau + p) H_p a_2 \phi\left(\dfrac{\eta_1}{R}\right) \right\rangle = 0$ we deduce from Remark 9.11 that

$$\langle\, \widetilde{\mu}\,,\, H_p a \,\rangle = \big\langle\, \widetilde{\mu}\,,\, H_p\big(a_0 + a_1\eta_1\big) \,\big\rangle$$

Then Proposition 9.12 will be proved if we can show that

$$\big\langle\, \widetilde{\mu}\,,\, H_p\big(a_0 + a_1\eta_1\big) \,\big\rangle = -\big\langle\, \widetilde{\nu}\,,\, a_{1|y_1} = 0 \,\big\rangle \tag{9.37}$$

By Lemma 9.14 the left hand side of (9.36) tends to zero when $k \to +\infty$.

Now by the semi classical symbolic calculus we can write

$$\frac{i}{h_k}\left[h_k^2\big(D_t + P_D\big)\left(a_0\big(\Lambda_k\big) + a_1\big(\Lambda_k\big) h_k D_1\right)\varphi\right] = \sum_{j=0}^{2} b_j\big(\Lambda_k\big)\varphi_1\big(h_k D_1\big)^j$$

where $b_j \in \mathcal{C}_0^\infty\big(U_0 \times \mathbb{R}^{d+1}\big)$, $\varphi_1 = 1$ on $\operatorname{supp}\varphi$ and $\big\{p, a_0 + a_1\eta_1\big\} = \displaystyle\sum_{j=0}^{2} b_j \eta_1^j$. So using (9.36), Lemma 9.15 and Lemma 9.8 we obtain $0 = -\big\langle\, \widetilde{\mu}\,,\, H_p\big(a_0 + a_1\eta_1\big) \,\big\rangle - \big\langle\, \widetilde{\nu}\,,\, a_{1|y_1=0} \,\big\rangle$ which proves (9.37) and Proposition 9.12.

□

Proof of Lemma 9.13

To prove (9.36) we use integration by parts in D_1, $D_{y'}$ and distribution derivative in D_t. Only the terms containing D_1 give a boundary contribution. We treat them as follows for $j = 0, 1$.

$$\Big(a_j\big(\Lambda_k\big)\varphi\big(h_kD_1\big)^j\mathbb{1}_{[0,T]}v_k, h_k^2D_1^2\mathbb{1}_{[0,T]}v_k\Big)_{L^2_+} = \Big(h_k^2D_1^2a_j\big(\Lambda_k\big)\varphi\big(h_kD_1\big)^j\mathbb{1}_{[0,T]}v_k, \mathbb{1}_{[0,T]}v_k\Big)_{L^2_+}$$

$$+\frac{h_k}{i}\big(a_j\big(0,y',t,h_kD_{y'},h_k^2D_t\big)\varphi_{|y_1=0}\mathbb{1}_{[0,T]}\big(h_kD_1\big)^jv_{k|y_1=0}, \mathbb{1}_{[0,T]}\big(h_kD_1\big)^jv_{k|y_1=0}\big)_{L^2(\mathbb{R}^{d-1}_{y'}\times\mathbb{R}_t)}$$

Here we have used the boundary condition $v_{k|y_1=0}$. □

Proof of Lemma 9.14

It is enough to consider symbols b of the form $b = c\big(y,\eta'\big)\chi(t,\tau)$.

We have $h_k^2\big(D_t + P_D\big)\mathbb{1}_{[0,T]}v_k = \dfrac{h_k^2}{i}\big[v_k(0,\cdot)\delta_{t=0} - v_k(T,\cdot)\delta_{t=T}\big]$. It follows that I_k^j is a sum of two terms of the form

$$\widetilde{I}_k^{\,j} = \left(\chi\big(t,h_k^2D_t\big)\delta_{t=a}h_k^{\frac{1}{2}}c\big(y,h_kD_{y'}\big)\varphi\big(h_kD_1\big)^jv_k(a,\cdot), \mathbb{1}_{[0,T]}v_k\right)_{L^2_+}$$

We have with $a = 0$ or T

$$\Big|\widetilde{I}_k^{\,j}\Big| \le C\int_{\mathbb{R}_t}\big|\chi\big(t,h_k^2D_t\big)\delta_{t=a}\big|\left\|h_k^{\frac{1}{2}}\big(h_kD_1\big)^jv_k(a,\cdot)\right\|_{L^2(U_0^+)}\left\|h_k^{\frac{1}{2}}\mathbb{1}_{[0,T]}v_k(t,\cdot)\right\|_{L^2(U_0^+)}\mathrm{d}t$$

Since for $t\in\big[0,T\big]$, $\left\|h_k^{\frac{1}{2}}\big(h_kD_1\big)^jv_k(t,\cdot)\right\|_{L^2(U_0^+)} \le C\left\|\widetilde{u}_k^{\,0}\right\|_{L^2(\Omega)}$, $j = 0, 1$, we deduce from Lemma 9.5 with $\ell = 2$, $p = 1$ that $\lim\limits_{k\to+\infty}\widetilde{I}_k^{\,j} = \lim\limits_{k\to+\infty}I_k^j = 0$.

Let us consider the term J_k^j. It is sufficient to consider symbols b of the form

$$b = \psi(t)c\big(y,h_kD_{y'}\big)\chi\big(h_k^2D_\sigma\big)$$

As before we have $\big(D_t + P_D\big)\mathbb{1}_{[0,T]}v_k = \dfrac{1}{i}\left(v_k(0,\cdot)\delta_{t=0} - v_k(T,\cdot)\delta_{t=T}\right)$ so J_u^j will be a sum of two terms of the form

$$\widetilde{J}_k^j = \int_{U_0^+}\left\langle h_k^{-1}\psi c(\cdots)\chi(\cdots)\varphi\big(h_kD_1\big)^j\mathbb{1}_{[0,T]}v_k\,,\,\frac{h_k^2}{i}v_k\big(a,\cdot\big)\delta_{t=a}\right\rangle\mathrm{d}y$$

Writing

$$\widetilde{J}_k^{\,j} = \int_{U_0^+}\left\langle c\big(y,h_kD_{y'}\big)\varphi\big(h_kD_1\big)^j\mathbb{1}_{[0,T]}v_k\,,\,\frac{h^2}{i}\psi(a)v_k(a,\cdot)\chi\big(h_k^2D_t\big)\delta_{t=a}\right\rangle\mathrm{d}y$$

we see that $\widetilde{J}_k^j$ can be estimated exactly by the same method as the term $\widetilde{I}_k^j$ above. □

Proof of Lemma 9.15

The proof will be different for each j. We shall consider the case $j = 0$ then $j = 2$ and finally $j = 1$.

Case $j = 0$: we shall need the following Lemma.

Lemma 9.16. *Let* $\Phi \in \mathcal{C}_0^\infty(\mathbb{R})$, $\phi(\eta_1) = 1$ *if* $|\eta_1| \leq 1$.

Let $b = b(y, t, \eta', \tau) \in \mathcal{C}_0^\infty\big(U_0 \times \mathbb{R}^{d+1}\big)$ *and* $\varphi \in \mathcal{C}_0^\infty\big(U_0\big)$, $\varphi = 1$ *near* $\pi_y \operatorname{supp} b$. *Then*

$$\lim_{R\to+\infty} \overline{\lim_{k\to+\infty}} \left\| \left(I - \Phi\left(\frac{h_k D_1}{R}\right)\right) b\big(\Lambda_k\big)\varphi \mathbb{1}_{[0,T]} \mathbb{1}_{y_1>0} v_k \right\|_{L^2(\mathbb{R}^{d+1})} = 0$$

Let us assume this Lemma for a moment and show how it implies the case $j = 0$. We remark first that $b_0\big(\Lambda_k\big)\mathbb{1}_{[0,T]}\mathbb{1}_{y_1>0}v_k = \mathbb{1}_{y_1>0}b_0\big(\Lambda_k\big)\mathbb{1}_{[0,T]}v_k$ which allows us to write $V_k = \mathbb{1}_{[0,T]}\mathbb{1}_{y_1>0}v_k$

$$\begin{aligned} L_k^0 &= \big(b_0\big(\Lambda_k\big)\varphi V_k, \varphi_1 V_k\big)_{L^2_+} \\ &= \left(\left(I - \Phi\left(\frac{h_k D_1}{R}\right)\right) b_0\big(\Lambda_k\big)\varphi V_k, \varphi_1 V_k \right)_{L^2_+} + \left(\Phi\left(\frac{h_k D_1}{R}\right) b_0\big(\Lambda_k\big)\varphi V_k, \varphi_1 V_k \right)_{L^2_+} \\ &= A_k + B_k \end{aligned}$$

where $\varphi_1 \in \mathcal{C}_0^\infty\big(U_0\big)$, $\varphi_1 = 1$ on $\operatorname{supp}\varphi$. We have

$$|A_k| \leq \left\| \left(I - \Phi\left(\frac{h_k D_1}{R}\right)\right) b_0\big(\Lambda_k\big) V_k \right\|_{L^2(\mathbb{R}^{2d+2})} \left\| \varphi_1 \mathbb{1}_{[0,T]} \mathbb{1}_{y_1>0} v_k \right\|_{L^2(\mathbb{R}^{2d+2})}$$

Since by (4.1) $\left\|\varphi_1 \mathbb{1}_{[0,T]}\mathbb{1}_{y_1>0}v_k\right\|_{L^2} \leq C$ uniformly in k, Lemma 9.16 shows that $\lim\limits_{R\to+\infty} \overline{\lim\limits_{k\to+\infty}} A_k = 0$. Now by Proposition 5.1 we have $\lim\limits_{k\to+\infty} B_{\sigma(k)} = \left\langle \widetilde{\mu}\,,\, \Phi\left(\dfrac{\eta_1}{R}\right) b_0 \right\rangle$ so $\lim\limits_{R\to+\infty} \lim\limits_{k\to+\infty} B_{\sigma(k)} = \langle\, \widetilde{\mu}\,,\, b_0\,\rangle$.

Proof of Lemma 9.16

If $v \in H^1\big(\mathbb{R}^{d+1}\big)$ we can write

$$\int \left\| \left(I - \Phi\left(\frac{h_k D_1}{R}\right)\right) v \right\|^2_{L^2(\mathbb{R}^d)} \mathrm{d}t = \int \left(\int \left| \frac{1 - \Phi\left(\dfrac{h_k\eta_1}{R}\right)}{h_k\eta_1} \right|^2 \left| h_k \widehat{D_1 v}(\eta, t)\right|^2 \mathrm{d}\eta \right) \mathrm{d}t$$

$$\leq \frac{c}{r^2} \left\| h_k D_1 v \right\|^2_{L^2(\mathbb{R}^{d+1})}$$

Now

$$h_k D_1 b\big(\Lambda_k\big)\varphi \mathbb{1}_{[0,T]}\mathbb{1}_{y_1>0} v_k$$
$$= \frac{h_k}{i}\left(\frac{\partial b}{\partial y_1}\big(\Lambda_k\big)\varphi + b\frac{\partial \varphi}{\partial y_1}\right)\mathbb{1}_{[0,T]}\mathbb{1}_{y_1>0} v_k + b\big(\Lambda_k\big)\mathbb{1}_{[0,T]}\mathbb{1}_{y_1>0}\big(h_k D_1 v_k\big)$$
$$= A_k + B_k$$

Here we have used $D_1\big(\mathbb{1}_{y_1>0} v_k\big) = \mathbb{1}_{y_1>0} D_1 v_k$ since $v_{k|y_1=0} = 0$.

We have

$$\|A_k\|_{L^2(\mathbb{R}^{d+1})} \leq C\, h_k \int_0^T \|\varphi_1 w_k(t)\|^2_{L^2(\Omega)}\, \mathrm{d}t$$

where $\varphi_1 \in \mathcal{C}_0^\infty\big(\mathbb{R}^d\big)$. It follows from (4.1) that $\lim\limits_{k\to+\infty} A_k = 0$.

Now since $v_k = w_k \circ F^{-1}$ and $w_k = h_k^{-\frac{1}{2}}\theta\big(h_k^2 P_D\big)\widetilde{u}\,{}_k^0$ it follows from Lemma 6.3 that $\|B_k\|_{L^2(\mathbb{R}^{d+1})} \leq C\displaystyle\int_0^T \|\varphi_1 w_k(t)\|^2_{L^2(\Omega)}\,\mathrm{d}t \;\leq\; C'$ since

$$\left\|\left(I - \Phi\left(\frac{h_k D_1}{R}\right) b\big(\Lambda_k\big)\varphi \mathbb{1}_{[0,T]}\mathbb{1}_{y_1>0} v_k\right\|_{L^2(\mathbb{R}^{d+1})}\right) \leq \frac{C}{R_2}\left(\|A_k\|_{L^2} + \|B_k\|_{L^2}\right)$$

the Lemma follows.

□

Case $j = 2$ (in Lemma 9.15)

Since $\big(h_k D_1\big)^2 = h_k^2\big(D_t + P_D\big) - h_k^2 D_t - R_2\big(y, h_k D'\big) - h_k R_1\big(y, h_k D'\big) - h_k^2 R_0(y)$ we can write

$$\text{(9.38)}\qquad \begin{cases} L_k^2 = A_k + B_k \quad \text{where} \\ A_k = \big(b_2\big(\Lambda_k\big)\varphi h_k^2\big(D_t + P_D\big)\mathbb{1}_{[0,T]} v_k,\, \mathbb{1}_{[0,T]} v_k\big)_{L^2_+} \\ B_k = \big(c\big(\Lambda_k\big)\varphi \mathbb{1}_{[0,T]} v_k,\, \mathbb{1}_{[0,T]} v_k\big)_{L^2_+} \\ C_k = h_k\left(d\left(h_k, y, t, h_k D_{y'}, h_k^2 D_t\right)\varphi \mathbb{1}_{[0,T]} v_k,\, \mathbb{1}_{[0,T]} v_k\right)_{L^2_+} \\ \text{and } c = -b_2(\tau + r) \end{cases}$$

By Lemma 9.14 we have $\lim\limits_{k\to+\infty} A_k = 0$. By Lemma 9.15 for $j = 0$ (case proved above) we have $\lim\limits_{k\to+\infty} B_{\sigma(k)} = \langle\, \widetilde{\mu}\,,\, -b_2(\tau + r)\,\rangle = \big\langle\, \widetilde{\mu}\,,\, b_2\eta_1^2\,\big\rangle$ since $\left\langle\, \widetilde{\mu}\,,\, b_2\big(\tau + \eta_1^2 + r\big)\phi\left(\dfrac{\eta_1}{R}\right)\right\rangle = 0$ for all R large enough.

Finally $|C_k| \leq M\, h_k \displaystyle\int_0^T \|w_k(t)\|^2_{L^2(\Omega)}\,\mathrm{d}t \;\leq\; M'\, h_k$ so $\lim\limits_{k\to+\infty} C_k = 0$.

Case $j = 1$

We have $\mathbb{1}_{y_1>0}\big(h_kD_1\big)v_k = \big(h_kD_1\big)\mathbb{1}_{y_1>0}v_k$ because $v_{k\,|y_1=0} = 0$. It follows that with $V_k = \mathbb{1}_{[0,T]}\mathbb{1}_{y_1>0}v_k$

$$L_k^1 = \big(b_1\big(\Lambda_k\big)\varphi\big(h_kD_1\big)V_k, V_k\big)_{L^2(\mathbb{R}^{d+1})}$$

Let $\Phi \in \mathcal{C}_0^\infty(\mathbb{R})$, $\Phi\big(\eta_1\big) = 1$ if $|\eta_1| \leq 1$. Then we write with $\varphi_1 = 1$ on $\operatorname{supp}\varphi$,

$$\begin{aligned} L_k^1 &= \left(\Phi\left(\frac{h_kD_1}{R}\right) b_1\big(\Lambda_k\big)\varphi\big(h_kD_1\big)V_k, \varphi_1V_k\right)_{L^2(\mathbb{R}^{d+1})} \\ &\qquad + \left(\left(I - \Phi\left(\frac{h_kD_1}{R}\right)\right) b_1\big(\Lambda_k\big)\varphi\big(h_kD_1\big)V_k, \varphi_1V_k\right)_{L^2(\mathbb{R}^{d+1})} \\ &= A_k + B_k \end{aligned}$$

It is easy to see from Proposition 5.1 that

$$\lim_{k\to+\infty} A_{\sigma(k)} = \left\langle\, \widetilde{\mu}\,,\, \Phi\left(\frac{\eta_1}{R}\right) b_1\eta_1 \,\right\rangle$$

Now

$$|B_k| \leq C\left\|b_1\big(\Lambda_k\big)\varphi\big(h_kD_1\big)V_k\right\|_{L^2(\mathbb{R}^{d+1})} \left\|\left(I - \Phi\left(\frac{h_kD_1}{R}\right)\right)\varphi_1V_k\right\|_{L^2(\mathbb{R}^{d+1})}$$

As in the proof of Lemma 9.16 we have

$$\left\|\left(I - \Phi\left(\frac{h_kD_1}{R}\right)\right)\varphi_1v_k\right\|_{L^2(\mathbb{R}^{d+1})} \leq \frac{1}{R}\left(C + h_k^{\frac{1}{2}}\left\|\widetilde{u}_k^{\,0}\right\|_{L^2(\Omega)}\right)$$

and using the fact that h_kD_1 commutes with $\mathbb{1}_{y_1>0}$ on v_k, since $v_{k\,y_1=0} = 0$ and using Lemma 6.3$(i),(ii)$ we find easily that $\left\|b_1\big(\Lambda_k\big)\varphi\big(h_kD_1\big)V_k\right\|_{L^2(\mathbb{R}^{d+1})} \leq M$ uniformly in k.

Lemma 9.17. *With $\mathcal{G}_d$ and $\mathcal{G}^k$ introduced in Definition 9.2 we have*

$$\widetilde{\nu}\left(\mathcal{G}_d \cup \left(\bigcup_{k=3}^{+\infty} \mathcal{G}^k\right)\right) = 0$$

Proof

Let us take in Lemma 9.10 $a(y,t,\eta,\tau) = b\big(y,t,\eta',\tau\big)\eta_1$. Since $p = \eta_1^2 + r\big(y,\eta'\big)$ we will have

$$\left\langle\, \widetilde{\mu}\,,\, \eta_1H_pb - b\frac{\partial r}{\partial y_1} \,\right\rangle = -\left\langle\, \widetilde{\nu}\,,\, b_{|y_1=0} \,\right\rangle \tag{9.39}$$

Let us make the change of variables

$$(y,t,\eta,\tau)\overset{\Phi}{\mapsto}\big(z=y,s=t,\zeta=\eta,\sigma=\tau+r(y,\eta)\big) \tag{9.40}$$

It is easy to see that

$$\begin{cases} H_p b = X\big(b\circ\Phi^{-1}\big)\circ\Phi \quad \text{with} \\ \quad X = 2\zeta_1\dfrac{\partial}{\partial z_1}+2\zeta_1\dfrac{\partial r}{\partial z_1}\dfrac{\partial}{\partial\sigma}-\dfrac{\partial r}{\partial z_1}\dfrac{\partial}{\partial\zeta_1}+H'_r \quad \text{where} \\ \quad H'_r=\dfrac{\partial r}{\partial\zeta'}\dfrac{\partial}{\partial z'}-\dfrac{\partial r}{\partial z'}\dfrac{\partial}{\partial\zeta'} \end{cases} \tag{9.41}$$

It we denote by $\widetilde{\mu}_1$, $\widetilde{\nu}_1$ the pull back of $\widetilde{\mu}$ and $\widetilde{\nu}$ by Φ and $\widetilde{b}=b\circ\Phi^{-1}$, the equality (9.39) becomes

$$\left\langle\,\widetilde{\mu}_1\,,\,\zeta_1 X\widetilde{b}-\widetilde{b}\frac{\partial r}{\partial z_1}\,\right\rangle=-\left\langle\,\widetilde{\nu}_1,\,,\,\widetilde{b}\big(0,z',s,\zeta',\sigma\big)\,\right\rangle \tag{9.42}$$

Let us take $\widetilde{b}$ of the following form

$$\widetilde{b}\big(z,s,\zeta',\sigma\big)=b_0\left(\frac{z_1}{\sqrt{\varepsilon}},z',s,\zeta',\frac{\sigma}{\varepsilon}\right)\psi\left(\frac{1}{\sqrt{\varepsilon}}\frac{\partial r}{\partial z_1}(z,\zeta')\right)$$

where $b_0\in\mathcal{C}_0^\infty$, $b_0\geq 0$, and $\psi\in\mathcal{C}^\infty(\mathbb{R})$ such that $\psi(t)=1$, $t\in\big(-\infty,0\big]$, $\psi(t)=0$ for $t\geq 1$ and $\varepsilon>0$. According to (9.42) we can write

$$\begin{cases} \left\langle\,\widetilde{\mu}_1\,,\,\zeta_1 X\widetilde{b}-\widetilde{b}\dfrac{\partial r}{\partial z_1}\,\right\rangle=\ \textcircled{1}+\ \textcircled{2} \quad \text{with} \\ \textcircled{1}=\left\langle\,\widetilde{\mu}_1\,,\,-\dfrac{\partial r}{\partial z_1}b_0\psi\,\right\rangle \\ \textcircled{2}=\langle\,\widetilde{\mu}_1\,,\,f_\varepsilon\,\rangle \quad \text{with} \\ f_\varepsilon=2\dfrac{\zeta_1^2}{\sqrt{\varepsilon}}\dfrac{\partial b_0}{\partial z_1}\psi+2\dfrac{\zeta_1^2}{\varepsilon}\dfrac{\partial r}{\partial z_1}\dfrac{\partial b_0}{\partial\sigma}\psi+\zeta_1 H'_x b_0\psi+\zeta_1 b_0\dfrac{1}{\sqrt{\varepsilon}}\left(X\dfrac{\partial r}{\partial z_1}\right)\psi' \end{cases} \tag{9.43}$$

According to Theorem 5.2 and (9.40) we have $\operatorname{supp}\widetilde{\mu}_1\subset\big\{z_1\geq 0\text{ and }\zeta_1^2+\sigma=0\big\}$. Therefore on $\operatorname{supp}\widetilde{\mu}_1\cap\operatorname{supp}b_0$ we have $|\zeta_1|^2\leq|\sigma|\leq C\varepsilon$.

This implies that $f_\varepsilon\in\mathcal{C}_0^\infty$ is uniformly bounded in $\varepsilon\in\big]0,1\big]$.

Moreover the first and the third term in f_ε tend to zero uniformly with ε. The second term can be written on $\operatorname{supp}\widetilde{\mu}_1$

$$-2\frac{\sigma}{\varepsilon}\frac{\partial r}{\partial z_1}(z,\zeta')\frac{\partial b_0}{\partial\sigma}\left(\frac{z_1}{\sqrt{\varepsilon}},z',s,\zeta',\frac{\sigma}{\varepsilon}\right)\psi(\cdots)$$

Since b_0 has compact support in σ, for fixed $\sigma \neq 0$ this term is indentically zero for ε small enough and it also vanish when $\sigma = 0$.

Finally, since $\operatorname{supp} \psi' = \big[0,1\big]$ and $\psi'(0) = 0$, $\psi'\left(\frac{1}{\sqrt{\varepsilon}} \frac{\partial r}{\partial z_1}\big(z, \zeta'\big)\right)$ vanishes if ε is small enough. Therefore we can apply the Lebesgue dominated convergence theorem and conclude that

$$\lim_{\varepsilon \to 0} \ ② = 0 \tag{9.44}$$

Now let us set $A = \left\{(z, s, \zeta, \sigma) \ : z_1 = 0, \sigma = 0, \frac{\partial r}{\partial z_1}\big(z, \zeta'\big) \leq 0\right\}$ and write

$$① = \left\langle \widetilde{\mu}_1 \, , \, -\frac{\partial r}{\partial z_1} b_0 \psi \mathbb{1}_A \right\rangle + \left\langle \widetilde{\mu}_1 \, , \, -\frac{\partial r}{\partial z_1} b_0 \psi \mathbb{1}_{A^c} \right\rangle \tag{9.45}$$

If we are in A^c we have else $z_1 \neq 0$ or $\sigma \neq 0$ or $\frac{\partial r}{\partial z_1}\big(z, \zeta'\big) > 0$. In all these case we have

$$\lim_{\varepsilon \to 0} b_0 \left(\frac{z_1}{\sqrt{\varepsilon}}, z', s, \zeta', \frac{\sigma}{\varepsilon}\right) \psi \left(\frac{1}{\sqrt{\varepsilon}} \frac{\partial r}{\partial z_1}\big(z, \zeta'\big)\right) = 0$$

By the dominated convergence theorem the second term in the right hand side of (9.45) tends to zero. It follows that for ε small enough we have

$$① = \left\langle \widetilde{\mu}_1 \, , \, -\frac{\partial r}{\partial z_1}\big(0, z', \zeta'\big) b_0\big(0, z', s, \zeta', 0\big) \mathbb{1}_A \right\rangle + o(1)$$

Using (9.43), (9.44) we conclude that

$$\lim_{\varepsilon \to 0} \left\langle \widetilde{\mu}_1 \, , \, \zeta_1 X \widetilde{b} - \widetilde{b} \frac{\partial r}{\partial z_1} \right\rangle \geq 0 \tag{9.46}$$

On the other hand we have

$$\left\langle \widetilde{\nu}_1 \, , \, \widetilde{b}\big(0, z', s, \zeta', \sigma\big) \right\rangle = \left\langle \widetilde{\nu}_1 \, , \, b_0 \left(0, z', s, \zeta', \frac{\sigma}{\varepsilon}\right) \psi \left(\frac{1}{\sqrt{\varepsilon}} \frac{\partial r}{\partial z_1}\big(0, z', \zeta'\big)\right) \right\rangle$$

We introduce $B = \big\{\big(z', s, \zeta', \sigma\big)\big\} \ : \quad \sigma = 0, \ \frac{\partial r}{\partial z_1}\big(0, z', \zeta'\big) \leq 0\big\}$ and write as before $1 = \mathbb{1}_B + \mathbb{1}_{B^c}$. The term corresponding to $\mathbb{1}_{B^c}$ tends to zero. It follows that

$$\left\langle \widetilde{\nu}_1 \, , \, \widetilde{b}_{|z_1 = 0} \right\rangle = \left\langle \widetilde{\nu}_1 \, , \, b_0\big(0, z', s, \zeta', 0\big) \mathbb{1}_B \right\rangle + o(1)$$

Therefore we have

$$\lim_{\varepsilon\to 0}\Big\langle\, \widetilde{\nu}_1\,,\, \widetilde{b}_{|z_1=0}\,\Big\rangle = \Big\langle\, \widetilde{\nu}_1\,,\, b_0\big(0,z',s,\zeta',0\big)\mathbb{1}_{\left\{\sigma=0,\frac{\partial r}{\partial z_1}(0,z',\zeta')\leq 0\right\}}\,\Big\rangle \geq 0 \tag{9.47}$$

Using (9.42), (9.46) and (9.47) we conclude that both sides of (9.42) vanish. Coming back to the coordinates (y,t,η,τ) by (9.40) we conclude that

$$\Big\langle\, \widetilde{\nu}\,,\, b_0\big(0,y',t,\eta',0\big)\mathbb{1}_{\left\{\tau+r(0,y',\eta')=0,\frac{\partial r}{\partial y_1}(0,y',\eta')\leq 0\right\}}\,\Big\rangle = 0$$

for every $b_0\in\mathcal{C}_0^\infty$, $b_0\geq 0$.

Since $\mathcal{G}_d\cup\left(\bigcup_{k=3}^{+\infty}\mathcal{G}^k\right)=\left\{\big(y',t,\eta',\tau\big)\ :\ \tau+r\big(0,y',\eta'\big)=0,\frac{\partial r}{\partial y_1}\big(0,y',\eta'\big)\leq 0\right\}$, Lemma 9.17 follows.

□

Proof of the propagation theorem 5.3 (continued)

From now on we follow closely [B], [B-G], [G-L] and we give the details for the convenience of the reader.

Let us set, with the notations of section 9.1

$$\mathcal{G}^0=T^*M,\quad \mathcal{G}^1=\mathcal{H} \tag{9.48}$$

We introduce for $k\in\mathbb{N}$ the following proposition.

$$\big(P_k\big)\qquad \begin{cases} \text{Let } \zeta\in\Sigma_b. \text{ If there exists } T>0 \text{ such that for all } s\in\big[0,T\big] \text{ we have}\\ \Gamma(s,\zeta)\in\bigcup_{j=0}^{k}\mathcal{G}^j, \text{ then for all } s_1,s_2 \text{ in } \big[0,T\big] \text{ we have}\\ \pi^{-1}\big(\Gamma\big(s_1,\zeta\big)\big)\cap\operatorname{supp}\mu\neq\emptyset \iff \pi^{-1}\big(\Gamma\big(s_2,\zeta\big)\big)\cap\operatorname{supp}\mu\neq\emptyset \end{cases}$$

If $\big(P_k\big)$ is true for all $k\in\mathbb{N}$ then using Remark 9.4 and a compacity argument we will obtain the conclusion of Theorem 5.3.

Now $\big(P_k\big)$ is of global nature but as usual using a connexity argument we can reduce the proof by induction to the following result.

Proposition 9.18. *Let $k\geq 1$. Assume $\big(P_{k-1}\big)$ is true. Let $\zeta_0\in\mathcal{G}^k$. If there exists $\varepsilon>0$ such that $\pi^{-1}\big(\Gamma\big(-s,\zeta_0\big)\big)\cap\operatorname{supp}\mu=\emptyset$ for all $s\in\big]0,\varepsilon\big]$ then there exists $\delta>0$ such that $\pi^{-1}\big(\Gamma\big(s,\zeta_0\big)\big)\cap\operatorname{supp}\mu=\emptyset$ for all $s\in\big[0,\delta\big]$.*

Before giving the proof of Proposition 9.18 let us show that

$$\big(P_0\big) \text{ is true}$$

Let $\zeta \in \Sigma_b$ and assume $\Gamma(s,\zeta) \subset \mathcal{G}^0 = T^*M$ for all $s \in \big[0,T\big[$. Then

$$\zeta = (x,t,\xi,\tau) \text{ and } \Gamma(s,\zeta) = \big(x(s),t,\xi(s),\tau\big)$$

where $\big(x(s),\xi(s)\big) = \gamma\big(s,(x,\xi)\big)$ is the usual bicharacteristic of p in T^*M. Since by Proposition 9.9 we have ${}^tH_p\mu = 0$ in $\mathcal{D}'\big(T^*M\big)$, the result follows.

Proof of Proposition 9.18

Case 1 : $k = 1$

Let $\zeta_0 = (x'_0,t_0,\xi'_0,T_0) \in \mathcal{G}^1 = \mathcal{H}$. Then $\tau_0 + r\,(0,x'_0,\xi'_0) = -A < 0$.

Let us set $\xi^0_1 = \big(-\,(\tau + r\,(0,x'_0,\xi'_0))\big)^{\frac{1}{2}}$. For small $\delta > 0$ we set

$$V^{\pm} = \big\{(x',t,\xi_1,\xi',\tau)\ :\ |x'-x'_0| < \delta, |t-t_0| < \delta, |\xi'-\xi'_0| < \delta, |\tau-\tau_0| < \delta, \big|\xi_1 \mp \xi^0_1\big| < \delta\big\}$$

If δ is small enough and $\rho = (x_1,x',t,\xi_1,\xi',\tau) \in \big[0,\delta\big[\times \big(V^+ \cup V^-\big)$, we have $\tau + r\,(x_1,x',\xi') \leq -\frac{1}{2}A$. If $p(\rho) = 0$ and $\rho \in \big]0,\delta\big[\times V^-$ then $\rho \in T^*M = \mathcal{G}^0$ and $x_1(s) = x_1 + 2\xi_1 s + s^2 g(s)$ where $|g(s)| \leq C$ and C depends only on A and p. It follows that with $\varepsilon = \dfrac{1}{2C}\left(\dfrac{A}{2}\right)^{\frac{1}{2}}$ we have $x_1(s) > 0$ for $s \in \big]-\varepsilon,0\big]$. This shows that $\Gamma(-s,\rho) \in T^*M$ for $s \in \big]-\varepsilon,0\big]$. Now by the assumption in proposition 9.18 and continuity one can find $\beta \in \big]0,\varepsilon\big[$ and δ small such that $\pi^{-1}\big(\Gamma(-\beta,\rho)\big) \cap \operatorname{supp}\mu = \emptyset$ for all ρ in $\big]0,\delta\big[\times V^-$. It follows then from $\big(P_0\big)$ that

$$\rho \notin \operatorname{supp}\mu \text{ for all } \rho \text{ in } \big]0,\delta\big[\times V^- \tag{9.49}$$

Since the hypersurface $\big\{x_1 = 0\big\}$ is non characteristic for the vector field ${}^tH_p$ and ${}^tH_p\mu = 0$ for $x_1 > 0$ (Proposition 9.9) the measure μ has a trace $\mu_{|x_1=0}$ which belongs to $\mathcal{D}'\big(V^+ \cup V^-\big)$. It follows then that

$${}^tH_p\mu = 2\xi_1\mu_{|x_1=0} \otimes \delta_{x_1=0} \text{ in } \mathcal{D}'\left(\big]-\delta,\delta\big[\times \big(V^+UV^-\big)\right) \tag{9.50}$$

Moreover by (9.49) we have

$$\mu_{|x_1=0} = 0 \text{ in } V^- \tag{9.51}$$

Our aim is to show that

$$\mu_{|x_1=0} = 0 \text{ in } V^+ \tag{9.52}$$

Indeed let us consider $j\ :\ T^*\mathbb{R}^{d+1} \to T^*\mathbb{R}^{d+1}$, $j\big(x,t,(\xi_1,\xi'),\tau\big) = \big(x,t,(-\xi_1,\xi'),\tau\big)$. Its follows from Proposition 9.12 that $\langle\,\mu\,,\,H_p(a\circ j)\,\rangle = -\,\langle\,\mu\,,\,H_p a\,\rangle$. Since $j^{-1} = j$ and $|\det Dj| = 1$ we have

$$\big({}^tH_p\mu\big)\circ j = -\,{}^tH_p\mu$$

Using (9.50) we see that $-2\xi_1\big(\mu_{|x_1=0}\big)\circ j\otimes\delta_{x_1=0}=-2\xi_1\mu_{|x_1=0}\otimes\delta_{x_1=0}$ on $V^+\cup V^-$. Then (9.52) follows from (9.51). Using (9.50), (9.51) and (9.52) we conclude that ${}^tH_p\mu=0$ on $\big]-\delta,\delta\big[\times\big(V^+\cup V^-\big)$.

Let us set

$$V=\{(x',t,\xi_1,\xi',\tau)\ :\ |x'-x'_0|<\delta_1,|t-t_0|<\delta_1,|\xi'-\xi'_0|<\delta_1,|\tau-\tau_0|<\delta_1\}$$

where $0<\delta_1\ll\delta$. Since by theorem 5.2 we have

$$\operatorname{supp}\mu\subset\big\{(x,t,\xi,\tau)\ :\ x_1\geq 0\text{ and }\tau+\xi_1^2+r\big(x,\xi'\big)=0\big\}\text{ and}$$

$$\big(\big]-\delta,\delta\big[\times V\big)\cap\big\{x_1\geq 0,\tau+\xi_1^2+r\big(x,\xi'\big)=0\big\}\subset\big[0,\delta\big[\times\big(V^+\cup V^-\big)$$

if δ_1 is small enough we deduce that ${}^tH_p\mu=0$ on $\big]-\delta,\delta\big[\times V$.

Let now $\gamma^+(s)=\big(x^+(s),\xi^+(s)\big)$ be the bicharacteristic of p defined for $|s|\leq\varepsilon$ and starting at the point $\big(0,x'_0,\xi'_1,\xi'_0\big)$. Since $x_1^+(s)<0$ for $s\in\big]-\varepsilon,0\big[$ and $\operatorname{supp}\mu\subset\big\{x_1\geq 0\big\}$ we have $\big(x^+(s),t_0,\xi^+(s),\tau_0\big)\notin\operatorname{supp}\mu$ for $s\in\big]-\varepsilon,0\big[$ since ${}^tH_p\mu=0$ on $\big]-\delta,\delta\big[\times V$ one can find $\delta_0>0$ such that for $s\in\big[0,\delta_0\big]$, $\big(x^+(s),t_0,\xi^+(s),\tau_0\big)\notin\operatorname{supp}\mu$ which implies that $\pi^{-1}\big(\Gamma\big(s,\zeta_0\big)\big)\cap\operatorname{supp}\mu=\emptyset$, $s\in\big[0,\delta_0\big]$, and proves Proposition 9.1 in the case $k=1$.

Case 2 : $k\geq 2$

We shall need several preliminary results.

Lemma 9.19. *Let $(0,x'_0,t_0,0,\xi'_0,\tau_0)\in T^*\mathbb{R}^{d+1}$ and $\delta>0$. We set*

$$V=\big\{(x,t,\xi,\tau)\in T^*\mathbb{R}^{d+1},0\leq x_1<\delta,|x'-x'_0|<\delta,|t-t_0|<\delta,|\xi'-\xi'_0|<\delta,|\tau-\tau_0|<\delta\big\}$$

We assume that

$$\begin{aligned}&(i)\quad \operatorname{supp}\mu\cap V\subset\big\{(x,t,\xi,\tau)\in T^*\mathbb{R}^{d+1}\ :\ x_1=\xi_1=0\big\}\\&(ii)\quad \big(0,x'_g\big(s,x',\xi'\big),t,0,\xi'_g\big(s,x',\xi'\big),\tau\big)\in V\text{ for all }s\in I=\big]-s_*,s^*\big[\end{aligned}$$

Then for all $s_1,s_2\in I$

$$\begin{gathered}\big(0,x'_g\big(s_1,x',\xi'\big),t,0,\xi'_g\big(s_1,x',\xi'\big),\tau\big)\in\operatorname{supp}\mu\\\Longleftrightarrow\\\big(0,x'_g\big(s_2,x',\xi'\big),t,0,\xi'_2\big(s_2,x',\xi'\big),\tau\big)\in\operatorname{supp}\mu\end{gathered}$$

Proof

By assumption (i) there exists a measure $\mu_1=\mu_1\big(x',t,\xi',\tau\big)$ on $T^*\mathbb{R}^d$ such that $\mu=\mu_1\otimes\delta_{x_1=0}\otimes\delta_{\xi_1=0}$. Moreover we can extend that definition of $\langle\,\mu,a\,\rangle$ to smooth $a=$

$a\big(x_1, x', t, \xi_1, \xi', \tau\big)$ which have compact support in $\big(x_1, x', t, \xi', \tau\big)$ contained in V. Indeed if $\chi \in \mathcal{C}_0^\infty(\mathbb{R})$, $\chi\big(\xi_1\big) = 1$ for $|\xi_1| \leq \dfrac{1}{2}$, $\chi\big(\xi_1\big) = 0$ for $|\xi_1| \geq 1$ we can set $\langle\, \mu, a\,\rangle = \big\langle\, \mu\,,\, \chi\big(\xi_1\big)a\,\big\rangle$ and this definition does not depend on χ. In particular we can take $a = a\big(x, t, \xi', \tau\big)$. With the notation of Lemma 9.10, we have $a_1 = 0$ so it follows from Proposition 9.11 that $\langle\, \mu\,,\, H_p a\,\rangle = 0$. By the above remark we have

$$0 = \langle\, \mu\,,\, H_p a\,\rangle = \big\langle\, \mu_1\,,\, H_p a_{|x_1=\xi_1=0}\,\big\rangle = \big\langle\, \mu_1\,,\, H_{r_0} a_{|x_1=0}\,\big\rangle = \big\langle\, \mu_1\,,\, H_{r_0}\big(a_{|x_1=0}\big)\,\big\rangle$$

Therefore the Lemma follows.

□

Remark 9.20. *(i) Let* $\rho_0 = (0, x_0', t_0, 0, \xi_0', \tau_0) \in T^*\mathbb{R}^{d+1}$. *If* $\rho_0 \notin \operatorname{supp}\mu$ *and* $\tau_0 + r\,(0, x_0', \xi_0') = 0$ *then*

$$(x_0', t_0, \xi_0', \tau_0) \notin \operatorname{supp}\nu$$

Indeed one can find $\delta > 0$ *such that* $B\big(\rho_0, \delta\big) \cap \operatorname{supp}\mu = \emptyset$.

Let $a(x, t, \xi, \tau) = b\,(x', t, \xi', \tau)\,\chi\big(x_1\big)\xi_1$ *with support in*
$\{|x' - x_0'| + |t - t_0| + |\xi' - \xi_0'| + |\tau - \tau_0| < \delta_1, |x_1| < \delta_1\}$. *Since*

$$\operatorname{supp}\mu \subset \Big\{|\xi_1|^2 = \big|\tau + r\big(x, \xi'\big)\big|\Big\}$$

we will have our assumption

$$\operatorname{supp}\mu \cap \operatorname{supp} a \subset \operatorname{supp}\mu \cap \operatorname{supp} a \cap \Big\{|\xi_1| \leq C\sqrt{\delta_1}\Big\}$$

If δ_1 *is small enough we will have* $\operatorname{supp} a \cap \big\{|\xi_1| \leq C\sqrt{\delta_1}\big\} \subset B\big(\rho_0, \delta\big)$ *so* $\operatorname{supp}\mu \cap \operatorname{supp} a = \emptyset$. *With the notation of Lemma 9.10 we have* $a_1 = \chi b$ *so, by Proposition 9.12,* $0 = \langle\, \mu, H_p a\,\rangle = -\langle\, \nu, b\,\rangle$. *Since* b *is arbitrary this shows that* $(x_0', t_0, \xi_0', \tau_0) \notin \operatorname{supp}\nu$.

(ii) Let $(x_0', t_0, \xi_0', \tau_0) \in \mathcal{H}$ *and* $\rho_0^\pm = \big(0, x_0', t_0, \pm\xi_1^0, \xi_0', \tau_0\big)$ *where*

$\xi_1^0 = \big(-\big(\tau_0 + r\big(0, x_0', \xi_0'\big)\big)\big)^{\frac{1}{2}}$. *Then if* $\big\{\rho_0^+, \rho_0^-\big\} \cap \operatorname{supp}\mu = \emptyset$ *then* $(x_0', t_0, \xi_0', \tau_0) \notin \operatorname{supp}\nu$. *Indeed one can find* $\delta > 0$ *such that*

$$\big(B\big(\rho_0^+, \delta\big) \cup B\big(\rho_0^-, \delta\big)\big) \cap \operatorname{supp}\mu = \emptyset$$

If we take $a = b\chi\xi_1$ *as in* (i) *we will have*

$$\operatorname{supp}\mu \cap \operatorname{supp} a \subset \operatorname{supp}\mu \cap \operatorname{supp} a \cap \Big\{\Big|\xi_1^2 - \big(\xi_1^0\big)^2\Big| \leq C\delta_1\Big\}$$

Now, if $\delta_1 \ll \delta^2$, $\Big\{\Big|\xi_1^2 - \big(\xi_1^0\big)^2\Big| \leq C\delta_1\Big\} \subset \big\{\big|\xi_1 - \xi_1^0\big| < \delta\big\} \cup \big\{\big|\xi_1 + \xi_1^0\big| < \delta\big\}$ *so*

$$\operatorname{supp} a \cap \operatorname{supp}\mu = \emptyset$$

As in (i) we deduce that $0 = \langle \mu, H_p a \rangle = - \langle \nu, b \rangle$.

Lemma 9.21. *Let* $k \geq 2$ *and* $\zeta_0 = (x'_0, t_0, \xi'_0, \tau_0) \in \mathcal{G}^k$ *be given as in Proposition 9.1. Set* $H_{r_0}^{k-2}\left(\dfrac{\partial r}{\partial x_1}\right)(0, x'_0, \xi'_0) = A \neq 0$ *then*

(i) one can find $\delta > 0$ *such that* $\left| H_{r_0}^{k-2}\left(\dfrac{\partial r}{\partial x_1}\right)(0, x', \xi') \right| \geq \dfrac{|A|}{2}$ *in the set*

$$V_1 = \big\{ (x', t, \xi', \tau) \ : \ |x' - x'_0| < \delta, |\xi' - \xi'_0| < \delta, |t - t_0| < \delta, |\tau - \tau_0| < \delta \big\}$$

(ii) one can find $\delta' > 0$ *and* $\beta > 0$ *such that*

$$\begin{cases} \Gamma\big(s, \widetilde{U}\big) \cap T^*\partial M \subset V_1 \text{ for all } s \in \big[-\beta, 0 \big], \\ \pi^{-1}\big(\Gamma\big(-\beta, \widetilde{U}\big)\big) \cap \operatorname{supp} \mu = \emptyset, \text{ where} \end{cases}$$

$$\widetilde{U} = \Big[\big\{ (x', t, \xi', \tau) \in T^*\partial M \ : |x' - x'_0| < \delta', |t - t_0| < \delta', |\xi' - \xi'_0| < \delta', |\tau - \tau_0| < \delta' \big\} \cap \big\{ (x', \xi') \ : \ \tau_0 + r\big(0, x', \xi'\big) \leq 0 \big\} \Big]$$

$$\cup \Big[\{ (x, t, \xi, \tau) \in T^*M : 0 < x_1 < \delta', |x' - x'_0| < \delta', |\xi' - \xi'_0| < \delta', |t - t_0| < \delta', |\tau - \tau_0| < \delta' \} \cap (\tau + p)^{-1}(0) \Big]$$

Moreover
case ① : k **even,** $A > 0$
$\forall \zeta \in \mathcal{G}^k \cap V_1, \quad \Gamma(s, \zeta) \in \mathcal{G}g, \quad \forall s \in \big[-\beta, \beta \big] \setminus \{0\}, \quad (\forall s \in \big[-\beta, \beta \big] \text{ if } k = 2).$
case ② : k **even,** $A < 0$
$\forall \zeta \in \mathcal{G}^k \cap V_1, \quad \Gamma(s, \zeta) \in T^*M, \quad \forall s \in \big[-\beta, \beta \big] \setminus \{0\}, \quad (\zeta \in \mathcal{G}_d \text{ if } k = 2).$
case ③ : k **odd,** $k \geq 3$**,** $A < 0$
$\forall \zeta \in \mathcal{G}^k \cap V_1, \quad \Gamma(s, \zeta) \in T^*M, \quad \forall s \in \big[-\beta, 0 \big[, \quad \Gamma(s, \zeta) \in \mathcal{G}_g, \quad \forall s \in \big] 0, \beta \big].$
case ④ : k **odd,** $k \geq 3$**,** $A > 0$
$\forall \zeta \in \mathcal{G}^k \cap V_1, \quad \Gamma(s, \zeta) \in \mathcal{G}_g, \quad \forall s \in \big[-\beta, 0 \big[, \quad \Gamma(s, \zeta) \in T^*M, \quad \forall s \in \big] 0, \beta \big].$

Proof of Lemma 9.21

(i) $H_{r_0}^{k-2}\left(\dfrac{\partial r}{\partial x_1}\right)\big(0, x', \xi'\big)$ being continuous, the existence of δ is clear. Moreover it has the same sign as A. Let us set $e(s, \zeta) = \dfrac{\partial r}{\partial x_1}\big(0, x'_g(s, \zeta), \xi'_g(s, \zeta)\big)$. Then since $\zeta \in \mathcal{G}^k \cap V_1$ we have for small $|s|$ (see [H], Chap. 24)

$$e(s, \zeta) = \frac{1}{(k-2)!} H_{r_0}^{k-2}\left(\frac{\partial r}{\partial x_1}\right)\big(0, x', \xi'\big) s^{k-2} + s^{k-1} g(s)$$

where $|g(s)| \leq C$, C depending only on p. If $\varepsilon_1 = \dfrac{|A|}{2C}$, $e(s,\zeta)$ has a constant sign on each interval $\big[-\varepsilon_1, 0\big[$ and $\big]0, \varepsilon_1\big]$; moreover either $\Gamma(s,\zeta) \in \mathcal{G}_g$ or $\Gamma(s,\zeta) \in T^*M$ on each interval which gives the four cases described in the statement of the Lemma, (see [H], Chap. 24).

(ii) Let us prove that

$$\exists \beta_0, \exists \delta' \ : \ \forall \beta \in \big[0, \beta_0\big[, \forall x \in \big[-\beta, 0\big], \Gamma\big(s, \widetilde{U}\big) \cap T^*\partial M \subset V_1$$

Otherwise one can find sequences $\beta_j \to 0$, $\delta'_j \to 0$, $s_j \in \big]-\beta_j, 0\big]$ and $\xi_j \in \widetilde{U}$ such that $\Gamma\big(s_j, \zeta_j\big) \in T^*\partial M$ and $\Gamma\big(s_j, \zeta_j\big) \notin V_1$. If $\zeta_j \in T^*\partial M$, $\zeta_j = \big(x'_j, t_j, \xi'_j, \tau_j\big)$ and if $\zeta_j \in T^*M$, $\zeta_j = \Big(x_1^j, x'_j, t_j, \xi_1^j, \xi'_j, \tau_j\Big)$. In both cases $\big|x'_j - x'_0\big| < \delta'_j$, $\big|\xi'_j - \xi'_0\big| < \delta'_j$, $|t_j - t_0| < \delta'_j$, $|\tau_j - \tau_0| < \delta'_j$ and if $\zeta_j \in T^*M$, $0 < x_1^j < \delta'_j$. It follows that $\tau_j + r\left(x_1^j, x'_j, \xi'_j\right) \to \tau_0 + r\left(0, x'_0, \xi'_0\right) = 0$ which implies that $\xi_1^j \to 0$ (since $\zeta_j \in p^{-1}(0)$ then). Therefore $\zeta_j \to \zeta_0$ in T^*_bM. Moreover since $s_j \to 0$ we have $\Gamma\big(s_j, \zeta_j\big) \to \zeta_0$ in T^*_bM so in $T^*\partial M$ since $\Gamma\big(s_j, \zeta_j\big) \in T^*\partial M$. But $\zeta_0 \in V_1$ so $\Gamma\big(s_j, \zeta_j\big) \in V_1$ for large j and we obtain a contradiction.

Let $\beta = \inf\left(\beta_0, \dfrac{\varepsilon_1}{2}\right)$ and let us set that if δ' is small enough then $\pi^{-1}\big(\Gamma\big(-\beta, \widetilde{U}\big)\big) \cap \operatorname{supp}\mu = \emptyset$. We know that $\pi^{-1}\big(\Gamma\big(-\beta, \zeta_0\big)\big) \cap \operatorname{supp}\mu = \emptyset$. Let $V \subset T^*\mathbb{R}^{d+1}$ be such that $V \cap \operatorname{supp}\mu = \emptyset$ and $\pi^{-1}\big(\Gamma\big(-\beta, \zeta_0\big)\big) \subset V$. If

$$\forall \delta' > 0, \quad \pi^{-1}\big(\Gamma\big(-\beta, \widetilde{U}\big)\big) \cap \operatorname{supp}\mu \neq \emptyset$$

then there exists $\delta'_j \to 0$, $\zeta_j \in \widetilde{U}$ such that $\rho_j \in \pi^{-1}\big(\Gamma\big(-\beta, \zeta_j\big)\big)$, $\rho_j \in \operatorname{supp}\mu$.

We keep the notations in the beginning of (ii). Then $x_1^j \to 0$, $x'_j \to x'_0$, $t_j \to t_0$, $\xi'_j \to \xi'_0$, $\tau_j \to \tau_0$, so $\zeta_j \to \zeta_0$ which implies that $\Gamma\big(-\beta, \zeta_j\big) \to \Gamma\big(-\beta, \zeta_0\big)$. Let us set

$$\Gamma\big(-\beta, \zeta_j\big) = \begin{cases} \big(X'_j, T_j, \Xi'_j, \Lambda_j\big) \ \text{ if } \Gamma\big(-\beta, \zeta_j\big) \in T^*\partial M \\ \Big(X_1^j, X'_j, T_j, \Xi_1^j, \Xi'_j, \Lambda_j\Big) \ \text{ if } \Gamma(-\beta, \zeta_j) \in T^*M \end{cases}$$

for $j = 0$ and $j \geq 1$.

If $\Gamma(-\beta, \zeta_j) \in T^*\partial M$ one can find Ξ_1^j such that $\rho_j = \big(0, X'_j, T_j, \Xi'_1, \Xi'_j, \Lambda_j\big)$ with

$$\big(\Xi_1^j\big)^2 + r\left(0, X'_j, \chi'_j, \Xi'_j\right) + \Lambda_j = 0$$

since $\rho_j \in \operatorname{supp}\mu \cap (\tau + p)^{-1}(0)$.

If $\Gamma\big(-\beta, \zeta_j\big) \in T^*M$ we have the same thing with $\rho_j = \Big(X_1^j, X'_j, T_j, \Xi_1^j, \Xi'_j, \Lambda_j\Big)$.

Now in the case where $\Gamma\big(-\beta, \zeta_0\big) \in T^*M$ we have $X_1^j \to X_1^0 > 0$, $X'_j \to X'_0$, $T_j \to T_0$, $\Xi_1^j \to \Xi_1^0$, $\Xi'_j \to \Xi'_0$, $\Lambda_j \to \Lambda_0$. Then $\rho_j \in V$ if j is large enough which contradicts the fact that $\rho_j \in \operatorname{supp}\mu$ and $V \cap \operatorname{supp}\mu = \emptyset$.

In the case where $\Gamma\big(-\beta,\zeta_0\big)\in T^*\partial M$ we have $X_j'\to X_0'$, $\Xi_j'\to\Xi_0'$, $T_j\to T_0$, $\Lambda_j\to\Lambda_0$ and $X_1^j\to 0$ (for the indices j such that $\Gamma\big(-\beta,\zeta_j\big)\in T^*M$).

In the both cases we have $\big(\Xi_1^j\big)^2\to -r\,(0,X_0',\Xi_0')-\Lambda_0=\xi_1^2$.

Now $(0,X_0',T_0,\pm\xi_1,\Xi_0',\lambda_0)\in\pi^{-1}\big(\Gamma\big(-\beta,\zeta_0\big)\big)\cap(\tau+p)^{-1}(0)$; therefore $(0,X_0',T_0,\pm\xi_1,\Xi_0',\Lambda_0)\in V$ so $\rho_j\in V$ for j large enough which is again a contradiction. The proof of Lemma 9.21 is complete.

□

Proof of Proposition 9.18

We are going to consider separately the four cases introduced in Lemma 9.21.

Case ① : we have $\Gamma\big(s,\zeta_0\big)\in\mathcal{G}_g$ for $s\in\big[-\beta,\beta\big]\setminus\{0\}$ Therefore

$$\Gamma\big(s,\zeta_0\big)=\big(x_g'\big(s,x_0',\xi_0'\big),t_0,\xi_g'\big(s,x_0',\xi_0'\big),\tau_0\big)\,,\quad s\in\big[-\beta,\beta\big] \tag{9.53}$$

Let U be the following set.

$$U=\big\{\big(x',t,\xi',\tau\big)\in T^*\partial M,\ |x'-x_0'|<\delta',|t-t_0|<\delta,|\xi'-\xi_0'|<\delta',|\tau-\tau_0|<\delta'\big\}$$
$$\cup\big\{(x,t,\xi,\tau)\in T^*M,\ 0<x_1<\delta',|x'-x_0'|<\delta',|t-t_0|<\delta',|\xi'-\xi_0'|<\delta',|\tau-\tau_0|<\delta'\big\}$$

Then $\widetilde{U}=U\cap\Sigma_b$ is the set introduced in Lemma 9.21. Moreover U being an open subset of T_b^*M, $\pi^{-1}(U)$ is open in $T^*\mathbb{R}^{d+1}$.

By continuity one can find $\varepsilon_0>0$ such that $\Gamma\big(s,\zeta_0\big)\in U$ for s in$\big[-\varepsilon_0,\varepsilon_0\big]$. Then one can find $\delta>0$ such that if we set

$$V=\big\{(x,t,\xi,\tau)\in T^*\mathbb{R}^{d+1},0\le x_1<\delta,|x'-x_0'|<\delta,|t-t_0|<\delta,|\xi'-\xi_0'|<\delta,|\tau-\tau_0|<\delta\big\}$$

then for $s\in\big[-\varepsilon_0,\varepsilon_0\big]$ we have

$$\pi^{-1}\big(\Gamma\big(s,\zeta_0\big)\big)=\big(0,x_g'\big(s,x_0',\xi_0'\big),t_0,0,\xi_g'\big(s,x_0',\xi_0'\big),\tau_0\big)\subset V\subset\pi^{-1}(U)$$

Assume that we can prove

$$\operatorname{supp}\mu\cap V\subset\big\{(x,t,\xi,\tau)\in T^*\mathbb{R}^{d+1}\ :\ x_1=\xi_1=0\big\} \tag{9.54}$$

then Proposition 9.18 follows immediately from Lemma 9.19.

Let $\rho=(x,t,\xi,\tau)\in\operatorname{supp}\mu\cap\pi^{-1}(U)$. By Theorem 5.2 we have $\tau+p(x,\xi)=0$ i.e. $\rho\in\Sigma$.

Let $\zeta=\pi(\rho)\in U\cap\Sigma_b=\widetilde{U}$.

If $\big\{\Gamma(-s,\zeta)\ :\ s\in\big[0,\beta\big]\big\}\cap T^*\partial M\subset\bigcup_{j=0}^{k-1}\mathcal{G}^j$ then since by Lemma 9.4(ii) we have

$$\pi^{-1}\big(\Gamma(-\beta,\zeta)\big)\cap\operatorname{supp}\mu=\emptyset$$

the hypothesis $\big(P_{k-1}\big)$ implies that $\pi^{-1}(\zeta)\cap\operatorname{supp}\mu=\emptyset$ which contradicts our assumption $\rho\in\operatorname{supp}\mu$.

Therefore one can find $s_1\in\big[0,\beta\big]$ such that $\zeta_1=\Gamma\big(-s_1,\zeta\big)\in T^*\partial M$ but $\zeta_1\notin\bigcup_{j=0}^{k-1}\mathcal{G}^j$. Since

$$\zeta_1\in\Gamma\big(-s_1,\widetilde{U}\big)\cap T^*\partial M\subset V_1$$

by Lemma 9.21(*ii*) we have $\zeta_1\in\mathcal{G}^k$. Then $\Gamma\big(s,\zeta_1\big)\in\mathcal{G}_g$ if $s\in\big[-\beta+s_1,0\big[\cup\big]0,s_1\big]$.

If $s_1\neq0$ we have $\Gamma\big(s_1,\zeta\big)=\Gamma\big(s_1,\Gamma\big(-s_1,\zeta\big)\big)=\zeta\in\mathcal{G}_g$ and if $s_1=0$ we have $\zeta=\zeta_1\in\mathcal{G}^k$. In both cases we have $\zeta=\big(x',t,\xi',\tau\big)$ and $\rho=\big(0,x',t,0,\xi',\tau\big)$ because $\tau+r\big(0,x',\xi'\big)=0$. It follows that

$$\operatorname{supp}\mu\cap\pi^{-1}(U)\subset\big\{(x,t,\xi,\tau)\ :\ x_1=\xi_1=0\big\}$$

as claimed in (9.54).

Case ② : here for $\zeta\in\mathcal{G}^k\cap V_1$ we have $\Gamma(s,\zeta)\subset T^*M$ when $s\in\big[-\beta,\beta\big]\setminus\{0\}$

We shall show that

$$\nu=0\text{ on }\widetilde{U}\cap T^*\partial M\tag{9.55}$$

Since by Lemma 9.17 we have $\nu\left(\mathcal{G}_d\cup\left(\bigcup_{k=3}^{+\infty}\mathcal{G}^k\right)\right)=0$ it is enough to prove that

$$\operatorname{supp}\nu\cap\widetilde{U}\cap\big(\mathcal{H}\cup\mathcal{G}_g\big)=\emptyset$$

Let $\zeta\in\widetilde{U}\cap\big(\mathcal{H}\cup\mathcal{G}_g\big)\cap\operatorname{supp}\nu$.

$$\big\{\Gamma(-s,\zeta)\ :\ s\in\big[0,\beta\big]\big\}\cap T^*\partial M\subset\bigcup_{j=1}^{k-1}\mathcal{G}^j$$

since by Lemma 9.21(*ii*) we have $\pi^{-1}\big(\Gamma(-\beta,\zeta)\big)\cap\operatorname{supp}\mu=\emptyset$, by $\big(P_{k-1}\big)$ we have $\pi^{-1}(\zeta)\cap\operatorname{supp}\mu=\emptyset$ so, by Remark 9.20, we have $\zeta\notin\operatorname{supp}\nu$ which is a contradiction. It follows that we can find $s_1\in\big[0,\beta\big]$ such that $\zeta_1=\Gamma\big(-s_1,\zeta\big)\in\mathcal{G}^k$ ($\mathcal{G}_d$ if $k=2$) (since $\Gamma(-s,\zeta)\in V_1$ by Lemma 9.21). Moreover by Lemma 9.21, case ②, we have $\Gamma\big(s,\zeta_1\big)\in T^*M$ if $s\in\big[-\beta+s_1,0\big[\cup\big]0,s_1\big]$. If $s_1\neq0$ then $\zeta=\Gamma\big(s_1,\xi_1\big)\in T^*M$ which contradicts our assumption. If $s_1=0$ we have $\zeta_1=\zeta\in\mathcal{G}^k$ ($\mathcal{G}_d$ if $k=2$) which again is impossible. It follows that $\widetilde{U}\cap\operatorname{supp}\nu\cap\big(\mathcal{H}\cup\mathcal{G}_g\big)=\emptyset$ which proves (9.55).

It follows from Proposition 9.12 that ${}^tH_p\mu=0$ on $\pi^{-1}\big(\widetilde{U}\big)$. This implies that the support of μ propagates along the bicharacteristics of p (with (t,τ) =constant). Now by assumption $\pi^{-1}\big(\Gamma\big(-s,\zeta_0\big)\big)\cap\operatorname{supp}\mu=\emptyset$ and $\pi^{-1}\big(\Gamma\big(s,\zeta_0\big)\big)\cap(\tau+p)^{-1}(0)=\big(x(s),t_0,\xi(s),\tau_0\big)$.

It follows that for small $s>0$ we have $\pi^{-1}\big(\Gamma\big(s,\zeta_0\big)\big)\cap\operatorname{supp}\mu=\emptyset$ which proves Proposition 9.18.

Case ③ : here k is odd, $k\geq 3$

We claim that

$$\operatorname{supp}\mu\cap\pi^{-1}(U)\subset\big\{(x,t,\xi,\tau)\in T^*\mathbb{R}^{d+1}\ :\ x_1=\xi_1=0\big\} \tag{9.56}$$

where U is been defined in case ①.

Let $\rho=(x,t,\xi,\tau)\in\pi^{-1}(U)\cap\operatorname{supp}\mu$. Then $\tau+p(x,\xi)=0$. Let $\zeta=\pi(\rho)\in U\cap\Sigma_b=\widetilde{U}$. If

$$\big\{\Gamma(s,\zeta)\ :\ s\in\big[0,\beta\big]\big\}\subset\bigcup_{j=1}^{k-1}\mathcal{G}^j$$

then $\big(P_{k-1}\big)$ and the fact that $\pi^{-1}\big(\Gamma(-\beta,\zeta)\big)\cap\operatorname{supp}\mu=\emptyset$ imply that $\pi^{-1}(\zeta)\cap\operatorname{supp}\mu=\emptyset$ which is in contradiction with $\rho\in\operatorname{supp}\mu$.

Therefore one can find $s_1\in\big[0,\beta\big]$ such that $\zeta_1=\Gamma\big(-s_1,\zeta\big)\in\mathcal{G}^k$ (since $\zeta_1\in V_1$). Since we are in case ③ we have

$$\Gamma\big(s,\zeta_1\big)=\begin{cases}\big(x_g'(s),t,\xi_g'(s),\tau\big) & s\in\big]0,s_1\big]\\ \big(x(s),t,\xi(s),\tau\big) & s\in\big[-\beta+s_1,0\big[\end{cases}$$

If $s_1\neq 0$, $\zeta=\Gamma\Big(s_1,\Gamma\big(-s_1,\zeta\big)\Big)=\Gamma\big(s_1,\zeta_1\big)\in\mathcal{G}_g$ so $\rho=\big(0,x',t,0,\xi',\tau\big)$.
If $s_1=0$ then $\zeta=\zeta_1\in\mathcal{G}^k$ and $\rho=\big(0,x',0,\xi',\tau\big)$.

This proves (9.56). Therefore we can use Lemma 9.19 and its conclusion with V such that $\pi^{-1}\big(\Gamma\big(s,\zeta_0\big)\big)\subset V\subset\pi^{-1}(U)$ for $s\in\big[-\beta,\beta\big]$.

Now $\widetilde{\zeta}_0=\big(x_g'\left(-s,x_0',\xi_0'\right),t_0,\xi_g'\left(-s,x_0',\xi_0'\right)\big)\in\mathcal{G}_d$, when $s\in\big]0,\beta\big]$ and $\widetilde{\zeta}_0\to\zeta_0$ if $s\to 0$ so $\widetilde{\zeta}_0\in\widetilde{U}$ if s is small enough, it follows from Lemma 9.21(ii) that $\pi^{-1}\big(\Gamma\big(-\beta,\widetilde{\zeta}_0\big)\big)\cap\operatorname{supp}\mu=\emptyset$ since $\big\{\Gamma\big(s,\widetilde{\zeta}_0\big)\ :x\in\big[-\beta,0\big]\big\}\subset\mathcal{G}_d\cup T^*M$, it follows from $\big(P_2\big)$, $\pi^{-1}\big(\widetilde{\zeta}_0\big)\cap\operatorname{supp}\mu=\emptyset$. By Lemma 9.21 we deduce that $\big(0,x_g'\left(s,x_0',\xi_0'\right),t_0,0,\xi_g'\left(s,x_0',\xi_0'\right),\tau_0\big)\notin\operatorname{supp}\mu$ for small s, which proves Proposition 9.18 in this case.

Case ④

We claim in this case that

$$\operatorname{supp}\nu\cap\widetilde{U}=\emptyset \tag{9.57}$$

As in case ② it is enough to prove that $\operatorname{supp}\nu\cap\widetilde{U}\cap\big(\mathcal{H}\cup\mathcal{G}_g\big)=\emptyset$. Let $\zeta\in\operatorname{supp}\nu\cap\widetilde{U}\cap\big(\mathcal{H}\cup\mathcal{G}_g\big)$. If

$$\big\{\Gamma(-s,\zeta)\ :\ s\in\big[0,\beta\big]\big\}\cap T^*\partial M\subset\bigcup_{j=1}^{k-1}\mathcal{G}^j$$

then $\big(P_{k-1}\big)$ and the fact that $\pi^{-1}\big(\Gamma(-\beta,\zeta)\big)\cap \operatorname{supp}\mu = \emptyset$ (Lemma 9.21(ii)) imply that $\pi^{-1}(\zeta)\cap\operatorname{supp}\mu = \emptyset$. Then by Remark 9.20 we have $\zeta\notin\operatorname{supp}\nu$ which is a contradiction. Therefore one can find $s_1\in\big[0,\beta\big]$ such that $\zeta_1 = \Gamma\big(-s_1,\zeta\big)\in\mathcal{G}^k$ (since $\zeta_1\in V_1$) and

$$\Gamma\big(s,\zeta_1\big) = \begin{cases} \big(x(s),t,\xi(s),\tau\big)\subset T^*M, & s\in\big]0,s_1\big], \\ \big(x_g'(s),t,\xi_g'(s),\tau\big)\subset T^*M, & s\in\big[-\beta+s_1,0\big[. \end{cases}$$

If $s_1\neq 0$ then $\zeta = \Gamma\big(s_1,\zeta_1\big)\in T^*M$ which contradicts our assumption. If $s_1 = 0$ then $\zeta=\zeta_1\in\mathcal{G}^k$ which again contradicts the fact that $\zeta\in\mathcal{H}\cup\mathcal{G}_g$ since k is odd, $k\geq 3$, in this case. Thus (9.57) is proved. It follows then, from Proposition 9.12, that ${}^tH_p\mu = 0$ on $\pi^{-1}\big(\widetilde{U}\big)$ therefore on a complete neighborhood of $\pi^{-1}\big(\zeta_0\big)$ in $T^*\mathbb{R}^{d+1}$ since $\mu\equiv 0$ in $x_1<0$. Now $\big(x\big(s,x_0,\xi_0\big),t_0,\xi\big(s,x_0,\xi_0\big),\tau_0\big)$ is contained in $\big\{(x,t,\xi,\tau)\ : x_1<0\big\}$ when $\beta+s<0$. By propagation along the bicharacteristics of p (since ${}^tH_p\mu=0$) we deduce that $\Gamma\big(s,\zeta_0\big) = \big(x\big(s,x_0,\xi_0\big),t_0,\xi\big(s,x_0,\xi_0\big),\tau_0\big)$ does not intersect $\operatorname{supp}\mu$ when $s>0$ is small enough. The proof of Theorem 5.2 is complete.

□

9.3 Proofs of the technical Lemmas

We shall need the following elementary result.

Lemma 9.22. *Let* $P = \sum_{j,k=1}^{d} D_j a^{jk}(x) D_k + V$ *where* P *satisfies conditions (2.3), (2.5) and* $V\geq 1$. *Then there exists* $C\geq 0$ *such that for any* $z\in\mathbb{C}$ *such that* $\operatorname{Im} z\neq 0$, *any* h *in* $\big]0,1\big]$ *and any solution* $u\in H_0^1(\Omega)$ *of the problem* $h^2Pu - zu = f$ *with* $f\in L^2(\Omega)$ *we have*

$$\big\|h^2Pu\big\|^2 + \sum_{j=1}^{n}\| hD_ju\|^2 + \Big\|hV^{\frac{1}{2}}u\Big\|^2 + \| u\|^2 \leq C\frac{\langle\,|z|\,\rangle^2}{|\operatorname{Im} z|^2}\| f\|^2$$

where $\|\cdot\|$ *is the norm in* $L^2(\Omega)$.

Proof

Taking the scalar product in $L^2(\Omega)$ of the equation with u we obtain

$$\big(h^2Pu,u\big) - \operatorname{Re} z\| u\|^2 - i\operatorname{Im} z\| u\|^2 = (f,u) \tag{9.58}$$

Since $\big(h^2Pu,u\big)$ is real, taking the imaginary part of (9.58), we obtain

$$\| u\| \leq \frac{\| f\|}{|\operatorname{Im} z|} \tag{9.59}$$

Now we have $\big(h^2Pu,u\big) \geq C\left(\sum_{j=1}^{d}\|hD_ju\|^2 + \left\|hV^{\frac{1}{2}}u\right\|^2\right)$ so taking the real part in (9.58) and using (9.59) we obtain

$$\sum_{j=1}^{d}\|hD_ju\|^2 + \left\|hV^{\frac{1}{2}}u\right\|^2 \leq \frac{C\,|z|}{|\mathrm{Im}\,z|^2}\|\,f\,\|^2 \tag{9.60}$$

Finally we have $\left\|h^2Pu\right\|^2 \leq 2\left(\|\,z\,\|^2\|\,u\,\|^2 + \|\,f\,\|^2\right)$ so using (9.59) and (9.60) we obtain the claim in the Lemma.

□

In that follows, we shall make a great use of the so called Helffer-Sjöstrand formula (see [Da]) which will recall now.

Let $\theta \in \mathcal{C}_0^\infty(\mathbb{R})$. We defined an «almost analytic extension» of θ as follows. Let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$ be such that $\varphi(t) = 1$ if $|t| \leq 1$, $\varphi(t) = 0$ if $|t| \geq 2$.

We set

$$\widetilde{\theta}(x,y) = \sum_{\ell=1}^{2}\frac{\theta^{(\ell)}(x)}{\ell!}(iy)^\ell\varphi\left(\frac{y}{\langle\,x\,\rangle}\right) \tag{9.61}$$

Then $\widetilde{\theta}$ is a $\mathcal{C}^\infty$ function on $\mathbb{R}\times\mathbb{R}$ and satisfies

$$\begin{cases} \left|\overline{\partial}\,\widetilde{\theta}(x,y)\right| \leq C_N\,|y|^2 \text{ as } |y|\to 0, \text{ where} \\ \overline{\partial}\,\widetilde{\theta}(x,y) = \dfrac{1}{2}\left(\dfrac{\partial\widetilde{\theta}}{\partial x} + i\,\dfrac{\partial\widetilde{\theta}}{\partial y}\right)(x,y) \end{cases} \tag{9.62}$$

Let P_D be our self adjoint operator defined in (2.1). Then the Helffer-Sjöstrand formula asserts that

$$\theta\big(h^2P_D\big) = -\frac{1}{\pi}\int_{\mathbb{R}^2}\overline{\partial}\,\widetilde{\theta}(x,y)\left(z - h^2P_D\right)^{-1}\mathrm{d}x\mathrm{d}y \tag{9.63}$$

where $z = x + iy$.

Proof of Lemma 6.3

(i) According to (9.63) we have (writing P instead P_D)

$$\big[\theta\big(h^2P\big),\chi\big] = -\frac{1}{\pi}\int_{\mathbb{R}^2}\overline{\partial}\,\widetilde{\theta}(x,y)\left[\left(z - h^2P\right)^{-1},\chi\right]\mathrm{d}x\mathrm{d}y \tag{9.64}$$

Now $\left(z - h^2P\right)\left[\left(z - h^2P\right)^{-1},\chi\right]f = \chi f + h^2\big[P,\chi\big]\left(z - h^2P\right)^{-1}f - \chi f$. Thus

$$\left[\left(z - h^2P\right)^{-1},\chi\right] = \left(z - h^2P\right)^{-1}h^2\big[P,\chi\big]\left(z - h^2P\right)^{-1}f$$

Let us set $① = \left\| \left[\left(z - h^2 P\right)^{-1}, \chi \right] f \right\|_{L^2}$. By (9.59) we have

$$① \leq \frac{1}{|\mathrm{Im}\, z|} \left\| h^2 \left[P, \chi\right] \left(z - h^2 P\right)^{-1} f \right\|_{L^2}$$

Now $\left[P, \chi\right] = \sum_{j=1}^{d} b_j D_j + b_0$ where $b_j \in \mathcal{C}_0^\infty\left(\overline{\Omega}\right)$, $j = 0, \cdots, d$. It follows that

$$① \leq \frac{Ch}{|\mathrm{Im}\, z|} \left(\sum_{j=1}^{d} \left\| h D_j \left(z - h^2 P\right)^{-1} f \right\|_{L^2} + h \left\| \left(z - h^2 P\right)^{-1} f \right\|_{L^2} \right)$$

Using Lemma 9.22 we deduce that

$$① \leq \frac{C' h \left\langle\, |z| \,\right\rangle}{|\mathrm{Im}\, z|^2} \| \, f \, \|_{L^2}$$

It follows from (9.54) that, with $z = x + iy$, we have

$$\left\| \left[\theta\left(h^2 P\right), \chi\right] f \right\|_{L^2} \leq Ch \int_{\mathbb{R}^2} \frac{\left\langle\, |z| \,\right\rangle}{|\mathrm{Im}\, z|^2} \left| \overline{\partial}\, \widetilde{\theta}(x, y) \right| \mathrm{d}x\mathrm{d}y$$

Using formula (9.62) and the fact that $\widetilde{\theta}$ has compact support in x and y we obtain (i).

(ii) Again the formula (9.63) we have

$$\left\| h D_j \theta\left(h^2 P\right) f \right\|_{L^2} \leq \frac{1}{\pi} \int \left| \overline{\partial}\, \widetilde{\theta}(x, y) \right| \left\| h \partial_j \left(z - h^2 P\right)^{-1} f \right\|_{L^2} \mathrm{d}x\mathrm{d}y$$

so using Lemma 9.22 we obtain

$$\left\| h D_j \theta\left(h^2 P\right) f \right\|_{L^2} \leq C \int \left| \overline{\partial}\, \widetilde{\theta}(x, y) \right| \frac{\left\langle\, |z| \,\right\rangle}{|\mathrm{Im}\, z|} \mathrm{d}x\mathrm{d}y \| \, f \, \|_{L^2} \leq C' \| \, f \, \|_{L^2}$$

(iii) We have as above

$$h D_j \left[\theta\left(h^2 P\right), \chi\right] = -\frac{1}{\pi} \int_{\mathbb{R}^2} \overline{\partial}\, \widetilde{\theta}(x, y) h D_j \left[\left(z - h^2 P\right)^{-1}, \chi \right] \mathrm{d}x\mathrm{d}y$$

and

$$h D_j \left[\left(z - h^2 P\right)^{-1}, \chi \right] = h D_j \left(z - h^2 P\right)^{-1} h^2 \left[P, \chi\right] \left(z - h^2 P\right)^{-1}$$

Using again Lemma 9.22 we obtain

$$\left\| hD_j \left[\left(z-h^2P\right)^{-1}, \chi \right] f \right\|_{L^2} \leq C \frac{\langle\, |z|\,\rangle}{|\mathrm{Im}\, z|} \left\| h^2 \left[P, \chi\right] \left(z-h^2P\right)^{-1} f \right\|_{L^2}$$

$$\leq \frac{Ch\,\langle\, |z|\,\rangle}{|\mathrm{Im}\, z|} \left(\sum_{j=1}^{d} \left\| hD_j \left(z-h^2P\right)^{-1} f \right\|_{L^2} + h \left\| \left(z-h^2P\right)^{-1} f \right\|_{L^2} \right) \leq C' h \frac{\langle\, |z|\,\rangle^2}{|\mathrm{Im}\, z|} \| f \|_{L^2}$$

and we conclude as before.

Proof of Lemma 8.2

We proceed as above. We have using (9.63) and Lemma 9.22

$$\left\| \left[\theta\left(h^2P\right), \chi_0 P^{\frac{1}{4}} \right] v \right\|_{L^2}$$

$$\leq \frac{1}{\pi} \int_{\mathbb{R}^2} \left| \overline{\partial}\, \widetilde{\theta}(x,y) \right| \left\| \left(z-h^2P\right)^{-1} h^2 \left[P, \chi_0\right] P^{\frac{1}{4}} \left(z-h^2P\right)^{-1} v \right\|_{L^2} \mathrm{d}x\mathrm{d}y$$

$$\leq C \int_{\mathbb{R}^2} \left| \overline{\partial}\, \widetilde{\theta}(x,y) \right| \frac{h}{|\mathrm{Im}\, z|} \left(\sum_{j=1}^{n} \left\| hD_j P^{\frac{1}{4}} \left(z-h^2P\right)^{-1} v \right\|_{L^2} + \left\| hP^{\frac{1}{4}} \left(z-h^2P\right)^{-1} v \right\|_{L^2} \right) \mathrm{d}x\mathrm{d}y$$

Now we have with $u = \left(z-h^2P\right)^{-1} v \in D(P)$

$$\left\| hD_j P^{\frac{1}{4}} u \right\|_{L^2} = h^{-\frac{1}{2}} \left\| hD_j \left(h^2P\right)^{\frac{1}{4}} u \right\|_{L^2} \leq C h^{-\frac{1}{2}} \left\| \left(h^2P\right)^{\frac{1}{2}} \left(h^2P\right)^{\frac{1}{4}} u \right\|_{L^2}$$

$$\leq C h^{-\frac{1}{2}} \frac{\langle\, |z|\,\rangle}{|\mathrm{Im}\, z|} \| v \|_{L^2}$$

by interpolation using Lemma 9.22. It follows that

$$\left\| \left[\theta\left(h^2P\right), \chi_0 P^{\frac{1}{4}} \right] v \right\|_{L^2} \leq C \int_{\mathbb{R}^2} \left| \overline{\partial}\, \widetilde{\theta}(x,y) \right| h^{\frac{1}{2}} \frac{\langle\, |z|\,\rangle}{|\mathrm{Im}\, z|^2} \mathrm{d}x\mathrm{d}y \, \| v \|_{L^2} \leq C h^{\frac{1}{2}} \| v \|_{L^2}$$

□

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