Local zeta functions attached to the minimal spherical series for a class of symmetric spaces

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LOCAL ZETA FUNCTIONS ATTACHED TO THE MINIMAL SPHERICAL SERIES FOR A CLASS OF SYMMETRIC SPACES

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Introduction

Local Zeta functions play a fundamental role in the theory of global Zeta functions, or more generally in the theory of automorphic L-functions. More precisely, their meromorphic continuation as well as their functional equations are often needed to prove the meromorphic continuation and the functional equation of the L-functions. For all these aspects the reader is refered to the book by Gelbart and Shahidi ([Ge-Sha]–1988) more specially to what these authors call "the L-function machine" (*loc. cit.* section 1.7).

The most famous local Zeta function is certainly the so-called Tate local Zeta function. This Zeta function appears in Tate's thesis ([Ta]–1967). It is associated to any local field and is used to prove the functional equation of the global Zeta function for any global field. Among these, corresponding to the field $\mathbb{Q}$, there is of course the Riemann Zeta function.

Let us briefly expose the basic results concerning this local Zeta function. Let K be a local field. A character of K^* is a morphism from K^* into $\mathbb{C}^*$. Any character of K^* is of the form

$$x \longmapsto \chi\left(\frac{x}{|x|}\right)\,|x|^s\ ,$$

where $s \in \mathbb{C}$, and where χ is a unitary character of the subgroup $\mathcal{U}_K$ of elements of absolute value 1. The set of these unitary characters is denoted by $\mathcal{U}_K^*$. Let $\mathcal{S}(K)$ be the space of Schwartz-Bruhat functions on K. The Tate Zeta function is then defined for $\varphi \in \mathcal{S}(K)$, $\chi \in \mathcal{U}_K^*$ and $s \in \mathbb{C}$ by

$$Z(f,s,\chi) = \int_{K^*} f(x)\,\chi\left(\frac{x}{|x|}\right)\,|x|^s\,d^*x.$$

Here d^*x is a suitably choosen Haar measure on K^*. In fact it is easily seen that the integral defining $Z(f,s,\chi)$ is only convergent for $\mathrm{Re}(s) > 0$, but it can be proved that this holomorphic function in the s parameter extends meromorphically to the whole complex plane. Moreover if $\mathcal{F}$ denotes the (additive) Fourier transform on K, the Zeta function satisfies the following functional equation:

$$(1) \qquad Z(\varphi,s,\chi) = \rho(s,\chi)Z(\mathcal{F}\varphi,1-s,\chi^{-1}) \quad \text{for } \varphi \in \mathcal{S}(K)\ ,$$

where $\rho(s,\chi)$ is a meromorphic function in s, which is explicitely known if χ is unramified or if $K = \mathbb{R},\mathbb{C}$. Let us say a few words about the proof by Tate of this functional equation, in order to show below that this proof does not extend to higher dimensions. Tate's trick is the following. For $\varphi,\psi \in \mathcal{S}(K)$, the product

$$Z(\varphi,s,\chi)Z(\mathcal{F}\psi,1-s,\chi^{-1})$$

which is a convergent integral for $0 < \mathrm{Re}(s) < 1$, is shown to be symmetric in φ and ψ:

$$Z(\varphi, s, \chi)Z(\mathcal{F}\psi, 1-s, \chi^{-1}) = Z(\psi, s, \chi)Z(\mathcal{F}\varphi, 1-s, \chi^{-1}) \ . \tag{2}$$

This remark implies the functional equation and that the Tate factor

$$\rho(s, \chi) = \frac{Z(\psi, s, \chi)}{Z(\mathcal{F}\psi, 1-s, \chi^{-1})}$$

can be computed by using some well choosen function ψ.

In 1966 André Weil gave a Bourbaki conference ([We]) where he interpreted the functional equation (1) in terms of homogeneous distributions. He remarked that once the meromorphic continuation of $Z(\varphi, s, \chi)$ and of $Z(\mathcal{F}\varphi, 1-s, \chi^{-1})$ is proved, then as the two tempered distributions $\varphi \longmapsto Z(\varphi, s, \chi)$ and $\varphi \longmapsto Z(\mathcal{F}\varphi, 1-s, \chi^{-1})$ have the same degree of homogeneity and, as the dimension of the space of such distributions is one, there exists a "constant" $\rho(s, \chi)$ such that (1) is satisfied.

André Weil suggested also to generalize Tate's result, as well as the problem of the determination of dimension of homogeneous distributions to any simple algebra. In connection with the paper by André Weil, M. Raïs made deep investigations concerning several type of homogeneous distributions on various matrix spaces ([Raïs]–1972). On the other hand the theory of Zeta functions of simple algebras was achieved by the work of Godement-Jacquet ([G-J]–1972) in the spirit of Tate, but without using any result on homogeneous distributions. Previously to their work several partial (but important) results for simple algebras were obtained by Leptin ([Lep]–1955), Fujisaki ([Fu]–1958), Stein ([St]–1967), Jacquet-Langlands ([Ja-La]–1970), Gelbart ([Ge]–1971). See also the Bourbaki Seminars on the subject by Godement ([Go]–1958, [Go]–1959). The paper by Knapp ([Kn]–1994) is also relevant to understand how the work by Godement-Jacquet fits into the local Langlands correspondence.

Let us here give an outline of the results obtained by Godement and Jacquet, just for the simple algebra $M_n(\mathbb{R})$ of real $n \times n$ matrices. Let $\mathcal{S}\big(M_n(\mathbb{R})\big)$ denote the space of Schwartz functions on $M_n(\mathbb{R})$. Let (π, V) be an admisssible representation of $GL_n(\mathbb{R})$ and let (π^*, V^*) be the contragredient representation. Let $s \in \mathbb{C}$. For any $f, h \in \mathcal{S}\big(M_n(\mathbb{R})\big)$ the $(\mathrm{End}(V) - \text{valued})$ local Zeta functions are defined by

$$Z(f, s, \pi) = \int_{GL_n(\mathbb{R})} f(x) |\det(x)|^s \pi(x)\, d^*x \ ,$$

$$Z(h, s, \pi^*) = \int_{GL_n(\mathbb{R})} f(x) |\det(x)|^s \pi(x^{-1})\, d^*x \ ,$$

where $d^*x = \dfrac{dx}{|\det(x)|^n}$ is the Haar measure on $GL_n(\mathbb{R})$. Godement and Jacquet proved that the integrals defining $Z(f, s, \pi)$ and $Z(h, s, \pi^*)$ are convergent for $\mathrm{Re}(s)$ large enough, have meromorphic continuations to $\mathbb{C}$ which satisfy the following functional equation:

$$Z(f, s+n-1, \pi) = \gamma(s, \pi) Z(\mathcal{F}f, 1-s, \pi^*) \ , \tag{3}$$

where again $\mathcal{F}f$ is the Fourier transform of f, which is defined to map $\mathcal{S}\big(M_n(\mathbb{R})\big)$ onto itself

The functional equation (1) is just the functional equation (3) specialized to $n = 1$. Unfortunately, and this is one of the difficulties of Godement-Jacquet's work, Tate's trick does no longer work for $n > 1$. For example, even for the case where $\pi = \mathbf{1}$ is the trivial one-dimensional representation, and $f, h \in \mathcal{S}\big(M_n(\mathbb{R})\big)$, the two "functions" $Z(f, s+n-1, \mathbf{1})Z(\mathcal{F}h, 1-s, \mathbf{1})$ and $Z(h, s+n-1, \mathbf{1})Z(\mathcal{F}f, 1-s, \mathbf{1})$ can be "formally" proved to be equal by making the same change of variables as Tate did for $n = 1$, but there is no domain in the s parameter where these products are defined by convergent integrals, hence nothing makes sense... Therefore Godement and Jacquet's method uses a difficult induction on n (for a quick proof in the case $\pi = \mathbf{1}$ (for any local field), see [Ru]– 1975).

At this point one should notice that if $k = \mathbb{Q}$ there are basically two ways to obtain the functional equation of the Riemann Zeta function $\zeta(s)$ from the local Zeta function. The first one uses the local functional equation (1) on $\mathbb{R}$ (the only archimedean place of $\mathbb{Q}$) together with the Poisson summation formula, the second one, due to Tate, is to use adelic analysis. Of course due to some adelic evidence, it is well known that these two ways are basically the same.

However, the first method was extended to obtain generalized ("global") Zeta functions in situations where there was no satisfactory adelic theory at hand. This is due to Mikio Sato and is called the Theory of Prehomogeneous Vector Spaces. Let us give a very brief and simplified overview of this Theory. Let us start with a connected reductive algebraic group G defined over $\mathbb{R}$. We also denote by G the group of complex points . We suppose also that we are given a finite dimensional irreducible representation ρ of G in a complex vector space V. The basic assumption we make are the following:

• G has a Zariski open orbit Ω in V, (then (G, ρ, V) is called a *prehomogeneous vector space*);

• There exists on V a non trivial polynomial P which is relatively invariant under G, and such that the Hessian of P is nonzero on Ω (then (G, ρ, V) is called *regular*).

Then it can be proved, under these assumptions, that the contragredient representation (G, ρ^*, V^*) is again a regular prehomogeneous vector space, whose fundamental relative invariant is denoted by P^*. The polynomials P and P^* have the same degree denoted by d.

We further assume that (G, ρ, V) is defined over $\mathbb{R}$. Then the group $G_{\mathbb{R}}$ of real points has the same finite number of open orbits in $V_{\mathbb{R}}$ and $V_{\mathbb{R}}^*$, denoted by $\Omega_1, \ldots, \Omega_r$ and $\Omega_1^*, \ldots, \Omega_r^*$ respectively. For $f \in \mathcal{S}(V_{\mathbb{R}})$ (the space of Schwartz functions on $V_{\mathbb{R}}$) and $h \in \mathcal{S}(V_{\mathbb{R}}^*)$ one defines the local Zeta functions by

$$
\begin{aligned}
Z_p^+(f, s) &= \textstyle\int_{\Omega_p} f(x)|P(x)|^s \, d^*x \ , \\
Z_p^-(h, s) &= \textstyle\int_{\Omega_p^*} h(y)|P^*(y)|^s \, d^*y \ .
\end{aligned}
\tag{4}
$$

These Zeta functions are defined by convergent integrals (and are holomorphic) for $\mathrm{Re}(s) > 0$, and admit meromorphic continuation to the whole

complex plane. Then if we denote by $\mathcal{F} : \mathcal{S}(V_{\mathbb{R}}) \longrightarrow \mathcal{S}(V_{\mathbb{R}}^*)$ the Fourier transform, the local functional (matrix) equation can be written as follows

$$Z_p^+(f,s) = \sum_{q=1}^{r} \gamma_{pq}(s) Z_q^-(\mathcal{F}f, -s - \frac{n}{d}) \ . \tag{5}$$

This is the exact analogue of Tate's local functional equation. There are at least two proofs of the functional equation (5) (see [Sa-Sh]–1974 and [Sh]–1972) and both of them relies on the fact that there is at most one invariant distribution on each open $G_{\mathbb{R}}$-orbit. Therefore one can say in some weak sense that these proofs are related to André Weil's suggestion. One should also notice that in almost all published papers on the subject the γ_{pq}'s are obtained by case by case examination.

If moreover G is supposed to be defined over $\mathbb{Z}$, then M. Sato defined global Zeta functions $\zeta_{p,L}$ for any $G_{\mathbb{Z}}$–stable lattice L in $V_{\mathbb{R}}$. Very roughly speaking, let us just say that

$$\zeta_{p,L}(s) = \sum_{x \in L \cap \Omega_p} \frac{\mu_p(x)}{|P(x)|^s} \ ,$$

where $\mu_p(x)$ are some well defined coefficients called "local densities" (see [Sa-Sh]–1974 and [Sh]–1972). The functional equation (5), together with the Poisson summation formula implies a matricial functional equation for $\zeta_{p,L}$. This is a generalization (which needs to overcome some serious technicalities) of the non-adelic proof of the functional equation of the Riemann Zeta function, starting from Tate functional equation (1).

Of course, the Godement-Jacquet Zeta function

$$Z(f,s,\mathbf{1}) = \int_{GL_n(\mathbb{R})} f(x) |\det(x)|^s \, d^*x$$

is an example of a Zeta function associated to a prehomogeneous vector space. On the other hand the Zeta functions of prehomogeneous vector spaces, do not reach the generality of Godement-Jacquet's theory, in the sense that there is no higher dimensional representation of G involved in the Zeta functions defined in (4). Howewer in his work [Sat]-1994, F. Sato obtained a functional equation which can be interpreted as involving some non scalar representations of G for some cases.

The purpose of the present paper (whose results have been announced in [B-R]–1997) is to define local Zeta functions involving representations for a family of prehomogeneous vector spaces, in which all open orbits Ω_p (and Ω_p^*) are symmetric spaces, and to investigate their functional equations. To be more precise we shall consider here a family of prehomogeneous vector spaces such that if H_p denotes the isotropy subgroup of a generic point $I_p \in \Omega_p$, then H_p is a symmetric subgroup of $G_{\mathbb{R}}$.

In such a situation we can consider a spherical representation π on a Hilbert space $\mathcal{H}$, that is a representation which has a non zero H_p-invariant distribution vector $a_p \in (\mathcal{H}^{-\infty})^{H_p}$ for all $p = 0, \ldots, r$ (the representations in the minimal spherical series will satisfy this condition in our cases). Then the $(\mathcal{H}^{-\infty})^{H_p}$-valued function on $G_{\mathbb{R}}$ defined by $g \longmapsto \pi(g)a_p$ depends only on

the class mod (H_p), and defines therefore a function $\pi(x)a_p$ on $G/H_p \simeq \Omega_p$. Then, at least formally one can consider the following Zeta function

$$\mathbf{Z}_p^+(f,\pi,a_p) = \int_{\Omega_p} f(x)\pi(x)a_p\, d^*x \quad (f \in \mathcal{S}(V_{\mathbb{R}}))\ , \tag{6}$$

where d^*x is the $G_{\mathbb{R}}$-invariant measure on $V_{\mathbb{R}}$.

Similarly, on the dual side $V_{\mathbb{R}}^*$ one can define

$$\mathbf{Z}_p^-(h,\pi,a_p) = \int_{\Omega_p^*} h(y)\pi(y)a_p\, d^*y \quad (h \in \mathcal{S}(V_{\mathbb{R}}^*))\,. \tag{7}$$

In this paper we are dealing with representations $(\pi_{\tau,\lambda}, \mathcal{H}_{\tau,\lambda})$ from the minimal spherical series. Here (τ,λ) is the usual inducing parameter for such representations. As we mentioned above, such representations of G are H_p–spherical for all $p = 0,\dots,r$. This explains why the same representation π occurs for each p in the definitions (6) and (7). Moreover, for generic λ, the space $(\mathcal{H}_{\tau,\lambda})^{-\infty})^{H_p}$ is finite dimensional.

We prove that the integrals defining $\mathbf{Z}_p^+(f,\pi_{\tau,\lambda},a_p)$ and $\mathbf{Z}_p^-(h,\pi_{\tau,\lambda},a_i)$ are convergent for $\mathrm{Re}(\lambda)$ large enough. The Zeta functions $\mathbf{Z}_p^+(f,\pi_{\tau,\lambda},a_p)$ and $\mathbf{Z}_p^-(h,\pi_{\tau,\lambda},a_p)$ for generic λ are then defined by meromorphic continuation. These Zeta functions are closely related to the local Zeta functions of the prehomogeneous vector space $(P,\rho,V_{\mathbb{R}})$ where P is some parabolic subgroup of $G_{\mathbb{R}}$. In this situation there are more than one fundamental relative invariant.

Then, for $a = (a_p) \in \prod_{p=0}^r (\mathcal{H}_{\tau,\lambda}^{-\infty})^{H_p}$ we set

$$\begin{aligned} \mathbf{Z}^+(f,\pi_{\tau,\lambda},a) &= \sum_{p=0}^r \mathbf{Z}_p^+(f,\pi_{\tau,\lambda},a_p)\ , \\ \mathbf{Z}^-(h,\pi_{\tau,\lambda},a) &= \sum_{p=0}^r \mathbf{Z}_p^-(h,\pi_{\tau,\lambda},a_p)\ . \end{aligned} \tag{8}$$

Our main Theorem asserts that these functions satisfy the following functional equation:

$$\mathbf{Z}^-(\mathcal{F}f,\pi_{\tau,\lambda},a) = \mathbf{Z}^+(f,\chi_0^{-m}\otimes\pi_{\tau,\lambda},A^{\tau,\lambda}a)\ ,$$

where $\mathcal{F}: \mathcal{S}(V_{\mathbb{R}}) \longrightarrow \mathcal{S}(V_{\mathbb{R}}^*)$ is the Fourier transform, where χ_0 is the character of the fundamental relative invariant of $(G,\rho,V_{\mathbb{R}})$ and where A is an element of $\mathrm{End}\big(\prod_{p=0}^r(\mathcal{H}_{\tau,\lambda}^{-\infty})^{H_p}\big)$. Moreover we are able to compute explicitely the matrix A in the standard basis of $\prod_{p=0}^r(\mathcal{H}_{\tau,\lambda}^{-\infty})^{H_p}$, for generic λ.

We use together a Weil type argument relying on some results by Bruhat ([Bru]–1956) to prove the existence of an abstract functional equation and an induction on the rank of our symmetric spaces (similar to Godement-Jacquet) to obtain the explicit "gamma" factor (*i.e.* the explicit matrix $A^{\tau,\lambda}$).

From easy considerations this can be viewed as a generalization of the "real" part of the work of Godement-Jacquet. More precisely their results

correspond to the case of the symmetric space

$$GL_n(\mathbb{R}) \times GL_n(\mathbb{R})/\mathrm{diag}\big(GL_n(\mathbb{R})\big) \simeq GL_n(\mathbb{R}) \ ,$$

where $\mathrm{diag}\big(GL_n(\mathbb{R})\big)$ is the diagonal subgroup of $GL_n(\mathbb{R}) \times GL_n(\mathbb{R})$. This symmetric space occurs as a particular case in our paper. To be quite precise we should say that we deal in fact with the connected group $GL_n(\mathbb{R})^+$ rather than with $GL_n(\mathbb{R})$, this introduces slight changes with respect to Godement-Jacquet's result.

Another point we would like to emphasize here is that, contrary to $GL_n(\mathbb{R})$, in the general case considered in this paper the representations $\pi_{\tau,\lambda}$ have "multiplicities", which means that generically $\dim(\mathcal{H}_{\tau,\lambda}^{-\infty})^{H_p} > 1$. This is a source of technical complication.

We hope to be able in the future to associate global Zeta functions to this family of symmetric spaces.

Content

Let us give now a brief description of this article.

In **Chapter** 1 we describe first a class of prehomogeneous vector spaces related to some graded reductive Lie algebra. To be more precise we consider a real reductive Lie algebra $\widetilde{\mathfrak{g}}$ for which there exists an element $H_0 \in \widetilde{\mathfrak{g}}$ such that

$$\widetilde{\mathfrak{g}} = V^- \oplus \mathfrak{g} \oplus V^+ \ ,$$

where V^- (*resp.* $\mathfrak{g}$, *resp.* V^+) is the -2 (*resp.* 0, *resp.* $+2$) eigenspace for $\mathrm{ad}\, H_0$. We also assume that the adjoint representation of $\mathfrak{g}$ on the complexification of V^+ is irreducible and that there exist $Y \in V^-$ and $X \in V^+$ such that $\{Y, H_0, X\}$ is a $\mathfrak{sl}_2$–triple. Such a data will be called a *regular graded Lie algebra* in the sequel.

The class of (real) regular graded simple algebras is in bijection with the class of simple real Jordan algebras up to "mutations" (see [Kay]–1994 for example).

If G is the analytic subgroup of the adjoint group $\widetilde{G}$ of $\widetilde{\mathfrak{g}}$, the pair (G, V^+) is a prehomogeneous vector space *i.e.* G has a finite number of (real) Zariski–open orbits in V^+. Using the Killing form of $\widetilde{\mathfrak{g}}$, the dual prehomogeneous vector space is now identified whith (G, V^-).

We define a maximal sequence $(\lambda_0, \ldots, \lambda_k)$ of strongly orthogonal roots of $\widetilde{\mathfrak{g}}$ relatively to a maximal split abelian subalgebra $\mathfrak{a}^{\mathfrak{p}}$ whose corresponding root spaces are in V^+. This sequence will be of great importance in the whole paper. The integer $k+1$ is called the *rank* of the graded Lie algebra. We give also various decompositions of V^-, $\mathfrak{g}$ and V^+ according to the eigenvalues of $\mathrm{ad}\, H_{\lambda_j}$, H_{λ_j} being the co–root of λ_j.

In **Chapter** 2 we describe explicitly the orbit structure of V^+ under the action of G. This uses the decomposition mentioned before and also some involved constructions of Weyl group elements.

The classification of these orbits leads us to distinguish three different types among the regular graded Lie algebras (called Type I, II and III). Inside a given Type the G–orbits are described in an uniform way.

At the end of the chapter a table gives the classification of the regular graded simple Lie algebras by means of weighted SATAKE diagrams.

In **Chapter** 3 we develop the structure theory of the various symmetric spaces occuring in our framework. The key fact is that each open G–orbit Ω_p^+ $(p = 0, \ldots, r)$ inside V^+ is a symmetric space G/H_p. We describe precisely the involutions σ_p defining the symmetric spaces as well as some specific root systems characterizing each type.

A striking fact here is the existence of a Cartan involution θ commuting with each involution σ_p and of a parabolic subgroup P, minimal among the $\sigma_p\theta$–stable parabolic subgroups for all $p = 0, \ldots, r$. The Langlands decomposition of this P is given. It is well known that P has a finite number of open orbits in each $\Omega_p = G/H_p$.

Then we study the irreducible polynomials Δ_j $(j = 0, \ldots, k)$ which are the fundamental relative invariant of the prehomogeneous vector space (P, V^+). These polynomials are used to describe precisely the open P–orbits. They will play an important role in the further chapters.

In **Chapter** 4 we establish several integral formulas which will be used in the further chapters. To do this we have to make some delicate normalizations of measures on subspaces, subgroups, or symmetric spaces occuring in graded Lie algebras.

We give also a detailed and adapted version of Weil's formula for the Fourier transform of a quadratic character. Finally in section 4.8. we give a key result (due to Iris Muller ([Mu]–1986) for Type I and Type II) on some orbital integral which will allow us to use an induction process on the rank of the graded Lie algebra.

In **Chapter** 5 we give the explicit functional equation (for Type I and Type II) for the prehomogeneous vector space (P, V^+), where P is the parabolic subgroup defined in Chapter 3. Roughly speaking the Zeta functions occuring in Chapter 5 are of the form

$$Z^+(f, s) = \int_{V^+} f(x)|\Delta(x)|^s dx \ ,$$

where $s = (s_0, \ldots, s_k) \in \mathbb{C}^{k+1}$ and $|\Delta(x)|^s = |\Delta_0(x)|^{s_0} \ldots |\Delta_k(x)|^{s_k}$. Here the results from the previous Chapter play a crucial role.

We give also a new proof (another one can be found in [Sat]–1982) of the existence of the functional equation which certainly extends to the general case of any regular prehomogeneous vector space with several relative invariants and which is based on the existence of a Bernstein identity for several polynomials.

As a Corollary we obtain the explicit functional equation for the prehomogeneous vector space (G, V^+).

In **Chapter** 6 we similarly obtain the explicit functional equation for the prehomogeneous (P, V^+) in the Type III case. An important change here

is that if $P = MAN$ is the Langlands decomposition of P, then the group M has higher dimensional irreducible $(M \cap H_p)$–spherical representations $(\tau, \mathcal{H}_\tau)$. Therefore we give first a complete description of these representations which are essentially tensor products of classical spherical harmonics on the root spaces $\widetilde{\mathfrak{g}}^{\lambda_j}$. The Zeta functions occuring there involve in fact these $(M \cap H_p)$–spherical representations τ. They are $\mathcal{H}_\tau$–valued and roughly speaking again they are of the form

$$Z^+(f, s, \tau) = \int f(x)|\Delta(x)|^s b_\tau^+(x)dx \ ,$$

where b_τ^+ is a rational P–equivariant mapping from V^+ into $\mathcal{H}_\tau$.

Chapter 7 is the heart of the paper and contains the main result.

We first define the minimal spherical series $\pi_{\tau,\lambda}$ for the symmetric spaces G/H_p. Then we introduce some unified notation which enables us to write the functional equations obtained in Chapter 5 and Chapter 6 in a way which does not depend on the Type (see Theorem 7.2.6).

For each $p = 0, \ldots, r$ we construct a family $a_{\tau,\lambda}^{p,\gamma}$ of H_p–invariant distribution vectors (*i.e.* $a_{\tau,\lambda}^{p,\gamma} \in (\mathcal{H}_{\tau,\lambda}^{-\infty})^{H_p}$). Here γ belongs to a finite set $\mathcal{W}_p$ which parametrizes the open P–orbits in $G/H_p \simeq \Omega_p$. These vectors $a_{\tau,\lambda}^{p,\gamma}$ $(\gamma \in \mathcal{W}_p)$ form a basis of $(\mathcal{H}_{\tau,\lambda}^{-\infty})^{H_p}$ for generic λ and can also be described via the Poisson kernel on some subdomain of the parameter λ.

A part of our results is to prove that the Zeta integrals $\mathbf{Z}_p^+(f, \pi, a_p)$ and $\mathbf{Z}_p^-(h, \pi, a_p)$ defined in (6) and (7) make sense for $\pi = \pi_{\tau,\lambda}$ and for $a_p = a_{\tau,\lambda}^{p,\gamma}$. Then, for generic λ and $a = (a_0, \ldots, a_r) \in \prod_{p=0}^r (\mathcal{H}_{\tau,\lambda}^{-\infty})^{H_p}$, the Zeta functions $\mathbf{Z}^+(f, \pi_{\tau,\lambda}, a)$ and $\mathbf{Z}^-(h, \pi_{\tau,\lambda}, a)$ are well defined by (8). We prove the functional equation (see Theorem 7.9.6)

$$\mathbf{Z}^-(\mathcal{F}f, \pi_{\tau,\lambda}, a) = \mathbf{Z}^+(f, \chi_0^{-m} \otimes \pi_{\tau,\lambda}, A^{\tau,\lambda}(a)) \ ,$$

and compute explicitely the endomorphism $A \in \mathrm{End}(\prod_{p=0}^r (\mathcal{H}_{\tau,\lambda}^{-\infty})^{H_p})$ in the basis $(a_{\tau,\lambda}^{p,\gamma})_{p,\gamma}$.

Finally we give an interpretation of the Main Theorem in terms of a modified Fourier transform $\mathcal{F}^*$. Such a modified Fourier transform was first introduced by E. M. Stein ([St]–1967).

In the **Appendix** we treat the case of symmetric matrices in detail.

CHAPTER 1

A class of real prehomogeneous spaces

The aim of this chapter is to give the definition and to describe the real prehomogeneous vector spaces that we will consider in the sequel. The notion of prehomogeneous space was introduced by M. SATO in the 1970's (see the english version in [Sa]–1990). The class of prehomogeneous vector spaces that we will study was introduced by H. RUBENTHALER ([Ru]–1982) and these spaces are called regular prehomogeneous vector spaces of commutative parabolic type. They are associated to a graded reductive Lie algebra which satisfies some supplementary hypothesis. In the first part of this chapter (1.1 to 1.7) we give the consequences of hypothesis ($\mathbf{H_1}$) and ($\mathbf{H_2}$) and in the second part (1.7 to 1.13) those of hypothesis ($\mathbf{H_3}$) which is equivalent to the regularity of the associated prehomogeneous space. For the general theory of prehomogeneous vector spaces we refer the reader to the paper by M. SATO and T. KIMURA ([Sa-K]–1977).

1.1. A class of graded algebras

We consider a real reductive Lie algebra $\widetilde{\mathfrak{g}}$ which satisfies the following two hypothesis :

($\mathbf{H_1}$) *There exists an element* $H_0 \in \widetilde{\mathfrak{g}}$ *such that* $\operatorname{ad} H_0$ *defines a* $\mathbb{Z}$*-gradation of the form*

$$\widetilde{\mathfrak{g}} = V^- \oplus \mathfrak{g} \oplus V^+ \quad (V^+ \neq \{0\}) \ ,$$

$$\text{where } [H_0, X] = \begin{cases} 0 & \text{for } X \in \mathfrak{g} \ ; \\ 2X & \text{for } X \in V^+ \ ; \\ -2X & \text{for } X \in V^- \ . \end{cases}$$

($\mathbf{H_2}$) *The representation of* $\mathfrak{g}$ *on the complexification* $V^+_{\mathbb{C}}$ *of* V^+ *is irreducible.*

Troughout this paper a Lie algebra $\widetilde{\mathfrak{g}}$ satisfying these two hypothesis will be called *a graded Lie algebra.*

The following relations are trivial consequences from ($\mathbf{H_1}$) :

$$[\mathfrak{g}, V^+] \subset V^+ \ ; \ [\mathfrak{g}, V^-] \subset V^- \ ; \ [\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g} \ ; \ [V^+, V^-] \subset \mathfrak{g} \ .$$

Remark 1.1.1. The hypothesis ($\mathbf{H_2}$) implies that there is no complex structure on the Lie algebra $\widetilde{\mathfrak{g}}$. If there is a complex structure on $\widetilde{\mathfrak{g}}$ then the complexification $\widetilde{\mathfrak{g}}_{\mathbb{C}}$ of $\widetilde{\mathfrak{g}}$ is a complex Lie algebra isomorphic to $\widetilde{\mathfrak{g}} \times \widetilde{\mathfrak{g}}$. The complex subspace $W = \{(X, iX) | X \in V^+\}$ of $V^+_{\mathbb{C}} \subset \widetilde{\mathfrak{g}}_{\mathbb{C}}$ is then invariant

under the action of $\mathfrak{g}$ which contradicts $(\mathbf{H_2})$. The complex situation has been studied in [R]–1982, [M-R-S]–1986 and [B-R]–1993.

Since H_0 is a semisimple element of $\widetilde{\mathfrak{g}}$ with real eigenvalues, there exists a Cartan involution θ of $\widetilde{\mathfrak{g}}$ such that $\theta(H_0) = -H_0$. Moreover there exists a Cartan subspace $\mathfrak{a}^{\mathfrak{p}}$ of $(\widetilde{\mathfrak{g}}, \theta)$ such that $H_0 \in \mathfrak{a}^{\mathfrak{p}}$ *i.e.* a maximal abelian subspace of the set $\widetilde{\mathfrak{p}} = \{X \in \widetilde{\mathfrak{g}} \mid \theta(X) = -X\}$. The Killing form of the semisimple part of $\widetilde{\mathfrak{g}}$ has an extension $\widetilde{B}$ to $\widetilde{\mathfrak{g}}$ such that the quadratic form $X \in \widetilde{\mathfrak{g}} \mapsto \widetilde{B}_\theta(X, X) = -\widetilde{B}(X, \theta(X))$ is positive definite.

We will make once and for all a choice of such θ, $\mathfrak{a}^{\mathfrak{p}}$ and $\widetilde{B}$.

1.2. Root systems

Since $\theta(H_0) = -H_0$, θ interchanges V^+ and V^-, stabilizes $\mathfrak{g}$ and is a Cartan involution of $\mathfrak{g}$. Moreover $\mathfrak{a}^{\mathfrak{p}}$ is also a Cartan subspace of the pair $(\mathfrak{g}, \theta)$. Thus we consider the (restricted) root systems $\widetilde{\Sigma}$ and Σ of the pair $(\widetilde{\mathfrak{g}}, \mathfrak{a}^{\mathfrak{p}})$ and the pair $(\mathfrak{g}, \mathfrak{a}^{\mathfrak{p}})$ respectively. For $\lambda \in \widetilde{\Sigma}$ (*resp.* Σ) we will denote by $\widetilde{\mathfrak{g}}^\lambda$ (*resp.* $\mathfrak{g}^\lambda$) the corresponding root space in $\widetilde{\mathfrak{g}}$ (*resp.* $\mathfrak{g}$).

Theorem 1.2.1.
(1) *There exists a simple system $\widetilde{\Pi}$ in $\widetilde{\Sigma}$ such that*

$$\nu \in \widetilde{\Pi} \Longrightarrow \nu(H_0) = 0 \text{ or } 2 \ .$$

(2) *There exists an unique root $\lambda_0 \in \widetilde{\Pi}$ such that $\lambda_0(H_0) = 2$.*
(3) *If the decomposition of a positive root $\lambda \in \widetilde{\Sigma}$ in the basis $\widetilde{\Pi}$ is given by*

$$\lambda = m_0\lambda_0 + \sum_{\nu \in \widetilde{\Pi} \setminus \{\lambda_0\}} m_\nu \nu \ , \ m_0 \in \mathbb{Z}^+, m_\nu \in \mathbb{Z}^+$$

then $m_0 = 0$ or $m_0 = 1$. Moreover λ belongs to Σ if and only if $m_0 = 0$.

Proof. — Let S be the subset of $\widetilde{\Sigma}$ given by

$$S = \{\lambda \in \widetilde{\Sigma} \mid \lambda(H_0) = 0 \text{ or } 2\} \ .$$

It is easily seen from $(\mathbf{H_1})$ that S is a parabolic subset of $\widetilde{\Sigma}$, i.e. $\mathfrak{g} \oplus V^+$ is a parabolic subalgebra of $\widetilde{\mathfrak{g}}$. It is well known (see [Bou]–1968 chap. 6 Prop. 20) that there exists an order on $\widetilde{\Sigma}$ such that, if a root $\lambda \in \widetilde{\Sigma}$ is positive for this order, then $\widetilde{\mathfrak{g}}^\lambda$ is a subspace of $\mathfrak{g} \oplus V^+$. If $\widetilde{\Pi}$ denotes the set of simple roots of $\widetilde{\Sigma}$ related to this order, then $\widetilde{\Pi}$ satisfies (1).

There exists at least one root $\lambda_0 \in \widetilde{\Pi}$ such that $\lambda_0(H_0) = 2$ because $V^+ \neq \{0\}$. Moreover the commutativity of V^+ which is the nilradical of the parabolic algebra $\mathfrak{g} \oplus V^+$ implies (3).

Let us suppose that there exists in $\widetilde{\Pi}$ a root $\lambda_1 \neq \lambda_0$ such that $\lambda_1(H_0) = 2$. Let V_0 be the sum of the root spaces $\widetilde{\mathfrak{g}}^\lambda$ for the roots λ of the form

$$\lambda = \lambda_0 + \sum_{\nu \in \widetilde{\Pi}, \nu(H_0) = 0} m_\nu \nu \quad (m_\nu \in \mathbb{Z}^+) \ .$$

Since V_0 does not contain $\widetilde{\mathfrak{g}}^{\lambda_0}$, it is a non-trivial subspace of V^+ which is invariant under the action of $\mathfrak{g}$. This gives a contradiction with $(\mathbf{H_2})$ and the unicity of λ_0 such that $\lambda_0(H_0) = 2$ follows, hence (2) is proved.

□

Let Π be the set of roots $\nu \in \widetilde{\Pi}$ such that $\nu(H_0) = 0$. We have proved that

$$\widetilde{\Pi} = \Pi \bigcup \{\lambda_0\} \ ,$$

where $\widetilde{\Pi}$ (*resp.* Π) is a simple system in $\widetilde{\Sigma}$ (*resp.* Σ). We will denote by $\widetilde{\Sigma}^+$ (*resp.* Σ^+) the set of positive roots of $\widetilde{\Sigma}$ (*resp.* Σ) for the order relatively to the basis $\widetilde{\Pi}$ (*resp.* Π).

We will make once and for all the choice of an order on $\widetilde{\Sigma}$ such that Theorem 1.2.1 *is satisfied.*

Then we have the following characterization of λ_0 :

Corollary 1.2.2. *The root λ_0 is the unique root in $\widetilde{\Sigma}$ such that*

- $\lambda_0(H_0) = 2$;
- $\lambda \in \Sigma^+ \Longrightarrow \lambda_0 - \lambda \notin \widetilde{\Sigma}$.

1.3. Complexification

We will use an index $\mathbb{C}$ to denote the complexification of real vector spaces or real Lie algebras. For example $\widetilde{\mathfrak{g}}_{\mathbb{C}}$ is the complexification of the Lie algebra $\widetilde{\mathfrak{g}}$.

There exists a Cartan subalgebra of $\widetilde{\mathfrak{g}}$ which is stable under θ and which contains $\mathfrak{a}^{\mathfrak{p}}$. From now on we will choose such a Cartan subalgebra, denoted by $\mathfrak{j}$. This is also a Cartan subalgebra of $\mathfrak{g}$. Let $\widetilde{\mathcal{R}}$ and $\mathcal{R}$ be respectively the root systems of the pairs $(\widetilde{\mathfrak{g}}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}})$ and $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}})$. There exists an order on $\widetilde{\mathcal{R}}$ such that, if $\widetilde{\mathcal{R}}^+$ denotes the set of positive roots for this order and ρ denotes the restriction to $\mathfrak{a}^{\mathfrak{p}}$ of a root of $\widetilde{\mathcal{R}}$, we have

$$\rho(\widetilde{\mathcal{R}}^+) = \widetilde{\Sigma}^+ \ .$$

Proposition 1.3.1. *There exists an unique simple root α_0 in $\widetilde{\mathcal{R}}^+$ such that*

$$\rho(\alpha_0) = \lambda_0 \ .$$

Proof. — Let $\widetilde{\Psi}$ be the set of simple roots in $\widetilde{\mathcal{R}}^+$. It is well known (see [Wa]–1972 section 1.1.3) that ρ is a surjection from $\widetilde{\Psi}$ onto $\widetilde{\Pi}$ and that each element of $\widetilde{\Pi}$ is the image of one or two elements of $\widetilde{\Psi}$. Assume that α_0 and β_0 are two different elements of $\widetilde{\Psi}$ such that

$$\rho(\alpha_0) = \rho(\beta_0) = \lambda_0 \ .$$

The roots α_0 and β_0 are not in the same irreducible connected component of $\widetilde{\mathcal{R}}$. If this would be the case, the highest root in this component would

be of the form

$$\gamma = m_0\alpha_0 + m_1\beta_0 + \sum_{\alpha\in\widetilde{\Psi}\setminus\{\alpha_0,\alpha_1\}} m_\alpha\alpha \ ,$$

where $m_j \in \mathbb{Z}^+$, $m_j \geq 1$ for $j = 0, 1$ and $m_\alpha \in \mathbb{Z}^+$. This would imply that

$$\rho(\gamma) = (m_0 + m_1)\lambda_0 + \sum_{\nu\in\Pi} m_\nu\nu \ (m_\nu \in \mathbb{Z}^+) \ ,$$

which is impossible by Theorem 1.2.1 (3).

Let now ω_0 and ω_1 the highest roots of the (distincts) irreducible components of $\widetilde{\mathcal{R}}$ which contain respectively α_0 and β_0. Since we have $\omega_0(H_0) = \omega_1(H_0) = 2$, ω_0 and ω_1 are weights for the representation of $\mathfrak{g}$ on $V_{\mathbb{C}}^+$ and these weights are highest weights. This gives a contradiction with hypothesis $(\mathbf{H_2})$ and the unicity of α_0 is proved.

□

Remark 1.3.2. The highest weight of the (irreducible) representation of $\mathfrak{g}$ in $V_{\mathbb{C}}^+$ is the highest root of the irreducible component of $\widetilde{\mathcal{R}}$ which contains α_0. But hypothesis $(\mathbf{H_2})$ does not imply that $\widetilde{\mathcal{R}}$ is irreducible.

Remark 1.3.3. Proposition 1.3.1 implies that the complexification of $\widetilde{\mathfrak{g}}$ is a complex graded algebra which satisfies $(\mathbf{H_1})$ and $(\mathbf{H_2})$. These complex algebras are studied in ([Ru]–1982) and in ([M-R-S]–1986).

1.4. Highest root in $\widetilde{\Sigma}$

Proposition 1.4.1. *There exists an unique root $\lambda^0 \in \widetilde{\Sigma}$ such that*

- $\lambda^0(H_0) = 2$;
- $\lambda \in \Sigma^+ \Longrightarrow \lambda^0 + \lambda \notin \widetilde{\Sigma}$.

Proof. — We will show that the restriction to $\mathfrak{a}^{\mathfrak{p}}$ of the highest weight ω of the representation of $\mathfrak{g}$ on $V_{\mathbb{C}}^+$ is the unique root which satisfies the above property. Let $\lambda^0 = \rho(\omega)$ be this restriction. Then λ^0 is a root in $\widetilde{\Sigma}$ such that $\lambda^0(H_0) = \omega(H_0) = 2$.

Let λ be a positive root in Σ. If $\lambda + \lambda^0$ is a root then there exist two roots α and β in $\widetilde{\mathcal{R}}$ such that

$$\rho(\alpha) = \lambda \ \text{ and } \rho(\beta) = \lambda + \lambda^0 \ .$$

The root space of β is a subspace of $V_{\mathbb{C}}^+$. Thus β is a weight of the irreducible representation of $\mathfrak{g}$ on $V_{\mathbb{C}}^+$ and can be written

$$\beta = \omega - \sum_{\gamma\in\widetilde{\Psi}} m_\gamma\gamma \ (m_\gamma \in \mathbb{Z}^+) \ ,$$

where $\widetilde{\Psi}$ is the the set of simple roots in $\widetilde{\mathcal{R}}^+$. If we take the restriction to $\mathfrak{a}^{\mathfrak{p}}$ of the members of this equality we obtain

$$\lambda = -\sum_{\gamma\in\widetilde{\Psi}} m_\gamma\rho(\gamma) \ .$$

Since λ and $\rho(\gamma)$ for $\gamma \in \widetilde{\Psi}$ are positive roots in $\widetilde{\Sigma}$, this is impossible and we have proved that $\lambda + \lambda^0$ is not a root.

It remains to prove that λ^0 is the unique root of $\widetilde{\Sigma}$ which satisfies the properties of the Proposition.

Let λ^1 be a root in $\widetilde{\Sigma}$ which satisfies these properties and

$$S = \{\alpha \in \widetilde{\mathcal{R}}^+ \mid \rho(\alpha) = \lambda^1\} \ .$$

An element of S is a weight for the representation of $\mathfrak{g}$ on $V_{\mathbb{C}}^+$. Let ω^1 be a maximal element of S for a (lexicographic) order induced by the basis of $\widetilde{\mathcal{R}}^+$.

For $\beta \in \widetilde{\mathcal{R}}^+$ such that $\rho(\beta) = 0$, $\omega^1 + \beta$ is not a root. If this would be the case, $\omega^1 + \beta$ would belong to S which contradicts the maximality of ω^1.

For $\beta \in \widetilde{\mathcal{R}}^+$ such that $\rho(\beta) \neq 0$, $\omega^1 + \beta$ is not a root. If this would be the case, then $\lambda^1 + \rho(\beta)$ would be a root in $\widetilde{\Sigma}^+$ which contradicts the assumption on λ^1 because $\rho(\beta)$ is positive.

Thus ω^1 is a highest weight of the representation of $\mathfrak{g}$ on $V_{\mathbb{C}}^+$ and, by irreducibility of this representation, is equal to ω which implies that

$$\lambda^1 = \rho(\omega^1) = \rho(\omega) = \lambda^0 \ .$$

□

If $\mathfrak{j}_{\mathbb{R}}$ is the real form of $\mathfrak{j}_{\mathbb{C}}$ defined by

$$\mathfrak{j}_{\mathbb{R}} = \mathfrak{a}^{\mathfrak{p}} \oplus i\{H \in \mathfrak{j} \mid \theta(H) = H\} \ ,$$

then the roots of $\widetilde{\mathcal{R}}$ take real values on $\mathfrak{j}_{\mathbb{R}}$. On the other hand the roots of $\widetilde{\Sigma}$ take real values on $\mathfrak{a}^{\mathfrak{p}}$. The form $\widetilde{B}$ is positive definite on $\mathfrak{a}^{\mathfrak{p}}$ and on $\mathfrak{j}_{\mathbb{R}}$ and induces a scalar product on the corresponding real dual spaces which will be denoted by

$$(\alpha \mid \beta) \text{ for } \alpha \text{ and } \beta \in (\mathfrak{a}^{\mathfrak{p}})^* \text{ or } \mathfrak{j}_{\mathbb{R}}^* \ .$$

There exists for each root $\alpha \in \widetilde{\Sigma}$ (*resp.* $\widetilde{\mathcal{R}}$) an unique element $H_\alpha \in \mathfrak{a}^{\mathfrak{p}}$ (*resp.* $\mathfrak{j}_{\mathbb{R}}$) such that

$$\beta(H_\alpha) = 2\,\frac{(\beta \mid \alpha)}{(\alpha \mid \alpha)} \text{ for each } \beta \in (\mathfrak{a}^{\mathfrak{p}})^* \quad (\textit{resp.}\ \beta \in \mathfrak{j}_{\mathbb{R}}{}^*) \ .$$

The element H_α is called the *co-root* of α.

We will apply the following classical result to the root systems $\widetilde{\Sigma}$ and $\widetilde{\mathcal{R}}$.

Lemma 1.4.2 ([Bou]–(1968) Ch.VI §1 Prop. 9)**.**
Let α and β be two roots in a root system (reduced or not reduced) such that α is not proportional to β. Then the set $\{j \in \mathbb{Z} \mid \beta + j\alpha$ is a root $\}$ is of the form

$$(-q, -q+1, \ldots, p) \textit{ with } p \geq 0,\ q \geq 0 \textit{ and } p - q = -2\,\frac{(\beta \mid \alpha)}{(\alpha \mid \alpha)} = -\beta(H_\alpha) \ .$$

The set of roots of the form $\beta + j\alpha$ with $j \in \mathbb{Z}$ is called the *α-series of roots containing β*. The roots α and β are *orthogonal* for $(. \mid .)$ if and only if $p = q$ i.e. the α-series of roots containing β is symmetric with respect to

β. The roots α and β are *strongly orthogonal* if neither $\alpha+\beta$ nor $\alpha-\beta$ is a root i.e. $p=q=0$.

We will write

$\alpha \perp \beta$ if α and β are orthogonal roots ;

$\alpha \perp\!\!\!\perp \beta$ if α and β are strongly orthogonal roots .

1.5. The first step for the descent

The aim of this section is to construct a graded subalgebra of $\widetilde{\mathfrak{g}}$ wich satisfies the same properties.

Let $\widetilde{\mathfrak{l}}_0$ the Lie subalgebra of $\widetilde{\mathfrak{g}}$ generated by the root spaces $\widetilde{\mathfrak{g}}^{\lambda_0}$ and $\widetilde{\mathfrak{g}}^{-\lambda_0}$. This is a θ-stable semisimple real Lie algebra of split rank 1. It's Cartan subspace is generated by H_{λ_0}, the co-root of λ_0 in $\mathfrak{a}^{\mathfrak{p}}$.

Proposition 1.5.1. *Let $\widetilde{\mathfrak{g}}_1$ be the reductive subalgebra of $\widetilde{\mathfrak{g}}$ which is the centralizer of $\widetilde{\mathfrak{l}}_0$ in $\widetilde{\mathfrak{g}}$. If the intersection of $\widetilde{\mathfrak{g}}_1$ with V^+ is nonzero, then $\widetilde{\mathfrak{g}}_1$ satisfies the hypothesis $(\mathbf{H_1})$ and $(\mathbf{H_2})$ where the gradation is defined by the element*

$$H_1 = H_0 - H_{\lambda_0} \ .$$

Proof. —Since H_{λ_0} belongs to $\widetilde{\mathfrak{l}}_0$ the action of $\operatorname{ad} H_0$ and $\operatorname{ad} H_1$ on $\widetilde{\mathfrak{g}}_1$ are the same. Moreover $\widetilde{\mathfrak{l}}_0$ and $\widetilde{\mathfrak{g}}_1$ are stable under the action of $\operatorname{ad} H_0$ and of $\operatorname{ad} H_1$. Thus the eigenvalues of $\operatorname{ad} H_1$ acting on $\widetilde{\mathfrak{g}}_1$ are $-2, 0$ and 2 because $V^+ \bigcap \widetilde{\mathfrak{g}}_1 \neq \{0\}$. It follows that the hypothesis $(\mathbf{H_1})$ is satisfied and it remains to prove that the representation of $\mathfrak{g}_1 = \mathfrak{g} \bigcap \widetilde{\mathfrak{g}}_1$ on $V^+_{1,\mathbb{C}} = V^+_{\mathbb{C}} \bigcap \widetilde{\mathfrak{g}}_{1,\mathbb{C}}$ is irreducible.

We will complexify the situation as in section 1.3. Let

$$S_0 = \{\alpha \in \widetilde{\mathcal{R}}^+ \mid \rho(\alpha) = \lambda_0\} \ .$$

The complexification of $\widetilde{\mathfrak{l}}_0$ is the complex Lie algebra generated by the root spaces $\widetilde{\mathfrak{g}}^{\alpha}$ and $\widetilde{\mathfrak{g}}^{-\alpha}$ for all $\alpha \in S_0$. Therefore the complexification of $\widetilde{\mathfrak{g}}_1$ has the following decomposition

$$\widetilde{\mathfrak{g}}_{1,\mathbb{C}} = \mathfrak{j}_{1,\mathbb{C}} \oplus \big(\oplus_{\beta \in \widetilde{\mathcal{R}}_1} \widetilde{\mathfrak{g}}^{\beta}_{\mathbb{C}}\big) \text{ where}$$

$$\mathfrak{j}_{1,\mathbb{C}} = \{H \in \mathfrak{j}_{\mathbb{C}} \mid \alpha \in S_0 \Longrightarrow \alpha(H) = 0\} \ ,$$

$$\widetilde{\mathcal{R}}_1 = \{\beta \in \widetilde{\mathcal{R}} \mid \forall \alpha \in S_0, \ \beta \perp\!\!\!\perp \alpha\} \ .$$

If β belongs to $\widetilde{\mathcal{R}}_1$ then $\alpha(H_\beta) = 0$ for each root $\alpha \in S_0$. It follows that H_β belongs to $\mathfrak{j}_{1,\mathbb{C}}$ and that the restriction of β to $\mathfrak{j}_{1,\mathbb{C}}$ is non zero. Thus $\mathfrak{j}_{1,\mathbb{C}}$ is a Cartan subalgebra of $\widetilde{\mathfrak{g}}_{1,\mathbb{C}}$ and $\widetilde{\mathcal{R}}_1$ can be identified with the root system of the pair $(\widetilde{\mathfrak{g}}_{1,\mathbb{C}}, \mathfrak{j}_{1,\mathbb{C}})$. The order on $\widetilde{\mathcal{R}}$ defined by $\widetilde{\mathcal{R}}^+$ induces an order on $\widetilde{\mathcal{R}}_1$ and on the root system $\mathcal{R}_1$ of the pair $(\mathfrak{g}_{1,\mathbb{C}}, \mathfrak{j}_{1,\mathbb{C}})$ by setting

$$\widetilde{\mathcal{R}}_1^+ = \widetilde{\mathcal{R}}_1 \cap \widetilde{\mathcal{R}}^+ \quad ; \quad \mathcal{R}_1^+ = \mathcal{R}_1 \cap \widetilde{\mathcal{R}}^+ = \widetilde{\mathcal{R}}_1 \cap \mathcal{R}^+ \ .$$

Let now ω_1 be the highest weight (for this order) of one of the irreducible components of the representation of $\mathfrak{g}_{1,\mathbb{C}}$ on $V^+_{1,\mathbb{C}}$. It is a root of $\widetilde{\mathcal{R}}_1^+$ and we

will show that it is also a highest weight for the representation of $\mathfrak{g}$ on $V_{\mathbb{C}}^{+}$. For that purpose we take a root $\beta \in \mathcal{R}^{+}$ and we will prove that $\omega_1 + \beta$ is not a root of $\widetilde{\mathcal{R}}$.

(1) If β belongs to $\mathcal{R}_1^{+}$, then the definition of ω_1 implies that $\omega_1 + \beta$ does not belong to $\widetilde{\mathcal{R}}_1$. But $\widetilde{\mathfrak{g}}_{\mathbb{C}}^{\beta+\omega_1} = [\widetilde{\mathfrak{g}}_{\mathbb{C}}^{\omega_1}, \widetilde{\mathfrak{g}}_{\mathbb{C}}^{\beta}]$ is a subspace of $V_{1,\mathbb{C}}^{+}$. Therefore $\widetilde{\mathfrak{g}}_{\mathbb{C}}^{\beta+\omega_1}$ is equal to zero, which implies that $\omega_1 + \beta$ is not a root in $\widetilde{\mathcal{R}}$.

(2) If β does not belong to $\mathcal{R}_1^{+}$, there is a root $\alpha \in S_0$ such that α and β are not strongly orthogonal. Let us prove that in this case there is a root $\gamma \in S_0$ such that

$$(*) \quad \gamma + \beta \in \widetilde{\mathcal{R}} \text{ and } \gamma - \beta \notin \widetilde{\mathcal{R}} \ .$$

(2.1) If $\rho(\beta) \neq 0$, then $\alpha - \beta$ is not a root. If it would be the case, we would have

$$\rho(\alpha - \beta) = \rho(\alpha) - \rho(\beta) = \lambda_0 - \rho(\beta) \ ,$$

which contradicts the fact that $\rho(\beta)$ is a positive root different from λ_0 (because $\rho(\beta) \in \Sigma^{+}$) and λ_0 is a simple root in $\widetilde{\Sigma}^{+}$. Thus the root $\gamma = \alpha$ satisfies $(*)$.

(2.2) If $\rho(\beta) = 0$, we consider the β-series of roots containing α. These roots belong to S_0 and the series contains at least two roots due to the fact that α and β are not strongly orthogonal roots. Therefore there exists a root γ in this series such that $(*)$ is satisfied.

In the two cases (2.1) and (2.2) we have $(\beta \mid \gamma) < 0$ (Lemma 1.4.2). Moreover $(\omega_1 \mid \gamma) = 0$ since these roots are strongly orthogonal ($\omega_1 \in \widetilde{\mathcal{R}}_1$ and $\gamma \in S_0$). Now if $\omega_1 + \beta$ is a root, we have

$$(\omega_1 + \beta \mid \gamma) = (\omega_1 \mid \gamma) + (\beta \mid \gamma) = (\beta \mid \gamma) < 0 \ .$$

This implies (Lemma 1.4.2) that $\omega_1 + \beta + \gamma$ is a root. This is impossible because of $(\omega_1 + \beta + \gamma)(H_0) = 4$ due to $\omega_1(H_0) = 2$, $\beta(H_0) = 0$ and $\gamma(H_0) = \lambda_0(H_0) = 2$.

We have now proved, in the two cases (1) and (2), that ω_1 is a highest weight for the representation of $\mathfrak{g}$ on $V_{\mathbb{C}}^{+}$ which is irreducible by $(\mathbf{H_2})$. Thus ω_1 is uniquely determined and the representation of $\mathfrak{g}_1$ on $V_{1,\mathbb{C}}^{+}$ is irreducible. □

Since $\widetilde{\mathfrak{l}}_0$ is θ–stable, θ stabilizes $\widetilde{\mathfrak{g}}_1$ and a Cartan subspace of the pair $(\widetilde{\mathfrak{g}}_1, \theta)$ is given by the space orthogonal to H_{λ_0} in $\mathfrak{a}^{\mathfrak{p}}$ i.e.

$$\mathfrak{a}_1^{\mathfrak{p}} = \mathfrak{a}^{\mathfrak{p}} \bigcap \mathfrak{g}_1 = \{H \in \mathfrak{a}^{\mathfrak{p}} \mid \lambda_0(H) = 0\} \ .$$

We associate to $\widetilde{\mathfrak{g}}_1$ the same objects as to $\widetilde{\mathfrak{g}}$ and we will denote them by an index 1.

Proposition 1.5.2.
(1) *The root system* $\widetilde{\Sigma}_1$ *of the pair* $(\widetilde{\mathfrak{g}}_1, \mathfrak{a}_1^{\mathfrak{p}})$ *can be identified with*

$$\widetilde{\Sigma}_1 = \{\lambda \in \widetilde{\Sigma} \mid \lambda \perp\!\!\!\perp \lambda_0\} = \{\lambda \in \widetilde{\Sigma} \mid \lambda \perp \lambda_0\} \ .$$

(2) *The order given by Theorem* 1.2.1(1) *on* $\widetilde{\Sigma}_1$ *is such that*

$$\widetilde{\Sigma}_1^+ = \widetilde{\Sigma}^+ \bigcap \widetilde{\Sigma}_1 \ .$$

(3) *The simple system in* $\widetilde{\Sigma}_1^+$ *defined by this order is given by*

$$\widetilde{\Pi}_1 = \left(\Pi \cap \widetilde{\Sigma}_1\right) \cup \{\lambda_1\} \ ,$$

where λ_1 *is the unique root in* $\widetilde{\Pi}_1$ *such that* $\lambda_1(H_0) = 2$.

Proof. —

(1) We will first prove that a root $\lambda \in \widetilde{\Sigma}$ is strongly orthogonal to λ_0 if and only if it is orthogonal to λ_0 . For that let $\lambda \in \widetilde{\Sigma}$ be a root orthogonal to λ_0. Then the λ_0–series of roots containing λ is symmetric with respect to λ. This implies that

$$\lambda - \lambda_0 \in \widetilde{\Sigma} \Longleftrightarrow \lambda + \lambda_0 \in \widetilde{\Sigma} \ .$$

a) If $\lambda(H_0) = 2$ then $(\lambda + \lambda_0)(H_0) = 4$ therefore $\lambda + \lambda_0$ is not a root.

b) If $\lambda(H_0) = -2$ then $(\lambda - \lambda_0)(H_0) = -4$ therefore $\lambda - \lambda_0$ is not a root.

c) If $\lambda(H_0) = 0$ then λ belongs to Σ and is a sum of elements of the simple system Π (or the opposite). But λ_0 is an element of the simple system $\widetilde{\Pi}$ of $\widetilde{\Sigma}$ such that $\lambda_0 \notin \Pi$. Thus

$$\lambda \in \Sigma^+ \Longrightarrow \lambda - \lambda_0 \notin \widetilde{\Sigma} \ ;$$

$$\lambda \in \Sigma^- \Longrightarrow \lambda + \lambda_0 \notin \widetilde{\Sigma} \ .$$

We have proved that the roots orthogonal to λ_0 i.e. the roots λ such that $\lambda(H_{\lambda_0}) = 0$ are the roots which are strongly orthogonal to λ_0.

For such a root the associated root space belongs to the centralizer of $\widetilde{\mathfrak{l}}_0$, the co-root belongs to $\mathfrak{a}_1^{\mathfrak{p}}$ and the restriction to $\mathfrak{a}_1^{\mathfrak{p}}$ is a root of the pair $(\widetilde{\mathfrak{g}}_1, \mathfrak{a}_1^{\mathfrak{p}})$. Conversely a root λ of the pair $(\widetilde{\mathfrak{g}}_1, \mathfrak{a}_1^{\mathfrak{p}})$ can be extended to $\mathfrak{a}^{\mathfrak{p}}$ by taking $\lambda(H_{\lambda_0}) = 0$. This gives rise to a root of $\widetilde{\Sigma}$ orthogonal to λ_0, hence strongly orthogonal to λ_0 and thus belonging to $\widetilde{\Sigma}_1$.

(2) The set $\widetilde{\Sigma}_1^+ = \widetilde{\Sigma}^+ \cap \Sigma_1$ defines an order on $\widetilde{\Sigma}_1$. For $\lambda \in \widetilde{\Sigma}_1^+$ we have

$$\lambda(H_1) = \lambda(H_0) = 2 \text{ or } 0 \ .$$

Therefore property (1) of Theorem 1.2.1 is satisfied for this order.

(3) Let $\widetilde{\Pi}_1$ be the set of simple roots in $\widetilde{\Sigma}_1^+$. By Theorem 1.2.1 (2) we know that there exists an unique root $\lambda_1 \in \widetilde{\Pi}_1$ such that $\lambda_1(H_1) = 2$ and that $\Pi_1 = \widetilde{\Pi}_1 \backslash \{\lambda_1\}$ is a simple system in $\Sigma_1 = \{\lambda \in \widetilde{\Sigma}_1 \mid \lambda(H_1) = 0\}$. Since a root $\lambda \in \widetilde{\Sigma}_1$ verifies $\lambda(H_1) = \lambda(H_0)$, Σ_1 is also the set of roots $\lambda \in \widetilde{\Sigma}_1$ such that $\lambda(H_0) = 0$. Thus $\Pi \bigcap \widetilde{\Sigma}_1$ is a subset of Π_1. Let us prove the equality of these two sets.

Let μ be simple root in Σ_1^+. Since Π is the set of simple roots in $\Sigma^+ \supset \Sigma_1^+$ we have

$$\mu = \sum_{\nu \in \Pi} m_\nu \nu \quad (m_\nu \in \mathbb{Z}^+) \ .$$

Since μ belongs to Σ_1 we have $(\mu \mid \lambda_0) = 0$. For $\nu \in \Pi$ we have $(\nu \mid \lambda_0) \leq 0$ since ν and λ_0 are simple roots in $\widetilde{\Sigma}$. Thus

$$m_\nu \neq 0 \Longrightarrow (\nu \mid \lambda_0) = 0 \Longrightarrow \nu \in \widetilde{\Sigma}_1 \ .$$

Therefore μ is equal to one of the roots $\nu \in \Pi$. This proves that $\Pi_1 \subset \Pi$.

□

Remark 1.5.3. There is only one root $\lambda \in \widetilde{\Pi}$ such that $\lambda(H_0) = 2$, namely λ_0. Therefore the root λ_1 does not belong to the set $\widetilde{\Pi}$ of simple roots in $\widetilde{\Sigma}^+$.

1.6. The descent

Theorem 1.6.1. *There exists a sequence of strongly orthogonal roots in* $\widetilde{\Sigma}^+ \backslash \Sigma$ *denoted by* $\{\lambda_0, \lambda_1, \ldots, \lambda_k\}$ *and a sequence of reductive algebras* $\widetilde{\mathfrak{g}} \supset \widetilde{\mathfrak{g}}_1 \supset \cdots \supset \widetilde{\mathfrak{g}}_k$ *such that*

(1) $\widetilde{\mathfrak{g}}_j$ *is the centralizer in* $\widetilde{\mathfrak{g}}$ *of* $\widetilde{\mathfrak{l}}_0 \oplus \widetilde{\mathfrak{l}}_1 \oplus \cdots \oplus \widetilde{\mathfrak{l}}_{j-1}$ *where* $\widetilde{\mathfrak{l}}_i$ *is the subalgebra generated by* $\widetilde{\mathfrak{g}}^{\lambda_i}$ *and* $\widetilde{\mathfrak{g}}^{-\lambda_i}$.

(2) $\widetilde{\mathfrak{g}}_j$ *satisfies the hypothesis* $(\mathbf{H_1})$ *and* $(\mathbf{H_2})$ *where the gradation is defined by the element* $H_0 - H_{\lambda_0} - H_{\lambda_1} - \cdots - H_{\lambda_{j-1}}$.

(3) $V^+ \cap \mathfrak{z}_{\widetilde{\mathfrak{g}}}(\widetilde{\mathfrak{l}}_0 \oplus \widetilde{\mathfrak{l}}_1 \oplus \cdots \oplus \widetilde{\mathfrak{l}}_k) = \{0\}$.

Proof. — The proof of (1) , (2) and (3) is done by induction on j where $\widetilde{\mathfrak{g}}_j$ is the centralizer in $\widetilde{\mathfrak{g}}_{j-1}$ of $\widetilde{\mathfrak{l}}_{j-1}$. Then Proposition 1.5.1 applied to the algebra $\widetilde{\mathfrak{g}}_j$ gives the result . The construction ends for the index k such that

$$\mathfrak{z}_{\widetilde{\mathfrak{g}}_k}(\widetilde{\mathfrak{l}}_k) \bigcap V^+ = \{0\} \ .$$

This means that the centralizer in V^+ (and in V^-) of $\widetilde{\mathfrak{l}}_0 \oplus \widetilde{\mathfrak{l}}_1 \oplus \cdots \oplus \widetilde{\mathfrak{l}}_k$ is zero.

□

Definition 1.6.2. *The number of strongly orthogonal roots will be called the rank of the graded algebra. It is denoted by* $k+1$.

Since $\widetilde{\mathfrak{g}}_j$ satisfies the hypothesis $(\mathbf{H_1})$ and $(\mathbf{H_2})$ we can associate to $\widetilde{\mathfrak{g}}_j$ the same objects as to $\widetilde{\mathfrak{g}}$ and we will denote them by an index j. For example V_j^+ is the set of elements in $\widetilde{\mathfrak{g}}_j$ of weight 2 under the action of $\mathrm{ad}(H_0 - H_{\lambda_0} - \cdots - H_{\lambda_{j-1}})$, $\widetilde{\Sigma}_j$ is the set of roots in $\widetilde{\Sigma}$ which are strongly orthogonal to $\lambda_0, \ldots, \lambda_{j-1}$ (see Proposition 1.5.2).

1.7. Generic elements in V^+

We introduce now some Lie groups $\widetilde{G}$ and G associated to the Lie algebras $\widetilde{\mathfrak{g}}$ and $\mathfrak{g}$. We make once for all the following choices :

• $\widetilde{G}$ is the adjoint group of $\widetilde{\mathfrak{g}}$ *i.e.* the subgroup of $GL(\widetilde{\mathfrak{g}})$ generated by $\{e^{\operatorname{ad} X} \mid X \in \widetilde{\mathfrak{g}}\}$; the group $\widetilde{G}$ is semisimple with trivial center; the Lie algebra of $\widetilde{G}$ is $\operatorname{ad}\widetilde{\mathfrak{g}}$.

• G is the analytic subgroup of $\widetilde{G}$ whith Lie algebra $\operatorname{ad}_{\widetilde{\mathfrak{g}}}\mathfrak{g}$ *i.e.* the subgroup of $\widetilde{G}$ generated by $\{e^{\operatorname{ad} X} \mid X \in \mathfrak{g}\}$ or the identity component of the centralizer of H_0 in $\widetilde{G}$; the group G is connected reductive and the Lie algebra of the center of G is isomorphic to $\mathbb{R}H_0$.

From now on all the Lie groups under consideration will be subgroups of $\widetilde{G}$. If X is an element of $\widetilde{\mathfrak{g}}$, the two notations $\exp X$ or $e^{\operatorname{ad} X}$ will be used to denote the corresponding element of $\widetilde{G}$. In general $e^{\operatorname{ad} X}$ will be used instead of $\operatorname{Ad}(\exp X)$ to denote the adjoint action of the group $\widetilde{G}$ on $\widetilde{\mathfrak{g}}$.

Definition 1.7.1. *An element $X \in V^+$ is generic in V^+ if it satisfies one of the two equivalent conditions :*

(i) The G-orbit of X is open in V^+;

(ii) $\operatorname{ad}X$ *maps $\mathfrak{g}$ onto V^+ .*

Using the analogy between $\widetilde{\mathfrak{g}}$ and $\widetilde{\mathfrak{g}}_j$, we will say that an element X in V_j^+ is generic in V_j^+ if $\operatorname{ad} X$ maps $\mathfrak{g}_j$ (the centralizer of $H_0 - H_{\lambda_0} - \cdots - H_{\lambda_{j-1}}$ in $\widetilde{\mathfrak{g}}_j$) onto V_j^+.

Lemma 1.7.2. *Let $X \in V^+$ and $Y \in V^-$ such that $[Y, X] = H_0$. Then X is generic in V^+.*

Proof. — For $v \in V^+$ we have

$$2v = \operatorname{ad} H_0.v = -\operatorname{ad}[X, Y].v = -\operatorname{ad} X \operatorname{ad} Y.v \ .$$

Thus $\operatorname{ad} X$ is a surjective map from $\mathfrak{g}$ onto V^+ which proves the Lemma. □

The existence of such an element X is exactly the Hypothesis (**H3**) (see the definition 1.7.8 below) which is not satisfied in general. However we will construct generic elements in V^+ which, in general, do not satisfy the Lemma 1.7.2.

Definition 1.7.3. *If Y, H and X are elements of $\widetilde{\mathfrak{g}}$, then $\{Y, H, X\}$ is called a $\mathfrak{sl}_2$-triple if the following relations are satisfied*

$$[H, X] = 2X \ ; \ [H, Y] = -2Y \ ; \ [Y, X] = H \ .$$

Then the Lie subalgebra of $\widetilde{\mathfrak{g}}$ generated by these elements is isomorphic to $\mathfrak{sl}(2, \mathbb{R})$.

The following (classical) Lemma gives a family of $\mathfrak{sl}_2$-triples.

Lemma 1.7.4. *Let X be a nonzero element of a root space $\widetilde{\mathfrak{g}}^\lambda$ (with $\lambda \in \widetilde{\Sigma}$).*

(1) *There exists a constant $c > 0$ such that $\{\theta(cX), H_\lambda, cX\}$ is a $\mathfrak{sl}_2$-triple.*

(2) *There exists an element $Y \in \widetilde{\mathfrak{g}}^{-\lambda}$ such that $\{Y, H_\lambda, X\}$ is a $\mathfrak{sl}_2$-triple.*

Proof. — The element $H = [\theta(X), X]$ satisfies $\theta(H) = -H$ and belongs to $[\widetilde{\mathfrak{g}}^{-\lambda}, \widetilde{\mathfrak{g}}^{\lambda}]$ which is a subset of the centralizer of $\mathfrak{a}^{\mathfrak{p}}$. Hence H is an element of $\mathfrak{a}^{\mathfrak{p}}$. Moreover we have

$$\widetilde{B}([\theta(X), X], Z) = -\lambda(Z)\, \widetilde{B}(\theta(X), X) \text{ for } Z \in \mathfrak{a}^{\mathfrak{p}} .$$

Since $(Z, Z') \mapsto -\widetilde{B}(\theta(Z), Z')$ is positive definite on $\mathfrak{a}^{\mathfrak{p}}$ we obtain (using the definition of the co-root H_λ)

$$[\theta(X), X] = CH_\lambda \text{ where } C = 2\frac{-\widetilde{B}(\theta(X), X)}{\widetilde{B}(H_\lambda, H_\lambda)} > 0 .$$

We obtain (1) for $c = 1/\sqrt{C}$ and (2) for $Y = \frac{1}{C}\theta(X)$.

□

We choose now for each $j \in \{0, 1, \ldots, k\}$ an element $X_j \in \widetilde{\mathfrak{g}}^{\lambda_j} \setminus \{0\}$ and we denote by Y_j an element in $\widetilde{\mathfrak{g}}^{-\lambda_j}$ such that $\{Y_j, H_{\lambda_j}, X_j\}$ is a $\mathfrak{sl}_2$-triple.

Lemma 1.7.5. *The element X_k is generic in V_k^+.*

Proof. — Lemma 1.7.2 does not give the result since $\widetilde{\mathfrak{g}}_k$ is graded by $H_0 - H_{\lambda_0} - \cdots - H_{\lambda_{k-1}}$ but not (in general) by H_{λ_k}. Nevertheless X_k belongs to V_k^+ and we will show that $\operatorname{ad} X_k$ maps $\mathfrak{g}_k$ onto V_k^+.

Let λ be a root occuring in V_k^+ and Z an element of $\widetilde{\mathfrak{g}}^{\lambda}$. Then we have

$$\operatorname{ad} X_k \operatorname{ad} Y_k.Z = -\operatorname{ad} H_{\lambda_k}.Z = -\lambda(H_{\lambda_k})Z .$$

Since X_k belongs to V_k^+, the element $\operatorname{ad} Y_k.Z$ belongs to $\mathfrak{g}_k$. Thus it remains to prove that $\lambda(H_{\lambda_k}) \neq 0$ for each root λ occuring in V_k^+.

If $\lambda = \lambda_k$ then $\lambda_k(H_{\lambda_k}) = 2$. If λ is different from λ_k and $\lambda(H_{\lambda_k}) = 0$ then λ is strongly orthogonal to λ_k since $\lambda + \lambda_k$ is not a root. This contradicts the definition of V_k^+.

□

Lemma 1.7.6. *Let $j \in \{1, 2, \ldots, k\}$. If X is a generic element in V_j^+ then $X_0 + X_1 + \cdots + X_{j-1} + X$ is generic in V^+.*

Proof. — We will prove this result for $j = 1$ and the Lemma will follow by induction on j.

Let X be a generic element in V_1^+ and Z an element of one of the root spaces $\widetilde{\mathfrak{g}}^{\lambda}$ occuring in V^+. We will show that $Z \in \operatorname{ad}(X_0 + X)(\mathfrak{g})$ which will prove that $X_0 + X$ is a generic element in V^+.

If $\lambda = \lambda_0$ then $U = \operatorname{ad} Y_0.Z$ is an element of $\mathfrak{g} \bigcap \widetilde{\mathfrak{l}}_0$ and commutes with $X \in V_1^+$. Hence $\operatorname{ad}(X_0 + X).U = \operatorname{ad} X_0.U = -2Z$ and the result is proved.

If λ is strongly orthogonal to λ_0, then Z belongs to V_1^+. But, by hypothesis X is generic in V_1^+. It follows that there is an element U in $\mathfrak{g}_1$ such that $\operatorname{ad} X.U = Z$. Since U and X_0 commute, we have also $\operatorname{ad}(X_0 + X).U = Z$.

If λ is not strongly orthogonal to λ_0 and is different from λ_0 then $\mu = \lambda - \lambda_0$ is a root belonging to Σ^+ and $U = \operatorname{ad} Y_0.Z$ belongs to $\mathfrak{g}^{\mu}$. If $[\mathfrak{g}^{\mu}, V_1^+]$ is nonzero, there exists a root ν such that $\widetilde{\mathfrak{g}}^{\nu} \subset V_1^+$ and $\mu + \nu$ is a root. But $(\mu + \nu \mid \lambda_0) = (\mu \mid \lambda_0)$ is strictly negative due to the fact that $\mu + \lambda_0$ is a root and $\mu - \lambda_0$ is not a root. Hence $\mu + \nu + \lambda_0$ is a root which takes

the value 4 on H_0. This is impossible, hence $\mathfrak{g}^\mu$ and V_1^+ commute. Thus we obtain the equality $\mathrm{ad}(X_0+X).U = \mathrm{ad}\, X_0.U = -2Z$ which gives the result. □

The two previous Lemmas imply

Proposition 1.7.7. *The space V^+ contains generic elements. For example an element of the form $X_0 + \cdots + X_k$, where $X_j \in \widetilde{\mathfrak{g}}^{\lambda_j}\backslash\{0\}$ for $j = 0, \ldots, k$ is generic in V^+.*

Consequently there exist open G-orbits in V^+.

We define now the complexifications $\widetilde{G}_{\mathbb{C}}$ and $G_{\mathbb{C}}$ of the groups $\widetilde{G}$ and G as follows :

• $\widetilde{G}_{\mathbb{C}}$ is the adjoint group of $\widetilde{\mathfrak{g}}_{\mathbb{C}}$ *i.e.* the subgroup of $GL(\widetilde{\mathfrak{g}}_{\mathbb{C}})$ generated by $\{e^{\mathrm{ad}\, X} \mid X \in \widetilde{\mathfrak{g}}_{\mathbb{C}}\}$. Since its Lie algebra $\mathrm{ad}\,\widetilde{\mathfrak{g}}_{\mathbb{C}}$ is semisimple, the group $\widetilde{G}_{\mathbb{C}}$ is an algebraic semisimple group ([O-V] p. 138).

• $G_{\mathbb{C}}$ is the analytic subgroup of $\widetilde{G}_{\mathbb{C}}$ whith Lie algebra $\mathrm{ad}_{\widetilde{\mathfrak{g}}_{\mathbb{C}}}\mathfrak{g}_{\mathbb{C}}$. As centralizers of tori are connected ([Hu]-1981 §22.3) the group $G_{\mathbb{C}}$ is also the centralizer of H_0 in $\widetilde{G}_{\mathbb{C}}$.

If X is a generic element in V^+ then the $G_{\mathbb{C}}$-orbit of X is open in $V_{\mathbb{C}}^+$ and, since $G_{\mathbb{C}}$ is algebraic and the adjoint action of $G_{\mathbb{C}}$ rational, this orbit is a Zariski open subset of V^+. Therefore the pair $(G_{\mathbb{C}}, V_{\mathbb{C}}^+)$ is a *prehomogeneous vector space* in the sense of M. Sato ([Sa]–1990), which is *irreducible* by hypothesis $(\mathbf{H_2})$.

The prehomogeneous spaces defined this way are called *prehomogeneous vector spaces of commutative parabolic type* ([Ru]–1982) due to the fact that $\mathfrak{g}_{\mathbb{C}} \oplus V_{\mathbb{C}}^+$ is a parabolic subalgebra of $\widetilde{\mathfrak{g}}_{\mathbb{C}}$ with a commutative nil-radical $V_{\mathbb{C}}^+$.

The pair (G, V^+) is a real form of the prehomogeneous vector space $(G_{\mathbb{C}}, V_{\mathbb{C}}^+)$. The infinitesimal classification of all real forms of $(G_{\mathbb{C}}, V_{\mathbb{C}}^+)$ is known from ([Ru]–1986).

We will now introduce a last hypothesis on the graded Lie algebra considered here which restricts our study to the class of regular prehomogeneous spaces of commutative parabolic type.

Definition 1.7.8. *A graded reductive Lie algebra $\widetilde{\mathfrak{g}}$ which satisfies $(\mathbf{H_1})$ and $(\mathbf{H_2})$ is called* **regular** *if it satisfies the following hypothesis :*

$(\mathbf{H_3})$ *There exists an element $I^+ \in V^+$ and an element $I^- \in V^-$ such that*

$$\{I^-, H_0, I^+\} \text{ is a } \mathfrak{sl}_2\text{–triple} .$$

We know from lemma 1.7.2 that such an element I^+ is generic in V^+ and we shall prove that all generic elements in V^+ are of this form.

Proposition 1.7.9. *In a regular graded algebra an element $X \in V^+$ is generic in V^+ if and only if there exists an element $Y \in V^-$ such that $\{Y, H_0, X\}$ is a $\mathfrak{sl}_2$-triple. Moreover if such an Y exists, it is unique.*

Proof. — If $\{Y, H_0, X\}$ is a $\mathfrak{sl}_2$-triple then, by lemma 1.7.2, X is generic in V^+.

Conversely, if X is generic in V^+, the G-orbit of X is open in V^+. For the same reason the $G_{\mathbb{C}}$-orbit of X is an open subset of $V^+_{\mathbb{C}}$. But the complex group $G_{\mathbb{C}}$ is algebraic, the action of $G_{\mathbb{C}}$ on $V^+_{\mathbb{C}}$ is rational. Thus the $G_{\mathbb{C}}$–orbit of X which is a Zariski open orbit is unique and is equal to the $G_{\mathbb{C}}$–orbit of I^+. Then there exists an element g in $G_{\mathbb{C}}$ such that $X = g.I^+$. It follows that $\{g.I^-, g.H_0, g.I^+\}$ is a $\mathfrak{sl}_2$-triple equal to $\{g.I^-, H_0, X\}$. A classical lemma ([Bou]–1975 Ch.VII §11.1 lemme 1) asserts that $g.I^- = Y$. The same argument implies the unicity of Y in the Proposition.

□

From now on we will assume that the graded algebra $(\widetilde{\mathfrak{g}}, H_0)$ is regular.

1.8. Structure of the regular graded algebra $(\widetilde{\mathfrak{g}}, H_0)$

Let $(\lambda_0, \lambda_1, \ldots, \lambda_k)$ be the sequence of strongly orthogonal roots given by Theorem 1.6.1 and let $\{H_{\lambda_0}, H_{\lambda_1}, \ldots, H_{\lambda_k}\}$ be the associated co-roots. We define the subspace $\mathfrak{a}^0$ of $\mathfrak{a}^{\mathfrak{p}}$ by

$$\mathfrak{a}^0 = \oplus_{j=0}^k \mathbb{R} H_{\lambda_j} \subset \mathfrak{a}^{\mathfrak{p}} \ .$$

For $i, j \in \{0, 1, \ldots, k\}$ and for $p, q \in \mathbb{Z}$ we define the subspaces $E_{i,j}(p,q)$ of $\widetilde{\mathfrak{g}}$ by

$$E_{i,j}(p,q) = \left\{ X \in \widetilde{\mathfrak{g}} \;\middle|\; [H_{\lambda_\ell}, X] = \begin{cases} pX & \text{if } \ell = i \ ; \\ qX & \text{if } \ell = j \ ; \\ 0 & \text{if } \ell \notin \{i,j\} \ . \end{cases} \right\}$$

Theorem 1.8.1. *If a graded algebra is regular i.e. satisfies* $(\mathbf{H_3})$ *then*

$$H_0 = H_{\lambda_0} + H_{\lambda_1} + \cdots + H_{\lambda_k} \ .$$

We have the following decompositions :

$$(1) \qquad \mathfrak{g} = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}^0) \oplus \left(\oplus_{i \neq j} E_{i,j}(1,-1)\right) \ ;$$

$$(2) \qquad V^+ = \left(\oplus_{j=0}^k \widetilde{\mathfrak{g}}^{\lambda_j}\right) \oplus \left(\oplus_{i<j} E_{i,j}(1,1)\right) \ ;$$

$$(3) \qquad V^- = \left(\oplus_{j=0}^k \widetilde{\mathfrak{g}}^{-\lambda_j}\right) \oplus \left(\oplus_{i<j} E_{i,j}(-1,-1)\right) \ .$$

Proof. —For $j \in \{0, 1, \ldots, k\}$ let X_j be an element of $\widetilde{\mathfrak{g}}^{\lambda_j}$ such that $\{\theta(X_j), H_{\lambda_j}, X_j\}$ is a $\mathfrak{sl}_2$–triple. By Proposition 1.7.7 the element $X = X_0 + X_1 + \cdots + X_k$ is generic in V^+ and hypothesis $(\mathbf{H_3})$ implies, by Proposition1.7.9, that there exists $Y \in V^-$ such that $\{Y, H_0, X\}$ is a $\mathfrak{sl}_2$-triple. Thus $(\operatorname{ad} X)^2$ is one to one on V^- which is the subspace of weight (-2) for the representation of the subalgebra generated by Y, H_0 and X. But $Y' = \theta(X_0) + \theta(X_1) + \cdots + \theta(X_k)$ is an element of V^- such that $(\operatorname{ad} X)^2 Y' = 2X = (\operatorname{ad} X)^2 Y$. It follows that $Y = Y'$ and we obtain

$$H_{\lambda_0} + H_{\lambda_1} + \cdots + H_{\lambda_k} = [Y', X] = [Y, X] = H_0 \ .$$

Let X be an element of a root space in $\widetilde{\mathfrak{g}}$. There exist integers p_j such that

$$[H_{\lambda_j}, X] = p_j X \text{ for } j = 0, 1, \ldots, k \ .$$

The relation $H_0 = H_{\lambda_0} + H_{\lambda_1} + \cdots + H_{\lambda_k}$ implies that

$$p_0 + p_1 + \cdots + p_k = \begin{cases} 2 & \text{if } X \in V^+ \ ; \\ 0 & \text{if } X \in \mathfrak{g} \ ; \\ -2 & \text{if } X \in V^- \ . \end{cases}$$

Let $w_j = e^{\operatorname{ad} X_j} e^{\operatorname{ad} \theta(X_j)} e^{\operatorname{ad} X_j}$ be the nontrivial element of the Weyl group of the algebra $\mathfrak{sl}(2,\mathbb{R})$ generated by the triple $\{\theta(X_j), H_{\lambda_j}, X_j\}$. The strong orthogonality of the roots λ_j implies that w_i and w_j commute and that

$$w_j.H_{\lambda_i} = \begin{cases} H_{\lambda_i} \text{ for } i \neq j \ ; \\ -H_{\lambda_j} \text{ for } i = j \ . \end{cases}$$

Therefore there exists an element $w \in \widetilde{G}$, depending on X, product of some w_j's, such that

$$[H_{\lambda_j}, w.X] = |p_j| w.X \text{ for } j = 0, 1, \ldots, k \ .$$

This implies that

$$|p_0| + |p_1| + \cdots + |p_k| = 0 \text{ or } 2 \ .$$

Thus each p_j is zero except one or two among them which take the values $0, \pm 1$ or ± 2. Consequently $\widetilde{\mathfrak{g}}$ is the direct sum of spaces of the form $E_{i,j}(p,q)$. In particular X belongs to the centralizer of $\mathfrak{a}^0$ in $\mathfrak{g}$ if and only if each p_j is zero for $j = 0, \ldots, k$.

To obtain the announced decompositions it remains to show that

$$E_{i,j}(0,2) = \widetilde{\mathfrak{g}}^{\lambda_j} \text{ and } E_{i,j}(0,-2) = \widetilde{\mathfrak{g}}^{-\lambda_j} \text{ for } i \neq j \ .$$

A root space $\widetilde{\mathfrak{g}}^{\lambda}$ occurs in $E_{i,j}(0,2)$ if $\lambda(H_j) = 2$ and $\lambda(H_{\lambda_\ell}) = 0$ for $\ell \neq j$. It means that each root λ_ℓ is orthogonal to λ for $\ell \neq j$. Since $(\lambda + \lambda_\ell)(H_0) = 4$, $\lambda + \lambda_\ell$ is not a root and therefore λ_ℓ is strongly orthogonal to λ. In particular λ belongs to the centralizer of $\widetilde{\mathfrak{l}}_0 \oplus \cdots \oplus \widetilde{\mathfrak{l}}_{j-1}$. Thus λ is a root of $\widetilde{\Sigma}_j$. Since λ is a positive root and λ_j belongs to the basis of $\widetilde{\Sigma}_j$, it follows that $\lambda - \lambda_j$ is not a root. But the fact that $\lambda(H_{\lambda_j}) = 2 > 0$ implies that λ is a multiple of λ_j and, therefore, that $\lambda = \lambda_j$.

□

The following Corollary is a generalisation to each $j \in \{1 \ldots, k\}$ of the assertion (1) of Proposition 1.5.2

Corollary 1.8.2. *Let $j \in \{0, 1, \ldots, k\}$. A root $\lambda \in \widetilde{\Sigma}$ is strongly orthogonal to λ_j if and only if λ is orthogonal to λ_j.*

Proof. — Let λ be a root orthogonal to λ_j.

If $\lambda(H_0) = 2$ (*resp.* $\lambda(H_0) = -2$) then $\lambda + \lambda_j$ (*resp.* $\lambda - \lambda_j$) is not a root. It follows, using Lemma 1.4.2, that λ is strongly orthogonal to λ_j.

If $\lambda(H_0) = 0$, then λ belongs to $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}^0)$ or λ belongs to some space $E_{r,s}(1,-1)$ with $r \neq j$ and $s \neq j$. We prove that, in the two cases, $\lambda + \lambda_j$ is not a root.

In the first case $(\lambda + \lambda_j)(H_{\lambda_j}) = 2$ which implies by Theorem 1.8.1 that $\lambda + \lambda_j = \lambda_j$, which is impossible.

In the second case $\lambda + \lambda_j$ takes the values 2 on H_{λ_j}, 1 on H_{λ_r} and -1 on H_{λ_s} which is impossible.

Hence λ is strongly orthogonal to λ_j.

□

Remark 1.8.3. The decomposition in the spaces $E_{i,j}(p,q)$ will be much more useful than the root space decomposition of $\widetilde{\mathfrak{g}}$. In particular we will frequently use the brackets of the spaces $E_{i,j}(p,q)$ which can be easely computed using the Jacobi identity. We will prove later on (see section 3.3) that this decomposition is in fact a root space decomposition relatively to an other system of roots than $\widetilde{\Sigma}$.

Theorem 1.8.1 implies that the gradation of the algebra $\widetilde{\mathfrak{g}}_j$ (see Theorem 1.6.1 (2)) is in fact given by $H_{\lambda_j} + \cdots + H_{\lambda_k}$. Moreover there exists an element $I_j^+ \in V_j^+$ such that $\{\theta(I_j^+), H_{\lambda_j} + \cdots + H_{\lambda_k}, I_j^+\}$ is a $\mathfrak{sl}_2$–triple. It is obtained as a sum for $i = j$ to k of elements $X_i \in \widetilde{\mathfrak{g}}^{\lambda_i}$ such that $\{\theta(X_i), H_{\lambda_i}, X_i\}$ are $\mathfrak{sl}_2$–triples. And we deduce from Theorem 1.6.1 the next result.

Corollary 1.8.4. *For $j = 0, \ldots, k$, the graded algebra $(\widetilde{\mathfrak{g}}_j, H_{\lambda_j} + \cdots + H_{\lambda_k})$ satisfies the hypothesis* $(\mathbf{H_1})$, $(\mathbf{H_2})$ *and* $(\mathbf{H_3})$. *We have the following decompositions :*

$$(1) \qquad \mathfrak{g}_j = \left(\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}^0) \cap \mathfrak{g}_j\right) \oplus \left(\oplus_{r \neq s; j \leq r; j \leq s} E_{r,s}(1,-1)\right) ;$$

$$(2) \qquad V_j^+ = \left(\oplus_{s=j}^k \widetilde{\mathfrak{g}}^{\lambda_s}\right) \oplus \left(\oplus_{j \leq r < s} E_{r,s}(1,1)\right) ;$$

$$(3) \qquad V_j^- = \left(\oplus_{s=j}^k \widetilde{\mathfrak{g}}^{-\lambda_s}\right) \oplus \left(\oplus_{j \leq r < s} E_{r,s}(-1,-1)\right) .$$

It is easy to construct other graded subalgebras of $\widetilde{\mathfrak{g}}$ satisfying the hypothesis $(\mathbf{H_1})$, $(\mathbf{H_2})$ and $(\mathbf{H_3})$ as follows :

Proposition 1.8.5. *Let A be a nonempty subset of $\{0, 1, \ldots k\}$ and $H_A = \sum_{j \in A} H_{\lambda_j}$. We denote by*

$$\mathfrak{g}_A = \{X \in \mathfrak{g} \mid [H_A, X] = 0\} ;$$
$$V_A^+ = \{X \in V^+ \mid [H_A, X] = 2X\} ;$$
$$V_A^- = \{X \in V^+ \mid [H_A, X] = -2X\} ;$$

Then the graded algebra $(\widetilde{\mathfrak{g}}_A = V_A^- \oplus \mathfrak{g}_A \oplus V_A^+, H_A)$ satisfies $(\mathbf{H_1})$, $(\mathbf{H_2})$ *and* $(\mathbf{H_3})$.

Proof. — The hypothesis $(\mathbf{H_1})$ is satisfied by construction and the hypothesis $(\mathbf{H_3})$ can be proved with the argument used for $\widetilde{\mathfrak{g}}_j$.

It remains to prove that the representation of $\mathfrak{g}_A$ on $V_{A,\mathbb{C}}^+$ is irreducible. For that we consider a subspace U of $V_{A,\mathbb{C}}^+$ invariant under the action of $\mathfrak{g}_A$. We will study the image of U under the action of $\mathfrak{g}$. From Theorem 1.8.1 the action of H_A on $\mathfrak{g}$ decomposes as follows :

$$\mathfrak{g} = \mathfrak{g}(-1) \oplus \mathfrak{g}_A \oplus \mathfrak{g}(1) ,$$

where $\mathfrak{g}(-1) = \oplus_{i \notin A, j \in A} E_{i,j}(1,-1)$ and $\mathfrak{g}(1) = \oplus_{i \in A, j \notin A} E_{i,j}(1,-1)$.

We will consider now subspaces of $V_{\mathbb{C}}^+$ which are invariant under the action of $\operatorname{ad} H_A$. We remark that the only eigenvalues of $\operatorname{ad} H_A$ in V^+ are $0, 1$ or 2.

• U is, by definition, contained in the eigenspace of $\operatorname{ad} H_A$ for the eigenvalue 2;

• $U_1 = [\mathfrak{g}(-1), U]$, the $\mathbb{C}$–subspace of $V_{\mathbb{C}}^+$ generated by the brackets $[X, Y]$ with $X \in \mathfrak{g}(-1)$ and $Y \in U$, is contained in the eigenspace of $\operatorname{ad} H_A$ for the eigenvalue 1;

• $U_0 = [\mathfrak{g}(-1), U_1]$ is contained in the eigenspace of $\operatorname{ad} H_A$ for the eigenvalue 0;

• $[\mathfrak{g}(-1), U_0]$ is contained in the eigenspace of $\operatorname{ad} H_A$ for the eigenvalue -1 hence is reduced to $\{0\}$.

• $[\mathfrak{g}(1), U]$ is contained in the eigenspace of $\operatorname{ad} H_A$ for the eigenvalue 3 hence is reduced to $\{0\}$.

We will prove that $U_0 \oplus U_1 \oplus U$ is invariant under the action of $\mathfrak{g}$.

1) Since $[\mathfrak{g}_A, U] \subset U$, $[\mathfrak{g}(1), U] = \{0\}$ and $[\mathfrak{g}(-1), U] = U_1$ we have

$$[\mathfrak{g}, U] \subset U \oplus U_1 \ .$$

2) Since $[\mathfrak{g}, \mathfrak{g}(-1)] \subset \mathfrak{g}$ and $[\mathfrak{g}, U] \subset U \oplus U_1$, we have

$$[\mathfrak{g}, U_1] \subset [\mathfrak{g}, U] + [\mathfrak{g}(-1), U \oplus U_1] \subset U \oplus U_1 \oplus U_0 \ .$$

3) Since $[\mathfrak{g}, \mathfrak{g}(-1)] \subset \mathfrak{g}$ and $[\mathfrak{g}, U_1] \subset U \oplus U_1 \oplus U_0$ we have,

$$[\mathfrak{g}, U_0] \subset [\mathfrak{g}, U_1] + [\mathfrak{g}(-1), U \oplus U_1 \oplus U_0] \subset U \oplus U_1 \oplus U_0 \ .$$

Then $U_0 \oplus U_1 \oplus U$ is a $\mathbb{C}$–subspace of $V_{\mathbb{C}}^+$ invariant under the action of $\mathfrak{g}$. Since the representation of $\mathfrak{g}$ on $V_{\mathbb{C}}^+$ is irreducible, we have $U_0 \oplus U_1 \oplus U = V_{\mathbb{C}}^+$. But U is contained in the eigenspace with eigenvalue 2 of $\operatorname{ad} H_A$ *i.e.* in $V_{A,\mathbb{C}}^+$ and U_0 and U_1 are contained in eigenspaces for other eigenvalues. Hence we obtained the equality $U = V_{A,\mathbb{C}}^+$.

□

Remark 1.8.6. For $A = \{j, j+1, \ldots, k\}$ the subalgebra $\widetilde{\mathfrak{g}}_A$ has a gradation given by the same element than the subalgebra $\widetilde{\mathfrak{g}}_j$, namely $H_{\lambda_j} + \cdots + H_{\lambda_k}$. But these algebras are distinct : we have $\widetilde{\mathfrak{g}}_j \subset \widetilde{\mathfrak{g}}_A$ and the inclusion is strict; for example the subspace $E_{r,s}(1,-1)$ is contained in $\widetilde{\mathfrak{g}}_A$ for $s < j$ and $r < j$ and is not contained in $\widetilde{\mathfrak{g}}_j$. More precisely we have the following decompositions :

$$V_A^+ = \widetilde{\mathfrak{g}}_A \bigcap V^+ = \left(\oplus_{s=j}^k \widetilde{\mathfrak{g}}^{\lambda_s}\right) \oplus \left(\oplus_{j \le r < s} E_{r,s}(1,1)\right) = V_j^+ \ ;$$

$$\mathfrak{g}_A = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}^0) \oplus \left(\oplus_{r \neq s; j \le r; j \le s} E_{r,s}(1,-1)\right) \oplus \left(\oplus_{r \neq s; r < j; s < j} E_{r,s}(1,-1)\right) \supset \mathfrak{g}_j \ .$$

In Remark 1.3.2 we have noticed that the the graded algebra $\widetilde{\mathfrak{g}}$ is, in general, not simple. However the following Proposition shows that it is possible to obtain a regular graded subalgebra of $\widetilde{\mathfrak{g}}$, with the same "V^+" and "V^-" than $\widetilde{\mathfrak{g}}$, which is simple.

Proposition 1.8.7. *The subalgebra $\widetilde{\mathfrak{G}}$ of the regular graded algebra $\widetilde{\mathfrak{g}}$ generated by V^+ and V^- is a regular graded algebra which is simple.*

Proof. — The algebra $\widetilde{\mathfrak{G}}$ is also graded by $\operatorname{ad} H_0$ and can be written as

$$\widetilde{\mathfrak{G}} = V^- \oplus \mathfrak{G} \oplus V^+ ,$$

where $\mathfrak{G}$ is the vector subspace generated by the elements of the form $[Y,X]$ with $Y \in V^-$ and $X \in V^+$. It is clear that the hypothesis $(\mathbf{H_1})$, $(\mathbf{H_2})$ and $(\mathbf{H_3})$ are satisfied.

Moreover $\widetilde{\mathfrak{G}}$ is an ideal of $\widetilde{\mathfrak{g}}$ since $[\mathfrak{g}, V^{\pm}] \subset V^{\pm}$. Let $\mathfrak{G}'$ be the ideal of $\widetilde{\mathfrak{g}}$ orthogonal to $\widetilde{\mathfrak{G}}$ for $\widetilde{B}$. Then $\mathfrak{G}'$ is contained in $\mathfrak{g}$ and we have

$$\widetilde{\mathfrak{g}} = V^- \oplus \mathfrak{G} \oplus \mathfrak{G}' \oplus V^+ .$$

To prove that $\widetilde{\mathfrak{G}}$ is simple, we consider an ideal J of $\widetilde{\mathfrak{G}}$. The subspace $J \bigcap V^+$ is stabilized by $\operatorname{ad}\mathfrak{G}$ and annihilated by $\operatorname{ad}\mathfrak{G}'$. Since the representation of $\mathfrak{g}$ on $V_{\mathbb{C}}^+$ is irreducible by $(\mathbf{H_2})$ we have two possibilities :

Case 1: $J \bigcap V^+ = V^+$.

In this case J contains $\mathfrak{G} = [V^-, V^+] = [V^-, V^+ \bigcap J]$. In particular J contains H_0 which is by $(\mathbf{H_3})$ the bracket of an element of V^+ and an element of V^-. Therefore J contains $V^- = [H_0, V^-]$. We conclude that $J = \widetilde{\mathfrak{G}}$.

Case 2 : $J \bigcap V^+ = \{0\}$.

We consider in this case the orthogonal $J^{\perp}$ of J in $\widetilde{\mathfrak{G}}$ for the restriction of $\widetilde{B}$. It is also an ideal of $\widetilde{\mathfrak{G}}$ such that $J^{\perp} \bigcap V^+ = V^+$. Using the result of the first case, we obtain that $J^{\perp} = \widetilde{\mathfrak{G}}$ and therefore that $J = \{0\}$.

□

1.9. Properties of the spaces $E_{i,j}(p,q)$

Proposition 1.9.1. *Let λ be a root in Σ such that $\mathfrak{g}^{\lambda} \subset E_{i,j}(1,-1)$. Then λ is a positive root if and only if $i > j$.*

Proof. — Let λ be a root in Σ such that $\lambda(H_{\lambda_i}) = 1$ and $\lambda(H_{\lambda_j}) = -1$. By Theorem 1.8.1 we have $\lambda(H_{\lambda_s}) = 0$ for $s \neq i$ and $s \neq j$. If $i > j$, then λ is orthogonal, hence (by Corollary 1.8.2) strongly orthogonal, to each root λ_s for $s < j$. Thus λ is a root belonging to $\widetilde{\Sigma}_j$.

Since $(\lambda \mid \lambda_j)$ is strictly negative, $\lambda + \lambda_j$ is a root in $\widetilde{\Sigma}_j$. But the root λ belongs in fact to Σ_j and the Corollary 1.2.2 applied to $\widetilde{\Sigma}_j$ implies that λ is necessarily positive. It is thus a positive root in $\widetilde{\Sigma}$ (Proposition 1.5.2) and in Σ (Theorem 1.2.1).

The equivalence in the Proposition can be deduced from the following remark :

$$\mathfrak{g}^{\lambda} \subset E_{i,j}(1,-1) \Longleftrightarrow \mathfrak{g}^{-\lambda} \subset E_{j,i}(1,-1) .$$

□

Let $\widetilde{W}$ and W be the Weyl groups of the root systems $\widetilde{\Sigma}$ and Σ. The group W is the subgroup of $\widetilde{W}$ which is generated by the reflections along the roots $\lambda \in \Sigma$. In particular H_0 is fixed by each element of W.

Proposition 1.9.2. *Let s_0 be the element of W which interchanges Σ^+ and Σ^-. Then we have*

$$s_0.\lambda_j = \lambda_{k-j} \text{ for } j = 0, 1, \ldots, k \ .$$

Moreover the roots λ_i and λ_j are W-conjugate for each i and each j in $\{0, 1, \ldots, k\}$.

Proof. — We first prove that $s_0.\lambda_0 = \lambda_k$.

We know from Corollary 1.2.2 that the root λ_0 is characterized by the properties :

$$(1) \quad \begin{cases} \bullet & \lambda_0(H_0) = 2 \ ; \\ \bullet & \lambda \in \Sigma^+ \Longrightarrow \lambda_0 - \lambda \notin \widetilde{\Sigma} \ . \end{cases}$$

Therefore $s_0.\lambda_0$ is the unique root μ such that :

$$(2) \quad \begin{cases} \bullet & \mu(H_0) = 2 \ ; \\ \bullet & \lambda \in \Sigma^+ \Longrightarrow \mu + \lambda \notin \widetilde{\Sigma} \ . \end{cases}$$

Proposition 1.4.1 implies that $s_0.\lambda_0$ is the root λ^0 which is the restriction to $\mathfrak{a}^{\mathfrak{p}}$ of the highest weight ω of the representation of $\mathfrak{g}$ on $V_{\mathbb{C}}^+$.

We know from the proof of Proposition 1.5.1 that ω is the highest weight of the representation of $\mathfrak{g}_1$ on $V_{1,\mathbb{C}}^+$. We obtain by induction that ω is also the highest weight of the (irreducible) representation of $\mathfrak{g}_k$ on $V_{k,\mathbb{C}}^+$. Hence the restriction of ω to $\mathfrak{a}^{\mathfrak{p}}$ is a root occuring in V_k^+. It is a root strongly orthogonal to the roots $\lambda_0, \lambda_1, \ldots, \lambda_{k-1}$. The decomposition of V^+ shows that there is a unique root occuring in V_k^+, namely λ_k. We have proved that $\lambda_k = \lambda^0 = s_0.\lambda_0$.

We will now prove that $s_0.\lambda_1 = \lambda_{k-1}$. If we take the set $s_0.\widetilde{\Sigma}^+$ as set of positive roots on $\widetilde{\Sigma}$, we obtain an order such that the condition (1) of Theorem 1.2.1 is satisfied and we obtain also a sequence $\mu_0, \mu_1, \ldots$ of strongly orthogonal roots by the method of descent. By Corollary 1.2.2 the root μ_0 is characterized by the relations (2) and is therefore equal to λ_k. The centralizer of $\widetilde{\mathfrak{l}}_k$ satisfies the hypothesis ($\mathbf{H_1}$) and ($\mathbf{H_2}$) by Proposition 1.5.2. Then Corollary 1.2.2 applied to this graded algebra implies that the root μ_1 is characterized by

$$(3) \quad \begin{cases} \bullet & \mu_1(H_0) = 2 \ ; \\ \bullet & \mu_1 \perp\!\!\!\perp \lambda_k \ ; \\ \bullet & \lambda \in \Sigma^+ \text{ and } \lambda \perp\!\!\!\perp \lambda_k \Longrightarrow \mu_1 + \lambda \notin \widetilde{\Sigma} \ . \end{cases}$$

But the root λ_1 is characterized by the Corollary 1.2.2 applied in $\widetilde{\mathfrak{g}}_1$ *i.e.*

$$\begin{cases} \bullet & \lambda_1(H_0) = 2 \ ; \\ \bullet & \lambda_1 \perp\!\!\!\perp \lambda_0 \ ; \\ \bullet & \lambda \in \Sigma^+ \text{ and } \lambda \perp\!\!\!\perp \lambda_0 \Longrightarrow \lambda_1 - \lambda \notin \widetilde{\Sigma} \ . \end{cases}$$

Since $s_0.\lambda_0 = \lambda_k$ it follows that $\mu_1 = s_0.\lambda_1$.

On the other hand the root λ_{k-1} occurs in V^+ and is strongly orthogonal to λ_k. Let $\lambda \in \Sigma^+$ be a root strongly orthogonal to λ_k. If $\lambda(H_{\lambda_{k-1}}) = 0$ then λ is strongly orthogonal to λ_{k-1} by Corollary 1.8.2 and $\lambda + \lambda_{k-1}$ is not a

root. If $\lambda(H_{\lambda_{k-1}}) \neq 0$ there exists by Proposition 1.9.1 an element $j < k-1$ such that $\mathfrak{g}^{\lambda} \subset E_{k-1,j}(1,-1)$. Since $(\lambda+\lambda_{k-1})(H_{\lambda_{k-1}}) = 3$, $\lambda+\lambda_{k-1}$ is not a root. We have proved that λ_{k-1} satisfies the properties (3) which implies that $\lambda_{k-1} = \mu_1 = s_0.\lambda_1$.

An induction on j concludes the proof of the first part of the Proposition.

To obtain the second part we apply the previous result to the graded algebras $\widetilde{\mathfrak{g}}_i$ and $\widetilde{\mathfrak{g}}_j$ with λ_i or λ_j taking the place of λ_0. There exists an element $s_i \in W_i$ such that $s_i.\lambda_i = \lambda_k$ and an element $s_j \in W_j$ such that $s_j.\lambda_j = \lambda_k$. Since W_i and W_j are subgroups of W, $s = s_j s_i^{-1}$ is an element of W such that $s.\lambda_i = \lambda_j$.

□

Proposition 1.9.3.
(1) *For $j = 0,\ldots,k$ the root spaces $\widetilde{\mathfrak{g}}^{\lambda_j}$ have the same dimension.*
(2) *For $i \neq j$, the spaces $E_{i,j}(1,1), E_{i,j}(-1,-1)$ and $E_{i,j}(1,-1)$ have the same dimension which is different from 0 and independent of the pair $\{i,j\} \in \{0,1\ldots,k\}^2$.*

Proof. —
(1) By Proposition1.9.2 there exists an element $s \in W$ such that $s.\lambda_i = \lambda_j$. Thus the root spaces of λ_i and λ_j are isomorphic.

(2) Let $i \neq j$ be fixed in $\{0,1\ldots,k\}$. We choose an element X_i (*resp.* X_j) in $\widetilde{\mathfrak{g}}^{\lambda_i}$ (*resp.* $\widetilde{\mathfrak{g}}^{\lambda_j}$) such that $\{\theta(X_i), H_{\lambda_i}, X_i\}$ (*resp.* $\{\theta(X_j), H_{\lambda_j}, X_j\}$) is a $\mathfrak{sl}_2$-triple. Then $\operatorname{ad} X_i$ is an isomorphism from $E_{i,j}(-1,-1)$ onto $E_{i,j}(1,-1)$ with inverse -$\operatorname{ad}\theta(X_i)$ and $\operatorname{ad} X_j$ is an isomorphism from $E_{i,j}(1,-1)$ onto $E_{i,j}(1,1)$ with inverse $-\operatorname{ad}\theta(X_j)$. It follows that the spaces $E_{i,j}(\pm 1,\pm 1)$ are isomorphic for $i \neq j$ fixed.

To prove that the spaces $E_{i,j}(1,1)$ are isomorphic for the different pairs $\{i,j\}$ where $i \neq j$, we will use Weyl group elements which permute the roots λ_j.

More precisely we first prove that $E_{i,j}(1,1)$ and $E_{k,j}(1,1)$ are isomorphic if $j < i \leq k$. We apply Proposition 1.9.2 to the graded algebra $\widetilde{\mathfrak{g}}_i$. There exists an element s_i of W_i, the Weyl group of Σ_i, which permutes λ_i and λ_k. The group W_i is generated by the reflections along the roots λ which are strongly orthogonal to $\lambda_0,\ldots,\lambda_j,\ldots,\lambda_{i-1}$. Since $j < i$, λ_j is invariant under the action of the elements of W_i. Hence s_i gives rise to an isomorphism from $E_{i,j}(1,1)$ onto $E_{k,j}(1,1)$.

On the other hand the action of s_0 gives rise to an isomorphism from $E_{k,j}(1,1)$ (for $j < k$) onto $E_{k-j,0}(1,1)$ which is, due to the action of s_{k-j}, isomorphic to $E_{k,0}(1,1)$. Thus all the spaces $E_{i,j}(1,1)$ are isomorphic for $i \neq j$.

If these spaces were reduced to $\{0\}$ then the decomposition of V^+ would be of the form

$$V^+ = \oplus_{j=0}^{k} \widetilde{\mathfrak{g}}^{\lambda_j} \ .$$

But in this case $\widetilde{\mathfrak{g}}^{\lambda_0}$ would be invariant under the action of $E_{i,j}(1,-1)$ for $i \neq j$, hence under the action of $\mathfrak{g}$. This is impossible since the representation of $\mathfrak{g}$ on V^+ is irreducible by $(\mathbf{H_2})$. This concludes the proof.

□

Notation 1.9.4. *We will use in the sequel the following notations*

$$\ell = \dim \widetilde{\mathfrak{g}}^{\lambda_j} \text{ for } j = 0, \ldots, k \ ;$$
$$d = \dim E_{i,j}(\pm 1, \pm 1) \text{ for } i \neq j \in \{0, \ldots, k\} \ .$$

We deduce from the decomposition of V^+ (Theorem 1.8.1) the following relation between the dimensions of the subspaces :

Proposition 1.9.5.

$$\dim V^+ = (k+1)\left(\ell + \frac{kd}{2}\right) \ .$$

1.10. Normalization of the Killing form

Let us recall from section 1.1 that $\widetilde{B}$ is an extension to $\widetilde{\mathfrak{g}}$ of the restriction of the Killing form to the semisimple part of $\widetilde{\mathfrak{g}}$ such that the form $\widetilde{B_\theta}$ defined by

$$\widetilde{B_\theta}(X,Y) = -\widetilde{B}(X, \theta(Y)) \ ,$$

is positive definite on $\widetilde{\mathfrak{g}}$.

Definition 1.10.1. *We define for X and Y in $\widetilde{\mathfrak{g}}$*

$$b(X,Y) = -\frac{k+1}{4 \dim V^+}\widetilde{B}(X,Y) \ .$$

The first consequence of this normalization of $\widetilde{B}$ is the following Lemma.

Lemma 1.10.2. *For $j \in \{0, \ldots, k\}$ we have $b(H_{\lambda_j}, H_{\lambda_j}) = -2$. If X_j is an element of $\widetilde{\mathfrak{g}}^{\lambda_j}$ such that $[\theta(X_j), X_j] = H_{\lambda_j}$ then $b(X_j, \theta(X_j)) = 1$.*

Proof. —Since the elements H_{λ_j} are conjugate (Proposition 1.9.2) and orthogonal for the Killing form, we have

$$\widetilde{B}(H_{\lambda_j}, H_{\lambda_j}) = \frac{1}{k+1}\widetilde{B}(H_0, H_0) = \frac{1}{k+1}\mathrm{tr}_{\widetilde{\mathfrak{g}}}(\mathrm{ad}\, H_0)^2 = 8\,\frac{\dim V^+}{k+1} \ ,$$

which implies that $b(H_{\lambda_j}, H_{\lambda_j}) = -2$.

Since $\{\theta(X_j), H_{\lambda_j}, X_j\}$ is a $\mathfrak{sl}_2$–triple we get

$$\widetilde{B}(\theta(X_j), X_j) = \frac{1}{2}\widetilde{B}(\theta(X_j), [H_{\lambda_j}, X_j]) = -\frac{1}{2}\widetilde{B}(H_{\lambda_j}, H_{\lambda_j}) = -4\,\frac{\dim V^+}{k+1} \ .$$

□

The second consequence of this normalization concerns the family of graded algebras $\widetilde{\mathfrak{g}}_A$ where A is a subset of $\{\lambda_0, \lambda_1, \ldots, \lambda_k\}$, which are defined in Proposition 1.8.5. These algebras satisfy the hypothesis $(\mathbf{H_1})$, $(\mathbf{H_2})$ and $(\mathbf{H_3})$ and we can define the bilinear form b_A on $\widetilde{\mathfrak{g}}_A$ the same way as b is defined on $\widetilde{\mathfrak{g}}$.

Lemma 1.10.3. *If X and Y belong to the subalgebra of $\widetilde{\mathfrak{g}}_A$ generated by V_A^+ and V_A^- then*

$$b_A(X,Y) = b(X,Y) \ .$$

Proof. — The proof relies on the fact that the subalgebra $\widetilde{\mathfrak{G}}_A$ of $\widetilde{\mathfrak{g}}_A$ generated by V_A^+ and V_A^- is simple (Proposition 1.8.7). Therefore, up to scalar multiplication, the restrictions of b and b_A to $\widetilde{\mathfrak{G}}_A$ are the same. Lemma 1.10.2 applied to $\widetilde{\mathfrak{g}}_A$, implies that for $\lambda_j \in A$ and X_j such that $\{\theta(X_j), H_{\lambda_j}, X_j\}$ is a $\mathfrak{sl}_2$–triple, we have $b_A(\theta(X_j), X_j) = 1$. This concludes the proof.

□

1.11. The relative invariant Δ_0

Definition 1.11.1. *A polynomial function P on V^+ is a relative invariant under the action of the group G if there exists a nontrivial character χ of G such that*

$$P(g.X) = \chi(g)P(X) \text{ for each } g \in G \text{ and } X \in V^+ \ .$$

The extension to $V_{\mathbb{C}}^+$ of such a P is a relative invariant under the action of $G_{\mathbb{C}}$.

Theorem 1.11.2.

(1) *There exists an unique (up to scalar multiplication) relative invariant Δ_0 on V^+ under the action of G which is a homogeneous irreducible polynomial over $\mathbb{C}$.*

(2) *Every relative invariant under the action of G is (up to scalar multiplication) a power of Δ_0.*

(3) *An element X is generic in V^+ if and only if $\Delta_0(X) \neq 0$.*

Proof. — We first give a construction of some relative invariant P. We identify V^+ and V^- by making a choice of basis (for example using θ). Then the determinant of a linear map from V^- to V^+ is well defined and we set

$$P(X) = \det_{(V^-,V^+)}(\operatorname{ad} X)^2 \text{ for } X \in V^+ \ .$$

For $g \in G$ we obtain

$$P(g.X) = \det_{V^+} \operatorname{Ad}(g) \det_{V^-} \operatorname{Ad}(g^{-1}) P(X) \ ,$$

which implies that P is a relative invariant.

Let X be a generic element in V^+. The regularity of $(\widetilde{\mathfrak{g}}, H_0)$ implies by Proposition 1.7.9 that $(\operatorname{ad} X)^2$ is injective on V^-. This proves that $P(X)$ is nonzero. For $g = e^{t \operatorname{ad} H_0}$ (with $t \in \mathbb{R}$) we have $P(g.X) = e^{4t \dim V^+} P(X)$. Thus P is non constant and the character nontrivial.

We deduce from the results of Sato and Kimura ([Sa-K]–1977) the existence of an irreducible relative invariant on $V_{\mathbb{C}}^+$ (denoted by Δ_0) which is unique due to the irreducibility of the representation of $G_{\mathbb{C}}$ on $V_{\mathbb{C}}^+$. Moreover each relative invariant is a power of Δ_0 and, up to scalar multiplication, Δ_0 takes real values on V^+ (see for example [Ru]–1992 Proposition 1.2.11 and Lemme 2.1.1). This proves (1) and (2).

We deduce from (2) the following equivalence for $X \in V^+$

$$\Delta_0(X) \neq 0 \Longleftrightarrow P(X) \neq 0 \ .$$

We have proved that if X is generic in V^+ then $P(X) \neq 0$. Conversely if $P(X) \neq 0$, then $(\operatorname{ad} X)^2$ is surjective from V^- onto V^+, which implies that $\operatorname{ad} X$ is surjective from $\mathfrak{g}$ onto V^+ i.e. X is generic in V^+. This proves (3). □

1.12. The case $k = 0$

Theorem 1.12.1. *The graded algebra $(\widetilde{\mathfrak{l}}_0, H_{\lambda_0})$ is regular and the $\mathbb{Z}$-gradation defined by $\operatorname{ad} H_{\lambda_0}$ can be written*

$$\widetilde{\mathfrak{l}}_0 = \widetilde{\mathfrak{g}}^{-\lambda_0} \oplus \mathfrak{l}_0 \oplus \widetilde{\mathfrak{g}}^{\lambda_0} \text{ where } \mathfrak{l}_0 = [\widetilde{\mathfrak{g}}^{-\lambda_0}, \widetilde{\mathfrak{g}}^{\lambda_0}].$$

Let L_0 (resp. $L_{0,\mathbb{C}}$) be the analytic subgroup of $\widetilde{G}$ (resp. $\widetilde{G}_{\mathbb{C}}$) generated by $\{e^{\operatorname{ad} X}\}$ for $X \in \mathfrak{l}_0$ (resp. $X \in \mathfrak{l}_{0,\mathbb{C}}$). Denote by δ_0 the corresponding irreducible relative invariant.

- *If $\dim \widetilde{\mathfrak{g}}^{\lambda_0} = 1$, then δ_0 is of degree 1.*
  The number of L_0-orbits in $\widetilde{\mathfrak{g}}^{\lambda_0}$ is 3 : the orbit of 0 and two open orbits.
  The number of $L_{0,\mathbb{C}}$-orbits in $\widetilde{\mathfrak{g}}^{\lambda_0}_{\mathbb{C}}$ is 2 : the orbit of 0 and $\widetilde{\mathfrak{g}}^{\lambda_0}_{\mathbb{C}} \backslash \{0\}$.
- *If $\dim \widetilde{\mathfrak{g}}^{\lambda_0} > 1$ then δ_0 is of degree 2.*
  The number of L_0-orbit in $\widetilde{\mathfrak{g}}^{\lambda_0}$ is 2, the orbit of 0 and $\widetilde{\mathfrak{g}}^{\lambda_0} \backslash \{0\}$.
  The number of $L_{0,\mathbb{C}}$-orbits in $\widetilde{\mathfrak{g}}^{\lambda_0}_{\mathbb{C}}$ is 3, the orbit of 0, one orbit of codimension 1 and one open orbit.

Remark 1.12.2. This Theorem can be applied to the regular graded algebra $(\widetilde{\mathfrak{l}}_j, H_{\lambda_j})$ where

$$\widetilde{\mathfrak{l}}_j = \widetilde{\mathfrak{g}}^{-\lambda_j} \oplus \mathfrak{l}_j \oplus \widetilde{\mathfrak{g}}^{\lambda_j} \quad (\mathfrak{l}_j = [\widetilde{\mathfrak{g}}^{-\lambda_j}, \widetilde{\mathfrak{g}}^{\lambda_j}]).$$

In particular this gives the number of orbits in $\widetilde{\mathfrak{g}}^{\lambda_j}$ under the action of the analytic subgroup L_j of G generated by $\{e^{\operatorname{ad} X} \mid X \in \mathfrak{l}_j\}$.

Proof. —
1) We will first study the real situation.

Each element $X \in \widetilde{\mathfrak{g}}^{\lambda_0} \backslash \{0\}$ is generic since there exists an element $Y \in \widetilde{\mathfrak{g}}^{-\lambda_0}$ such that $\{Y, H_{\lambda_0}, X\}$ is a $\mathfrak{sl}_2$-triple. It follows that $(\widetilde{\mathfrak{l}}_0, H_{\lambda_0})$ is a regular graded algebra. Moreover the orbit under the action of L_0 of such an element X is a connected open subset of $\widetilde{\mathfrak{g}}^{\lambda_0} \backslash \{0\}$. It follows that $\widetilde{\mathfrak{g}}^{\lambda_0} \backslash \{0\}$ is the disjoint union of open orbits.

If the dimension of $\widetilde{\mathfrak{g}}^{\lambda_0}$ is 1, $\widetilde{\mathfrak{g}}^{\lambda_0} \backslash \{0\}$ has two connected components and the result is proved. The action of L_0 on $\widetilde{\mathfrak{g}}^{\lambda_0}$ is scalar and, if X_0 is a nonzero element in $\widetilde{\mathfrak{g}}^{\lambda_0}$, the polynomial $\delta_0 : tX_0 \mapsto t$ is an irreducible relative invariant.

If the dimension of $\widetilde{\mathfrak{g}}^{\lambda_0}$ is strictly greater than 1, then $\widetilde{\mathfrak{g}}^{\lambda_0} \backslash \{0\}$ is connected and there is only one nonzero orbit. Let δ_0 be the polynomial of degree 2 defined by

$$\delta_0(X) = b(X, \theta(X)) \text{ for } X \in \widetilde{\mathfrak{g}}^{\lambda_0} \ .$$

We want to prove that δ_0 is a relative invariant. The Cartan decomposition of $\mathfrak{l}_0$ is given by

$$\mathfrak{l}_0 = \mathfrak{m}_0 \oplus \mathbb{R}H_{\lambda_0} \text{ where } \mathfrak{m}_0 = \{X \in \mathfrak{l}_0 \mid \theta(X) = X\} .$$

It follows that the group L_0 is of the form $L_0 = M_0 e^{\mathbb{R}\,\mathrm{ad}\,H_{\lambda_0}}$ where M_0 is the subgroup of elements invariant under θ. For an element $l = m_0 e^{t\,\mathrm{ad}\,H_{\lambda_0}} \in L_0$ we have

$$\delta_0(l.X) = b(m_0.e^{2t}X, m_0 e^{2t}\theta(X)) = e^{4t}\delta_0(X) ,$$

which implies that δ_0 is a relative invariant. If δ_0 is not irreducible , it is the square of a relative invariant P of degree 1. Then there exists a character χ of the connected group L_0 such that $P(l.X) = \chi(l)P(X)$. But there exists an element $\ell \in L_0$ such that $\ell.X = -X$. Since P is homogeneous of degree 1, this implies that $P(\ell.X) = -P(X)$. And, as character of a connected group, χ is positive which is impossible. Thus δ_0 is irreducible.

2) We will consider now the complex situation. If $\dim \mathfrak{g}^{\lambda_0} = 1$ the result is trivial. If it is not the case we will apply the results of ([M-R-S]–1986) which describe the structure of the complex regular graded algebras *i.e.* the complex Lie algebras which satisfy the hypothesis $(\mathbf{H_1}), (\mathbf{H_2})$ and $(\mathbf{H_3})$. Let $\mathfrak{j}_{0,\mathbb{C}} = \mathfrak{j}_{\mathbb{C}} \bigcap \mathfrak{l}_{0,\mathbb{C}}$ be a Cartan subalgebra of $\widetilde{\mathfrak{l}}_{0,\mathbb{C}}$ and $\widetilde{\mathcal{R}}_0$ be the associated system of roots with the order induced by $\widetilde{\mathcal{R}}^+$. By the method of descent a maximal sequence of strongly orthogonal roots $\alpha_0, \alpha_1, \ldots, \alpha_n$ is obtained among the roots occuring in $\mathfrak{g}_{\mathbb{C}}^{\lambda_0}$. But Proposition 2.16 in [M-R-S] asserts that the number of these roots is equal to the degree of the irreducible invariant, which is here of degree 2. Hence the sequence of strongly orthogonal roots is reduced to two roots α_0 and α_1 in $\widetilde{\mathcal{R}}_0$ such that $\alpha_j(H_{\lambda_0}) = 2$ and such that $\rho(\alpha_0) = \rho(\alpha_1) = \lambda_0$ where ρ is the restriction to $\mathfrak{a}^{\mathfrak{p}}$ (see section 1.3). If X_{α_0} and X_{α_1} denote nonzero elements in the complex root spaces $\widetilde{\mathfrak{g}}_{\mathbb{C}}^{\alpha_0}$ and $\widetilde{\mathfrak{g}}_{\mathbb{C}}^{\alpha_1}$, then Theorem 2.12 in [M-R-S] asserts that the number of orbits of $L_{0,\mathbb{C}}$ in $\widetilde{\mathfrak{g}}_{\mathbb{C}}^{\lambda_0}$ is 3 with $0, X_{\alpha_0}, X_{\alpha_0} + X_{\alpha_1}$ as representatives. Moreover the orbit of $X_{\alpha_0} + X_{\alpha_1}$ is open and the orbit of X_{α_0} is of codimension 1 in $\widetilde{\mathfrak{g}}_{\mathbb{C}}^{\lambda_0}$ by Lemma 2.15 in [M-R-S].

□

Notation 1.12.3. *We will denote by κ the degree of δ_0. The previous theorem implies that*

$$\ell = 1 \Longrightarrow \kappa = 1 ;$$
$$\ell > 1 \Longrightarrow \kappa = 2 .$$

We will now study in the general case ($k \geq 0$) the codimension of the G–orbit of an element of the form

$$X^1 + X \text{ where } \begin{cases} X^1 = X_1 + X_2 + \cdots + X_k \quad (X_j \in \widetilde{\mathfrak{g}}^{\lambda_j} \backslash \{0\}) , \\ X \in \widetilde{\mathfrak{g}}^{\lambda_0} . \end{cases}$$

Lemma 1.12.4. *For each $X \in \widetilde{\mathfrak{g}}^{\lambda_0}$ we have*

$$V^+ = [\mathfrak{g}, X^1 + X] + \widetilde{\mathfrak{g}}^{\lambda_0} .$$

The codimension in V^+ of the G–orbit of $X + X^1$ is given by

$$\operatorname{codim}[\mathfrak{g}, X^1 + X] = \begin{cases} \ell \text{ if } X = 0 \ ; \\ 0 \text{ if } X \neq 0 \ . \end{cases}$$

Proof. —

If $X \neq 0$, we know from Proposition 1.7.7 that $X + X^1$ is generic in V^+. Consequently we have

$$V^+ = [\mathfrak{g}, X + X^1] \ ,$$

and therefore the codimension of the G–orbit of $X^1 + X$ is 0.

If $X = 0$ we study the tangent space of the G–orbit of X^1. The decompositions given in Corollary 1.8.4 imply that

$$V^+ = V_1^+ \oplus \left(\oplus_{j=1}^k E_{0,j}(1,1) \right) \oplus \widetilde{\mathfrak{g}}^{\lambda_0} \ .$$

Since X^1 is generic in V_1^+ by Proposition 1.7.7 applied to $\widetilde{\mathfrak{g}}_1$, we have

$$V_1^+ = [\mathfrak{g}_1, X^1] \subset [\mathfrak{g}, X^1] \ .$$

On the other hand let A be an element of $E_{0,j}(1,1)$ and $Y_j \in \widetilde{\mathfrak{g}}^{-\lambda_j}$ such that $\{Y_j, H_{\lambda_j}, X_j\}$ is a $\mathfrak{sl}_2$-triple. Then $B = [Y_j, A]$ is an element of $E_{0,j}(1,-1)$ and we have

$$[B, X^1] = [B, X_j] = [[Y_j, X_j], A] = A \ .$$

It follows that

$$\oplus_{j=1}^k E_{0,j}(1,1) \subset [\mathfrak{g}, X^1] \ .$$

And we deduce from the given decomposition of V^+ that

$$V^+ = [\mathfrak{g}, X^1] + \widetilde{\mathfrak{g}}^{\lambda_0} \ .$$

The relations

$$[\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}^0), X^1] \subset \oplus_{j=1}^k \widetilde{\mathfrak{g}}^{\lambda_j} \subset V_1^+ \ ,$$

$$[E_{i,j}(1,-1), X^1] \subset \begin{cases} \{0\} \text{ if } j = 0, \\ E_{i,j}(1,1) \text{ if } j \neq 0 \ , \end{cases}$$

imply that

$$[\mathfrak{g}, X^1] \subset V_1^+ \oplus \left(\oplus_{j=1}^k E_{0,j}(1,1) \right) \ .$$

It follows that

$$[\mathfrak{g}, X^1] = V_1^+ \oplus \left(\oplus_{j=1}^k E_{0,j}(1,1) \right) \ ,$$

and consequently that

$$V^+ = [\mathfrak{g}, X^1] \oplus \widetilde{\mathfrak{g}}^{\lambda_0} \ .$$

Therefore the codimension of the G–orbit of X^1 is equal to $\dim \widetilde{\mathfrak{g}}^{\lambda_0} = \ell$.

□

If we consider the complex situation, the situation is quite different in the case $\ell > 1$ and we deduce from Theorem 1.12.1 the following result.

Corollary 1.12.5. *Let X_j be a sequence of nonzero elements in $\widetilde{\mathfrak{g}}^{\lambda_j}$ for $j = 1, \dots, k$ and $X^1 = X_1 + X_2 + \cdots + X_k$. For $X \in \widetilde{\mathfrak{g}}_{\mathbb{C}}^{\lambda_0}$ we have*

$$V_{\mathbb{C}}^+ = [\mathfrak{g}_{\mathbb{C}}, X^1 + X] + \widetilde{\mathfrak{g}}_{\mathbb{C}}^{\lambda_0} .$$

Moreover there exists a non generic element $X \in \widetilde{\mathfrak{g}}_{\mathbb{C}}^{\lambda_0}$ such that the codimension of the $G_{\mathbb{C}}$-orbit of $X + X^1$ in $V_{\mathbb{C}}^+$ is 1.

Proof. — The decomposition of $V_{\mathbb{C}}^+$ is a direct consequence of the real decomposition given in the previous Lemma. Since X^1 centralizes $\mathfrak{l}_0$, we have

$$[\mathfrak{g}_{\mathbb{C}}, X + X^1] \supset [\mathfrak{l}_{0,\mathbb{C}}, X] .$$

It follows that the codimension of the $G_{\mathbb{C}}$-orbit of $X + X^1$ is less or equal to the codimension of the $L_{0,\mathbb{C}}$-orbit of X in $\widetilde{\mathfrak{g}}_{\mathbb{C}}^{\lambda_0}$. We know from Theorem 1.12.1 that there exists an element $X \in \widetilde{\mathfrak{g}}_{\mathbb{C}}^{\lambda_0}$ such that the codimension in $\widetilde{\mathfrak{g}}_{\mathbb{C}}^{\lambda_0}$ of its $L_{0,\mathbb{C}}$-orbit is 1 (if $\ell = 1$, we take $X = 0$). Therefore, for this element X, the codimension of the $G_{\mathbb{C}}$-orbit of $X + X^1$ is 1 or 0.

Since X is not generic in $\widetilde{\mathfrak{g}}_{\mathbb{C}}^{\lambda_0}$ the polynomial $\widetilde{\mathfrak{g}}_{\mathbb{C}}^{\lambda_0} \ni U \mapsto \Delta_0(U + X^1)$, which is a nontrivial relative invariant for the action of $L_{0,\mathbb{C}}$, takes the value 0 on X. But $\Delta_0(X + X^1) = 0$ implies that $X + X^1$ is not generic in $V_{\mathbb{C}}^+$. Thus the $G_{\mathbb{C}}$-orbit is not open and consequently of codimension 1 in $V_{\mathbb{C}}^+$. □

1.13. Properties of Δ_0

Let Δ_0 be an irreducible relative invariant which takes real values on V^+ and χ_0 the associated character of G. We will later on make a choice to normalize Δ_0.

Theorem 1.13.1.
(1) *Δ_0 is a homogeneous polynomial of degree $\kappa(k+1)$.*
(2) *For $j = 0, \dots, k$, let X_j be an element of $\widetilde{\mathfrak{g}}^{\lambda_j} \setminus \{0\}$ and x_j a real number. Then we have*

$$\Delta_0\Big(\sum_{j=0}^{k} x_j X_j\Big) = \prod_{j=0}^{k} x_j^{\kappa} \,.\Delta_0\Big(\sum_{j=0}^{k} X_j\Big) .$$

(3) *For each $H \in \mathfrak{a}^{\mathfrak{p}}$ we have*

$$\chi_0(e^{\operatorname{ad} H}) = e^{\kappa(\lambda_0(H)+\lambda_1(H)+\cdots+\lambda_k(H))} .$$

Proof. — For $j = 0, \dots, k$, let $X_j \in \widetilde{\mathfrak{g}}^{\lambda_j} \setminus \{0\}$ and $Y_j \in \widetilde{\mathfrak{g}}^{-\lambda_j} \setminus \{0\}$ such that $\{Y_j, H_{\lambda_J}, X_j\}$ is a $\mathfrak{sl}_2$–triple. If I^+ (*resp.* I^-) denotes the element $X_0 + X_1 + \cdots + X_k$ (*resp.* $Y_0 + \cdots + Y_k$) then $\{I^-, H_0, I^+\}$ is a $\mathfrak{sl}_2$-triple and Lemma 1.7.2 implies that I^+ is generic in V^+. Therefore, by Theorem 1.11.2, $\Delta_0(I^+) \neq 0$.

Let d_0 be the degree of the homogeneous polynomial Δ_0. For $t \in \mathbb{R}$, the action of $e^{t \operatorname{ad} H_0}$ is scalar on V^+ and we have

$$\Delta_0(e^{t \operatorname{ad} H_0}.I^+) = \Delta_0(e^{2t} I^+) = e^{2td_0} \Delta_0(I^+) ;$$
$$\Delta_0(e^{t \operatorname{ad} H_0}.I^+) = \chi_0(\exp tH_0) \Delta_0(I^+) .$$

Since $\Delta_0(I^+) \neq 0$, it follows that

$$\chi_0(\exp tH_0) = e^{2td_0} \text{ and } 2d_0 = d\chi_0(H_0) \ .$$

Since two roots of the family $\{\lambda_0, \lambda_1, \ldots, \lambda_k\}$ are conjugate by an element of W which has a realization as a subgroup of G, the character χ_0 takes the same values on the elements $\exp tH_{\lambda_j} (j = 0, \ldots, k)$. Therefore

$$\chi_0(\exp tH_0) = \chi_0(\exp tH_{\lambda_0})^{k+1} \text{ and } 2d_0 = (k+1)d\chi_0(H_{\lambda_0}) \ .$$

Consider the polynomial function on $\widetilde{\mathfrak{g}}^{\lambda_0}$ given by

$$\delta : X \mapsto \Delta_0(X + X^1) \text{ where } X^1 = X_1 + X_2 + \cdots + X_k \ .$$

It is a relative invariant under the action of L_0. It is not a constant since we have, for $t \in \mathbb{R}$,

$$\delta(e^{t \operatorname{ad} H_{\lambda_0}}.X_0) = \Delta_0(e^{t \operatorname{ad} H_{\lambda_0}}.I^+) = \chi_0(\exp tH_{\lambda_0})\Delta_0(I^+) = e^{\frac{2td_0}{k+1}}\Delta_0(I^+) \ .$$

Therefore there exists $p \in \mathbb{N}^*$ and $C_1 \in \mathbb{R}^*$ such that

$$\delta(X) = \Delta_0(X^1 + X) = C_1\delta_0(X)^p \text{ for } X \in \widetilde{\mathfrak{g}}^{\lambda_0} \ .$$

If $p = 1$ we obtain, using the degre of δ_0 which is denoted by κ,

$$\chi_0(\exp tH_{\lambda_0}) = e^{2t\kappa} (t \in \mathbb{R}) \text{ and } d\chi_0(H_{\lambda_0}) = 2\kappa \ .$$

From the last relation we deduce (1).
We have also for $(t_0, \ldots, t_k) \in \mathbb{R}^{k+1}$ the relation

$$\chi_0\big(\exp \sum_{j=0}^{k} t_j H_{\lambda_j}\big) = \prod_{j=0}^{k} e^{2t_j\kappa} \ .$$

It follows that

$$\Delta_0\Big(\sum_{j=0}^{k} e^{2t_j} X_j\Big) = \prod_{j=0}^{k} e^{2t_j\kappa} \ .\Delta_0(I^+) \ ,$$

and, therefore that assertion (2) is satisfied for $x_j = e^{2t_j}$ Since the function $(x_0, , \ldots, x_k) \in \mathbb{R}^{k+1} \mapsto \Delta_0(x_0X_0 + \cdots + x_kX_k)$ is polynomial, (2) is true for $x_j \in \mathbb{R}$ $(j = 0, \ldots, k)$.

For $H \in \mathfrak{a}^{\mathfrak{p}}$ we have $e^{\operatorname{ad} H}.I^+ = \sum_{j=0}^{k} \lambda_j(H)X_j$. It follows that (3) is a consequence of (2).

To conclude the proof, we will prove that the irreducibility of Δ_0 implies that $p = 1$. This result relies on the study of the singularities of the hypersurface $S = \{X \in V_{\mathbb{C}}^+ \mid \Delta_0(X) = 0\}$. We suppose that $p > 1$.

Let $X \in \widetilde{\mathfrak{g}}_{\mathbb{C}}^{\lambda_0}$ such that

$$\Delta_0(X^1 + X) = 0 \ ,$$

which is equivalent to $\delta_0(X) = 0$. Then we will prove that the differential $\Delta_0'(X + X^1)$ of Δ_0 at the point $X + X^1$ is zero *i.e.* that $X + X^1$ is a singular point of S.
For $X' \in \widetilde{\mathfrak{g}}_{\mathbb{C}}^{\lambda_0}$ and $t \in \mathbb{C}$ we have

$$\Delta_0(tX' + X + X^1) = C_1\delta_0(tX' + X)^p \ .$$

As $p > 1$, derivation with respect to t at the point 0 implies

$$\langle \Delta_0'(X + X^1), X' \rangle = 0 \text{ for } X' \in \widetilde{\mathfrak{g}}_{\mathbb{C}}^{\lambda_0} .$$

On the other hand, for $Z \in \mathfrak{g}_{\mathbb{C}}$ and $t \in \mathbb{C}$ we have $\Delta_0(e^{\operatorname{ad} tZ}(X + X^1)) = \chi_0(\exp tZ)\Delta_0(X + X^1) = 0$. Again by derivation, it follows that

$$\langle \Delta_0'(X + X^1), [Z, X + X^1] \rangle = 0 \text{ for } Z \in \mathfrak{g}_{\mathbb{C}} .$$

But from Corollary 1.12.5 we know that $V_{\mathbb{C}}^+ = \widetilde{\mathfrak{g}}_{\mathbb{C}}^{\lambda_0} + [\mathfrak{g}_{\mathbb{C}}, X + X^1]$. Thus, for $p > 1$, we have

$$\Delta_0'(X + X^1) = 0 .$$

Moreover for $g \in G_{\mathbb{C}}$ we have

$$\langle \Delta_0'\big(g(X + X^1)\big), U \rangle = \chi_0(g)\langle \Delta_0'(X + X^1), g^{-1}U \rangle = 0 \text{ for } U \in V_{\mathbb{C}}^+ .$$

In other words, if X is a non generic point of $\widetilde{\mathfrak{g}}_{\mathbb{C}}^{\lambda_0}$, the $G_{\mathbb{C}}$–orbit of $X + X^1$ is contained in the singular set of S and is therefore of codimension ≥ 2 in $V_{\mathbb{C}}^+$ ([Hu]–1981, §3.2). But this gives a contradiction with Corollary 1.12.5 which asserts that there exists an non generic element in $\widetilde{\mathfrak{g}}_{\mathbb{C}}^{\lambda_0}$ with a $G_{\mathbb{C}}$–orbit of codimension 1.

□

Remark 1.13.2. In the case $\ell > 1$, the existence of an element $X \in \widetilde{\mathfrak{g}}_{\mathbb{C}}^{\lambda_0}$ such that the $L_{0,\mathbb{C}}$-orbit of X is of codimension 1 (Theorem 1.12.1) is deduced from the study of the complex regular graded algebras in [M-R-S]. The proof of Theorem 1.13.1 given here relies, for the case $\ell = 1$, on the same arguments than the proof given in [M-R-S] for the complex case.

1.14. The polynomials Δ_j

Let j be an element of $\{1, 2, \ldots, k\}$. From Theorem 1.11.2 applied to the regular graded algebras $(\widetilde{\mathfrak{g}}_j, H_{\lambda_j} + H_{\lambda_{j+1}} + \cdots + H_{\lambda_k})$ we deduce the existence of an irreducible polynomial P_j on V_j^+ which is relatively invariant under the action of G_j (the analytic subgroup of $\widetilde{G}$ generated by $\{e^{\operatorname{ad} X} \mid X \in \mathfrak{g}_j\}$). Let χ_j be the associated character of G_j. We will define an extension of this polynomials to V^+ using the following decomposition

$$V^+ = V_j^+ \oplus V_j^\perp \text{ where } V_j^\perp = \Big(\bigoplus_{r<s, r<j} E_{r,s}(1,1) \Big) \oplus \Big(\bigoplus_{r<j} \widetilde{\mathfrak{g}}^{\lambda_r} \Big) .$$

Definition 1.14.1. *We denote by Δ_j the unique (up to scalar multiplication) polynomial such that*

$$\Delta_j(X + Y) = P_j(X) \text{ for } X \in V_j^+, Y \in V_j^\perp ,$$

where P_j is an irreducible relative invariant on V_j^+ under the action of G_j.

In the following Theorem X_s $(s = 0, 1, \ldots, k)$ denotes a nonzero element of $\widetilde{\mathfrak{g}}^{\lambda_s}$.

Theorem 1.14.2. *Let $j \in \{0, 1, \dots, k\}$.*
(1) *Δ_j is a homogeneous polynomial of degree $\kappa(k+1-j)$.*
(2) *For $X \in V^+$ we have*

$$\Delta_j(g.X) = \chi_j(g)\Delta_j(X) \text{ for } g \in G_j \ ;$$
$$\Delta_j(g.X) = \Delta_j(X) \text{ for } g = \exp Z \text{ where } Z \in \bigoplus_{r<s} E_{r,s}(1,-1) \ .$$

(3) *Let $x_s \in \mathbb{R}$ for $s = 0, \dots, k$. Then*

$$\Delta_j\Big(\sum_{s=0}^{k} x_s X_s\Big) = \prod_{s=j}^{k} x_s^{\kappa} \,.\Delta_j\Big(\sum_{s=j}^{k} X_s\Big) \ .$$

(4) *Let $H \in \mathfrak{a}^{\mathfrak{p}}$. Then*

$$\chi_j(e^{\operatorname{ad} H}) = e^{\kappa(\lambda_j(H)+\lambda_{j+1}(H)+\cdots+\lambda_k(H))} \ .$$

(5) *The polynomial $X \in V_j^+ \mapsto \Delta_0(X_0 + X_1 + \cdots + X_{j-1} + X)$ is equal (up to scalar multiplication) to the restriction of Δ_j to V_j^+.*

Proof. — From Corollary 1.8.4 the Lie algebra $\mathfrak{g}_j$ can be written as

$$\mathfrak{g}_j = \mathfrak{z}_{\mathfrak{g}_j}(\mathfrak{a}^0) \oplus \Big(\bigoplus_{r\neq s;\, r\geq j;\, s\geq j} E_{r,s}(1,-1)\Big) \ .$$

Thus the space $V_j^{\perp}$ is invariant under the action of G_j and the definition of Δ_j implies the first part of (2). For $s < j$ the root spaces $\widetilde{\mathfrak{g}}^{\lambda_s}$ are subsets of $V_j^{\perp}$ and $\{\lambda_j, \dots, \lambda_k\}$ is the maximal sequence of strongly orthogonal roots for the graded algebra $(\widetilde{\mathfrak{g}}_j, H_{\lambda_j})$. Then Theorem 1.13.1 applied to the graded algebra $\widetilde{\mathfrak{g}}_j$ implies (1), (3) and (4).

To prove (5) we remark that the polynomial Q_j defined on V_j^+ by

$$Q_j(X) = \Delta_0(X_0 + \cdots + X_{j-1} + X) \text{ for } X \in V_j^+ \ ,$$

is relatively invariant under the action of G_j because this group centralizes $X_0, \dots, X_{j-1}$. If $g = \exp\big(t(H_{\lambda_j} + \cdots + H_{\lambda_k})\big)$ $(t \in \mathbb{R})$ we have

$$g.X = e^{2t}X \text{ for } X \in V_j^+ \ ; \ g.X_s = X_s \text{ for } s < j \ ; \ \chi_0(g) = e^{2t(k+1-j)\kappa} \ ,$$

hence

$$Q_j(e^{2t}X) = e^{2t(k+1-j)\kappa} Q_j(X) \text{ for } X \in V_j^+ \ .$$

It follows that de degree of Q_j is equal to $\kappa(k+1-j)$ hence to the degree of the irreducible invariant on V_j^+ under the action of G_j. Therefore Q_j is (up to a constant) the restriction of Δ_j to V_j^+.

It remains to prove that Δ_j is invariant under the action of $\exp \mathfrak{n}$ where

$$\mathfrak{n} = \bigoplus_{r<s} E_{r,s}(1,-1) \ .$$

From Proposition 1.9.1, $\mathfrak{n}$ is the sum of the root spaces related to the negative roots of Σ whose restrictions to $\mathfrak{a}^0$ are nonzero. In particular $\mathfrak{n}$ is contained in the semisimple part of $\mathfrak{g}$ which implies that the character χ_0 is trivial on

$\mathfrak{n}$. Therefore the result is proved for $j = 0$. By the same argument Δ_j is invariant under the action of $\exp \mathfrak{n}_j$ which is defined by

$$\mathfrak{n}_j = \bigoplus_{j \le r < s} E_{r,s}(1, -1) \ .$$

We consider the decomposition

$$\mathfrak{n} = \mathfrak{n}_j \oplus \mathfrak{n}_j^{\perp} \quad \text{where } \mathfrak{n}_j^{\perp} = \bigoplus_{r<s;\, r<j} E_{r,s}(1, -1) \ .$$

Since $[\mathfrak{n}_j, \mathfrak{n}_j^{\perp}] \subset \mathfrak{n}_j^{\perp}$, each element of $\exp \mathfrak{n}$ can be written as $\exp Z' \exp Z$ with $Z' \in \mathfrak{n}_j$ and $Z \in \mathfrak{n}_j^{\perp}$. It is therefore sufficient to prove that Δ_j is invariant under the action of $\exp \mathfrak{n}_j^{\perp}$. Since $[E_{r,s}(1, -1), V^+]$ is a subset of $\oplus_{s'=0,\dots,k} E_{r,s'}(1, 1)$, then $[\mathfrak{n}_j^{\perp}, V^+]$ is a subset of $V_j^{\perp}$. It follows that

$$X \in V^+ \text{ and } Z \in \mathfrak{n}_j^{\perp} \Longrightarrow e^{\operatorname{ad} Z}.X \in X + V_j^{\perp} \ ,$$

which implies that $\Delta_j(e^{\operatorname{ad} Z}.X) = \Delta_j(X)$.

□

CHAPTER 2

The orbits of G in V^+

The aim of this chapter is the description of the G–orbits in V^+ for the real prehomogeneous spaces related to a graded algebra $(\widetilde{\mathfrak{g}}, H_0)$ which satisfies the hypothesis $(\mathbf{H_1})$,$(\mathbf{H_2})$ and $(\mathbf{H_3})$.

The result is given and proved by I. MULLER ([Mu]–1998) in a more general situation. She studies graded algebras over a field of characteristic 0 and also some cases where the gradation is longer, in fact $\widetilde{\mathfrak{g}} = \oplus_{j=-2}^{2}\widetilde{\mathfrak{g}}_j$ with $[\widetilde{\mathfrak{g}}_j, \widetilde{\mathfrak{g}}_i] \subset \widetilde{\mathfrak{g}}_{i+j}$. On the other hand S. KANEYUKI ([Ka]–1998) gives also a description of the G–orbits for the real prehomogeneous spaces associated to a simple graded algebra $(\widetilde{\mathfrak{g}}, H_0)$ which satisfies only the hypothesis $(\mathbf{H_1})$ and $(\mathbf{H_2})$. His proof needs some explicit case by case calculations on root systems.

For completeness we will give here the details of a proof which is mostly inspired by ([Mu]–1998) and which is analogous to the proof in the complex case ([Ru]–1982, [M–R–S]–1986). This proof leads us naturally to distinguish three types among the prehomogeneous spaces under consideration. This partition is known in the setting of Jordan algebras ([Kay]–1994, [Sah]–1995). It is obtained from the study of the types of some root system. We will consider these root systems in section 3.3.

The Type I case (Definition 2.7.4) corresponds, in the setting of Jordan algebras, to the case of euclidean Jordan algebras. In this case the description of the open orbits is given by SATAKE ([S]–1984) and the description of all the orbits by KANEYUKI ([Ka]–1991). The last argument of our proof (section 2.10) in the Type I case is the argument used by these two authors.

In this chapter we will use some basic properties of representations of $\mathfrak{sl}(2, \mathbb{C})$ which we now recall (see for example [Bou]–1975 ch.VIII §1).

2.1. Representations of $\mathfrak{sl}(2, \mathbb{C})$

Let $\{Y, H, X\}$ be a $\mathfrak{sl}(2)$-triple (Notation 1.7.3) in $\widetilde{\mathfrak{g}}$ and E a subspace of $\widetilde{\mathfrak{g}}$ invariant under the action of this $\mathfrak{sl}(2)$-triple. Then E is a sum of finite dimensional irreducible $\mathfrak{sl}_2$-modules.

If an irreducible $\mathfrak{sl}_2$–module is of dimension $m+1$ ($m \in \mathbb{N}$), it is generated by a primitive element e_0 (*i.e.* such that $X.e_0 = 0$) of integer weight m under the action of H. A basis of this module is given by the elements

$$e_p = \frac{(-1)^p}{p!}(Y)^p.e_0 \quad (p = 0, \ldots, m) \ ,$$

which are elements of weight $m-2p$ under the action of H.

The action on E of the nontrivial element w of the Weyl group of $SL_2(\mathbb{R})$ is given by :

$$w = e^X e^Y e^X = e^Y e^X e^Y \ .$$

Each irreducible submodule F of E is invariant under the action of w. If the dimension of F is $m+1$ then the restriction of w to the subspace of weight $m-2p$ is a bijection onto the subspace of weight $2p-m$. More precisely the action on the basis defined above is given by

$$w.e_p = (-1)^{m-p} e_{m-p} \ .$$

If E_p is the set of elements of E of weight p ($p \in \mathbb{Z}$) under the action of H then the action of w^2 on $Z \in E_p$ is given by

$$w^2.Z = \begin{cases} Z & \text{if } p \text{ is even} \ , \\ -Z & \text{if } p \text{ is odd} \ . \end{cases}$$

Moreover if j is a positive integer, the application

$$(X)^j : E_p \longrightarrow E_{p+2j} \text{ is } \begin{cases} \text{one to one for} & j \leq -p \ , \\ \text{bijective for} & j = -p \ , \\ \text{onto for} & j \geq -p \ , \end{cases}$$

and the application

$$(Y)^j : E_p \longrightarrow E_{p-2j} \text{ is } \begin{cases} \text{one to one for} & j \leq p \ , \\ \text{bijective for} & j = p \ , \\ \text{onto for} & j \geq p \ . \end{cases}$$

2.2. First reduction

We recall the definition of V_1^+ :

$$V_1^+ = \widetilde{\mathfrak{g}}_1 \bigcap V^+ = \{X \in V^+ \mid [H_{\lambda_0}, X] = 0\} \ .$$

The eigenspaces decomposition of V^+ relatively to $\operatorname{ad} H_{\lambda_0}$ is given by

$$V^+ = V_1^+ \oplus \widetilde{\mathfrak{g}}^{\lambda_0} \oplus W_1^+ \text{ where } W_1^+ = \{X \in V^+ \mid [H_{\lambda_0}, X] = X\} \ .$$

Proposition 2.2.1. *Let X be an element of V^+.*
(1) *If $\Delta_0(X) = 0$, then X is conjugate under G to an element of V_1^+.*
(2) *If $\Delta_1(X) \neq 0$, then X is conjugate under G to an element of $\widetilde{\mathfrak{g}}^{\lambda_0} \oplus V_1^+$.*

Proof. — (see [M-R-S]–1986 p. 102)

(1) Let X be a non generic element of V^+. An element X of V^+ belongs to the semisimple part $[\widetilde{\mathfrak{g}}, \widetilde{\mathfrak{g}}]$ of $\widetilde{\mathfrak{g}}$ and satisfies $(\operatorname{ad} X)^3 = 0$. Thus the theorem of JACOBSON–MOROSOV implies the existence of a $\mathfrak{sl}_2$-triple of the form $\{Y, H, X\}$. Writing down the decomposition of these elements according to the decomposition $V^- \oplus \mathfrak{g} \oplus V^+$, we see easily that it is possible to take Y in V^- and H in $\mathfrak{g}$. The eigenvalues of $\operatorname{ad} H$ are the weights of the representation of this $\mathfrak{sl}_2$-triple on $\widetilde{\mathfrak{g}}$. Thus H is semi-simple in $\mathfrak{g}$ with real eigenvalues

and there exists therefore an element $g \in G$ such that $g.H$ belongs to $\mathfrak{a}^{\mathfrak{p}}$. We can choose an element w in $N_G(\mathfrak{a}^{\mathfrak{p}})$ (which is a representative of an element of the Weyl group of Σ) such that $wg.H$ belongs to the Weyl chamber defined by Σ^+ *i.e.* such that

$$\lambda(wg.H) \geq 0 \text{ for all } \lambda \in \Sigma^+ .$$

Let $\{Y', H', X'\}$ be the $\mathfrak{sl}_2$–triple $wg.\{Y, H, X\}$. We will show that $\lambda_0(H') = 0$.

Each element of $\widetilde{\mathfrak{g}}^{\lambda_0}\backslash\{0\}$ is a primitive element (of weight $\lambda_0(H')$) under the action of this $\mathfrak{sl}_2$-triple, which implies that $\lambda_0(H') \geq 0$. Moreover we have, by definition, $\lambda(H') \geq 0$ for $\lambda \in \Sigma^+$.

If we suppose $\lambda_0(H') > 0$, then for any root $\lambda \in \widetilde{\Sigma}^+\backslash\Sigma$ we have $\lambda(H') > 0$ because $\lambda = \lambda_0 + \sum_i m_i\lambda_i \quad (m_i \in \mathbb{N}, \lambda_i \in \Sigma^+)$. But in this case, for any $Z \in \widetilde{\mathfrak{g}}^{\lambda} \subset V^+$ we have

$$Z = -\frac{1}{\lambda(H')} \operatorname{ad} X'(\operatorname{ad} Y'.Z) .$$

This implies that $\operatorname{ad} X' : \mathfrak{g} \to V^+$ is onto , *i.e.* that X' is generic in V^+ and therefore that X is generic in V^+. Hence we should have $\Delta_0(X) \neq 0$. Thus we have proved that $\lambda_0(H') = 0$.

It remains to show that X' belongs to V_1^+. The $\mathfrak{sl}_2$–module generated by an element $Y_0 \in \widetilde{\mathfrak{g}}^{-\lambda_0}\backslash\{0\}$ under the action of $\{Y', H', X'\}$ has lowest weight 0 (which is the weight of Y_0) and therefore is the trivial module. Hence we have $\operatorname{ad} X'.Y_0 = 0$. But X' is also commuting with $\widetilde{\mathfrak{g}}^{\lambda_0}$, hence with H_{λ_0} and so belongs to V_1^+.

(2) Let now X be an element of V^+ such that $\Delta_1(X) \neq 0$. If we write

$$X = X_0 + X_1 + X_2 \text{ with } X_0 \in \widetilde{\mathfrak{g}}^{\lambda_0},\ X_1 \in W_1^+\ X_2 \in V_1^+ ,$$

then the definition (1.14.1) of Δ_1 implies that

$$\Delta_1(X) = \Delta_1(X_2) .$$

The hypothesis implies that X_2 is generic in V_1^+ and Proposition 1.7.9 applied to the graded algebra $\widetilde{\mathfrak{g}}_1$ implies the existence of $Y_2 \in V_1^-$ such that $\{Y_2, H_{\lambda_1} + \cdots + H_{\lambda_k}, X_2\}$ is a $\mathfrak{sl}_2$-triple.

The weights for the action of this $\mathfrak{sl}_2$-triple on V^+ are 2 on V_1^+, 1 on W_1^+ and 0 on $\widetilde{\mathfrak{g}}^{\lambda_0}$. Define $\mathfrak{g}_{-1}$ by

$$\mathfrak{g}_{-1} = \{X \in \mathfrak{g} \mid [H_{\lambda_1} + \cdots + H_{\lambda_k}, X] = -X\} .$$

Then the map $\operatorname{ad} X_2$ is a bijection from $\mathfrak{g}_{-1}$ onto W_1^+. Thus there exists an element $Z \in \mathfrak{g}_{-1}$ such that $[X_2, Z] = X_1$. Writing the decomposition of $e^{\operatorname{ad} Z}X$ with respect to the weight spaces of V^+, we obtain

$$\begin{aligned} e^{\operatorname{ad} Z}X =& X_0 + [Z, X_1] + \frac{1}{2}[Z, [Z, X_2]] + X_1 + [Z, X_2] + X_2 \\ =& X_0 + [Z, X_1] + \frac{1}{2}(\operatorname{ad} Z)^2.X_2 + X_2 . \end{aligned}$$

Thus $e^{\operatorname{ad} Z}X$ belongs to $X_2 \oplus \widetilde{\mathfrak{g}}^{\lambda_0}$ *i.e.* to $V_1^+ \oplus \widetilde{\mathfrak{g}}^{\lambda_0}$ which proves (2).

□

Theorem 2.2.2. *Each element of V^+ belongs to the G-orbit of an element of $\widetilde{\mathfrak{g}}^{\lambda_0} \oplus \cdots \oplus \widetilde{\mathfrak{g}}^{\lambda_k}$.*

Proof. — Each element X of V^+ is conjugate under G to an element of $\widetilde{\mathfrak{g}}^{\lambda_0} \oplus V_1^+$. If X is not generic, it is the first part of Proposition 2.2.1. If X is generic in V^+, the G–orbit of X in V^+ is open and intersects the open dense set $\{Y \in V^+ \mid \Delta_1(Y) \neq 0\}$. Then the second part of Proposition 2.2.1 gives the result.

The proof is finished by induction. In fact the same proof applied in $\widetilde{\mathfrak{g}}_j$ shows that if X belongs to V_j^+ $(j = 1, \ldots, k-1)$, it is conjugate under an element of G_j to some element of $\widetilde{\mathfrak{g}}^{\lambda_j} \oplus V_{j+1}^+$. Moreover each element of G_j stabilizes the set $\widetilde{\mathfrak{g}}^{\lambda_0} \oplus \cdots \oplus \widetilde{\mathfrak{g}}^{\lambda_{j-1}}$. By Corollary 1.8.4 we have the equality $V_k^+ = \widetilde{\mathfrak{g}}^{\lambda_k}$. This gives the result after k steps.

□

2.3. An involution which permutes the roots in $E_{i,j}(\pm 1, \pm 1)$

We introduce now the compact groups $\widetilde{K}$ and K which are respectively the subgroups of $\widetilde{G}$ and G whose points are fixed by the Cartan involution θ (which is well defined on the adjoint group of $\widetilde{\mathfrak{g}}$). The action on $\mathfrak{a}^{\mathfrak{p}}$ of the Weyl group W (*resp.* $\widetilde{W}$) of the root system Σ (*resp.* $\widetilde{\Sigma}$) is given by the elements of the normalizer $N_K(\mathfrak{a}^{\mathfrak{p}})$ (*resp.* $N_{\widetilde{K}}(\mathfrak{a}^{\mathfrak{p}})$) of $\mathfrak{a}^{\mathfrak{p}}$ in K (*resp.* $\widetilde{K}$).

Let X_i be an element of $\widetilde{\mathfrak{g}}^{\lambda_i}$ such that $\{\theta(X_i), H_{\lambda_i}, X_i\}$ is a $\mathfrak{sl}_2$-triple. The action of the nontrivial element of the Weyl group of this triple is given by

$$w_i = e^{\operatorname{ad} X_i} e^{\operatorname{ad} \theta(X_i)} e^{\operatorname{ad} X_i} \ .$$

The equality $\theta \circ w_i = e^{\operatorname{ad} \theta(X_i)} \circ e^{\operatorname{ad} X_i} \circ e^{\operatorname{ad} \theta(X_i)} \circ \theta = w_i \circ \theta$ implies that w_i belongs to $\widetilde{K}$.

Lemma 2.3.1. *Let $j \neq i \in \{0, \ldots, k\}$ and $p = \pm 1$.*
a) If H belongs to $\mathfrak{a}^{\mathfrak{p}}$ then $w_i.H = H - \lambda_i(H) H_{\lambda_i}$;
b) If X belongs to $E_{i,j}(1,p)$ then $w_i.X = \operatorname{ad}\theta(X_i)X \in E_{i,j}(-1,p)$;
c) If X belongs to $E_{i,j}(-1,p)$ then $w_i.X = \operatorname{ad}(X_i)X \in E_{i,j}(1,p)$;
d) If X belongs to $E_{i,j}(\pm 1,p)$ then $w_i^2.X = -X$.

Proof. — The element w_i is a representative in $N_{\widetilde{K}}(\mathfrak{a}^{\mathfrak{p}})$ of the reflexion of $\mathfrak{a}^{\mathfrak{p}}$ along the root λ_i and the assertion a) follows.

Since each $E_{i,j}(\pm 1, p)$ is contained in a weight space for the action of the $\mathfrak{sl}_2$–triple on $\widetilde{\mathfrak{g}}$, the assertions b), c) and d) are easy consequences of the elementary properties of $\mathfrak{sl}_2$–modules listed in section 2.1.

□

The previous Lemma implies that the element

$$w_{i,j} = w_i w_j = w_j w_i \ ,$$

satisfies the following properties :

Proposition 2.3.2. *For $i \neq j \in \{0, \ldots, k\}$ and $p, q \in \{\pm 1\}$, $w_{i,j}$ is an element of $N_{\widetilde{K}}(\mathfrak{a}^{\mathfrak{p}})$ such that*

(1) *$w_{i,j}$ is an isomorphism from $E_{i,j}(p,q)$ onto $E_{i,j}(-p,-q)$;*

(2) *The restriction of $w_{i,j}^2$ to $E_{i,j}(p,q)$ is the identity.*

Notation 2.3.3. *The involution $w_{i,j} \circ \theta = \theta \circ w_{i,j}$ of $E_{i,j}(\pm 1, \pm 1)$ will be denoted by $\tau_{i,j}$.*

This involution permutes the roots λ such that $\widetilde{\mathfrak{g}}^{\lambda} \subset E_{i,j}(\pm 1, \pm 1)$. More precisely we have

$$\widetilde{\mathfrak{g}}^{\lambda} \subset E_{i,j}(1,-1) \Longrightarrow \tau_{i,j}(\lambda) = -\lambda + \lambda_i - \lambda_j \ ,$$
$$\widetilde{\mathfrak{g}}^{\lambda} \subset E_{i,j}(1,1) \Longrightarrow \tau_{i,j}(\lambda) = -\lambda + \lambda_i + \lambda_j \ ,$$
$$\widetilde{\mathfrak{g}}^{\lambda} \subset E_{i,j}(-1,-1) \Longrightarrow \tau_{i,j}(\lambda) = -\lambda - \lambda_i - \lambda_j \ .$$

It follows that a root λ such that $\widetilde{\mathfrak{g}}^{\lambda} \subset E_{i,j}(1,-1)$ satisfies $\tau_{i,j}(\lambda) = \lambda$ if and only if $\lambda = \frac{\lambda_i - \lambda_j}{2}$. In some cases $\frac{\lambda_i - \lambda_j}{2}$ is not a root. It will appear that the situation is quite different if $\tau_{i,j}$ does or does not admit a non fixed point among the roots occuring in $E_{i,j}(1,-1)$.

2.4. Construction of elements interchanging λ_i and λ_j

We suppose in this section that i and j are distinct elements of $\{0, \ldots, k\}$. We know from Proposition 1.9.2 that the roots λ_i and λ_j are W-conjugate. We want to obtain explicitly an element in K which interchanges λ_i and λ_j and to know how it acts on $\widetilde{\mathfrak{g}}$. The element we are looking for will be a representative of the nontrivial element of the Weyl group of the $\mathfrak{sl}(2)$–triple given in the following Lemma.

Lemma 2.4.1. *There exists an element $X \in E_{i,j}(1,-1)$ such that*

$$\{\theta(X), H_{\lambda_i} - H_{\lambda_j}, X\}$$

is a $\mathfrak{sl}(2)$-triple. More precisely we can choose for X a multiple of any nonzero element of $E_{i,j}(1,-1)$ such that $\tau_{i,j}(X) = X$ or $\tau_{i,j}(X) = -X$.

Proof. — Since $\tau_{i,j}$ is an involution of $E_{i,j}(1,-1)$, there exists an element $X \in E_{i,j}(1,-1)$ such that $X \neq 0$ and $\tau_{i,j}(X) = \pm X$. It is clear that

$$[H_{\lambda_i} - H_{\lambda_j}, X] = 2X \text{ and } [H_{\lambda_i} - H_{\lambda_j}, \theta(X)] = -2\theta(X) \ .$$

It remains to study the element $Y = [\theta(X), X]$ which belongs to $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}^0)$ and can be written as

$$Y = Z + \sum_{\lambda \in \Sigma, \lambda_{|\mathfrak{a}^0} = 0} Y_{\lambda} \quad \text{where } Y_{\lambda} \in \mathfrak{g}^{\lambda} \text{ and } Z \in \mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}^{\mathfrak{p}}) \ .$$

Since $\theta(Y) = -Y$, we have $\theta(Z) = -Z$ which implies that Z belongs to $\mathfrak{a}^{\mathfrak{p}}$ and $\theta(Y_{\lambda}) = -Y_{-\lambda}$. By Corollary 1.8.2 the roots λ occuring in the decomposition of Y are strongly orthogonal to all the roots λ_s $(s = 0, \ldots, k)$. Thus the action of $w_{i,j}$ on Y_{λ} is trivial and we obtain

$$\tau_{i,j}(Y) = \tau_{i,j}(Z) + \sum(-Y_{-\lambda}) \ .$$

But the choice of X implies that $\tau_{i,j}(Y) = Y$ and consequently that $Y_\lambda = 0$ for each λ. Thus

$$Y = Z \in \mathfrak{a}^{\mathfrak{p}} \text{ and } \tau_{i,j}(Y) = Y \ .$$

Lemma 2.3.1 implies that $\tau_{i,j}(Y) = -(Y - \lambda_i(Y)H_{\lambda_i} - \lambda_j(Y)H_{\lambda_j})$, hence Y belongs to $\mathbb{R}H_{\lambda_i} \oplus \mathbb{R}H_{\lambda_j}$. Now a classical calculus shows that for $H \in \mathfrak{a}^o$

$$\widetilde{B}(H,Y) = \frac{-\lambda_i(H) + \lambda_j(H)}{2}\widetilde{B}(\theta(X),X) \ ,$$

which proves that there exists a strictly positive constants C_1 such that

$$\widetilde{B}(H,Y) = C_1\frac{\lambda_i(H) - \lambda_j(H)}{2} \text{ for } H \in \mathfrak{a}^0 \ .$$

Furthermore the roots λ_i and λ_j are W–conjugate (Proposition 1.9.2). Then we have $\widetilde{B}(H_{\lambda_i}, H_{\lambda_i}) = \widetilde{B}(H_{\lambda_j}, H_{\lambda_j})$. This implies that there exists a strictly positive constant C_2 such that

$$\widetilde{B}(H, H_{\lambda_i} - H_{\lambda_j}) = C_2\frac{\lambda_i(H) - \lambda_j(H)}{2} \text{ for } H \in \mathfrak{a}^0 \ .$$

Since $\widetilde{B}$ is non-degenerate on $\mathfrak{a}^0$, we have

$$Y = \frac{C_1}{C_2}(H_{\lambda_i} - H_{\lambda_j}) \ ,$$

and if we replace X by $\sqrt{\frac{C_2}{C_1}}X$, we obtain now that

$$[\theta(X), X] = H_{\lambda_i} - H_{\lambda_j} \ .$$

□

Remark 2.4.2. The previous construction is exactly the starting point of the proof by ROSSMANN ([Ro]–1979) of the following result : if τ and θ are commuting involutions on a semisimple algebra $\mathfrak{g}$ (θ being the Cartan involution) and $\mathfrak{b}$ is a maximal abelian subspace of the set

$$\{X \in \mathfrak{g} \mid \tau(X) = -X\} \bigcap \{X \in \mathfrak{g} \mid \theta(X) = -X\} \ ,$$

then the set of roots $\Sigma(\mathfrak{g}, \mathfrak{b})$ is a root system.

Proposition 2.4.3. *For $i \neq j \in \{0, \ldots, k\}$ there exists $\gamma_{i,j} \in N_K(\mathfrak{a}^0)$ such that*

$$\textit{(i)}\ \gamma_{i,j}(H_{\lambda_s}) = \begin{cases} H_{\lambda_i} & \textit{for } s = j \ , \\ H_{\lambda_j} & \textit{for } s = i \ , \\ H_{\lambda_s} & \textit{for } s \notin \{i,j\} \ . \end{cases}$$

(ii) The action of $\gamma_{i,j}$ is trivial on each root space $\widetilde{\mathfrak{g}}^{\lambda_s}$ for $s \notin \{i,j\}$ and is an involutive isomorphism from $\widetilde{\mathfrak{g}}^{\lambda_i}$ onto $\widetilde{\mathfrak{g}}^{\lambda_j}$.

Proof. — Let $\{\theta(X), H_{\lambda_i} - H_{\lambda_j}, X\}$ be the $\mathfrak{sl}(2)$-triple obtained in the previous Lemma . The action of the nontrivial element of the Weyl group of this $\mathfrak{sl}(2)$-triple is given by

$$\gamma_{i,j} = e^{\operatorname{ad} X}e^{\operatorname{ad}\theta(X)}e^{\operatorname{ad} X} = e^{\operatorname{ad}\theta(X)}e^{\operatorname{ad} X}e^{\operatorname{ad}\theta(X)} \in K \ .$$

Since X belongs to $E_{i,j}(1,-1)$, we have for $H \in \mathfrak{a}^0$

$$\gamma_{i,j}(H) = H - \frac{\lambda_i(H) - \lambda_j(H)}{2}(H_{\lambda_i} - H_{\lambda_j}) \ ,$$

which proves *(i)*.

For $\{s,\ell\}\bigcap\{i,j\} = \emptyset$, we have $[E_{s,\ell}(p,q), E_{i,j}(p,q)] = 0$. Thus the action of $\gamma_{i,j}$ is trivial on the spaces $E_{s,\ell}(p,q)$ and in particular on $\widetilde{\mathfrak{g}}^{\lambda_s}$ for $s \notin \{i,j\}$. Moreover *(i)* implies that $\gamma_{i,j}$ is an isomorphism from $\widetilde{\mathfrak{g}}^{\lambda_i}$ onto $\widetilde{\mathfrak{g}}^{\lambda_j}$. It is involutive since the action of $\gamma_{i,j}^2$ is trivial on the spaces of even weight for $H_{\lambda_i} - H_{\lambda_j}$ (see section 2.1).

□

The action of $\gamma_{i,j}^2$ is not trivial on $\widetilde{\mathfrak{g}}$. In fact for X in a odd weight space for $H_{\lambda_i} - H_{\lambda_j}$ we have $\gamma_{i,j}^2(X) = -X$. It is the case for $X \in E_{i,s}(\pm 1, \pm 1)$ and $X \in E_{j,s}(\pm 1, \pm 1)$ if $s \notin \{i,j\}$. We will modify slightly $\gamma_{i,j}$ by setting

$$\widetilde{\gamma_{i,j}} = \gamma_{i,j} w_i^2 \ ,$$

where w_i is defined in section 2.3. We obtain now an element of $\widetilde{K}$ and not of K with the following properties.

Proposition 2.4.4. *For $i \neq j \in \{0,\dots,k\}$ there exists $\widetilde{\gamma_{i,j}} \in N_{\widetilde{K}}(\mathfrak{a}^0)$ such that*

(i) $\widetilde{\gamma_{i,j}}(H_{\lambda_s}) = \begin{cases} H_{\lambda_i} & \text{for } s = j \ , \\ H_{\lambda_j} & \text{for } s = i \ , \\ H_{\lambda_s} & \text{for } s \notin \{i,j\} \ . \end{cases}$

(ii) $\widetilde{\gamma_{i,j}}^2 = \mathrm{Id}_{\widetilde{\mathfrak{g}}}$.

Proof. — We know (section 2.1) that the action of w_i^2 on the spaces $E_{i,s}(\pm 1, \pm 1)$ for $s \neq i$ is given by the product by (-1) and that this action is trivial on the orthogonal of these subspaces. On the other hand the action of $\gamma_{i,j}$ is trivial on the spaces $E_{s,\ell}(p,q)$ for $\{s,\ell\}\bigcap\{i,j\} = \emptyset$ and is an isomorphism from $E_{i,s}(p,q)$ onto $E_{j,s}(p,q)$. Then

$$\widetilde{\gamma_{i,j}}^2(X) = \begin{cases} -\gamma_{i,j}^2(X) & \text{for } X \in \oplus_{s\notin\{i,j\}}(E_{i,s}(\pm 1,\pm 1) \oplus E_{j,s}(\pm 1,\pm 1)) \ , \\ \gamma_{i,j}^2(X) & \text{for } X \text{ in the other } E_{s,\ell}(p,q) \ . \end{cases}$$

But the subspace $\oplus_{s\notin\{i,j\}}(E_{i,s}(\pm 1,\pm 1) \oplus E_{j,s}(\pm 1,\pm 1))$ is exactly the sum of the odd weight spaces for $H_{\lambda_i} - H_{\lambda_j}$ which implies *(ii)*.

□

The following Proposition will show that the situation is more easy to handle if the involution $\tau_{i,j}$ admits a non fixed root among the roots of $E_{i,j}(1,-1)$ *i.e.* a root distinct from $\frac{\lambda_i - \lambda_j}{2}$

Proposition 2.4.5. *We assume that there exists a root μ distinct of $\frac{\lambda_i-\lambda_j}{2}$ such that $\mathfrak{g}^\mu \subset E_{i,j}(1,-1)$. Then there exists an element $m_{i,j} \in Z_K(\mathfrak{a}^0)$ such that for $Z \in \widetilde{\mathfrak{g}}^{\lambda_s}$ we have*

$$m_{i,j}(Z) = \begin{cases} Z & \text{for } s \notin \{i,j\} \ , \\ -Z & \text{for } s \in \{i,j\} \ . \end{cases}$$

Proof. — Let $Y \neq 0$ be an element of $\mathfrak{g}^\mu$. Then $\tau_{i,j}(Y)$ belongs to $\mathfrak{g}^{-\mu+\lambda_i-\lambda_j}$. Since the roots μ and $-\mu + \lambda_i - \lambda_j$ are distinct by hypothesis,

we have

$$X' = Y + \tau_{i,j}(Y) \neq 0 \ ,$$
$$X'' = Y - \tau_{i,j}(Y) \neq 0 \ .$$

Thus there exist (see the proof of Lemma 2.4.1) $C' > 0$ and $C'' > 0$ such that $\{\theta(C'X'), H_{\lambda_i} - H_{\lambda_j}, C'X'\}$ and $\{\theta(C''X''), H_{\lambda_i} - H_{\lambda_j}, C''X''\}$ are $\mathfrak{sl}(2)$-triple. Let us denote by $\gamma'_{i,j}$ and $\gamma''_{i,j}$ the Weyl group elements associated to these $\mathfrak{sl}_2$–triples as in Proposition 2.4.3. We define $m_{i,j}$ by

$$m_{i,j} = \gamma'_{i,j}\gamma''_{i,j} \ .$$

Proposition 2.4.3 implies that $m_{i,j}$ is an element of K which centralizes $\mathfrak{a}^0$ and which acts trivially on $\widetilde{\mathfrak{g}}^{\lambda_s}$ for $s \notin \{i, j\}$. It remains to study the action of m_{ij} on $\widetilde{\mathfrak{g}}^{\lambda_i}$ and on $\widetilde{\mathfrak{g}}^{\lambda_j}$.

An element Z in $\widetilde{\mathfrak{g}}^{\lambda_i}$ is a primitive element of weight 2 for these $\mathfrak{sl}(2)$-triples. Thus we have (see section 2.1)

$$\gamma'_{i,j}(Z) = \frac{1}{2}(\operatorname{ad} C'\theta(X'))^2 Z = \frac{C'^2}{2}\Big(\operatorname{ad}(Y + \tau_{ij}(Y))\Big)^2 Z \ ,$$
$$\gamma''_{i,j}(Z) = \frac{1}{2}(\operatorname{ad} C''\theta(X''))^2 Z = \frac{C''^2}{2}\Big(\operatorname{ad}(Y - \tau_{ij}(Y))\Big)^2 Z \ .$$

Since Y belongs to $\mathfrak{g}^{\mu}$, then $(\operatorname{ad}\theta(Y))^2 Z$ belongs to $\widetilde{\mathfrak{g}}^{\lambda_i - 2\mu}$. Since $\mathfrak{g}^{\mu}$ is a subset of $E_{i,j}(1,-1)$, then $(\operatorname{ad}\theta(Y))^2 Z$ belongs to $E_{i,j}(0,2) = \widetilde{\mathfrak{g}}^{\lambda_j}$. As $\mu \neq \frac{\lambda_i - \lambda_j}{2}$ we have $(\operatorname{ad}\theta(Y))^2.Z = 0$. Using the same argument, we show that $(\operatorname{ad}(w_{i,j}Y))^2.Z = 0$. Therefore we obtain

$$\gamma'_{i,j}(Z) = C'^2 \operatorname{ad} Y \operatorname{ad} \tau_{i,j}(Y) Z \ ,$$
$$\gamma''_{i,j}(Z) = -C''^2 \operatorname{ad} Y \operatorname{ad} \tau_{i,j}(Y) Z \ ,$$

which implies that

$$\gamma'_{i,j}(Z) = -\frac{C''^2}{C'^2}\gamma''_{i,j}(Z) \ .$$

But $\gamma'_{i,j}$ and $\gamma''_{i,j}$ are isometries for the product $(X, Y) \mapsto -\widetilde{B}(\theta(X), Y)$. Thus $C' = C''$ and therefore $\gamma'_{i,j}\gamma''_{i,j}(Z) = -{\gamma'_{i,j}}^2(Z)$. Proposition 2.4.3 (ii) implies that $\gamma'_{i,j}\gamma''_{i,j}(Z) = -Z$.

The same argument gives the result for $Z \in \widetilde{\mathfrak{g}}^{\lambda_j}$.

□

2.5. Quadratic forms

To each element $X \in V^+$, we associate the quadratic form Q_X on V^- defined by

$$Q_X(Y) = b(e^{\operatorname{ad} X}Y, Y) \text{ for } Y \in V^- \ .$$

For $g \in G$, the quadratic forms Q_X and $Q_{g.X}$ are equivalent. Therefore we will study these forms for $X \in \oplus_{j=0}^{k} \widetilde{\mathfrak{g}}^{\lambda_j}$ (see Theorem 2.2.2). More precisely we will determine the rank of these forms because it is sufficient, in some cases, to separate the G–orbits of the elements $X \in V^+$.

Using the gradation of $\widetilde{\mathfrak{g}}$ which is an orthogonal decomposition for $\widetilde{B}$ and b we obtain

$$Q_X(Y) = \frac{1}{2} b((\operatorname{ad} X)^2 Y, Y) \text{ for } X \in V^+, Y \in V^- \ .$$

Let X be an element of V^+ of the form

$$X = \sum_{j=0}^{k} X_j \text{ where } X_j \in \widetilde{\mathfrak{g}}^{\lambda_j} \ .$$

There is only one subspace of the form $E_{s,\ell}(p,q)$ of V^- where $\operatorname{ad} X_i \operatorname{ad} X_j$ is nonzero, namely $E_{i,j}(-1,-1)$ for $i \neq j$ and $\widetilde{\mathfrak{g}}^{-\lambda_j}$ for $i = j$. Therefore we introduce the quadratic forms q_{X_i,X_j} defined on $E_{i,j}(-1,-1)$ (*resp.* $\widetilde{\mathfrak{g}}^{-\lambda_j}$) for $i \neq j$ (*resp.* $i = j$) by

$$q_{X_i,X_j}(Y) = -\frac{1}{2} b(\operatorname{ad} X_i.Y, \operatorname{ad} X_j.Y) \quad \text{for } Y \in E_{i,j}(-1,-1) \ (\textit{resp.}\ \widetilde{\mathfrak{g}}^{-\lambda_j}) \ .$$

Then, for an element X of the given form, the quadratic form Q_X is equivalent to

$$\big(\bigoplus_{j=0}^{k} q_{X_j,X_j}\big) \oplus \big(\bigoplus_{i<j} 2\, q_{X_i,X_j}\big) \ .$$

Theorem 2.5.1. *Let $X = \sum_{j=0}^{k} X_j$ where $X_j \in \widetilde{\mathfrak{g}}^{\lambda_j}$ and m the number of indices j such that $X_j \neq 0$. Then*

$$\operatorname{rank} Q_X = m\ell + \frac{m(m-1)}{2} d \ .$$

We recall (Notation 1.9.4) that $\ell = \dim \widetilde{\mathfrak{g}}^{\lambda_j}$ and $d = \dim E_{i,j}(-1,-1)$ for $i \neq j$.

Proof. —The decomposition of the quadratic form Q_X given above shows that it is sufficient to prove that the forms q_{X_i,X_j} are non-degenerate on $E_{i,j}(-1,-1)$ for $i \neq j$ (*resp.* $\widetilde{\mathfrak{g}}^{-\lambda_j}$ for $i = j$).

The bilinear form associated to q_{X_i,X_j} is given for $u, v \in E_{i,j}(-1,-1)$ (*resp.* $\widetilde{\mathfrak{g}}^{-\lambda_j}$) by

$$L_{X_i,X_j}(u,v) = -\frac{1}{2} b(\operatorname{ad} X_i.u, \operatorname{ad} X_j.v) \ .$$

In the case $i = j$, we know from section 2.1 that $(\operatorname{ad} X_j)^2$ is an isomorphism from $\widetilde{\mathfrak{g}}^{-\lambda_j}$ onto $\widetilde{\mathfrak{g}}^{\lambda_j}$ and it is well known that $\widetilde{B}$ is non-degenerate on $\widetilde{\mathfrak{g}}^{-\lambda_j} \times \widetilde{\mathfrak{g}}^{\lambda_j}$.

In the case $i \neq j$, let λ and μ be two roots in the decomposition of $E_{i,j}(-1,-1)$. The only case where L_{X_i,X_j} is nonzero on $\widetilde{\mathfrak{g}}^{\lambda} \times \widetilde{\mathfrak{g}}^{\mu}$ occurs when the roots $\lambda + \lambda_i$ and $\lambda + \lambda_j$ are opposite roots, *i.e.* $\mu = \tau_{i,j}(\lambda)$ ($\tau_{i,j}$ is defined in section 2.3). It remains to show that L_{X_i,X_j} is non-degenerate on $\widetilde{\mathfrak{g}}^{\lambda} \times \widetilde{\mathfrak{g}}^{\tau_{i,j}(\lambda)}$. For that purpose let Y be an element of $\widetilde{\mathfrak{g}}^{\lambda}$ such that

$$b(\operatorname{ad} X_i \operatorname{ad} X_j.Y, Z) = 0 \text{ for all } Z \in \widetilde{\mathfrak{g}}^{\tau_{i,j}(\lambda)} \ .$$

In particular the element $Z = \theta(\operatorname{ad} X_i \operatorname{ad} X_j.Y)$ belongs to $\widetilde{\mathfrak{g}}^{\tau_{i,j}(\lambda)}$, which implies that $\operatorname{ad} X_i \operatorname{ad} X_j.Y = 0$. From section 2.1 we know that $\operatorname{ad} X_i$ is an isomorphism from $E_{i,j}(-1,-1)$ onto $E_{i,j}(1,-1)$ and that $\operatorname{ad} X_j$ is an

isomorphism from $E_{i,j}(1,-1)$ onto $E_{i,j}(1,1)$. Therefore $Y = 0$ and the Theorem is proved. □

The study of the G–orbits in V^+ induces a classification of the regular real graded algebras into three types that we will call Type I, Type II and Type III . The description of the G-orbits is easy to obtain in Type II and Type III cases and will be done in sections 2.6 and 2.7.

2.6. The G-orbits for Type III

We recall that all the spaces $\widetilde{\mathfrak{g}}^{\lambda_i}$ $(j = 0, \ldots, k)$ have the same dimension, denoted by ℓ.

Definition 2.6.1 (Type III). *We say that the regular graded algebra $(\widetilde{\mathfrak{g}}, H_0)$ is of Type III if the dimension of the root spaces $\widetilde{\mathfrak{g}}^{\lambda_j}$ is different from 1 i.e. $\ell > 1$.*

Theorem 2.6.2 (Type III). *If $\ell > 1$, the number of G-orbits in V^+ is $k+2$ with only one open orbit. Assume that X_j $(j = 0, \ldots, k)$ are nonzero elements in $\widetilde{\mathfrak{g}}^{\lambda_j}$. Then the G-orbits in V^+ are the orbits of the elements*

$$0 \text{ and } X_0 + \cdots + X_m \text{ for } m = 0, 1, \ldots, k \ .$$

The open orbit is the orbit of the element $X_0 + X_1 + \cdots + X_k$

Proof. — Theorem 2.2.2 shows that each element of V^+ is conjugate under G to an element of the form

$$Y = Y_0 + Y_1 + \cdots + Y_k \text{ where } Y_j \in \widetilde{\mathfrak{g}}^{\lambda_j} \ .$$

If $Y \neq 0$, let $m > 0$ be the number of indices j such that $Y_j \neq 0$. Then Y is G-conjugate to an element of the form

$$Z_0 + Z_1 + \cdots + Z_m \text{ where } Z_j \in \widetilde{\mathfrak{g}}^{\lambda_j} \setminus \{0\} \ .$$

To prove this assertion, it is sufficient to use the element $\gamma_{i,j}$ obtained in Proposition 2.4.3 which interchanges the spaces $\widetilde{\mathfrak{g}}^{\lambda_i}$ and $\widetilde{\mathfrak{g}}^{\lambda_j}$ whereas the spaces $\widetilde{\mathfrak{g}}^{\lambda_s}$ are fixed for $s \notin \{i, j\}$.

Furthermore, we proved (Theorem 1.12.1) that, in case $\ell > 1$, two nonzero elements of $\widetilde{\mathfrak{g}}^{\lambda_j}$ are L_j-conjugate with $L_j \subset G$. Moreover the action of L_j on the spaces $\widetilde{\mathfrak{g}}^{\lambda_s}$ is trivial for $s \neq j$. Therefore $Z_0 + \cdots + Z_m$ is G-conjugate to $X_0 + \cdots + X_m$.

It only remains to prove that for $m \neq m'$ the elements $X = X_0 + \cdots + X_m$ and $X' = X_0 + \cdots + X_{m'}$ are not G-conjugate. If they were G-conjugate, the quadratic forms Q_X and $Q_{X'}$ would have the same rank. But from Theorem 2.5.1 we have

$$\operatorname{rank} Q_X - \operatorname{rank} Q_{X'} = (m - m') \left(\ell + \frac{(m + m' - 1)d}{2} \right) \ ,$$

which is nonzero for $m' \neq m$.

□

2.7. The G-orbits for Type II

Definition 2.7.1 (Type II). *We say that the regular graded algebra $(\widetilde{\mathfrak{g}}, H_0)$ is of Type II if the dimension of each root spaces $\widetilde{\mathfrak{g}}^{\lambda_j}$ is equal to 1 i.e. $\ell = 1$ and if, for any pair $\{i,j\}$ with $i \neq j$, there exists a root $\mu \neq \frac{\lambda_i+\lambda_j}{2}$ such that $\widetilde{\mathfrak{g}}^{\mu} \subset E_{i,j}(1,1)$.*

Remark 2.7.2. Using the elements $\gamma_{i,j} \in N_K(\mathfrak{a}^0)$, we can prove that if the assumption is verified for one pair $\{i_0, j_0\}$ it is also verified for each pair $\{i,j\} \subset \{0,\dots,k\}^2$. Therefore the graded algebra $(\widetilde{\mathfrak{g}}, H_0)$ is of Type II if and only if

$$E_{0,1}(1,1) \neq \widetilde{\mathfrak{g}}^{\frac{\lambda_0+\lambda_1}{2}} \ .$$

It may happen that $\frac{\lambda_0+\lambda_1}{2}$ is not a root and then the assumption is automatically verified.

Moreover by Lemma 2.3.1 there exist isomorphisms from $E_{i,j}(1,1)$ onto $E_{i,j}(-1,1)$ and from $E_{i,j}(-1,1)$ onto $E_{i,j}(-1,-1)$ which stabilize $\mathfrak{a}^{\mathfrak{p}}$. The assumption is also equivalent to

$$E_{i,j}(-1,1) \neq \widetilde{\mathfrak{g}}^{\frac{-\lambda_i+\lambda_j}{2}} \quad \text{or} \quad E_{i,j}(-1,-1) \neq \widetilde{\mathfrak{g}}^{\frac{-\lambda_i-\lambda_j}{2}} \ ,$$

for one or each pair $\{i,j\} \subset \{0,\dots,k\}^2$.

Theorem 2.7.3 (Type II). *If $\ell = 1$ and $E_{0,1}(1,1) \neq \widetilde{\mathfrak{g}}^{\frac{\lambda_0+\lambda_1}{2}}$ the number of G–orbits in V^+ is $k+3$ with exactly two open orbits. Assume that X_j $(j = 0,\dots,k)$ are nonzero elements in $\widetilde{\mathfrak{g}}^{\lambda_j}$. Then the G-orbits in V^+ are the orbits of the elements*

$$X_0 + \cdots + X_k \quad \text{and} \quad -X_0 + X_1 + \cdots + X_k \qquad \text{for the open orbits ,}$$
$$0 \text{ and } X_0 + \cdots + X_m \ (m = 0,\dots,k-1) \qquad \text{for the non open orbits .}$$

Let ε_j, $\varepsilon'_j \in \{0,1,-1\}$. Then the elements $\sum_{j=0}^k \varepsilon_j X_j$ and $\sum_{j=0}^k \varepsilon'_j X_j$ are G–conjugate if and only if

$$\prod_{j=0}^{k} \varepsilon_j = \prod_{j=0}^{k} \varepsilon'_j \quad \text{and} \quad \#\{j \mid \varepsilon_j \neq 0\} = \#\{j \mid \varepsilon'_j \neq 0\}$$

(where $\#A$ denotes the number of elements of the set A).

Proof. — The same proof as in Theorem 2.6.2 shows that each nonzero element X of V^+ is conjugate under G to an element of the form

$$Z_0 + \cdots + Z_m \ \text{ with } Z_j \in \widetilde{\mathfrak{g}}^{\lambda_j} \setminus \{0\} \text{ and } m \in \{0,\dots,k\} \ .$$

But, now the dimension of $\widetilde{\mathfrak{g}}^{\lambda_j}$ is one and Z_j is conjugate by an element of the form $e^{t \operatorname{ad} H_{\lambda_j}}$ $(t \in \mathbb{R})$ to X_j or $-X_j$. Moreover the action of $e^{t \operatorname{ad} H_{\lambda_j}}$ is trivial on the spaces $\widetilde{\mathfrak{g}}^{\lambda_s}$ for $s \neq j$. Thus X is G-conjugate to an element of the form

$$\varepsilon_0 X_0 + \cdots + \varepsilon_m X_m \text{ with } \varepsilon_j = \pm 1 \ .$$

We are here (Type II) in a situation where the hypothesis of Proposition 2.4.5 is satisfied. It is then possible to replace simultaneously ε_j by $-\varepsilon_j$ and ε_k by $-\varepsilon_k$ whereas the other ε_i's are fixed using the conjugation by

the element $m_{j,k}$ from Proposition 2.4.5. Consequently, if $m < k$ then X is G-conjugate to $X_0 + \cdots + X_m$ and if $m = k$, then X is G-conjugate to $\pm X_0 + \cdots + X_k$.

It remains to prove that the elements given in the Theorem are not G-conjugate. If m and m' are distinct, the calculus of the rank (as in Theorem 2.6.2) of the quadratic form Q_X shows that $X_0 + \cdots + X_m$ and $X_0 + \cdots + X_{m'}$ are not conjugate. To prove that there exist effectively two open orbits *i.e.* that $X_0 + \cdots + X_k$ and $-X_0 + X_1 + \cdots + X_k$ are not conjugate, we will use the relative invariant Δ_0 (see section 1.11). Since G is a connected group, the character χ_0 takes only strictly positive values and the sign of Δ_0 is constant on an open orbit of G. Since the dimension of $\widetilde{\mathfrak{g}}^{\lambda_0}$ equals 1, Theorem 1.13.1 (2) shows that

$$\Delta_0(\varepsilon_0 X_0 + X_1 + \cdots + X_k) = \varepsilon_0 \Delta_0(X_0 + \cdots + X_k) \ .$$

Thus Δ_0 takes different sign on the orbits of $X_0 + X_1 + \cdots + X_k$ and of $-X_0 + X_1 + \cdots + X_k$. This implies the result.

□

It remains to study the G-orbits in the following case :

Definition 2.7.4 (Type I). *We say that the regular graded algebra $(\widetilde{\mathfrak{g}}, H_0)$ is of Type I if the dimension of the root spaces $\widetilde{\mathfrak{g}}^{\lambda_j}$ is equal to 1 i.e. $\ell = 1$ and if, for any pair $\{i, j\}$ with $i \neq j$, there is only one root μ such that $\widetilde{\mathfrak{g}}^{\mu} \subset E_{i,j}(1,1)$ namely $\mu = \frac{\lambda_i + \lambda_j}{2}$.*

If there is only one root μ occuring in $E_{ij}(1,1)$, we have necessarily $\tau_{ij}(\mu) = \mu$ which implies that $\mu = \frac{\lambda_i + \lambda_j}{2}$.

2.8. Signature of the quadratic forms q_{X_i,X_j}

The aim of this section is the determination of the signature of the quadratic forms q_{X_i,X_j} defined in section 2.5 on $E_{ij}(-1,-1)$ for $i \neq j$ and on $\widetilde{\mathfrak{g}}^{\lambda_j}$ for $i = j$. This result will be used in section 2.9 and 2.10 to determine the G–orbit in Type I case. It will be also used later on in the other cases (see Lemma 5.3.9 and Lemma 6.6.5).

Notation 2.8.1 (Condition C). *We say that a sequence of elements $(X_j)_{j=0,\ldots,k}$ satisfies the condition* (C) *if, for $j = 0, \ldots, k$, X_j belongs to $\widetilde{\mathfrak{g}}^{\lambda_j} \setminus \{0\}$ and $\{\theta(X_j), H_{\lambda_j}, X_j\}$ is a $\mathfrak{sl}_2$–triple.*

In this section we choose once for all such a sequence.

Proposition 2.8.2. *The signature of the quadratic form q_{X_j,X_j} on $\widetilde{\mathfrak{g}}^{-\lambda_j}$ is $(1, \ell - 1)$.*

Proof. — From the proof of Theorem 2.5.1 we know that this quadratic form is non-degenerate. The root space $\widetilde{\mathfrak{g}}^{-\lambda_j}$ is contained in the subspace of weight -2 for the action of the $\mathfrak{sl}_2$-triple $\{\theta(X_j), H_{\lambda_j}, X_j\}$. It follows that $\operatorname{ad} X_j$ is one to one on this subspace and that q_{X_j,X_j} is equivalent to the restriction of the form $u \mapsto -b(u,u)$ to the space $U_j = \operatorname{ad} X_j . \widetilde{\mathfrak{g}}^{-\lambda_j}$.

If the dimension of $\widetilde{\mathfrak{g}}^{\lambda_j}$ is one, then we have $U_j = \mathbb{R}H_{\lambda_j}$ and the quadratic form is positive definite.

In general U_j is a subspace of dimension ℓ in $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}^{\mathfrak{p}})$. Since $(\operatorname{ad} X_j)^2$ is an isomorphism from $\widetilde{\mathfrak{g}}^{-\lambda_j}$ onto $\widetilde{\mathfrak{g}}^{\lambda_j}$ it is easily seen that we also have $U_j = \operatorname{ad}\theta(X_j)\widetilde{\mathfrak{g}}^{\lambda_j}$. It follows that U_j is invariant under the action of θ. If $u \in U_j$ satisfies $\theta(u) = -u$ then we have $u \in \mathfrak{a}^{\mathfrak{p}}$ and $u = [X_j, Y]$ with $Y \in \widetilde{\mathfrak{g}}^{-\lambda_j}$. Therefore for each $H \in \mathfrak{a}^{\mathfrak{p}}$ we have

$$\widetilde{B}(H,u) = \lambda_j(H)\widetilde{B}(X_j,Y) \ ,$$

which implies that u belongs to $\mathbb{R}H_{\lambda_j}$. It follows that

$$U_j = (U_j \cap \mathfrak{k}) \oplus \mathbb{R}H_{\lambda_j} \ ,$$

where $\mathfrak{k} = \{X \in \mathfrak{g} \mid \theta(X) = X\}$. Since the Killing form is negative definite on $\mathfrak{k}$, the signature of the restriction of $-b$ to U_j is $(1, \ell - 1)$.

□

We suppose now that $k > 0$. Let i and j be distinct elements of $\{0, 1, \ldots, k\}$.

Lemma 2.8.3. *If there exists an element $m \in Z_K(\mathfrak{a}^0)$ such that $mX_j = -X_j$ and $mX_i = X_i$ then the signature of q_{X_i,X_j} is $(\frac{d}{2}, \frac{d}{2})$.*

Proof. — An element m of the centralizer of $\mathfrak{a}^0$ induces an automorphism of $E_{i,j}(-1,-1)$. The hypothesis implies that the quadratic form $q_{X_i,-X_j} = -q_{X_i,X_j}$ is equivalent to q_{X_i,X_j}. This proves the result since d is the dimension of $E_{i,j}(-1,-1)$.

□

Corollary 2.8.4. *If $(\widetilde{\mathfrak{g}}, H_0)$ is a regular graded algebra of Type II with $k > 1$ or of Type III with $k > 0$ then the signature of the quadratic form q_{X_i,X_j} is $(\frac{d}{2}, \frac{d}{2})$.*

Proof. —

In Type III case we know from Theorem 1.12.1 the existence of an element $m \in L_j$ such that $m.X_j = -X_j$. Since L_j centralizes $\widetilde{\mathfrak{g}}^{\lambda_i}$, the previous Lemma implies the result.

In Type II case with $k > 1$ let $s \in \{0, 1, \ldots, k\}$ be such that $s \notin \{i, j\}$. We know from Proposition 2.4.5 the existence of an element $m \in Z_K(\mathfrak{a}^0)$ such that

$$mX_s = -X_s \ ; \ mX_j = -X_j \ ; \ mX_i = X_i \ .$$

And the result is a consequence of the previous Lemma.

□

We know from the proof of Theorem 2.5.1 that q_{X_i,X_j} is the direct sum of its restriction to the subspaces $\widetilde{\mathfrak{g}}^{\mu} + \widetilde{\mathfrak{g}}^{\tau_{i,j}(\mu)}$ where μ is a root such that $\widetilde{\mathfrak{g}}^{\mu} \subset E_{i,j}(-1,-1)$ and $\tau_{i,j}$ is defined in section 2.3. Moreover $\tau_{i,j}(\mu) = \mu$ if and only if $\mu = -\frac{\lambda_i + \lambda_j}{2}$.

Lemma 2.8.5. *Let $(\widetilde{\mathfrak{g}}, H_0)$ be a regular graded algebra of Type I or II i.e. such that $\ell = 1$.*

(i) If $-\frac{\lambda_i+\lambda_j}{2}$ is a root, the restriction of q_{X_i,X_j} to $\widetilde{\mathfrak{g}}^{-\frac{\lambda_i+\lambda_j}{2}}$ is a positive definite or negative definite quadratic form.

(ii) Let μ be a root such that $\widetilde{\mathfrak{g}}^{\mu} \subset E_{i,j}(-1,-1)$ and $\tau_{i,j}(\mu) \neq \mu$. The restriction of q_{X_i,X_j} to $\widetilde{\mathfrak{g}}^{\mu} \oplus \widetilde{\mathfrak{g}}^{\tau_{i,j}(\mu)}$ is a quadratic form of signature (p,p) where $p = \dim \widetilde{\mathfrak{g}}^{\mu}$.

Proof. —

(i) Since q_{X_i,X_j} is a non-degenerate quadratic form it is sufficient to prove that we have $q_{X_i,X_j}(Y) \neq 0$ for each $Y \in \widetilde{\mathfrak{g}}^{-\frac{\lambda_i+\lambda_j}{2}} \backslash \{0\}$ *i.e.* that

$$b((\operatorname{ad} Y)^2 X_i, X_j) \neq 0 \ .$$

Up to scalar multiplication we can suppose that $\{Y, H_{\frac{\lambda_i+\lambda_j}{2}}, \theta(Y)\}$ is a $\mathfrak{sl}_2$-triple. Let μ be the root $\frac{\lambda_i+\lambda_j}{2}$. Since $[\lambda_i+\mu](H_{\lambda_i}) = 3$ (*resp.* $[\lambda_i-3\mu](H_{\lambda_i}) = -3$), $\lambda_i + \mu$ (*resp.* $\lambda_i - 3\mu$) is not a root. It follows that the μ-series of roots containing λ_i is the set $\{\lambda_i, \lambda_i - \mu = \frac{\lambda_i-\lambda_j}{2}, \lambda_i - 2\mu = -\lambda_j\}$. Then, by Lemma 1.4.2, we have

$$\lambda_i(H_\mu) = 2 \ .$$

Thus X_i is primitive element of weight 2 for the $\mathfrak{sl}_2$-triple defined above and consequently $(\operatorname{ad} Y)^2$ is one to one on $\widetilde{\mathfrak{g}}^{\lambda_i}$. In particular $(\operatorname{ad} Y)^2 X_i$ is a nonzero element of $\widetilde{\mathfrak{g}}^{-\lambda_j}$. Since the dimension of $\widetilde{\mathfrak{g}}^{-\lambda_j}$ is 1, there exists $x \in \mathbb{R}^*$ such that $(\operatorname{ad} Y)^2 X_i = x\theta(X_j)$. It follows that

$$b((\operatorname{ad} Y)^2 X_i, X_j) = xb(\theta(X_j), X_j) \neq 0 \ .$$

(ii) If $\tau_{i,j}(\mu)$ is distinct from μ then the subspaces $\widetilde{\mathfrak{g}}^{\mu+\lambda_j}$ and $\widetilde{\mathfrak{g}}^{\mu+\lambda_i}$ are orthogonal for the form b. Then for $Y \in \widetilde{\mathfrak{g}}^{\mu}$ and $Y' \in \widetilde{\mathfrak{g}}^{\tau_{i,j}(\mu)}$ we have $q_{X_i,X_j}(Y+Y') = L_{X_i,X_j}(Y,Y')$ where L_{X_i,X_j} is the bilinear form associated to q_{X_i,X_j}. Since $Y + Y' \mapsto Y - Y'$ is an automorphism of $\widetilde{\mathfrak{g}}^{\mu} \oplus \widetilde{\mathfrak{g}}^{\tau_{i,j}(\mu)}$ the quadratic form q_{X_i,X_j} is equivalent to $-q_{X_i,X_j}$ on $\widetilde{\mathfrak{g}}^{\mu} \oplus \widetilde{\mathfrak{g}}^{\tau_{i,j}(\mu)}$, hence of signature (p,p) where $p = \dim \widetilde{\mathfrak{g}}^{\mu}$.

□

Notation 2.8.6 (Definition of e)**.** *For $i \neq j$, the dimension of the root space $\widetilde{\mathfrak{g}}^{\frac{\lambda_i+\lambda_j}{2}}$ will be denoted by e ($e \geq 0$).*

The definitions 2.7.1 and 2.7.4 imply that

$$(\widetilde{\mathfrak{g}}, H_0) \text{ is of Type I } \iff \ell = 1 \text{ and } e = d \ ,$$
$$(\widetilde{\mathfrak{g}}, H_0) \text{ is of Type II } \iff \ell = 1 \text{ and } e < d \ .$$

Corollary 2.8.7. *If $(\widetilde{\mathfrak{g}}, H_0)$ is a regular graded algebra of*

(i) Type I, then the signature of q_{X_i,X_j} is $(d,0)$ or $(0,d)$.

(ii) Type II, then the signature of q_{X_i,X_j} is $(\frac{d+e}{2}, \frac{d-e}{2})$ or $(\frac{d-e}{2}, \frac{d+e}{2})$. Moreover in this case

$$k > 1 \Longrightarrow e = 0 \ .$$

Proof. — In Type I case we have $e = d$ and the result follows from *(i)* of the previous Lemma . In Type II case we have $e < d$ and the result follows from *(ii)* and eventually (if $e \neq 0$) *(i)* of the same Lemma. If k is strictly greater than 1, the Corollary 2.8.4 implies that $e = 0$.

□

We will prove that it is possible to choose the elements X_j in $\widetilde{\mathfrak{g}}^{\lambda_j}$ such that the signature of the quadratic forms q_{X_i,X_j} is well defined. By Corollary 2.8.7 there is some indetermination only in the case where $e \neq 0$.

Since $\tau_{i,j}$ is an involution of $\widetilde{\mathfrak{g}}^{\frac{\lambda_i-\lambda_j}{2}}$ there exists by Lemma 2.4.1 an element $U \in \widetilde{\mathfrak{g}}^{\frac{\lambda_i-\lambda_j}{2}} \backslash \{0\}$ with $\tau_{i,j}(U) = U$ or $\tau_{ij}(U) = -U$ such that $\{\theta(U), H_{\lambda_i} - H_{\lambda_j}, U\}$ is a $\mathfrak{sl}_2$-triple. Let $\gamma_{i,j} \in N_K(\mathfrak{a}^0)$ be the non trivial element of the Weyl group of this $\mathfrak{sl}_2$–triple. This element satisfies the conditions of Proposition 2.4.3.

Lemma 2.8.8. *If $\ell = 1$ and if $\frac{\lambda_i+\lambda_j}{2}$ is a root i.e. $e \neq 0$, then the quadratic form $q_{X_i,\gamma_{i,j}(X_i)}$ is positive definite on $\widetilde{\mathfrak{g}}^{-\frac{\lambda_i+\lambda_j}{2}}$.*

Proof. — Since X_i is a primitive element of weight 2 for the $\mathfrak{sl}_2$-triple $\{\theta(U), H_{\lambda_i} - H_{\lambda_j}, U\}$ we have (section 2.1)

$$\gamma_{i,j}(X_i) = \frac{1}{2}(\operatorname{ad}\theta(U))^2 X_i \ .$$

Let now $V = [\theta(X_i), U]$ which is an element of $\widetilde{\mathfrak{g}}^{-\frac{\lambda_i+\lambda_j}{2}}$. Since we have by hypothesis $[\theta(X_i), X_i] = H_{\lambda_i}$ we obtain

$$\begin{aligned} (\operatorname{ad} V)X_i &= U \ , \\ (\operatorname{ad} V)^2 X_i &= (\operatorname{ad} U)^2 \theta(X_i) = 2\theta(\gamma_{i,j}(X_i)) \ . \end{aligned}$$

Thus we have

$$\begin{aligned} q_{X_i,\gamma_{i,j}(X_i)}(V) &= \frac{1}{2} b\Big((\operatorname{ad} V)^2 X_i, \gamma_{i,j}(X_i)\Big) \\ &= b\Big(\theta\big(\gamma_{i,j}(X_i)\big), \gamma_{i,j}(X_i)\Big) > 0 \ . \end{aligned}$$

Together with Lemma 2.8.5 *(i)* this implies that $q_{X_i,\gamma_{i,j}(X_i)}$ is positive definite on $\widetilde{\mathfrak{g}}^{-\frac{\lambda_i+\lambda_j}{2}}$.

□

Notation 2.8.9 (Condition (D))**.** *We will say that a sequence of elements $(X_j)_{j=0,\dots,k}$ satisfies the condition* (D) *if*

$$\text{(D)} \begin{cases} \text{(C)} & X_j \in \widetilde{\mathfrak{g}}^{\lambda_j} \text{ and } \{\theta(X_j), H_{\lambda_j}, X_j\} \text{ is a } \mathfrak{sl}_2\text{-triple} \\ & \text{for } j = 0, \dots, k \ ; \\ \text{(C}') & \text{in the case } \ell = 1 \text{ and } e \neq 0 \text{ the restriction of } q_{X_i,X_j} \text{ to} \\ & \widetilde{\mathfrak{g}}^{-\frac{\lambda_i+\lambda_j}{2}} \text{ is positive definite for each pair } \{i,j\}\,(i \neq j) \ . \end{cases}$$

Proposition 2.8.10. *There exists a sequence $(X_j)_{j=0,\dots,k}$ which satisfies the condition* (D).

Proof. — If $\ell \neq 1$ or $e = 0$ the condition (D) is reduced to (C) and the result is proved by Lemma 1.7.4.

If $\ell = 1$ and $e \neq 0$ we will apply the Lemma 2.8.8 as follows.

First we choose $X_0 \in \widetilde{\mathfrak{g}}^{\lambda_0}$ such that $\{\theta(X_0), H_{\lambda_0}, X_0\}$ is a $\mathfrak{sl}_2$-triple. Then we choose elements $\gamma_{0,j}$ such that Proposition 2.4.3 and Lemma 2.8.8 are satisfied for $i = 0$ and $j = 1, \ldots, k$. We set $X_j = \gamma_{0,j}(X_0)$. It is clear by conjugation that $\{\theta(X_j), H_{\lambda_j}, X_j\}$ is a $\mathfrak{sl}_2$-triple.

It remains to prove that, if $k > 1$, the restriction of q_{X_i,X_j} to $\widetilde{\mathfrak{g}}^{-\frac{\lambda_i+\lambda_j}{2}}$ is positive definite for $i \neq j$, $i \neq 0$ and $j \neq 0$. The element $\gamma_{0,i}$ is an isomorphism from $E_{0,j}(-1,-1)$ onto $E_{i,j}(-1,-1)$ and X_j is invariant under the action of $\gamma_{0,i}$. It follows that for $Y \in E_{i,j}(-1,-1)$ we have

$$\begin{aligned} q_{X_i,X_j}(Y) &= -\frac{1}{2} b([\gamma_{0,i}(X_0), Y], [X_j, Y]) \\ &= -\frac{1}{2} b([X_0, \gamma_{0,i}^{-1}(Y)], [X_j, \gamma_{0,i}^{-1}(Y)]) = q_{X_0,X_j}(\gamma_{0,i}^{-1}(Y)) \ . \end{aligned}$$

Hence q_{X_i,X_j} is equivalent to q_{X_0,X_j} which proves the result.

□

Remark 2.8.11. The situation where $k > 1$ and $e \neq 0$ appears only in Type I case by Corollary 2.8.7 namely in the case where $\widetilde{\mathfrak{g}}^{-\frac{\lambda_i+\lambda_j}{2}} = E_{i,j}(-1,-1)$.

We summarize now the results obtained in this section.

Theorem 2.8.12. *Let $(X_j)_{j=0,\ldots,k}$ be a sequence which satisfies the condition* (D). *The signature of the quadratic form q_{X_i,X_j} is given for $i \neq j$ by*

$$\begin{aligned} (d,0) \quad & \textit{for Type I} \quad (\ell = 1 \ ; e = d) \ , \\ (\frac{d+e}{2}, \frac{d-e}{2}) \quad & \textit{for Type II} \quad (\ell = 1 \ ; 0 \leq e < d) \ , \\ (\frac{d}{2}, \frac{d}{2}) \quad & \textit{for Type III} \quad (\ell > 1 \ ) \ . \end{aligned}$$

Remark 2.8.13. If $\ell = 1$ (Type I and II) X_j is determined up to the sign by the condition (C). If $e \neq 0$, there are only two sequences $(X_j)_{j=0,\ldots,k}$ which satisfy the condition (D). It is a consequence of the equality $q_{\varepsilon_i X_i, \varepsilon_j X_j} = \varepsilon_i \varepsilon_j q_{X_i,X_j}$ for $\varepsilon_i = \pm 1$ and $\varepsilon_j = \pm 1$.

2.9. Action of $Z_G(I^+)$ for Type I

In this section we assume that the regular graded algebra is of Type I and that $(X_j)_{j=0,\ldots,k}$ satisfies the condition (D). We have seen (Remark 2.8.13) that there are only two choices for such sequences namely $(X_j)_{j=0,\ldots,k}$ and $(-X_j)_{j=0,\ldots,k}$. Moreover we will prove that two elements of such a sequence are conjugate.

Lemma 2.9.1. *If the sequence $(X_j)_{j=0,\dots,k}$ satisfies the condition* (D), *then for $i \neq j$ there exists an element $\gamma_{i,j} \in N_K(\mathfrak{a}^0)$ which satisfies the conditions (i) and (ii) of Proposition 2.4.3 and such that $\gamma_{i,j}(X_i) = X_j$.*

Proof. — The sequence (X_j) (or $(-X_j)$) has been obtained in Proposition 2.8.10 by setting $X_j = \gamma_{0,j}(X_0)$ where $\gamma_{0,j}$ satisfies *(i)* and *(ii)* of Proposition 2.4.3. For i and j distinct from 0 let $\gamma_{i,j}$ be defined by

$$\gamma_{i,j} = \gamma_{0,i}\gamma_{0,j}\gamma_{0,i} \ .$$

Then $\gamma_{i,j}$ is an element of $N_K(\mathfrak{a}^0)$ which interchanges H_{λ_i} and H_{λ_j} and which interchanges X_i and X_j.

The action of $\gamma_{i,j}$ is trivial on each root space $\widetilde{\mathfrak{g}}^{\lambda_s}$ where $s \notin \{0, i, j\}$ since it is the case for $\gamma_{0,i}$ and $\gamma_{0,j}$. The action of $\gamma_{i,j}$ is trivial on $\widetilde{\mathfrak{g}}^{\lambda_0}$ since the action of $\gamma_{0,i}^2$ is trivial on $\widetilde{\mathfrak{g}}^{\lambda_i}$. Moreover the action of $\gamma_{i,j}^2$ is trivial on $\oplus_{j=0,\dots,k}\widetilde{\mathfrak{g}}^{\lambda_j}$ since it is the case for $\gamma_{0,j}^2$ and $\gamma_{0,i}^2$. This proves that $\gamma_{i,j}$ satisfies (ii) of Proposition 2.4.3

□

As in Theorem 2.7.3 it is easy to show that each nonzero element of V^+ is G–conjugate to an element of the form

$$X = \sum_{j=0}^{k} \varepsilon_j X_j \quad \text{where } \varepsilon_j \in \{-1, 0, 1\} \ .$$

Let p be the number of j such that $\varepsilon_j = 1$ and q the number of j such that $\varepsilon_j = -1$. The decomposition of Q_X into a sum of q_{X_i,X_j} , Proposition 2.8.2 and Theorem 2.8.12 prove that the signature of the quadratic form Q_X is (P, Q) where

$$P = p + q + \frac{d}{2}(p(p-1) + q(q-1)) \ ,$$
$$Q = dpq \ .$$

But this result is not sufficient to separate the orbits of G in V^+. If $X = \sum \varepsilon_j X_j$ and $X' = -X$ then, from the above formula, Q_X and $Q_{X'}$ have the same signature. However we will show that, if $p \neq q$, these elements are not in the same G–orbit.

We set

$$I^+ = X_0 + X_1 + \cdots + X_k \text{ where } (X_j)_{j=0,\dots,k} \text{ satisfies (D)} \ .$$

Then the quadratic form Q_{I^+} is positive definite. This result is the main tool to study the G-orbits in Type I case.

Lemma 2.9.2. *The centralizer $Z_G(I^+)$ of I^+ in G is a compact group. Moreover we have*

$$Z_G(I^+) = Z_G(\theta(I^+)) \ .$$

Proof. — The definition of Q_X implies that

$$Q_X(g.Y) = Q_{g^{-1}.X}(Y) \text{ for } X \in V^+, Y \in V^- \text{ and } g \in G \ .$$

In particular, for $X = I^+$ and $g \in Z_G(I^+)$ we have

$$Q_{I^+}(g.Y) = Q_{I^+}(Y) \ .$$

Since Q_{I^+} is positive definite , the group $Z_G(I^+)$ is compact.

Moreover $\{\theta(I^+), H_0, I^+\}$ is a $\mathfrak{sl}_2$-triple due to the choice of I^+ (see condition (C)). If g belongs to $Z_G(I^+)$ then $\{g.\theta(I^+), H_0, I^+\}$ is a $\mathfrak{sl}_2$-triple and the unicity in Proposition 1.7.9 implies that

$$g.\theta(I^+) = \theta(I^+) \ .$$

This concludes the proof.

□

We will give now a more precise version of Theorem 2.2.2 in Type I case.

Theorem 2.9.3 (Type I)**.**
If the graded algebra is of Type I, each element of V^+ is conjugate under $Z_G(I^+)$ to an element of $\oplus_{j=0}^k \widetilde{\mathfrak{g}}^{\lambda_j}$.

Proof. — Let X and X' be nonzero elements of V^+, X being fixed during the proof and $X' \in \oplus\widetilde{\mathfrak{g}}^{\lambda_j}$. More precisely we take

$$X' = \sum_{j=0}^{k} t_j X_j \text{ with } t_i \neq t_j \text{ for all } i \neq j \ .$$

We denote by Z (*resp.* Z') the element $[\theta(I^+), X]$ (*resp.* $[\theta(I^+), X']$) of $\mathfrak{g}$.

We first prove that $[Z, Z']$ centralizes I^+. We have $\operatorname{ad} I^+.Z = -2X$ and $\operatorname{ad} I^+.Z' = -2X'$. Thus we obtain

$$\begin{aligned} \operatorname{ad} I^+[Z, Z'] &= -2\Big([X, \operatorname{ad}\theta(I^+).X'] + [\operatorname{ad}\theta(I^+).X, X']\Big) \\ &= -2\operatorname{ad}\theta(I^+)[X, X'] = 0 \ . \end{aligned}$$

Since $Z_G(I^+)$ is compact the map $k \in Z_G(I^+) \mapsto \widetilde{B}(Z', k.Z)$ reaches a minimum, say for k_0. Using derivation we obtain

$$\widetilde{B}(U, [Z', k_0.Z]) = 0 \ \text{ for all } U \in \mathfrak{z}_{\mathfrak{g}}(I^+) \ .$$

Since k_0 belongs to $Z_G(I^+) = Z_G(\theta(I^+))$, we have $k_0.Z = \operatorname{ad}\theta(I^+).k_0X$. It follows that $[Z', k_0.Z]$ belongs to $\mathfrak{z}_{\mathfrak{g}}(I^+)$. Then, by the previous Lemma, $U = \theta([Z', k_0.Z])$ is also an element of $\mathfrak{z}_{\mathfrak{g}}(I^+)$. Therefore we have

$$\widetilde{B}(\theta([Z', k_0.Z]), [Z', k_0.Z]) = 0 \ , \text{ hence } [Z', k_0.Z] = 0 \ .$$

If we write the decomposition of $k_0.X$ as

$$k_0.X = \sum_{j=0}^{k} x_j X_j + \sum_{0 \leq i < j \leq k} X_{i,j} \quad \text{where } x_j \in \mathbb{R} \text{ and } X_{i,j} \in E_{i,j}(1,1) \ ,$$

then we have

$$k_0.Z = \operatorname{ad}\theta(I^+)k_0.X = \sum_{j=0}^{k} x_j H_{\lambda_j} + \sum_{0 \leq i < j \leq k} [\theta(X_i), X_{i,j}] + [\theta(X_j), X_{i,j}] \ .$$

Since $Z' = \sum_{j=0}^{k} t_j H_{\lambda_j}$ we thus obtain

$$0 = [Z', k_0.Z] = \sum_{i<j} (t_i - t_j)\big([\theta(X_i), X_{i,j}] - [\theta(X_j), X_{i,j}]\big) \ .$$

Since the real numbers t_j are distincts, the equality $[Z', k_0.Z] = 0$ implies that

$$\mathrm{ad}\,\theta(X_i).X_{i,j} = 0 \text{ and } \mathrm{ad}\,\theta(X_j).X_{i,j} = 0 \ .$$

But $\mathrm{ad}\,\theta(X_i)$ is one to one on the space $E_{i,j}(1,-1)$. This proves that $X_{i,j}$ equals zero for each $\{i,j\}$ with $i \neq j$ which means that $k_0.X$ belongs to $\oplus\widetilde{\mathfrak{g}}^{\lambda_j}$.

□

Remark 2.9.4. We recognize here the starting point of the classical proof of the following result : if $K \exp \mathfrak{p}$ is the Cartan decomposition of a semisimple Lie group and $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p}$ then each element of $\mathfrak{p}$ is K-conjugate to an element of $\mathfrak{a}$. We will see in section 3.4 that we are effectively in this situation (with $\mathfrak{a}^0 = \mathfrak{a}$ and $K = Z_G(I^+)$).

2.10. The G–orbits for Type I

Theorem 2.10.1 (Type I)**.**
If $\ell = 1$ and $E_{0,1}(1,1) = \widetilde{\mathfrak{g}}^{\frac{\lambda_0+\lambda_1}{2}}$ then the number of G–orbits in V^+ is $(k+2)(k+3)$ with $k+2$ open orbits. Assume that $(X_j)_{j=0,\dots,k}$ is a sequence of elements in $\widetilde{\mathfrak{g}}^{\lambda_j}$ which satisfies condition (D)*. Then a set of representatives of the G–orbits is given by the elements*

$$\sum_{j=0}^{k} \varepsilon_j X_j \text{ where } \begin{cases} \varepsilon_j = 1 & \text{for } j = 0, \dots p-1 \\ \varepsilon_j = -1 & \text{for } j = p, \dots p+q-1 \\ \varepsilon_j = 0 & \text{for } j = p+q, \dots, k \end{cases} \text{ with } 0 \leq p+q \leq k+1$$

The corresponding orbit is open if and only if $p+q = k+1$.

Let ε_j, $\varepsilon'_j \in \{0,1,-1\}$. Then the elements $\sum_{j=0}^k \varepsilon_j X_j$ and $\sum_{j=0}^k \varepsilon'_j X_j$ are G–conjugate if and only

$$\#\{j \mid \varepsilon_j = 0\} = \#\{j \mid \varepsilon'_j = 0\} \quad \text{and} \quad \#\{j \mid \varepsilon_j = 1\} = \#\{j \mid \varepsilon'_j = 1\} \ .$$

Proof. — The same proof as in Theorem 2.7.3 shows that each nonzero element of V^+ is G–conjugate to an element of the form

$$\varepsilon_0 X_0 + \cdots + \varepsilon_k X_k \text{ where } \varepsilon_j \in \{1,-1,0\} \ .$$

Using the elements $\gamma_{i,j}$ which interchanges X_i and X_j (Lemma 2.9.1), we conclude that each element of V^+ is G–conjugate to one of the representative elements given in the Theorem.

It remains to prove that these elements are not G–conjugate. Let X and X' given by

$$X = \sum \varepsilon_j X_i \text{ and } X' = \sum \varepsilon'_j X_j \ ,$$

where the p (*resp.* p') first ε_j (*resp.* ε'_j) are equal to 1, the following q (*resp.* q') ε_j (*resp.* ε'_j) are equal to -1 and the last are equal to 0. If $p+q$ is different from $p'+q'$ then the rank of Q_X and $Q_{X'}$ are distinct (Theorem 2.5.1) and therefore X and X' are not G–conjugate. It remains to prove that if (p,q) and (p',q') are distinct with $p+q = p'+q'$ then X and X' are not G–conjugate.

To each element $Z \in V^+$ we associate the polynomial P with real coefficients defined by

$$P_Z(\lambda) = \Delta_0(\lambda I^+ - Z) \ ,$$

where $I^+ = X_0 + \cdots + X_k$. If Z belongs to the diagonal $\oplus \widetilde{\mathfrak{g}}^{\lambda_j}$ *i.e.* $Z = \sum z_j X_j$ ($z_j \in \mathbb{R}$) then, by Theorem 1.13.1 (2), we have

$$P_Z(\lambda) = \prod_{j=0}^{k} (\lambda - z_j) \Delta_0(I^+) \ .$$

Hence the roots of P_Z are real. We know from Theorem 2.9.3 that, for Type I, each element of V^+ is conjugate by an element $g \in Z_G(I^+)$ to an element of the diagonal. But for $g \in Z_G(I^+)$ we have

$$P_{g.Z}(\lambda) = \Delta_0\big(g.(\lambda I^+ - Z)\big) = \chi_0(g) P_Z(\lambda) \ .$$

This proves that the roots of P_Z are real for each element $Z \in V^+$.

We will prove now that the set

$$\Omega_n = \{Z \in V^+ \mid P_Z \text{ has exactly } n \text{ roots } \neq 0\} \ .$$

is G–invariant. Let Z be an element of Ω_n and Z' be G–conjugate to Z. Then there exist Y and Y' belonging to the diagonal and respectively $Z_G(I^+)$–conjugate to Z and Z' such that P_Y and P_Z (*resp.* $P_{Y'}$ and $P_{Z'}$) have the same roots. However Y and Y' are G–conjugate, hence the quadratic forms Q_Y and Q'_Y have the same rank. It follows that the polynomials P_Y and P'_Y have the same number of nonzero roots. But P_Y has n nonzero roots (as P_Z), hence P'_Z too, which implies that Z' belongs to Ω_n.

Now we consider the set

$$\Omega_{p,q} = \{Z \in V^+ \mid P_Z \text{ has exactly } p \text{ roots } > 0 \text{ and } q \text{ roots } < 0\} \ .$$

Then $\Omega_{p,q}$ is contained in Ω_{p+q} and, moreover, $\Omega_{p,q}$ is an open subset of Ω_{p+q}. Indeed, as a consequence of Rouché's Theorem, there exists for $Z_0 \in \Omega_{p,q}$ an open neighborhood U of Z_0 in V^+ such that for $Z \in U$ the polynomial P_Z, which has only real roots, admits at less p strictly positive roots and q strictly negative roots. Therefore $U \cap \Omega_{p+q}$ is a subset of $\Omega_{p,q}$ and an open neighboroud of Z_0 in Ω_{p+q}.

Since $(p,q) \neq (p',q')$ and $p+q = p'+q'$, X and X' belong to two disjoints open subsets of Ω_{p+q}. If they were G-conjugate the connectivity of G would imply the existence of a path in Ω_{p+q} (which is G-invariant) with starting point in $\Omega_{p,q}$ and ending point in $\Omega_{p',q'}$ which is impossible. This concludes the proof of the Theorem.

□

Corollary 2.10.2 (Type I)**.**
(1) *The centralizer $Z_K(\mathfrak{a}^0)$ of $\mathfrak{a}^0$ in K acts trivially on $\oplus_{j=0}^{k} \widetilde{\mathfrak{g}}^{\lambda_j}$.*
(2) *Let γ_{ij} $(i \neq j)$ be an element of $N_K(\mathfrak{a}^0)$ such that*

$$\gamma_{ij}(H_{\lambda_i}) = H_{\lambda_j}; \ \gamma_{ij}(H_{\lambda_j}) = H_{\lambda_i}; \ \gamma_{ij}(H_{\lambda_s}) = H_{\lambda_s} \text{ if } s \notin \{i,j\} \ .$$

If $(X_j)_{j=0,\ldots,k}$ is a sequence which satisfies condition (D) *then we have*

$$\gamma_{ij}(X_i) = X_j \ ; \ \gamma_{ij}(X_j) = X_i \ ; \ \gamma_{ij}(X_s) = X_s \text{ if } s \notin \{i,j\} \ .$$

Proof. —

(1) If m belongs to the centralizer of $\mathfrak{a}^0$ then m stabilizes each space $\widetilde{\mathfrak{g}}^{\lambda_j}$. Moreover if m belongs to K, then m is an element of the orthogonal group for the positive definite form $X \mapsto -\widetilde{B}(X, \theta(X))$. Hence the action of m on the 1–dimensional space $\widetilde{\mathfrak{g}}^{\lambda_j}$ is either the identity or the product by -1. But, we know from the previous Theorem that a nonzero element X in $\widetilde{\mathfrak{g}}^{\lambda_j}$ is not G–conjugate to $-X$ which proves that m acts trivially on $\oplus\widetilde{\mathfrak{g}}^{\lambda_j}$.

(2) By the same argument we have

$$\gamma_{ij}(X_i) = \pm X_j;\ \gamma_{ij}(X_j) = \pm X_i;\ \gamma_{ij}(X_s) = \pm X_s \text{ if } s \notin \{i, j\}\ .$$

By Lemma 2.9.1 the element X_i and X_j are G–conjugate and the previous Theorem implies that X_s and $-X_s$ are not G–conjugate for $s = 0, \ldots, k$. The result follows.

□

Remark 2.10.3. The first part of the previous Corollary shows that the situation in Type I case is, in some sense, the opposite of the situation in Type II case where Proposition 2.4.5 is satisfied. The second part shows that if $\gamma_{ij} \in N_K(\mathfrak{a}^0)$ satisfies only condition *(i)* of Proposition 2.4.3 then it satisfies automatically the second condition of the same Proposition.

2.11. The classification

We will now describe all the real regular graded algebras when $\widetilde{\mathfrak{g}}$ is a simple Lie algebra and determine the constants k, d, ℓ, e in the different cases.

We know from Proposition 1.3.1 and Proposition 1.7.7 that the complexified graded algebra is a regular complex prehomogeneous space of commutative parabolic type. The classification of these spaces together with the degree of Δ_0 (it is the same polynomial in the complexified situation) and the dimension of V^+ is given in ([M-R-S]–1986).

H. Rubenthaler ([Ru]–1986) classified all real forms of the irreducible prehomogeneous vector spaces of parabolic type, not necessarily commutative. As a consequence of this classification, the simple graded Lie algebras considered in this paper are in one to one correspondance with the Satake diagrams of real simple Lie algebras with a single simple root α_0 which is circled. This root α_0 is a white root which has coefficient 1 in the highest root and which is not related by an arrow to another simple root. The corresponding simple Lie algebra is $\widetilde{\mathfrak{g}}$ (or its simple part) and α_0 is the unique (complex) simple root whose restriction to $\mathfrak{a}^{\mathfrak{p}}$ is λ_0 (see Proposition 1.3.1).

The dimension of $\widetilde{\mathfrak{g}}^{\lambda_0}$ is given in ([Wa]–1972 p. 30) or in ([O-V]–1990 p. 312). However $\dim \widetilde{\mathfrak{g}}^{\lambda_0}$ is strictly greater than 1 (and therefore the corresponding graded Lie algebra of Type III) if and only if there is a black

root related to α_0. We can now obtain k and d from the formulas

$$k+1 = \begin{cases} \text{degree } \Delta_0 & \text{if } \dim \mathfrak{g}^{\lambda_0} = 1 \\ \frac{1}{2}\text{degree } \Delta_0 & \text{if } \dim \mathfrak{g}^{\lambda_0} > 1 \end{cases} \quad (\text{ Theorem 1.13.1 })$$

$$d = \frac{2}{k}\left(\frac{\dim V^+}{k+1} - \dim \widetilde{\mathfrak{g}}^{\lambda_0}\right) \text{ (Proposition 1.9.5)}$$

The determination of the Type of $(\widetilde{\mathfrak{g}}, H_0)$ goes as follows. The graded Lie algebra $(\widetilde{\mathfrak{g}}, H_0)$ is of Type III (Definition 2.6.1) if and only if $\dim \widetilde{\mathfrak{g}}^{\lambda_0} > 1$. If $\dim \widetilde{\mathfrak{g}}^{\lambda_0} = 1$, then $(\widetilde{\mathfrak{g}}, H_0)$ is of Type I (Definition 2.7.4) if and only if $\widetilde{\Sigma} = \{\frac{\pm\lambda_i \pm \lambda_j}{2}\}$ *i.e.* if the restricted root system $\widetilde{\Sigma}$ is of type C_{k+1}. As the type of $\widetilde{\Sigma}$ is available from the classical tables ([Wa]–1972, [O-V]–1990) it is easy to determine the Type of $(\widetilde{\mathfrak{g}}, H_0)$.

In the Type I case we have $e = d$ (see Notation 2.8.6). It remains to determine e for Type II and III. In section 2.3 we defined an involution $\tau_{i,j}$ which interchanges the root spaces contained in $E_{i,j}(1,1)$. Consequently if d is odd, then $\tau_{i,j}$ has a fixed point and e is strictly positive. Moreover if $\widetilde{\mathfrak{g}}$ is the split real form of $\widetilde{\mathfrak{g}}_{\mathbb{C}}$ *i.e.* all the roots of the SATAKE diagram are white and there is no arrows between roots, then $\dim \widetilde{\mathfrak{g}}^{\frac{\lambda_i+\lambda_j}{2}}$ is equal to 0 or 1. In this case $e = 1$ if and only if d is odd. For the remaining cases we have to use the explicit decomposition of the positive roots in the basis of $\widetilde{\mathcal{R}}^+$ given in ([Bou]–1968). If ρ denotes the restriction to $\mathfrak{a}^{\mathfrak{p}}$ of a root in $\widetilde{\mathcal{R}}$, then λ_0 is equal to $\rho(\alpha_0)$ where α_0 is the circled root. For two simple roots α and β we have $\rho(\alpha) = 0$ if and only if α is a black root and $\rho(\alpha) = \rho(\beta)$ if and only if there is an arrow relating α and β. Moreover $\widetilde{\Sigma}^+ \backslash \Sigma$ is the set of restrictions of the roots which contain α_0 in the decomposition in the basis. It is then easy to determine the set $\{\lambda \in \widetilde{\Sigma}^+ \backslash \Sigma \mid \lambda \perp\!\!\!\perp \lambda_0\}$ and the lowest root of this set which is λ_1. Then we can determine the number e of roots $\alpha \in \mathcal{R}$ such that $\rho(\alpha) = \frac{\lambda_0+\lambda_1}{2}$.

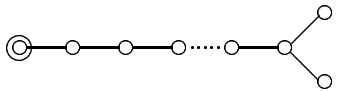
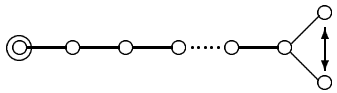
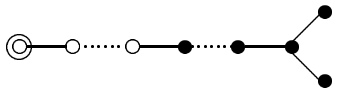
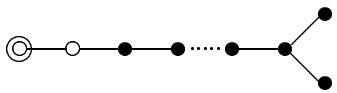
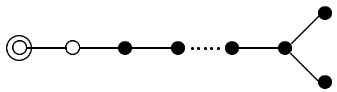
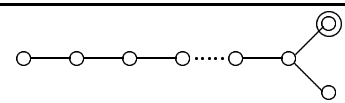
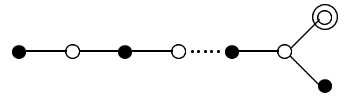
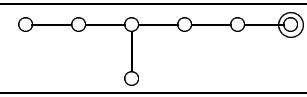
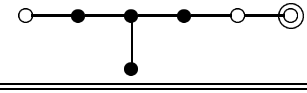
The results are given in table I where the type of the complex root system $\widetilde{\mathcal{R}}$ and the type of the restricted root $\widetilde{\Sigma}$ are given with the notations of ([Bou]–1968).

Remark 2.11.1. We will explain in section 3.1 (see remark 3.1.2) that the regular graded algebras are in one to one correspondance with simple Jordan algebras. Modulo this correspondance our classification in Type I, II and III is coherent with the Type I, II and III partition of the classification of simple real Jordan algebras as given by J. FARAUT and S. GINDIKIN ([F-G]–1996). Except in the case where $\widetilde{\mathfrak{g}}$ is of type B_m or D_m, they give (p. 153) the description of $\widetilde{\mathfrak{g}}$, V^+ , $\mathfrak{g}$, and of $\mathfrak{h}$ (see section 3.1 for these notations) which they denote respectively by $\mathfrak{g}$, V, $\mathfrak{l}$, and $\mathfrak{a}$ and which we reproduce in table I.

TABLE I

SIMPLE REGULAR GRADED LIE ALGEBRAS

$\widetilde{\mathfrak{g}}$	$\mathfrak{g}'$	V^+	$\widetilde{\mathcal{R}}$	$\widetilde{\Sigma}$	Satake diagram	k	ℓ	d	e	Type
$\mathfrak{sl}(2(k+1),\mathbb{R})$	$\mathfrak{sl}(k+1,\mathbb{R})$ $\oplus$ $\mathfrak{sl}(k+1,\mathbb{R})$	$\mathrm{M}(k+1,\mathbb{R})$	A_{2k+1} $k\geq 0$	A_{2k+1}		k	1	2	0	II
$\mathfrak{sl}(2(k+1),\mathbb{H})$	$\mathfrak{sl}(k+1,\mathbb{H})$ $\oplus$ $\mathfrak{sl}(k+1,\mathbb{H})$	$\mathrm{M}(k+1,\mathbb{H})$	A_{4k+3} $k\geq 0$	A_{2k+1}		k	4	8	0	III
$\mathfrak{su}(k+1,k+1)$	$\mathfrak{sl}(k+1,\mathbb{C})$	$\mathrm{Herm}(k+1,\mathbb{C})$	A_{2k+1} $k\geq 1$	C_{k+1}		k	1	2	2	I
$\mathfrak{so}(2m+1-p,p)$	$\mathfrak{so}(2m-p,p-1)$	$\mathbb{R}^{2m-1}$	B_m $m>2$	B_p $2<p\leq m$		1	1	$2m-3$	$2m-2p+1$	II
$\mathfrak{so}(2m-1,2)$	$\mathfrak{so}(2m-2,1)$	$\mathbb{R}^{2m-1}$	B_m $m\geq 2$	$B_2=C_2$		1	1	$2m-3$	$2m-3$	I
$\mathfrak{so}(2m,1)$	$\mathfrak{so}(2m-1)$	$\mathbb{R}^{2m-1}$	B_m $m\geq 2$	$B_1=A_1$		0	$2m-1$	$--$	$--$	III
$\mathfrak{sp}(k+1,\mathbb{R})$	$\mathfrak{sl}(k+1,\mathbb{R})$	$\mathrm{Sym}(k+1,\mathbb{R})$	C_{k+1} $k\geq 2$	C_{k+1}		k	1	1	1	I
$\mathfrak{sp}(k+1,k+1)$	$\mathfrak{sl}(k+1,\mathbb{H})$	$\mathrm{M}(k+1,\mathbb{H})\cap$ $\mathrm{Sym}(2(k+1),\mathbb{C})$	C_{2k+2} $k\geq 1$	C_{k+1}		k	3	4	4	III

$\widetilde{\mathfrak{g}}$	$\mathfrak{g}'$	V^+	$\widetilde{\mathcal{R}}$	$\widetilde{\Sigma}$	Satake diagram	k	ℓ	d	e	Type
$\mathfrak{so}(m,m)$	$\mathfrak{so}(m-1,m-1)$	$\mathbb{R}^{2m-2}$	D_m $m\geq 4$	D_m		1	1	$2m-4$	0	II
$\mathfrak{so}(m+1,m-1)$	$\mathfrak{so}(m,m-2)$	$\mathbb{R}^{2m-2}$	D_m $m\geq 4$	B_{m-1}		1	1	$2m-4$	2	II
$\mathfrak{so}(2m-p,p)$	$\mathfrak{so}(2m-p-1,p-1)$	$\mathbb{R}^{2m-2}$	D_m $m\geq 4$	B_p $2<p\leq m-2$		1	1	$2m-4$	$2m-2p$	II
$\mathfrak{so}(2m-2,2)$	$\mathfrak{so}(2m-3,1)$	$\mathbb{R}^{2m-2}$	D_m $m\geq 4$	$B_2=C_2$		1	1	$2m-4$	$2m-4$	I
$\mathfrak{so}(2m-1,1)$	$\mathfrak{so}(2m-2)$	$\mathbb{R}^{2m-2}$	D_m $m\geq 4$	$B_1=A_1$		0	$2m-2$	$--$	$--$	III
$\mathfrak{so}(2(k+1),2(k+1))$	$\mathfrak{sl}(2(k+1),\mathbb{R})$	$\mathrm{Skew}(2(k+1),\mathbb{R})$	$D_{2(k+1)}$ $k\geq 2$	$D_{2(k+1)}$		k	1	4	0	II
$\mathfrak{so}^*(4(k+1))$	$\mathfrak{sl}(k+1,\mathbb{H})$	$\mathrm{Herm}(k+1,\mathbb{H})$	$D_{2(k+1)}$ $k\geq 2$	C_{k+1}		k	1	4	4	I
$\mathfrak{e}_{7,7}$	$\mathfrak{e}_{6,6}$	$\mathrm{Herm}(3,\mathbb{O}_s)$	E_7	E_7		2	1	8	0	II
$\mathfrak{e}_{7,-25}$	$\mathfrak{e}_{6,-26}$	$\mathrm{Herm}(3,\mathbb{O})$	E_7	C_3		2	1	8	8	I

CHAPTER 3

The symmetric spaces G/H

Let $(\widetilde{\mathfrak{g}}, H_0)$ be a regular graded algebra. The open orbits of G in V^+, which are described in Chapter 2, are realizations of some homegeneous spaces of the group G. We will show that these spaces are symmetric spaces and we will describe their structure.

3.1. The involutions

Let I^+ be a generic element in V^+. Proposition 1.7.9 implies the existence of some element I^- in V^- such that $\{I^-, H_0, I^+\}$ is a $\mathfrak{sl}_2$–triple. The action on $\widetilde{\mathfrak{g}}$ of the nontrivial element of the Weyl group of this $\mathfrak{sl}_2$-triple is given by the element $w \in \widetilde{G}$ (or w_{I^+}) where

$$w = e^{\operatorname{ad} I^+} e^{\operatorname{ad} I^-} e^{\operatorname{ad} I^+} = e^{\operatorname{ad} I^-} e^{\operatorname{ad} I^+} e^{\operatorname{ad} I^-} \ .$$

It will be proved in the following Theorem that this action is an involution and in order to use a standard notation we will denote it by σ (or σ_{I^+}) :

$$\sigma(X) = w.X \text{ for } X \in \widetilde{\mathfrak{g}} \ .$$

As usual we denote also by σ the automorphism of $\widetilde{G}$ induced by σ *i.e.*

$$\sigma(g) = wgw^{-1} \text{ for } g \in \widetilde{G} \ \text{ or } \ \sigma(e^{\operatorname{ad} X}) = e^{\operatorname{ad}(\sigma(X))} \text{ for } X \in \widetilde{\mathfrak{g}} \ .$$

Theorem 3.1.1. *The automorphism σ is an involution of $\widetilde{\mathfrak{g}}$ which satisfies the following properties:*

(1) $\sigma(H_0) = -H_0$ *and* $\sigma(\mathfrak{g}) = \mathfrak{g}$;
$\sigma(X) = X + \operatorname{ad} I^- \operatorname{ad} I^+ X$ *for* $X \in \mathfrak{g}$.

(2) $\sigma(I^+) = I^-$ *and* $\sigma(V^+) = V^-$;
$\sigma(X) = \frac{1}{2}(\operatorname{ad} I^-)^2 X$ *for* $X \in V^+$;
$\sigma(X) = \frac{1}{2}(\operatorname{ad} I^+)^2 X$ *for* $X \in V^-$.

(3) *Let* $\mathfrak{h} = \{X \in \mathfrak{g} \mid \sigma(X) = X\}$;
then $\mathfrak{h} = \mathfrak{z}_{\mathfrak{g}}(I^+) = \mathfrak{z}_{\mathfrak{g}}(I^-)$.

(4) *Let* $\mathfrak{q} = \{X \in \mathfrak{g} \mid \sigma(X) = -X\}$;
then $\operatorname{ad} I^+$ *is an isomorphism from* $\mathfrak{q}$ *onto* V^+,
and $\operatorname{ad} I^-$ *is an isomorphism from* $\mathfrak{q}$ *onto* V^-.

(5) *If* $I^- = \theta(I^+)$ *then* σ *and* θ *commute.*

Proof. — The proof relies on the properties of the representation of the $\mathfrak{sl}_2$-triple $\{I^-, H_0, I^+\}$ on $\widetilde{\mathfrak{g}}$ (see section 2.1). The irreducible components of this $\mathfrak{sl}_2$-module are subspaces of dimension 3 or 1 since the primitive elements are of weight 2 or 0. In the case of odd dimensions the action of w^2 is trivial on these irreducible components, hence σ is an involution.

Moreover $\mathfrak{g}$ is the sum of the 0–weight spaces of these irreducible components, hence invariant under the action of w. If the dimension of the component is 1 (*resp.* 3), then the action of w is trivial (*resp.* the product by (-1)). It follows that $\mathfrak{h}$ is the sum of the irreducible components of dimension 1 which implies (3) and that $\mathfrak{q}$ is the sum of the 0–weight spaces of the irreducible components of dimension 3 which implies (4).

The action of w on V^+ is given by (2) since V^+ is the sum of subspaces of primitive elements in irreducible modules of dimension 3 (see section 2.1) and therefore $\sigma(I^+) = I^-$.

If X belongs to V^-, then $Y = (\operatorname{ad} I^+)^2 X$ belongs to V^+ and we have $\sigma(Y) = (\operatorname{ad}\sigma(I^+))^2\sigma(X) = (\operatorname{ad} I^-)^2\sigma(X)$. But we know that $w.Y = \frac{1}{2}(\operatorname{ad} I^-)^2 Y$ which implies, since $(\operatorname{ad} I^-)^2$ is one to one on V^+, that

$$\sigma(X) = \frac{1}{2}(\operatorname{ad} I^+)^2 X \ .$$

If X belongs to $\mathfrak{h}$ we have $\operatorname{ad} I^+ X = 0$ and $\operatorname{ad} I^- X = 0$. It follows that

$$w.X = \sigma(X) = X = X + \operatorname{ad} I^- \operatorname{ad} I^+ X \ .$$

If X belongs to $\mathfrak{q}\backslash\{0\}$ then $\operatorname{ad} I^+ X$ is a nonzero element of V^+ *i.e.* a primitive element of an irreducible component of dimension 3. It follows that $\operatorname{ad} I^+ \operatorname{ad} I^- \operatorname{ad} I^+ X = -2 \operatorname{ad} I^+ X$. Since $\operatorname{ad} I^+$ is one to one on $\mathfrak{q}$ we obtain $\operatorname{ad} I^- \operatorname{ad} I^+ X = -2X$, hence we have

$$w.X = -X = X + \operatorname{ad} I^- \operatorname{ad} I^+ X \ .$$

This proves (1) and (2).

If we assume that $\theta(I^+) = I^-$ then it is easily seen from (1) that σ and θ commute on $\mathfrak{g}$ since $\operatorname{ad} I^+ \operatorname{ad} I^- = \operatorname{ad} I^- \operatorname{ad} I^+$ on $\mathfrak{g}$. Moreover (2) implies that σ and θ commute on V^+ and on V^-.

□

Remark 3.1.2. The proof of this Theorem relies uniquely on the hypothesis $(\mathbf{H_1})$ and $(\mathbf{H_3})$ *i.e.* on the existence of a $\mathfrak{sl}_2$-triple $\{I^-, H_0, I^+\}$ associated to the gradation of $\widetilde{\mathfrak{g}}$. The product defined on V^+ by

$$V^+ \times V^+ \longrightarrow V^+$$
$$(X, Y) \mapsto \frac{1}{2} \operatorname{ad} X \operatorname{ad} Y I^-$$

endows V^+ with a structure of real Jordan algebra (this structure depends on the choice of I^+ which becomes the unit element). The proof of this result is easily obtained by an adaptation of the proof of Theorem III.3.1 ([F-K]–(1994)) where the scalar product used by FARAUT and KORANYI is replaced by the bilinear form given on $V^+ \times V^+$ by

$$\widetilde{B}_\sigma(X, Y) = -\widetilde{B}(X, \sigma(Y)) \text{ for } X \in V^+, Y \in V^+ \ .$$

Conversely each real simple Jordan algebra is the part V^+ of some regular graded algebra by the KANTOR–TITS–KOECHER construction (see [F-K]–1994 p. 218 for references)

Let $(X_j)_{j=0,\dots,k}$ be a sequence of elements in $\widetilde{\mathfrak{g}}^{\lambda_j}$ which satisfies the condition (C) (Notation 2.8.1). If $\varepsilon = (\varepsilon_j)_{j=0,\dots k}$ is a sequence such that $\varepsilon_j = \pm 1$ for $j = 0, \dots k$, we set

$$I_\varepsilon^+ = \varepsilon_0 X_0 + \cdots + \varepsilon_k X_k \text{ and } I_\varepsilon^- = \theta(I_\varepsilon^+) = \varepsilon_0 \theta(X_0) + \cdots + \varepsilon_k \theta(X_k) \ .$$

It should be noticed that the sequence $(\varepsilon_j X_j)_{j=0,\dots,k}$ satisfies also the condition (C). We know from Proposition 1.7.7 that I_ε^+ is generic in V^+ and we may therefore associate to it the involution σ_ε given by $e^{\operatorname{ad} I_\varepsilon^+} e^{\operatorname{ad} I_\varepsilon^-} e^{\operatorname{ad} I_\varepsilon^+}$.

Proposition 3.1.3. *If σ_ε and $\sigma_{\varepsilon'}$ are the involutions associated with two sequences ε and ε' then σ_ε and $\sigma_{\varepsilon'}$ commute on $\widetilde{\mathfrak{g}}$.*

Proof. — Since the roots λ_j are strongly orthogonal, we deduce from Theorem 3.1.1(2) that

$$\sigma_\varepsilon(X_j) = \sigma_{\varepsilon'}(X_j) = \theta(X_j) \ .$$

It follows that

$$\sigma_\varepsilon(I_{\varepsilon'}^+) = I_{\varepsilon'}^- \ .$$

Thus the commutativity of σ_ε and $\sigma_{\varepsilon'}$ relies on the same arguments as the commutativity of σ and θ (Theorem 3.1.1 (5)).

□

Let now $I^+ = X_0 + \cdots + X_k$ where the sequence $(X_j)_{j=0,\dots,k}$ satisfies the condition (C) and let σ be the involution associated to I^+. We recall that $\mathfrak{a}^0$ is the subspace of $\mathfrak{a}^\mathfrak{p}$ defined by

$$\mathfrak{a}^0 = \bigoplus_{j=0}^{k} \mathbb{R} H_{\lambda_j} \ .$$

We have $\mathfrak{a}^0 \subset \mathfrak{q}$ since $\sigma(H_{\lambda_j}) = -H_{\lambda_j}$. We recall also that $\mathfrak{a}^\mathfrak{p}$ is a maximal abelian subspace of $\mathfrak{p} = \{X \in \mathfrak{g} \mid \theta(X) = -X\}$.

Proposition 3.1.4. *The space $\mathfrak{a}^0$ is maximal abelian in $\mathfrak{p} \bigcap \mathfrak{q}$ for any choice of σ. Moreover the Cartan subspace $\mathfrak{a}^\mathfrak{p}$ of $(\mathfrak{g}, \theta)$ can be decomposed as $\mathfrak{a}^\mathfrak{p} = \mathfrak{a}^\mathfrak{p} \bigcap \mathfrak{h} \oplus \mathfrak{a}^0$ where*

$$\mathfrak{a}^\mathfrak{p} \bigcap \mathfrak{h} = \{H \in \mathfrak{a}^\mathfrak{p} \mid \lambda_j(H) = 0 \text{ for } j = 0, \dots, k\} \ .$$

Proof. — The choice of I^+ implies (Theorem 3.1.1(1)) that for $H \in \mathfrak{a}^\mathfrak{p}$ we have

$$\sigma(H) = H - \sum_{j=0}^{k} \lambda_j(H) H_{\lambda_j} \ .$$

It follows that $\mathfrak{a}^\mathfrak{p}$ is invariant under the action of σ and we obtain the decomposition given in the Proposition.

It remains to prove that $\mathfrak{a}^0$ is maximal abelian in $\mathfrak{p} \bigcap \mathfrak{q}$. Let X be an element of $\mathfrak{p} \bigcap \mathfrak{q}$ which centralizes $\mathfrak{a}^0$. Using the root space decomposition relatively to the root system Σ we obtain

$$X = U + \sum_{\lambda \in \Sigma} X_\lambda \text{ where } U \in \mathfrak{z}_\mathfrak{g}(\mathfrak{a}^\mathfrak{p}) \text{ and } X_\lambda \in \mathfrak{g}^\lambda \ .$$

Since X centralizes $\mathfrak{a}^0$, each root λ takes the value 0 on $\mathfrak{a}^0$ if X_λ is nonzero. It follows by Corollary 1.8.2 that such a λ is strongly orthogonal to each

root λ_j which implies that $\operatorname{ad} I^+ X_\lambda = 0$. Thus X_λ is an element of $\mathfrak{h}$. But $\sigma(U)$ belongs to $\mathfrak{z}_\mathfrak{g}(\mathfrak{a}^\mathfrak{p})$ and $\sigma(X) = -X$. Hence X equals U and belongs to $\mathfrak{z}_\mathfrak{g}(\mathfrak{a}^\mathfrak{p}) \bigcap \mathfrak{p}$ which is by definition equal to $\mathfrak{a}^\mathfrak{p}$. It follows that X is an element of $\mathfrak{a}^\mathfrak{p} \bigcap \mathfrak{q}$ *i.e.* of $\mathfrak{a}^0$.

□

3.2. In Type I case θ is one of the σ's

In this section we suppose that $(\widetilde{\mathfrak{g}}, H_0)$ is a regular graded algebra of Type I. The choice of the Cartan involution θ in section 1.1 is arbitrary on the center $\widetilde{\mathfrak{z}}$ of $\widetilde{\mathfrak{g}}$. In Type I case we will choose θ trivial on $\widetilde{\mathfrak{z}}$ and the following Lemma proves that this choice is coherent with the descent (see section 1.5).

Lemma 3.2.1. *If the action of θ on the center of $\widetilde{\mathfrak{g}}$ is trivial, then the action of θ on the center of $\widetilde{\mathfrak{g}}_1$ is trivial.*

Proof. —Recall from section 1.5 that $\widetilde{\mathfrak{g}}_1$ is the centralizer of $\widetilde{\mathfrak{g}}^{\lambda_0}$ and $\widetilde{\mathfrak{g}}^{-\lambda_0}$. It is therefore θ–stable and the center $\widetilde{\mathfrak{z}}_1$ of this reductive algebra is also θ–stable. We will prove that an element $X \in \widetilde{\mathfrak{z}}_1$ such that $\theta(X) = -X$ is equal to zero.

Since X centralizes $\widetilde{\mathfrak{l}}_0$ and $\widetilde{\mathfrak{l}}_j$ for $j \in \{1, \ldots, k\}$, X belongs to the centralizer in $\mathfrak{g}$ of $\mathfrak{a}^0$. The decomposition of X in the root system Σ can be written

$$X = H + \sum_{\lambda \in \Sigma} X_\lambda \quad (H \in \mathfrak{z}_\mathfrak{g}(\mathfrak{a}^\mathfrak{p}), X_\lambda \in \mathfrak{g}^\lambda) \ ,$$

where each root λ occuring in this decomposition is strongly orthogonal to λ_0 by Corollary 1.8.2, hence is a root of Σ_1 by Proposition 1.5.2. Since X belongs to the center of $\widetilde{\mathfrak{g}}_1$, each X_λ occuring in the decomposition is zero which implies that X belongs to $\mathfrak{z}_\mathfrak{g}(\mathfrak{a}^\mathfrak{p})$. Then $\theta(X) = -X$ implies that X belongs to $\mathfrak{a}^\mathfrak{p}$.

We will prove now that X centralizes $\widetilde{\mathfrak{g}}$. Let λ be a root in $\widetilde{\Sigma}$.

- If $\lambda(H_{\lambda_0}) = 0$, then λ is a root of $\widetilde{\Sigma}_1$ and $\lambda(X) = 0$ since X centralizes $\widetilde{\mathfrak{g}}_1$.
- If $\lambda(H_{\lambda_0}) = \pm 2$, then $\lambda = \pm\lambda_0$ and $\lambda_0(X) = 0$ since X centralizes $\widetilde{\mathfrak{g}}^{\lambda_0}$.
- If $\lambda(H_{\lambda_0}) = \pm 1$ then there exists $j \neq 0$ such that $\lambda(H_{\lambda_j}) = \pm 1$. But in Type I case this implies that $\lambda = \frac{\pm\lambda_0 \pm \lambda_j}{2}$ and, therefore, that $\lambda(X) = 0$ since X centralizes $\widetilde{\mathfrak{g}}^{\lambda_0}$ and $\widetilde{\mathfrak{g}}^{\lambda_j}$.

It follows that X is an element of the center of $\widetilde{\mathfrak{g}}$ such that $\theta(X) = -X$. The hypothesis implies that $X = 0$.

□

Proposition 3.2.2 (Type I)**.** *If the Cartan involution θ is trivial on the center $\widetilde{\mathfrak{z}}$ of $\widetilde{\mathfrak{g}}$ and if σ is the involution associated to $I^+ = X_0 + \cdots + X_k$ where $(X_j)_{j=0,\ldots,k}$ satisfies the condition* (D) *then we have*

$$\theta = \sigma \quad \textit{and} \ \mathfrak{a}^0 = \mathfrak{a}^\mathfrak{p} \ .$$

Proof. — The choice of I^+ implies that the quadratic form Q_{I^+} is positive definite (section 2.9). Let us prove that the bilinear form defined by

$$\widetilde{B}_\sigma(X,Y) = -\widetilde{B}(X,\sigma(Y)) \text{ for } X \in \widetilde{\mathfrak{g}}, Y \in \widetilde{\mathfrak{g}} ,$$

is positive definite. For that we will study the restriction of this form to each space of the decomposition

$$\widetilde{\mathfrak{g}} = V^- \oplus \mathfrak{h} \cap \widetilde{\mathfrak{z}} \oplus \mathfrak{h} \cap [\widetilde{\mathfrak{g}}, \widetilde{\mathfrak{g}}] \oplus \mathfrak{q} \oplus V^+ ,$$

which is an orthogonal decomposition for $\widetilde{B}_\sigma$.

• For $X \in V^- \backslash \{0\}$ we have

$$\widetilde{B}_\sigma(X,X) = -\widetilde{B}(X, \frac{1}{2}(\operatorname{ad} I^+)^2 X) = \frac{4 \dim V^+}{k+1} Q_{I^+}(X) > 0 .$$

• For $X \in V^+ \backslash \{0\}$ we have, since $\sigma(X) \in V^-$,

$$\widetilde{B}_\sigma(X,X) = \widetilde{B}_\sigma(\sigma(X),\sigma(X)) = \frac{4 \dim V^+}{k+1} Q_{I^+}(\sigma(X)) > 0 .$$

• For $X \in \mathfrak{q} \backslash \{0\}$ there exists by Theorem 3.1.1 an element $Y \in V^- \backslash \{0\}$ such that $X = [I^+, Y]$. Therefore

$$\widetilde{B}_\sigma(X,X) = \widetilde{B}(X,X) = \frac{8 \dim V^+}{k+1} Q_{I^+}(Y) > 0 .$$

• For $X \in (\mathfrak{h} \cap \widetilde{\mathfrak{z}}) \backslash \{0\}$ we have $\theta(X) = X$ due to the choice of θ. Therefore

$$\widetilde{B}_\sigma(X,X) = -\widetilde{B}(X,X) = -\widetilde{B}(X,\theta(X)) > 0 ,$$

since $\widetilde{B}$ is the extension of the Killing form such that $X \mapsto -\widetilde{B}(X,\theta(X))$ is positive definite.

• For $X \in (\mathfrak{h} \cap [\widetilde{\mathfrak{g}}, \widetilde{\mathfrak{g}}]) \backslash \{0\}$ we have

$$\widetilde{B}_\sigma(X,X) = -\widetilde{B}(X,X) = -\operatorname{tr}(\operatorname{ad}_{\widetilde{\mathfrak{g}}} X)^2 .$$

Since X centralizes I^+, the closed subgroup of G generated by $\{e^{t \operatorname{ad} X} \mid t \in \mathbb{R}\}$ is a closed subgroup of the centralizer in G of I^+. Since Q_{I^+} is positive definite, this group is compact (Lemma 2.9.2). By a classical method (see for example ([H]–1978) Ch.II Prop. 6.8) it is easy to prove that $-\operatorname{tr}(\operatorname{ad} X)^2$ is strictly positive, which ends the proof of the fact that $\widetilde{B}_\sigma$ is positive definite.

It follows that $\widetilde{B}$ is negative definite on the set $\{X \in \widetilde{\mathfrak{g}} \mid \sigma(X) = X\}$ and positive definite on the set $\{X \in \widetilde{\mathfrak{g}} \mid \sigma(X) = -X\}$. By definition $\widetilde{B}$ is negative definite on the set $\{X \in \widetilde{\mathfrak{g}} \mid \theta(X) = X\}$ and positive definite on the set $\{X \in \widetilde{\mathfrak{g}} \mid \theta(X) = -X\}$. Therefore, since σ and θ commute, we have $\mathfrak{p} = \mathfrak{p} \cap \mathfrak{q} = \mathfrak{q}$ and $\mathfrak{k} = \mathfrak{k} \cap \mathfrak{h} = \mathfrak{h}$, which implies that σ and θ are the same involutions. Moreover $\mathfrak{a}^0$ which is maximal abelian in $\mathfrak{p} \bigcap \mathfrak{q}$ is equal to $\mathfrak{a}^{\mathfrak{p}}$ which is maximal abelian in $\mathfrak{p}$.

□

Remark 3.2.3. If the Jordan Algebra structure associated to I^+ is defined on V^+ (remark 3.1.2) then, in Type I case, this Jordan algebra is

euclidean. To prove this we denote, as in [F–K]–1994, by $L(Z)$ the Jordan product by the element $Z \in V^+$ which is given by

$$L(Z)X = ZX = \frac{1}{2}\operatorname{ad} Z \operatorname{ad} X.I^- = \operatorname{ad}\Big(\frac{1}{2}[I^-, Z]\Big).X \text{ for } X \in V^+ ,$$

where $I^- = \theta(I^+)$. Since $[I^-, Z]$ is an element of $\mathfrak{q}$ for $Z \in V^+$ we have

$$\widetilde{B}_\sigma(L(Z)X, Y) = \widetilde{B}_\sigma(X, L(Z)Y) \text{ for } X, Y, Z \in V^+ .$$

And the fact that the bilinear form $\widetilde{B}_\sigma$ is positive definite implies that the Jordan algebra is euclidean ([F–K] p. 42).

Remark 3.2.4. If $k \neq 0$, the Cartan involution θ and an involution σ associated to a sequence $(X_j)_{j=0,\dots,k}$ satisfying the condition (C) never coincide, except under the hypothesis of the previous Proposition.

The equality $\theta = \sigma$ would imply that the quadratic form $\widetilde{B}_\sigma$ is positive definite, in particular on $E_{i,j}(-1,-1)$ for $i \neq j$. But for $X \in E_{i,j}(-1,-1)$ we obtain $\sigma(X) = \frac{1}{2}(\operatorname{ad} I^+)^2 X = (\operatorname{ad} X_i \operatorname{ad} X_j)\, X$ and therefore

$$\widetilde{B}_\sigma(X, X) = \widetilde{B}(\operatorname{ad} X_i.X, \operatorname{ad} X_j.X) = \frac{4 \dim V^+}{k+1} q_{X_i, X_j}(X) .$$

The quadratic form q_{X_i,X_j} is not positive definite in Type III case by Corollary 2.8.4 and in Type II case by Corollary 2.8.7. In Type I case, if the sequence (X_j) does not satisfy the condition (D), there exists $i \neq j$ such that q_{X_i,X_j} is not positive definite (see Notation 2.8.9).

3.3. Root systems and Types I, II and III

In this section we will give a new characterisation, in terms of some restricted root systems, of the Type of a regular graded algebra.

Let σ be the involution associated as in section 3.1 to some generic element $I^+ = X_0 + \cdots + X_k$ where $(X_j)_{j=0,\dots,k}$ satisfies the condition (C). We consider a new involution $\underline{\sigma}$ defined on $\widetilde{\mathfrak{g}}$ by

$$\underline{\sigma}(X) = \begin{cases} \sigma(X) & \text{for } X \in \mathfrak{g} , \\ -\sigma(X) & \text{for } X \in V^+ \oplus V^- , \end{cases}$$

The involution $\underline{\sigma}$ commutes with σ and θ. The decomposition of $\widetilde{\mathfrak{g}}$ relatively to $\underline{\sigma}$ will be written as

$$\widetilde{\mathfrak{g}} = \underline{\widetilde{\mathfrak{h}}} \oplus \underline{\widetilde{\mathfrak{q}}} \text{ where } \begin{cases} \underline{\widetilde{\mathfrak{h}}} & = \{X \in \widetilde{\mathfrak{g}} \mid \underline{\sigma}(X) = X\} , \\ \underline{\widetilde{\mathfrak{q}}} & = \{X \in \widetilde{\mathfrak{g}} \mid \underline{\sigma}(X) = -X\} . \end{cases}$$

Lemma 3.3.1. *For each choice of an involution σ, the space $\mathfrak{a}^0$ is maximal abelian in $\widetilde{\mathfrak{p}} \bigcap \underline{\widetilde{\mathfrak{q}}}$ where $\widetilde{\mathfrak{p}} = \{X \in \widetilde{\mathfrak{g}} \mid \theta(X) = -X\}$.*

Proof. — Let X be an element of $\widetilde{\mathfrak{p}} \bigcap \underline{\widetilde{\mathfrak{q}}}$ which centralizes $\mathfrak{a}^0$. Since H_0 belongs to $\mathfrak{a}^0$, X is an element of $\mathfrak{g}$ and therefore of $(\widetilde{\mathfrak{p}} \cap \underline{\widetilde{\mathfrak{q}}}) \cap \mathfrak{g}$ which is $\mathfrak{p} \bigcap \mathfrak{q}$. Hence, by Proposition 3.1.4, X belongs to $\mathfrak{a}^0$.

□

From a Theorem of ROSSMANN ([Ro]–1979), the previous Lemma implies that the set of roots $\Sigma(\widetilde{\mathfrak{g}}, \mathfrak{a}^0)$ is a root system and we set

$$\widetilde{\Sigma}^0 = \Sigma(\widetilde{\mathfrak{g}}, \mathfrak{a}^0) \ .$$

On the other hand we may consider the reductive Lie algebra

$$\widetilde{\mathfrak{g}}_{\underline{\sigma}\theta} = \{X \in \widetilde{\mathfrak{g}} \mid \underline{\sigma}\theta(X) = X\} \ .$$

Since $\mathfrak{a}^0$ is a Cartan subspace of $(\widetilde{\mathfrak{g}}_{\underline{\sigma}\theta}, \theta)$, the set of roots $\Sigma(\widetilde{\mathfrak{g}}_{\underline{\sigma}\theta}, \mathfrak{a}^0)$ is a root system and we set

$$\widetilde{\Sigma}^{0\underline{\sigma}} = \Sigma(\widetilde{\mathfrak{g}}_{\underline{\sigma}\theta}, \mathfrak{a}^0) \ .$$

Theorem 3.3.2. *The root systems $\widetilde{\Sigma}^0$ and $\widetilde{\Sigma}^{0\underline{\sigma}}$ are reduced and irreducible. More precisely we obtain, using the notations of ([Bou]–1968) :*

(1) *The root system $\widetilde{\Sigma}^0$ is always of type C_{k+1} .*

(2) *The root system $\widetilde{\Sigma}^{0\underline{\sigma}}$ is of type A_k if $(\widetilde{\mathfrak{g}}, H_0)$ is of Type I ;*
D_{k+1} if $(\widetilde{\mathfrak{g}}, H_0)$ is of Type II ;
C_{k+1} if $(\widetilde{\mathfrak{g}}, H_0)$ is of Type III .

Proof. — We will denote by $\eta_j (j = 0, \ldots, k)$ the linear forms defined on $\mathfrak{a}^0$ by

$$\eta_j(H_{\lambda_i}) = \delta_{i,j} = \begin{cases} 1 & \text{if } i = j \ ; \\ 0 & \text{if } i \neq j \ . \end{cases}$$

(1) The decomposition of $\widetilde{\mathfrak{g}}$ given in section 1.8 by

$$\widetilde{\mathfrak{g}} = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}^0) \oplus \left(\bigoplus_{0 \leq i < j \leq k} E_{i,j}(\pm 1, \pm 1) \right) \oplus \left(\bigoplus_{j=0}^{k} \widetilde{\mathfrak{g}}^{\pm \lambda_j} \right) \ ,$$

is the root space decomposition associated to the root system $\widetilde{\Sigma}_0$. Then we have

$$\widetilde{\Sigma}^0 = \{\pm\eta_i \pm \eta_j (0 \leq i < j \leq k) \ ; \pm 2\eta_j (0 \leq j \leq k)\} \ .$$

Therefore $\widetilde{\Sigma}^0$ is a root system of type C_{k+1}.

(2) The root system $\widetilde{\Sigma}^{0\underline{\sigma}}$ is the set of roots $\mu \in \widetilde{\Sigma^0}$ such that the intersection of $\widetilde{\mathfrak{g}}^{\mu}$ with $\widetilde{\mathfrak{g}}_{\underline{\sigma}\theta}$ is nonzero. Using (1) and (2) of Theorem 3.1.1 we obtain that for $i \neq j$

$$X \in E_{i,j}(1,1) \Longrightarrow \sigma(X) = \operatorname{ad}\theta(X_i) \operatorname{ad}\theta(X_j).X \ ;$$

$$X \in \widetilde{\mathfrak{g}}^{\lambda_j} \Longrightarrow \sigma(X) = \frac{1}{2}(\operatorname{ad}\theta(X_j))^2 X \ ;$$

$$X \in E_{i,j}(1,-1) \Longrightarrow \sigma(X) = \operatorname{ad}\theta(X_i) \operatorname{ad} X_j.X \ .$$

We will use this calculus to study in the different cases the intersection of $E_{i,j}(\pm 1, \pm 1)$ with $\widetilde{\mathfrak{g}}_{\underline{\sigma}\theta}$. Since $E_{i,j}(p,q)$ is invariant under the action of $\underline{\sigma}\theta$, this intersection will be nonzero if and only if there exists an element X in $E_{i,j}(p,q)$ such that $\underline{\sigma}\theta(X) \neq -X$.

Type I. — There exists a sequence $(\varepsilon_j)_{j=0,\ldots,k}$ with $\varepsilon_j = \pm 1$ such that the sequence $(\varepsilon_j X_j)_{j=0,\ldots,k}$ satisfies the condition (D). Let σ_ε be the involution

defined by $I_\varepsilon^+ = \varepsilon_0 X_0 + \cdots + \varepsilon_k X_k$. It follows from Proposition 3.2.2 that θ and σ_ε are equal on the semisimple part of $\widetilde{\mathfrak{g}}$. Hence we obtain

$$X \in E_{i,j}(1,1) \Longrightarrow \theta \circ \underline{\sigma}(X) = -\sigma_\varepsilon \circ \sigma(X) = -\varepsilon_i \varepsilon_j X \text{ for } i \leq j \ ;$$
$$X \in E_{i,j}(1,-1) \Longrightarrow \theta \circ \underline{\sigma}(X) = \sigma_\varepsilon \circ \sigma(X) = \varepsilon_i \varepsilon_j X \text{ for } i \neq j \ .$$

It follows that

$$\varepsilon_i \varepsilon_j = 1 \Longrightarrow E_{i,j}(1,-1) \subset \widetilde{\mathfrak{g}}_{\underline{\sigma}\theta} \text{ and } E_{i,j}(1,1) \cap \widetilde{\mathfrak{g}}_{\underline{\sigma}\theta} = \{0\} \ ;$$
$$\varepsilon_i \varepsilon_j = -1 \Longrightarrow E_{i,j}(1,1) \subset \widetilde{\mathfrak{g}}_{\underline{\sigma}\theta} \text{ and } E_{i,j}(1,-1) \cap \widetilde{\mathfrak{g}}_{\underline{\sigma}\theta} = \{0\} \ .$$

Therefore the root system $\widetilde{\Sigma}^{0\underline{\sigma}}$ is of the form

$$\widetilde{\Sigma}^{0\underline{\sigma}} = \{\varepsilon_i \eta_i - \varepsilon_j \eta_j \mid 0 \leq i \leq k \, ; \ 0 \leq j \leq k \, ; i \neq j\} \ ,$$

hence of type A_k.

Type II. — Since each element of $\widetilde{\mathfrak{g}}^{\lambda_j}$ is a multiple of X_j we obtain, as in Type I case, that

$$X \in \widetilde{\mathfrak{g}}^{\lambda_j} \Longrightarrow \sigma(X) = \theta(X) \Longrightarrow \underline{\sigma} \circ \theta(X) = -X \ ,$$

which implies that $\widetilde{\mathfrak{g}}^{\lambda_j} \bigcap \widetilde{\mathfrak{g}}_{\underline{\sigma}\theta} = \{0\}$. On the other hand for $i \neq j$ there exists in $E_{i,j}(1,1)$ a nonzero element $X \in \widetilde{\mathfrak{g}}^{\lambda}(\lambda \in \widetilde{\Sigma})$ such that $\lambda \neq \frac{\lambda_i + \lambda_j}{2}$. From the previous calculus we get that $\underline{\sigma} \circ \theta(X)$ is an element of $\widetilde{\mathfrak{g}}^{-\lambda+\lambda_i+\lambda_j}$ which is distinct from $\widetilde{\mathfrak{g}}^{\lambda}$. Then we have

$$X + \underline{\sigma} \circ \theta(X) \neq 0 \ ,$$

and consequently $E_{i,j}(1,1) \bigcap \widetilde{\mathfrak{g}}_{\underline{\sigma}\theta} \neq \{0\}$. For the same reasons the intersection of $\widetilde{\mathfrak{g}}_{\underline{\sigma}\theta}$ with $E_{i,j}(1,-1)$ and with $E_{i,j}(-1,-1)$ is nonzero for $i \neq j$. It follows that the root system $\widetilde{\Sigma}^{0\underline{\sigma}}$ is of the form

$$\widetilde{\Sigma}^{0\underline{\sigma}} = \{\pm\eta_i \pm \eta_j \mid 0 \leq i < j \leq k\} \ ,$$

hence of type D_{k+1}.

Type III. — We will prove that each root space of the system $\widetilde{\Sigma}^0$ has a nonzero intersection with $\widetilde{\mathfrak{g}}_{\underline{\sigma}\theta}$

a) $\widetilde{\mathfrak{g}}^{\lambda_j} \cap \widetilde{\mathfrak{g}}_{\underline{\sigma}\theta} \neq \{0\}$.

The Lie algebra generated by $\widetilde{\mathfrak{g}}^{\lambda_j}$ and $\widetilde{\mathfrak{g}}^{-\lambda_j}$ is a semisimple algebra of split rank one and the Cartan subspace is generated by H_{λ_j}. Therefore, if X is an element of $\widetilde{\mathfrak{g}}^{\lambda_j}$, there exists a real t such that

$$[\theta(X_j), X] - tH_{\lambda_j} \in \mathfrak{k} \ ,$$

which implies that

$$[\theta(X_j), X] - [X_j, \theta(X)] = 2tH_{\lambda_j} \ .$$

If we apply $\frac{1}{2} \operatorname{ad} \theta(X_j)$ to this equality we obtain

$$-\underline{\sigma}(X) + \theta(X) = 2t\theta(X_j) \ .$$

Therefore $\underline{\sigma}\theta(X) = -X$ if and only if $X = tX_j$. Since the dimension of $\widetilde{\mathfrak{g}}^{\lambda_j}$ is strictly greater than one, this implies that there exists an nonzero element X in $\widetilde{\mathfrak{g}}^{\lambda_j} \bigcap \widetilde{\mathfrak{g}}_{\underline{\sigma}\theta}$.

b) $E_{i,j}(1,-1) \cap \widetilde{\mathfrak{g}}_{\underline{\sigma}\theta} \neq \{0\}$ for $i \neq j$.

If X is a nonzero element of $\widetilde{\mathfrak{g}}^{\lambda_j}$ then $\operatorname{ad} X$ is an isomorphism from $E_{i,j}(1,-1)$ onto $E_{i,j}(1,1)$. Moreover if $\underline{\sigma\theta}(X) = X$ (*resp.* $\underline{\sigma\theta}(X) = -X$), then $\operatorname{ad} X$ is an isomorphism from $E_{i,j}(1,-1) \bigcap \widetilde{\mathfrak{g}}_{\underline{\sigma\theta}}$ onto $E_{i,j}(1,1) \bigcap \widetilde{\mathfrak{g}}_{\underline{\sigma\theta}}$ (*resp.* onto $\{X \in E_{i,j}(1,1) \mid \underline{\sigma\theta}(X) = -X\}$). We know that for $X = X_j$ we have $\underline{\sigma\theta}(X) = -X$ and that there exists an element $X \in \widetilde{\mathfrak{g}}^{\lambda_j} \setminus \{0\}$ such that $\underline{\sigma\theta}(X) = X$. Therefore if we have $E_{i,j}(1,-1) \cap \widetilde{\mathfrak{g}}_{\underline{\sigma\theta}} = \{0\}$ then $E_{i,j}(1,1)$ is also reduced to zero. This is impossible by Proposition 1.9.3.

c) $E_{i,j}(1,1) \cap \widetilde{\mathfrak{g}}_{\underline{\sigma\theta}} \neq \{0\}$.

Let X be a nonzero element of $\widetilde{\mathfrak{g}}^{\lambda_j}$ such that $\underline{\sigma\theta}(X) = X$. Then $\operatorname{ad} X$ is an isomorphism from $E_{i,j}(1,-1) \cap \widetilde{\mathfrak{g}}_{\underline{\sigma\theta}}$ onto $E_{i,j}(1,1) \cap \widetilde{\mathfrak{g}}_{\underline{\sigma\theta}}$. Thus the result is a consequence of b).

In conclusion, the root systems $\widetilde{\Sigma}^0$ and $\widetilde{\Sigma}^{0\underline{\sigma}}$ are the same in this case. □

Let us now go back to the involution σ and to the root systems and Weyl groups which are related to the structure of the symmetric space associated to the pair $(\mathfrak{g}, \sigma)$.

Since $\mathfrak{a}^0$ is maximal abelian in $\mathfrak{p} \bigcap \mathfrak{q}$ (Proposition 3.1.4) we consider the root systems

$$\Sigma^0 = \Sigma(\mathfrak{g}, \mathfrak{a}^0) \quad \text{and} \quad \Sigma^{0\sigma} = \Sigma(\mathfrak{g}_{\sigma\theta}, \mathfrak{a}^0) \ ,$$

where $\mathfrak{g}_{\sigma\theta} = \{X \in \mathfrak{g} \mid \sigma\theta(X) = X\}$. We denote by W^0 and $W^{0\sigma}$ the corresponding Weyl groups.

The restrictions to $\mathfrak{g}$ of the involutions σ and $\underline{\sigma}$ are the same. Therefore it is easy to describe Σ^0 and $\Sigma^{0\sigma}$ using Theorem 3.3.2. One only has to determine which root spaces of $\widetilde{\Sigma}^0$ and $\widetilde{\Sigma}^{0\underline{\sigma}}$ are contained in $\mathfrak{g}$. The Weyl group of irreducible root systems are given in ([Bou]–1968). Consequently we obtain the following result.

Theorem 3.3.3.
(1) *The root system Σ^0 is always of type A_k and the Weyl group W^0 is isomorphic to the group $\mathfrak{S}_{k+1}$ of permutations of $k+1$ elements.*

(2) *In Type II and Type III cases, the root system $\Sigma^{0\sigma}$ is also of type A_k and the group $W^{0\sigma}$ is isomorphic to $W^0 \simeq \mathfrak{S}_{k+1}$.*

(3) *In Type I case we consider the involution σ associated to an element $I_\varepsilon^+ = \sum_{j=0}^k \varepsilon_j X_j$ where the sequence $(X_j)_{j=0,\dots,k}$ satisfies the condition* (D) *and $\varepsilon_j = \pm 1$. If p (resp. $q = k+1-p$) denotes the number of indices j such that $\varepsilon_j = 1$ (resp. $\varepsilon_j = -1$), then $\Sigma^{0\sigma}$ is a system of type $A_{p-1} \times A_{q-1}$ and $W^{0\sigma}$ is isomorphic to $\mathfrak{S}_p \times \mathfrak{S}_q$.*

3.4. The groups K, H and G^σ.

In this section we will study the connectedness of some subgroups of G. Let us recall from section 1.7 that $\widetilde{G}$ is the adjoint group of $\widetilde{\mathfrak{g}}$ and that G is the identity component of the centralizer of H_0 in $\widetilde{G}$.

In this section σ is again an involution associated as in section 3.1 to a generic element $I^+ = X_0 + \cdots + X_k$ where $(X_j)_{j=0,\dots,k}$ satisfies the condition

(C). The involutions θ and σ are well defined on the adjoint group $\widetilde{G}$. More precisely σ is given by the conjugation by the element

$$w = e^{\mathrm{ad}\, I^+} e^{\mathrm{ad}\, I^-} e^{\mathrm{ad}\, I^+} .$$

Since $\theta(H_0) = \sigma(H_0) = -H_0$ the group G is invariant under the action of θ and σ and we define

$$K = G^\theta = \{g \in G \mid \theta(g) = g\} \text{ and } G^\sigma = \{g \in G \mid \sigma(g) = g\} .$$

The Lie algebra of G^σ is equal to $\mathfrak{h} = \mathfrak{z}_{\mathfrak{g}}(I^+)$ (Theorem 3.1.1 (3)). The centralizer of I^+ in G has the same Lie algebra and is therefore an open subgroup of G^σ.

Definition 3.4.1. *The group H is the open subgroup of G^σ given by*

$$H = Z_G(I^+) = \{g \in G \mid g.I^+ = I^+\} .$$

This group is in general neither connected (see Appendix), nor equal to G^σ (see the next Theorem).

Theorem 3.4.2.

(1) *The group K is connected. It is a maximal compact subgroup of G.*

(2) *The group H is also the centralizer of I^- in G. If H^0 denotes the identity component of H we have*

$$H = Z_{H\cap K}(\mathfrak{a}^0)H^0 .$$

(3) *The following properties are equivalent :*

(i) H is strictly contained in G^σ;

(ii) There exists $g \in G$ such that $g.I^+ = -I^+$;

(iii) The graded regular algebra is of one of the following form :

- *Type III;*
- *Type II with $k+1$ even;*
- *Type I with $k+1 (= 2p)$ even and with I^+ belonging to the G–orbit of $\Sigma_{j=0}^{p-1} X_j - \Sigma_{j=p}^{2p-1} X_j$ where the sequence $(X_j)_{j=0,\dots,k}$ satisfies the condition* (D) *(*Notation 2.8.9*).*

Remark 3.4.3. Property (2) means that H is essentially connected in the sense of the definition given by van den Ban ([vdB]–1988)

Proof. — A Cartan decomposition of the Lie algebra $\widetilde{\mathfrak{g}}$ is given by

$$\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{k}} \oplus \widetilde{\mathfrak{p}} \text{ where } \begin{cases} \widetilde{\mathfrak{k}} = \{X \in \widetilde{\mathfrak{g}} \mid \theta(X) = X\} , \\ \widetilde{\mathfrak{p}} = \{X \in \widetilde{\mathfrak{g}} \mid \theta(X) = -X\} . \end{cases}$$

Since the group $\widetilde{G}$ is semisimple, connected, with a trivial center, the subgroup $\widetilde{G}^\theta$ of elements in $\widetilde{G}$ fixed under θ is maximal compact and connected ([H]–1978). Moreover the Cartan decomposition of $\widetilde{G}$ is defined by the application

$$\begin{aligned} \Phi : \widetilde{G}^\theta \times \mathrm{ad}_{\widetilde{\mathfrak{g}}}\, \widetilde{\mathfrak{p}} &\longrightarrow \widetilde{G} \\ (k, \mathrm{ad}\, X) &\mapsto k\, e^{\mathrm{ad}\, X} , \end{aligned}$$

which is an diffeomorphism.

In the sequel we consider some reductive, not always connected, closed subgroups of G. Let us denote by L such a subgroup and by $\mathfrak{l}$ its Lie algebra. If the group L is stable under the action of θ we say that $(\widetilde{G}^\theta \cap L)e^{\mathrm{ad}(\mathfrak{l}\cap\widetilde{\mathfrak{p}})}$ is a Cartan decomposition of L if the restriction of Φ to $(\widetilde{G}^\theta \cap L) \times \mathrm{ad}(\mathfrak{l} \cap \widetilde{\mathfrak{p}})$ is a diffeomorphism onto L. In order to obtain such a decomposition it is sufficient to verify that this restriction is surjective *i.e.* that we can write

$$L = (\widetilde{G}^\theta \cap L)e^{\mathrm{ad}(\mathfrak{l}\cap\widetilde{\mathfrak{p}})} .$$

(1) We will prove that

$$G = Ke^{\mathrm{ad}\,\mathfrak{p}}$$

is the Cartan decomposition of G. Let g be an element of G and $\Phi^{-1}(g) = (k, \mathrm{ad}\,X)$ with $k \in \widetilde{G}^\theta$ and $X \in \widetilde{\mathfrak{p}}$. Since G is stable under the action of θ, the element $e^{2\,\mathrm{ad}\,X} = \theta(g)^{-1}g$ belongs to G and thus centralizes H_0. Since X belongs to $\widetilde{\mathfrak{p}}$, the map $\mathrm{ad}\,X : \widetilde{\mathfrak{g}} \longrightarrow \widetilde{\mathfrak{g}}$ is symmetric for the positive definite form $\widetilde{B}_\theta$, hence diagonalisable with real eigenvalues. It follows that X centralizes H_0 hence belongs to $\mathfrak{g}$. Thus X belongs to $\mathfrak{g} \cap \widetilde{\mathfrak{p}} = \mathfrak{p}$ and k to $\widetilde{G}^\theta \cap G = K$, which proves the result.

The group K is compact since $\widetilde{G}^\theta$ is compact. It is connected since it is the projection on the first component of $\Phi^{-1}(G)$ which is connected as G. Moreover it is easy to deduce from the Cartan decomposition of G that K is maximal compact.

(2) We will first prove that $Z_G(I^+)$ and $Z_G(I^-)$ are the same groups. Let h be an element of G which centralizes I^+. Then $\{I^-, H_0, I^+\}$ and $\{h.I^-, H_0, I^+\}$ are $\mathfrak{sl}_2$-triples. The unicity in Proposition 1.7.9 implies that h centralizes I^-. The same argument shows that, if h centralizes I^-, then it centralizes I^+.

This result, together with the relation $\theta(I^+) = I^-$, implies that the group H is invariant under the action of θ. Moreover, by definition, H is the centralizer of some element in $\widetilde{\mathfrak{g}}$. The same argument as for G shows that the Cartan decomposition of H is given by $H = (H \cap K)e^{\mathrm{ad}(\mathfrak{p}\cap\mathfrak{h})}$.

In order to prove that the (eventual) non connectedness of H is located in the subgroup $Z_{H\cap K}(\mathfrak{a}^0)$ we proceed by steps.

We deduce from the Cartan decomposition that

$$H = (H \cap K)H^0 ,$$

where H^0 is the identity component of H.

Let G_+ be the analytic subgroup of G with Lie algebra $\mathfrak{g}_{\sigma\theta} = \mathfrak{h}\cap\mathfrak{k}\oplus\mathfrak{p}\cap\mathfrak{q}$. Since the involutions θ and σ commute, the group K is invariant under the action of σ and it is easy to prove , using the involution $\sigma\theta$, that the Cartan decomposition of G_+ is given by

$$G_+ = (G_+ \cap K)e^{\mathrm{ad}(\mathfrak{p}\cap\mathfrak{q})} .$$

The same argument as for K shows that $G_+ \cap K$ is connected. Since it has the same Lie algebra as $H \cap K$ we obtain

$$G_+ \cap K = (H \cap K)^0 ,$$

where $(H \cap K)^0$ is the identity component of $H \cap K$.

If k is an element of $H \cap K$, then $k.\mathfrak{a}^0$ is still maximal abelian in $\mathfrak{p} \cap \mathfrak{q}$. By a classical result ([Kn]–1996 Prop.7.29) there exists an element k_1 in $G_+ \cap K$ such that $k_1.\mathfrak{a}^0 = k.\mathfrak{a}^0$. Thus $k_1^{-1}k$ belongs to $N_{H\cap K}(\mathfrak{a}^0)$. Since we have proved that $G_+ \cap K$ is a subgroup of H^0, it follows that k belongs to $N_{H\cap K}(\mathfrak{a}^0)H^0$. Therefore $H \cap K$ is a subset of $N_{H\cap K}(\mathfrak{a}^0)H^0$, which implies that

$$H = N_{H\cap K}(\mathfrak{a}^0)H^0 \ .$$

It remains to study $N_{H\cap K}(\mathfrak{a}^0)$.

For that purpose we will use the determination of the Weyl groups W^0 of $\Sigma(\mathfrak{g}, \mathfrak{a}^0)$ and $W^{0\sigma}$ of $\Sigma(\mathfrak{g}_{\sigma\theta}, \mathfrak{a}^0)$ obtained in Theorem 3.3.3. It is well known ([H]–1978 Ch.VII §2) that, since G_+ is connected, the Weyl group $W^{0\sigma}$ is given by

$$W^{0\sigma} = N_{(G_+\cap K)}(\mathfrak{a}^0)/Z_{(G_+\cap K)}(\mathfrak{a}^0) \ .$$

On the other hand ROSSMANN ([Ro]–1979) proved that

$$W^0 = N_K(\mathfrak{a}^0)/Z_K(\mathfrak{a}^0) \ .$$

Let $k \in N_{H\cap K}(\mathfrak{a}^0)$. Then there is an element $s \in W^0$ such that $s.H = k.H$ for each $H \in \mathfrak{a}^0$.

Let us now suppose first that we are in Type I case. There exists a sequence $(Y_j)_{j=0,\dots,k}$ satisfying the condition (D) and a sequence $(\varepsilon_j)_{j=0,\dots,k}$ with $\varepsilon_j = \pm 1$ such that $X_j = \varepsilon_j Y_j$ for $j = 0, \dots, k$.

By Theorem 3.3.3 (1) there exists a permutation τ of $\{0, 1, \dots, k\}$ such that $s.H_{\lambda_j} = k.H_{\lambda_j} = H_{\lambda_{\tau(j)}}$. As the dimension of $\widetilde{\mathfrak{g}}^{\lambda_j}$ is equal to 1, we have

$$k.Y_j = t_j Y_{\tau(j)} \text{ where } t_j \in \mathbb{R} \text{ for } j = 0, \dots, k \ .$$

We have proved (Proposition 3.2.2) that θ is the involution associated to the generic element $J^+ = \sum_{j=0,\dots,k} Y_j$. Therefore an element of the connected group K centralizes J^+. It follows that

$$k.J^+ = \sum_{j=0}^{k} Y_j = \sum_{j=0}^{k} t_j Y_{\tau(j)} \ ,$$

and that, therefore, the constants t_j are equal to 1. On the other hand k is also an element of H, hence centralizes I^+ which implies that

$$\sum_{j=0}^{k} \varepsilon_j Y_{\tau(j)} = \sum_{j=0}^{k} \varepsilon_j Y_j \ .$$

Therefore for each $j \in \{0, \dots, k\}$ we have

$$\varepsilon_{\tau(j)} = \varepsilon_j \ .$$

This proves that τ is a product of a permutation of the indices j for which $\varepsilon_j = 1$ and of a permutation of the indices j for which $\varepsilon_j = -1$. Hence Theorem 3.3.3 (3) implies that k acts on $\mathfrak{a}^0$ as an element of $W^{0\sigma}$. As $G_+ \cap K = (H \cap K)^0$, there exists an element $k_0 \in N_{(H\cap K)^0}(\mathfrak{a}^0) \subset H^0$ such that $k_0.H = k.H$ for each $H \in \mathfrak{a}^0$. Thus $k_0^{-1}k$ belongs to $Z_{H\cap K}(\mathfrak{a}^0)$ and we have proved that

$$N_{H\cap K}(\mathfrak{a}^0) \subset Z_{H\cap K}(\mathfrak{a}^0)H^0 \ .$$

which concludes the proof in Type I case.

In Type II or Type III cases we know from Theorem 3.3.3 (2) that the groups W^0 and $W^{0\sigma}$ are equal. Therefore we obtain directly the existence of an element $k_0 \in N_{(H\cap K)^0}(\mathfrak{a}^0) \subset H^0$ such that $k_0.H = k.H$ for each $H \in \mathfrak{a}^0$. And the result follows.

(3) The description of the G-orbits in V^+ given in sections 2.6 , 2.7 and 2.10 shows that *(ii)* and *(iii)* are equivalent.

$(ii) \Longrightarrow (i)$

Let g be an element of G such that $g.I^+ = -I^+$. Such an element does not belong to H and we will prove that it belongs to G^σ. Considering the $\mathfrak{sl}_2$–triples $\{-I^-, H_0, -I^+\}$ and $\{g.I^-, H_0, g.I^+\}$, we deduce from the unicity in Proposition 1.7.9 that $g.I^- = -I^-$. Since $w = e^{\operatorname{ad} I^+} e^{\operatorname{ad} I^-} e^{\operatorname{ad} I^+}$, it follows that

$$gwg^{-1} = w^{-1} .$$

As σ is an involution we have $w^{-1} = w$ and we obtain

$$\sigma(g) = wgw^{-1} = g .$$

$(i) \Longrightarrow (ii)$

We assume that we are not in the situation described in *(iii)* and we will prove that $G^\sigma = H$. In particular the graded algebra is necessarily of Type I or II in this situation.

The same argument as the one used for H in (2) shows that

$$G^\sigma = N_{G^\sigma\cap K}(\mathfrak{a}^0)H^0 ,$$

since H^0 is also the identity component of G^σ. It is therefore sufficient to prove that an element $N_{G^\sigma\cap K}(\mathfrak{a}^0)$ centralizes I^+.

Let $k \in N_{G^\sigma\cap K}(\mathfrak{a}^0)$. Since k normalizes $\mathfrak{a}^0$, we obtain, using the same arguments as in (2), a permutation τ of $\{0, \ldots, k\}$ such that

$$k.H_{\lambda_j} = H_{\lambda_{\tau(j)}} \text{ for } j = 0, \ldots, k .$$

The dimension of each $\widetilde{\mathfrak{g}}^{\lambda_j}$ is one. Therefore there exist some real numbers t_j such that

$$k.X_j = t_j X_{\tau(j)} \text{ for } j = 0, \ldots, k .$$

Moreover, as k belongs to K, we have $\theta(k.X_j) = k.\theta(X_j)$. It follows that

$$H_{\lambda_{\tau(j)}} = [k\theta(X_j), k.X_j] = [t_j\theta(X_{\tau(j)}), t_j X_{\tau(j)}] = t_j^2 H_{\lambda_{\tau(j)}} ,$$

which implies that $t_j = \pm 1$. We recall (see the proof of Theorem 3.3.2) that, if X is an element of $E_{i,j}(-1,-1)$, then $\sigma(X) = \operatorname{ad} X_i \operatorname{ad} X_j.X$, which implies that

$$k.\sigma(X) = t_i t_j \operatorname{ad} X_{\tau(i)} \operatorname{ad} X_{\tau(j)}.kX .$$

On the other hand $k.X$ belongs to $E_{\tau(i),\tau(j)}(-1,-1)$ which implies that

$$\sigma(k.X) = \operatorname{ad} X_{\tau(i)} \operatorname{ad} X_{\tau(j)}.kX .$$

But k belongs to G^σ, hence $k\sigma(X) = \sigma(kX)$ and the previous equalities imply that the real t_j are all equal to 1 or all equal to -1. In the first case k centralizes I^+, hence belongs to H. In the second case we have $k.I^+ = -I^+$

which is impossible since *(ii)* and *(iii)* are equivalent . Therefore we have proved that $G^\sigma = H$.

□

Remark 3.4.4. In Type I case, if the sequence $(X_j)_{j=0,\dots,k}$ satisfies the condition (D), then we have $\sigma = \theta$, $\mathfrak{a}^0 = \mathfrak{a}^\mathfrak{p}$ (Proposition 3.2.2) and $K = Z_G(I^+)$. Moreover $\operatorname{ad} I^+$ is an isomorphism from $\mathfrak{q} = \mathfrak{p}$ onto V^+ and from $\mathfrak{a}^\mathfrak{p}$ onto the diagonal $\oplus_{j=0}^k \widetilde{\mathfrak{g}}^{\lambda_j}$ of V^+. Therefore, in that case, the fact that each element of V^+ is $Z_G(I^+)$–conjugate to an element of the diagonal (Theorem 2.9.3) is equivalent to the classical following result : each element of $\mathfrak{p}$ is K–conjugate to an element of $\mathfrak{a}^\mathfrak{p}$.

3.5. A parabolic subgroup of G

A symmetric space G/H is associated to each $\mathfrak{sl}_2$–triple of the form $\{\theta(I^+), H_0, I^+\}$. This symmetric space is either realized as the open G–orbit of I^+ or of $\theta(I^+)$. We choose a representative for each open G–orbit in V^+ of the form $I^+ = \sum_{j=0}^k X_j$ where $(X_j)_{j=0,\dots,k}$ satisfies the condition (C). Thus we obtain a family of symmetric spaces (which reduces to one element in Type III case and two elements in Type II case).

In this section we will construct a parabolic subgroup of G, which will be the same for each of these symmetric spaces, such that PH is an open subset of G. This parabolic subgroup will be a minimal $\sigma\theta$–stable parabolic subgroup in the sense of van den BAN ([vdB]–1988). The description of the open P–orbits in G/H *i.e.* the IWASAWA decomposition of these symmetric spaces will be given in section 3.6.

The Lie algebra of P will be the subalgebra $\mathfrak{P}$ of $\mathfrak{g}$ defined by

$$\mathfrak{P} = \mathfrak{z}_\mathfrak{g}(\mathfrak{a}^0) \oplus \big(\underset{i<j}{\oplus} E_{i,j}(1,-1) \big) \ .$$

Let again σ be the involution associated to some generic element $I^+ = \sum_{j=0}^k X_j$ where $(X_j)_{j=0,\dots,k}$ satisfies the condition (C).

Proposition 3.5.1. *For each such σ, the subspace $\mathfrak{P}$ is a minimal $\sigma\theta$–stable parabolic subalgebra of $\mathfrak{g}$. It's* LANGLANDS *decomposition is given by*

$$\mathfrak{P} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n} \ ,$$

where

$$\begin{cases} \mathfrak{n} = \oplus_{i<j} E_{i,j}(1,-1) \ ; \\ \mathfrak{m} \oplus \mathfrak{a} = \mathfrak{z}_\mathfrak{g}(\mathfrak{a}^0) \ ; \\ \mathfrak{a} = \{H \in \mathfrak{a}^\mathfrak{p} \mid \big(\lambda \in \Sigma, \lambda(\mathfrak{a}^0) = 0\big) \Longrightarrow \lambda(H) = 0\} \\ \quad = \mathfrak{p} \cap \big(\textit{center of } \mathfrak{m} \oplus \mathfrak{a}\big) \ ; \\ \mathfrak{m} \textit{ is the orthogonal of } \mathfrak{a} \textit{ in } \mathfrak{m} \oplus \mathfrak{a} \ . \end{cases}$$

Proof. — Proposition 1.9.1 implies that $\mathfrak{P}$ contains all the root-spaces of the negative roots, hence contains a minimal parabolic subalgebra. Moreover each element of $\mathfrak{a}^0$ is invariant under the action of $\sigma\theta$, hence each space $E_{i,j}(1,-1)$ is invariant under this action. Therefore $\mathfrak{P}$ is a $\sigma\theta$–stable parabolic subalgebra of $\mathfrak{g}$.

Let Γ be the set of roots λ in Σ such that $\mathfrak{g}^\lambda \bigcap \mathfrak{P} \neq 0$ (or equivalently $\mathfrak{g}^\lambda \subset \mathfrak{P}$). Then, from the definition of $\mathfrak{P}$, we have

$$\Gamma = \Sigma^- \cup \{\lambda \in \Sigma^+ \mid \lambda(\mathfrak{a}^0) = 0\} ,$$

$$\mathfrak{P} = \mathfrak{z}_\mathfrak{g}(\mathfrak{a}^\mathfrak{p}) \oplus \left(\underset{\lambda\in\Gamma}{\oplus} \mathfrak{g}^\lambda \right) .$$

Therefore we have

$$\Gamma \cap -\Gamma = \{\lambda \in \Sigma \mid \lambda(\mathfrak{a}^0) = 0\} .$$

Since $\mathfrak{P}$ is a parabolic subalgebra, there exists an unique subset Π' of the set Π of simple roots in Σ^+ such that

$$\Gamma = \Sigma^- \cup \langle \Pi' \rangle ,$$

where $\langle \Pi' \rangle$ is the set of roots in Σ linearly generated by Π (see [Kn]–1996 Prop. 7.76). We will say that $\mathfrak{P}$ is the standard parabolic subalgebra associated to Π'. The definition of Γ implies that, here, we have

$$\Pi' = \{\lambda \in \Pi \mid \lambda(\mathfrak{a}^0) = 0\} .$$

By a result of van den Ban ([VdB]–1988 Cor.2.7) such a standard parabolic subalgebra associated to the subset Π' of simple roots taking the value 0 on a maximal abelian subspace of $\mathfrak{p} \cap \mathfrak{q}$ is minimal $\sigma\theta$–stable. Therefore $\mathfrak{P}$ is minimal $\sigma\theta$–stable.

In the Langlands decomposition $\mathfrak{P} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$, $\mathfrak{n}$ is given by the sum of root-spaces related to the roots $\lambda \in \Gamma \backslash (\Gamma \cap -\Gamma)$, $\mathfrak{a}$ by the intersection of the kernels of the roots $\lambda \in \Gamma \cap -\Gamma$ and $\mathfrak{m} \oplus \mathfrak{a}$ by the centralizer in $\mathfrak{g}$ of $\mathfrak{a}$ (see [Kn]–1996 p. 415–417). Moreover this decomposition is orthogonal with respect to $\widetilde{B}_\theta$. It follows that $\mathfrak{n}$ is the sum of the spaces $E_{i,j}(1,-1)$ for $i < j$ and that $\mathfrak{a}$ is the intersection of the kernels of the roots λ such that $\lambda(\mathfrak{a}^0) = 0$. Since the decomposition $\mathfrak{P} = \mathfrak{z}_\mathfrak{g}(\mathfrak{a}^0) \oplus \mathfrak{n}$ is also orthogonal with respect to $\widetilde{B}_\theta$ we obtain that

$$\mathfrak{m} \oplus \mathfrak{a} = \mathfrak{z}_\mathfrak{g}(\mathfrak{a}) = \mathfrak{z}_\mathfrak{g}(\mathfrak{a}^0) .$$

To conclude the proof it remains to notice that $\mathfrak{a}$ is also given by the intersection of the center of $\mathfrak{m} \oplus \mathfrak{a}$ with $\mathfrak{p}$ (see [Kn]–1996 Prop. 7.82(a)).

□

Remark 3.5.2. The inclusions $\mathfrak{a}^0 \subset \mathfrak{a} \subset \mathfrak{a}^\mathfrak{p}$ can be strict. For example in Type II case we have

• if $\widetilde{\Sigma} = A_{2k+1}$ then $\mathfrak{a}^\mathfrak{p} = \mathfrak{a}$, $\dim \mathfrak{a} = 2k+1$ and $\dim \mathfrak{a}^0 = k+1$. In this case $\mathfrak{P}$ is a minimal parabolic subalgebra.

• if $\widetilde{\Sigma} = D_m$ then $\mathfrak{a} = \mathfrak{a}^0$, $\dim \mathfrak{a} = 2$ and $\dim \mathfrak{a}^\mathfrak{p} = m$. In this case $\mathfrak{P}$ is not a minimal parabolic subalgebra.

Let $N = e^{\mathrm{ad}\,\mathfrak{n}}$ and $A = e^{\mathrm{ad}\,\mathfrak{a}}$ be the analytic subgroups of G wich have respectively $\mathfrak{n}$ and $\mathfrak{a}$ for Lie algebras. Using the Cartan decomposition of G it can be proved that the multiplication is a diffeomorphism from the product $Z_K(\mathfrak{a}) \times e^{\mathrm{ad}(\mathfrak{m}\cap\mathfrak{p})} \times A$ onto $Z_G(\mathfrak{a})$. The parabolic subgroup P associated to $\mathfrak{P}$ is defined by it's LANGLANDS decomposition $P = MAN$ where

$$MA = Z_G(\mathfrak{a}) \quad \text{and} \quad M = Z_K(\mathfrak{a})e^{\mathrm{ad}(\mathfrak{m}\cap\mathfrak{p})} .$$

Moreover, since $Z_{G_{\mathbb{C}}}(\mathfrak{a})$ and $Z_{G_{\mathbb{C}}}(\mathfrak{a}^0)$ are connected groups with the same Lie algebras, namely $\mathfrak{z}_{\mathfrak{g}_{\mathbb{C}}}(\mathfrak{a}) = \mathfrak{z}_{\mathfrak{g}_{\mathbb{C}}}(\mathfrak{a}^0)$, they are equal. Thus we have the equalities

$$Z_G(\mathfrak{a}) = Z_G(\mathfrak{a}^0) \quad \text{and} \quad Z_K(\mathfrak{a}) = Z_K(\mathfrak{a}^0) .$$

Van den BAN proved that this group P is a minimal $\sigma\theta$–stable parabolic subgroup of G ([vdB]–1988, Cor. 2.7). Moreover he proved in the same paper (Prop. B.1) that there is a strictly positive finite number of open P–orbits in G/H (where H is the centralizer in G of I^+) and that they are parametrised by $W^0/W^0_{K\cap H}$, where $W^0_{K\cap H}$ denotes the image of $N_{K\cap H}(\mathfrak{a}^0)$ in W^0. It follows from Lemma 1.4 in ([vdB]–1988) and from the fact that H is essentially connected (Theorem 3.4.2) that $W^0_{K\cap H} = W^{0\sigma}$. Thus the open P–orbits in G/H are parametrized by $W^0/W^{0\sigma}$. Using the description of the groups W^0 and $W^{0\sigma}$ given in Theorem 3.3.3 we deduce from this result the following Theorem.

Theorem 3.5.3. *The number of open P–orbits in G/H is equal to*

- 1 *in Type II and Type III cases;*
- $\binom{k+1}{p}$ *in Type I case,*

where p is the number of ε_j such that $\varepsilon_j = 1$ if $I^+ = \sum_{j=0}^k \varepsilon_j Y_j$ with $(Y_j)_{j=0,\dots,k}$ satisfying the condition (D).

We will in the next section describe the open orbits of P in V^+ and obtain a direct proof of the previous Theorem. Let us first study the action of M on $\oplus_{j=0}^k \widetilde{\mathfrak{g}}^{\lambda_j}$.

Theorem 3.5.4. *For any Type and any σ, the space $M/M\cap H$ is compact.*

- *For Type I, the action of M on $\oplus_{j=0}^k \widetilde{\mathfrak{g}}^{\lambda_j}$ is trivial and $M = M\cap H$.*
- *For Type II, the action of $M\cap H$ on $\oplus_{j=0}^k \widetilde{\mathfrak{g}}^{\lambda_j}$ is trivial and $M/M\cap H$ is isomorphic to the group*

$$\{(m_0, m_1, \dots, m_k) \in \{\pm 1\}^{k+1} \mid m_0 m_1 \cdots m_k = 1\} .$$

- *For Type III, the action of M is transitive on the product for $j = 0, \dots, k$ of spheres of same radius in $\widetilde{\mathfrak{g}}^{\lambda_j}$ (for the metric defined by b). Moreover $M/M\cap H$ is homeomorphic to the product of these spheres.*

Proof. — Since M is θ-stable, the Cartan decomposition of M is of the form

$$M = (M\cap K)e^{\mathrm{ad}(\mathfrak{m}\cap\mathfrak{p})} .$$

Furthermore $\mathfrak{m}$ centralizes $\mathfrak{a}^0$ which is maximal abelian in $\mathfrak{p} \cap \mathfrak{q}$. Hence we have

$$\mathfrak{m} \cap \mathfrak{p} \cap \mathfrak{q} \subset \mathfrak{a}^0 \ .$$

Since $\mathfrak{m}$ is orthogonal to $\mathfrak{a} \supset \mathfrak{a}^0$ we obtain

$$\mathfrak{m} \cap \mathfrak{p} \subset \mathfrak{m} \cap \mathfrak{p} \cap \mathfrak{h} \ .$$

Therefore $M/M \cap H$ is homeomorphic to $(M \cap K)/(M \cap K \cap H)$ which implies that $M/M \cap H$ is compact.

The root-spaces $\widetilde{\mathfrak{g}}^{\lambda_j}$ are stable under the action of M which is a subgroup of $Z_G(\mathfrak{a}^0)$. We will study this action in the different cases.

Type I and Type II : Each root-space $\widetilde{\mathfrak{g}}^{\lambda_j}$ is of the form $\mathbb{R}X_j$ where $I^+ = \sum_{j=0}^k X_j$. An element of $M \cap H$ centralizes I^+ and stabilizes $\mathbb{R}X_j$, hence centralizes $\widetilde{\mathfrak{g}}^{\lambda_j}$. Since $M/M \cap H$ is compact we have for $j \in \{0, \ldots, k\}$ and $m \in M$

$$m.X_j = m_j X_j \text{ where } m_j = \pm 1 \ .$$

In Type I case we have $X_j = \varepsilon_j Y_j (\varepsilon_j = \pm 1)$ where $(Y_j)_{j=0,\ldots,k}$ satisfies the condition (D). Thus $m.Y_j = m_j Y_j$ for each $j \in \{0, \ldots, k\}$ which implies that $\sum_{j=0}^k Y_j$ and $\sum_{j=0}^k m_j Y_j$ are G-conjugate. By Theorem 2.10.1 this is possible only if $m_j = 1$ for each j. Therefore the action of M is trivial on $\oplus_{j=0}^k \widetilde{\mathfrak{g}}^{\lambda_j}$ and $M = M \cap H$.

In Type II case we consider the morphism φ from M to the group $\{\pm 1\}^{k+1}$ which associates to $m \in M$ the sequence $(m_0, \ldots, m_k)$ defined above. The kernel of φ is $M \cap H$. From Proposition 2.4.5 we know the existence of elements of $Z_K(\mathfrak{a}^0) \subset M$ which act trivially on each $\widetilde{\mathfrak{g}}^{\lambda_j}$ except on two of them whose sign is changed. Therefore each sequence $(m_0, \ldots, m_k)$ such that $m_0 m_1 \cdots m_k = 1$ belongs to the image of φ. If a sequence $(m_0, \ldots, m_k)$ with $m_0 m_1 \cdots m_k = -1$ belongs to the image of φ, then I^+ is conjugate under M to the element $-X_0 + X_1 + \cdots + X_k$, which is impossible by Theorem 2.7.3. Therefore the image of φ is the set of sequences $(m_0, \ldots, m_k)$ such that $m_j = \pm 1$ and $m_0 m_1 \cdots m_k = 1$. It follows, in particular, that M is not connected.

Type III : Let φ be the application from M to $\oplus_{j=0}^k \widetilde{\mathfrak{g}}^{\lambda_j}$ which associates to $m \in M$ the element $m.I^+ = m.X_0 + \cdots + m.X_k$. As the stabilizer of I^+ in M is $M \cap H$, the image of φ is homeomorphic to $M/M \cap H$. Moreover we have $\varphi(M) = \varphi(M \cap K)$.

If m belongs to $M \cap K$ it follows from Lemma 1.10.2 that for $j \in \{0, \ldots, k\}$

$$b(m.X_j, \theta(m.X_j)) = b(X_j, \theta(X_j)) = 1 \ .$$

Therefore the image of φ is contained in the product for $j = 0$ to k of the spheres

$$S_j = \left\{ X \in \widetilde{\mathfrak{g}}^{\lambda_j} \mid b(X, \theta(X)) = 1 \right\} \ .$$

It remains to prove that the image of φ is equal to this product.

Let us recall that L_j is the analytic subgroup of G with Lie algebra $\mathfrak{l}_j = [\widetilde{\mathfrak{g}}^{-\lambda_j}, \widetilde{\mathfrak{g}}^{\lambda_j}]$. This is a subgroup of $Z_G(\mathfrak{a}^0)$ since an element of L_j centralizes

H_0 and each H_{λ_s} for $s \neq j$, hence H_{λ_j}. It follows that $L_j \cap K$ is a subgroup of M. The Cartan decomposition of L_j is given by

$$L_j = (L_j \cap K) e^{\mathbb{R} \operatorname{ad} H_{\lambda_j}} .$$

We have proved (Theorem 1.12.1) that the action of L_j is transitive on $\widetilde{\mathfrak{g}}^{\lambda_j} \setminus \{0\}$. Thus the action of $L_j \cap K$ is transitive on the sphere S_j. Since L_j centralizes the spheres S_s for $s \neq j$, the action of M is transitive on the product of the spheres S_j.

□

3.6. The prehomogeneous vector space (P, V^+)

There is a finite number of open orbits of P in V^+ by Proposition 3.5.3 We will prove in this section that the polynomials Δ_j (defined in 1.14.1) are relatively invariant under the action of P (Theorem 3.6.1) and moreover that they are the fundamental invariants (Theorem 3.6.3) for the prehomogeneous vector space (P, V^+).

We will now normalize the polynomials Δ_j as follows.

We will make once and for all the choice of a sequence $(X_j)_{j=0,\dots,k}$ which satisfies the condition (D) *and we denote by I^+ the generic element $I^+ = \sum_{j=0}^{k} X_j$. We let*

$$\Delta_j(I^+) = 1 \text{ for } j = 0, \dots, k .$$

We recall (Notation 1.12.3) that $\kappa = 1$ in Type I and Type II cases and $\kappa = 2$ in Type III case.

Theorem 3.6.1. *The polynomials Δ_j are irreducible relatively invariant under the action of the parabolic group P. More precisely we have for $j = 0, \dots, k$*

$$\Delta_j(p.X) = \chi_j(p)\Delta_j(X) \text{ for } X \in V^+,\ p \in P ,$$

where χ_j is a character of $P = MAN$. For $n \in N$, $H \in \mathfrak{a}$ and $m \in M$ we have

- $\chi_j(n) = 1$;
- $\chi_j(e^{\operatorname{ad} H}) = e^{\kappa(\lambda_j(H)+\cdots+\lambda_k(H))}$;
- $\chi_j(m) = \begin{cases} 1 & \text{in Type I and Type III cases ;} \\ m_j \cdots m_k & \text{if } m.X_j = m_j X_j \ (m_j = \pm 1) \text{ in Type II case .} \end{cases}$

Proof. — We have proved (Theorem 1.14.2) that Δ_j is invariant under the action of N and relatively invariant under the action of $\exp \mathfrak{a}^{\mathfrak{p}}$. The formula for $\chi_j(a)$ if $a \in \exp \mathfrak{a}$ is a consequence of (4) in Theorem 1.14.2 It remains to prove that Δ_j is relatively invariant under the action of M, which is not a consequence of Theorem 1.14.2 (2) since M is not a subset of G_j.

But the definition of Δ_j, Theorem 1.14.2 (5) and the normalization done above imply that, if we write an element of V^+ as $X + Y$ where $X \in V_j^+$ and $Y \in V_j^{\perp}$, then we have

$$\Delta_j(X + Y) = \Delta_0(X_0 + \cdots + X_{j-1} + X) .$$

An element of the centralizer of $\mathfrak{a}^0$ (which contains M) stabilizes V_j^+, $V_j^\perp$ and each root-space $\widetilde{\mathfrak{g}}^{\lambda_j}$. As Δ_0 is G relatively invariant, it follows that, for $m \in M$ we have

$$\begin{aligned}\Delta_j(m.(X+Y)) &= \Delta_0(X_0 + \cdots + X_{j-1} + m.X) \\ &= \chi_0(m)\Delta_j(m^{-1}.X_0 + \cdots + m^{-1}.X_{j-1} + X) \ .\end{aligned}$$

Since m^{-1} stabilizes the spaces $\widetilde{\mathfrak{g}}^{\lambda_s}$, we know from Theorem 1.14.2 (5) that the polynomial $X + Y \mapsto \Delta_j(m^{-1}.X_0 + \cdots + m^{-1}.X_{j-1} + X)$ is a multiple of the polynomial $X \mapsto \Delta_j(X+Y) = \Delta_j(X)$. Therefore Δ_j is relatively invariant under the action of M and the calculus of $\Delta_j(m.I^+)$ gives the value of $\chi_j(m)$. Using Theorem 3.5.4 we obtain :

Type I : M centralizes I^+, thus $\chi_j(m) = 1$.

Type II : $m.I^+ = \sum_{j=0}^k m_j X_j$ and by Theorem 1.14.2 (3) we have

$$\Delta_j\left(\sum_{s=0}^k m_s X_s\right) = m_j \cdots m_k \Delta_j(I^+) = m_j \cdots m_k \ .$$

Type III : Since χ_j is trivial on $M \cap H$ and since $M/M \cap H$ is compact, the character χ_j takes only the values ± 1. But in this case the space $M/M \cap H$ is connected as product of spheres, hence χ_j is trivial.

□

We denote by $\mathcal{O}^+$ the open dense subset of V^+ given by

$$\mathcal{O}^+ = \{X \in V^+ \mid \Delta_0(X)\Delta_1(X)\cdots\Delta_k(X) \neq 0\} \ .$$

We will prove that $\mathcal{O}^+$ is the union of the open orbits of P in V^+. Therefore, since the polynomials Δ_j are irreducible on $\mathbb{C}$, they are the fundamental invariants of the prehomogeneous vector space (P, V^+).

We start the proof with the following Lemma which proves that an element of $\mathcal{O}^+$ is not only G–conjugate to an element of the "diagonal" (Theorem 2.2.2) but also N–conjugate to such an element.

Lemma 3.6.2. *Each element of $\mathcal{O}^+$ is N–conjugate to an element of $\oplus_{j=0}^k \widetilde{\mathfrak{g}}^{\lambda_j}$.*

Proof. — Let X be an element in $\mathcal{O}^+$. Since $\Delta_1(X) \neq 0$, we know from the proof of Proposition 2.2.1(2) that there exists an element $Z \in \mathfrak{g}$ such that

$$[H_{\lambda_0}, Z] = Z \quad \text{and} \quad e^{\operatorname{ad} Z}.X \in V_1^+ \oplus \widetilde{\mathfrak{g}}^{\lambda_0} \ .$$

It was proved that the element Z belongs to $\mathfrak{g}_{-1} = \oplus_{j=1}^k E_{0,j}(1,-1)$ hence to $\mathfrak{n}$. If X^1 denotes the element of V_1^+ such that $X \in X^1 \oplus \widetilde{\mathfrak{g}}^{\lambda_0}$, we have for $j \geq 1$

$$\Delta_j(X^1) = \Delta_j(X) \ .$$

Since X belongs to $\mathcal{O}^+$, it follows that

$$\Delta_j(X^1) \neq 0 \quad \text{for } j = 1, \ldots, k \ .$$

The proof is finished by induction. In fact the same proof applied to $\widetilde{\mathfrak{g}}_j$ shows that if X belongs to V_j^+ with $\Delta_s(X) \neq 0$ for $s \geq j$ then X is N–conjugate to an element of the form $X^{j+1} + Y_j$ where $X^{j+1} \in V_{j+1}^+$ and

$Y_j \in \widetilde{\mathfrak{g}}^{\lambda_j}$. Moreover we have $\Delta_s(X^{j+1}) = \Delta_s(X) \neq 0$ for $s > j$ and the induction follows.

□

Theorem 3.6.3. *The open dense subset $\mathcal{O}^+$ of V^+ is the union of the open orbits of P in V^+.*
Type I. — *The number of open P–orbits in V^+ is 2^{k+1}. Representatives of these orbits are given by the elements $I_\varepsilon^+ = \sum_{j=0}^k \varepsilon_j X_j$ $(\varepsilon_j = \pm 1)$. Moreover we have*

$$P.I_\varepsilon^+ = \{X \in V^+ \mid \Delta_j(X)\varepsilon_j \cdots \varepsilon_k > 0 \text{ for } j = 0, \ldots, k\} \ .$$

Type II. — *There are two open P–orbits in V^+. Representatives of these orbits are given by the elements $I^+ = \sum_{j=0}^k X_j$ and $I_1^+ = \left(\sum_{j=0}^{k-1} X_j\right) - X_k$. Moreover we have*

$$P.I^+ = \{X \in \mathcal{O}^+ \mid \Delta_0(X) > 0\} \text{ and } P.I_1^+ = \{X \in \mathcal{O}^+ \mid \Delta_0(X) < 0\} \ .$$

Type III. — *There is only one open P–orbit in V^+. Moreover we have*

$$X \in \mathcal{O}^+ \Longleftrightarrow \Delta_j(X) > 0 \text{ for } j = 0, \ldots, k \ .$$

Proof. — The relative invariance of the Δ_j under the action of P implies that $\mathcal{O}^+$ is a union of P–orbits. Let now X be an element of $\mathcal{O}^+$. By Lemma 3.6.2 it is P–conjugate to an element of the form $\sum_{j=0}^k Y_j$ where $Y_j \in \widetilde{\mathfrak{g}}^{\lambda_j}$. Since X is generic, the element $\sum_{j=0}^k Y_j$ is also generic which implies that each Y_j is nonzero. Then there exists an element $a = \exp(\sum_{j=0}^k t_j H_{\lambda_j}) \in A$ such that $a.\sum_{j=0}^k Y_j$ belongs to the product of spheres S_j given by

$$S_j = \left\{ X \in \widetilde{\mathfrak{g}}^{\lambda_j} \mid b(X, \theta(X)) = 1 \right\} \ .$$

Type I : Since the dimension of $\widetilde{\mathfrak{g}}^{\lambda_j}$ is one, it follows that each element of $\mathcal{O}^+$ is P–conjugate to one of the elements I_ε^+ where $\varepsilon = (\varepsilon_0, \ldots, \varepsilon_k)$ with $\varepsilon_j = \pm 1$. By Theorem 1.14.2 (3) we have for $j = 0, \ldots, k$

$$\Delta_j(I_\varepsilon^+) = \varepsilon_j \cdots \varepsilon_k \ .$$

On the other hand the character χ_j is strictly positive on P (Theorem 3.6.1). Therefore the elements I_ε^+ et $I_{\varepsilon'}^+$ are not P-conjugate for $\varepsilon \neq \varepsilon'$ and the result follows.

Type II : For the same reasons each element of $\mathcal{O}^+$ is in fact AN–conjugate to an element of the form I_ε^+. But in this case each I_ε^+ is M-conjugate (Proposition 2.4.5) to I^+ or I_1^+. These two elements are not G–conjugate thus a fortiori not P–conjugate. Therefore we obtain two P–orbits which are distinguished by the sign of Δ_0. This is a consequence of the connectedness of G which implies that χ_0 takes positive values on G and that (by Theorem1.14.2 (3))

$$\Delta_0(I^+) = 1 \text{ and } \Delta_0(I_1^+) = -1 \ .$$

Type III : Since the action of M is transitive on the product of spheres of $\oplus\widetilde{\mathfrak{g}}^{\lambda_j}$, each element of $\mathcal{O}^+$ is P–conjugate to the element I^+. Therefore $\mathcal{O}^+$ is the unique open P-orbit. Since in this case χ_j takes only positive values

on P (Theorem 3.6.1) the sign of Δ_j is constant on $\mathcal{O}^+$, namely strictly positive due to the choice of the normalization.

□

Remark 3.6.4. Using the description of the G–orbits in V^+ given in chapter 2 it is easy to obtain here Theorem 3.5.3 as a consequence of Theorem 3.6.3.

The group M is not so easy to handle (for example, as we have seen in Theorem 3.5.4, M is not connected in Type II case) so we will in the sequel use the AN–orbits in V^+ which are described for Type I and Type II cases in the following Corollary.

Corollary 3.6.5 (Type I and Type II)**.**
The number of open orbits of AN in V^+ is 2^{k+1}. Representatives of these orbits are given by the elements $I_\varepsilon^+ = \sum_{j=0}^k \varepsilon_j X_j$ $(\varepsilon_j = \pm 1)$. The corresponding orbits are described as follows :

$$\mathcal{O}_\varepsilon^+ = AN.I_\varepsilon^+ = \{X \in V^+ \mid \Delta_j(X)\varepsilon_j \cdots \varepsilon_k > 0 \text{ for } j = 0, \dots k\} \ .$$

Notice that for Type I the P–orbits and the AN–orbits are the same.

Proof. — The Corollary is an easy consequence of the identity

$$\Delta_j(anI_\varepsilon^+) = \chi_j(an)\varepsilon_j \cdots \varepsilon_k \ ,$$

which is a consequence of Theorem 1.14.2 and Theorem 3.6.1.

□

Remark 3.6.6. For Type III, it may be noticed that the action of AN on V^+ has no open orbits.

3.7. The involution γ of $\widetilde{\mathfrak{g}}$

In order to characterize the open orbits of P in V^- by using some polynomials analogous to the polynomials Δ_j, we are looking for an element $\gamma \in \widetilde{G}$ such that

(i) γ stabilizes $\mathfrak{g}$ and interchanges V^+ and V^- ;

(ii) γ normalizes P *i.e.* $\gamma P \gamma^{-1} = P$.

It would be nice if moreover

(iii) γ normalizes $\mathfrak{a}^0$,

which means that γ is a representative of an element of the Weyl group of $\widetilde{\Sigma}^0$. By Theorem 3.3.2 the set $\widetilde{\Sigma}^0$ is a root system of type C_{k+1} with a basis given by the forms $\pm\eta_j \pm \eta_i$ and $\pm 2\eta_j$ where

$$\eta_j(H_{\lambda_i}) = \delta_{ij} \ .$$

From the description of the Weyl group for a root system of type C_{k+1} in ([Bou]–1968 Planche III), we know that, if such a γ exists, then there is a permutation τ of $\{0, \dots, k\}$ such that

$$\gamma.\eta_i = \varepsilon_i \eta_{\tau(i)} \text{ where } \varepsilon_i = \pm 1 \text{ for } i = 0, \dots, k \ .$$

Since γ interchanges V^+ and V^- we must have

$$\gamma(\eta_i + \eta_j) = -\eta_{\tau(i)} - \eta_{\tau(j)} ,$$

which implies that each ε_i is be equal to -1. Moreover if γ normalizes P, it normalizes $\mathfrak{n}$. Therefore, since $\mathfrak{n}$ is the sum of the root spaces for the roots $\eta_i - \eta_j$ $(i < j)$ we necessarily have

$$i < j \Longrightarrow \tau(i) > \tau(j) .$$

It follows that, if such a γ exists, it must satisfy

$$(iv)\ \gamma.\eta_i = -\eta_{k-i} \text{ for } i = 0, \ldots, k .$$

But, from ([Bou]–1968), it is known that the Weyl group of C_{k+1} contains an element γ satisfying *(iv)*. Since this group is isomorphic with $N_{\widetilde{K}}(\mathfrak{a}^0)/Z_{\widetilde{K}}(\mathfrak{a}^0)$, the existence of an element $\gamma \in N_{\widetilde{K}}(\mathfrak{a}^0)$ which satisfies *(i)* to *(iv)* is proved.

In the following Theorem we construct explicitely such an element which, moreover, will be an involution of $\widetilde{\mathfrak{g}}$.

Theorem 3.7.1. *There exists an element $\gamma \in N_{\widetilde{K}}(\mathfrak{a}^0)$ such that*

$$(a) \qquad \gamma.H_{\lambda_j} = -H_{\lambda_{k-j}} \text{ for } j = 0, \ldots, k ;$$
$$(b) \qquad \gamma.X_j = \theta(X_{k-j}) \text{ for } j = 0, \ldots, k ;$$
$$(c) \qquad \gamma^2 = \mathrm{Id}_{\widetilde{\mathfrak{g}}} .$$

Such an element γ stabilizes $\mathfrak{g}$, interchanges V^+ and V^- and normalizes G, P, M, A and N.

Proof. —
Step 1 : We first construct an element $\widetilde{\gamma}$ defined as follows if $k > 0$. We recall (Proposition 2.4.4) the existence of elements $\widetilde{\gamma_{ij}} \in N_{\widetilde{K}}(\mathfrak{a}^0)$ such that

$$\widetilde{\gamma_{ij}}^2 = \mathrm{Id}_{\widetilde{\mathfrak{g}}} \text{ and } \widetilde{\gamma_{ij}}(H_{\lambda_s}) = \begin{cases} H_{\lambda_i} & \text{for } s = j , \\ H_{\lambda_j} & \text{for } s = i , \\ H_{\lambda_s} & \text{for } s \notin \{i, j\} . \end{cases}$$

Let n be the integer defined by $k = 2n+2$ if k even and $k = 2n+1$ if k odd. Then we define

$$\widetilde{\gamma} = \widetilde{\gamma_{0,k}} \circ \widetilde{\gamma_{1,k-1}} \circ \cdots \circ \widetilde{\gamma_{n,k-n}} .$$

Thus we have

$$\widetilde{\gamma}(H_{\lambda_j}) = H_{\lambda_{k-j}} \text{ for } j = 0, \ldots, k .$$

On the other hand the elements $\widetilde{\gamma_{i,k-i}}$ $(i = 0, \ldots, n)$ which are associated to strongly orthogonal $\mathfrak{sl}_2$–triples commute. It follows that $\widetilde{\gamma}$ is an involution of $\widetilde{\mathfrak{g}}$. We will now prove that in the different cases it is possible to choose $\widetilde{\gamma}$ such that

$$\widetilde{\gamma}(I^+) = I^+ .$$

More precisely we will prove that we can choose $\widetilde{\gamma}$ such that

$$\widetilde{\gamma}(X_j) = X_{k-j} .$$

The elements $\widetilde{\gamma_{ij}}$ were defined as the product $\gamma_{ij}w_i^2$ where $\gamma_{i,j}(\widetilde{\mathfrak{g}}^{\lambda_i}) = \widetilde{\mathfrak{g}}^{\lambda_j}$ and w_i is the nontrivial Weyl group element associated to the $\mathfrak{sl}_2$–triple $\{\theta(X_i), H_{\lambda_i}, X_i\}$. The action of w_i^2 is trivial on $\oplus_{j=0}^k \widetilde{\mathfrak{g}}^{\lambda_j}$ (section 2.1).

Type I : Since the sequence $(X_j)_{j=0,\dots,k}$ satisfies the condition (D) we have by Corollary 2.10.2

$$\gamma_{ij}(X_i) = X_j \text{ for } i \neq j \ ,$$

and the result follows.

Type II : In this case the choice of the sign of X_j is arbitrary and we have $\gamma_{ij}(X_i) = \pm X_j$ (Proposition 2.4.3). But we proved (Proposition 2.4.5) that there exists $m_{ij} \in Z_K(\mathfrak{a}^0)$ such that $m_{ij}(Z) = -Z$ for $Z \in \widetilde{\mathfrak{g}}^{\lambda_i} \oplus \widetilde{\mathfrak{g}}^{\lambda_j}$ and $m_{ij}(Z) = Z$ for $Z \in \oplus_{s \notin \{i,j\}} \widetilde{\mathfrak{g}}^{\lambda_s}$. Therefore if $\gamma_{i,k-i}(X_i) = -X_{k-i}$ it is sufficient to replace $\gamma_{i,k-i}$ by $m_{i,k-i}\gamma_{i,k-i}$. Since again the action of w_i^2 is trivial on $\oplus_{j=0}^k \widetilde{\mathfrak{g}}^{\lambda_j}$, we obtain the result for such a $\widetilde{\gamma}$.

Type III : In this case two nonzero elements of $\widetilde{\mathfrak{g}}^{\lambda_j}$ are conjugate by an element of L_j (Theorem 1.12.1). More precisely two elements of the same sphere S_j are conjugate under an element of $L_j \cap K \subset Z_K(\mathfrak{a}^0)$ (see the proof of Theorem 3.5.4). Therefore there exists $m_{k-i} \in L_{k-i} \cap K$ such that we have

$$\widetilde{\gamma_{i,k-i}}(X_i) = m_{k-i}X_{k-i} \text{ and } \widetilde{\gamma_{i,k-i}}(m_{k-i}X_{k-i}) = X_i \ .$$

The action of m_{k-i} is trivial on $\oplus_{j \neq k-i} \widetilde{\mathfrak{g}}^{\lambda_j}$. Therefore if we replace, for each $i \leq n$, $\widetilde{\gamma_{i,k-i}}$ by $m_{k-i}^{-1}\widetilde{\gamma_{i,k-i}}m_{k-i}$ we obtain again an element $\widetilde{\gamma}$ of $N_{\widetilde{K}}(\mathfrak{a}^0)$ which is an involution of $\widetilde{\mathfrak{g}}$ and interchanges X_i and X_{k-i}. This implies the result.

Remark. — The element $\widetilde{\gamma}$ centralizes H_0 but does not belong to G in general. As shown in Appendix , this is the case in Type I case with $\widetilde{\mathcal{R}} = C_{k+1}$ and n even *i.e.* $k \equiv 1$ (4) or $k \equiv 2$ (4).

It remains to choose $\widetilde{\gamma}$ if $k = 0$ and we take for $\widetilde{\gamma}$ the identity.

Step 2 : Let now $w = e^{\operatorname{ad} I^+} e^{\operatorname{ad} I^-} e^{\operatorname{ad} I^+}$ be the element of $\widetilde{G}$ which defines σ and let γ be given by

$$\gamma = \widetilde{\gamma}w = w\widetilde{\gamma} \in \widetilde{K} \ .$$

It is clear that γ commutes with w since $\widetilde{\gamma}$ centralizes I^+ and $I^- = \theta(I^+)$ ($\widetilde{\gamma} \in \widetilde{K}$). It follows that γ is an involution of $\widetilde{\mathfrak{g}}$ and that

$$\gamma(H_{\lambda_j}) = \sigma(H_{\lambda_{k-j}}) = -H_{\lambda_{k-j}} \text{ for } j = 0, \dots, k \ ;$$
$$\gamma(X_j) = \sigma(X_{k-j}) = \theta(X_{k-j}) \text{ for } j = 0, \dots, k \ .$$

We deduce from *(a)* that $\gamma(H_0) = -H_0$. This implies that γ interchanges V^+ and V^- ans stabilizes $\mathfrak{g}$. Moreover, since G is connected, it follows that γ normalizes G.

It remains to prove that γ normalizes P, M, A and N. From (a) it follows that γ stabilizes $\mathfrak{a}^0$ and interchanges $E_{i,j}(1,-1)$ and $E_{k-i,k-j}(-1,1)$. Therefore γ stabilizes $\mathfrak{n}$, $\mathfrak{a}^0$ and consequently $\mathfrak{m} \oplus \mathfrak{a} = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}^0)$, $\mathfrak{a} = \mathfrak{p} \cap$ (center of $\mathfrak{m} \oplus \mathfrak{a}$) and $\mathfrak{m}$. It follows that γ normalizes N, A and the identity component M^0 of M. Since $M = Z_K(\mathfrak{a}^0)e^{\operatorname{ad}(\mathfrak{m} \cap \mathfrak{p})} = Z_K(\mathfrak{a}^0)M^0$, γ normalizes M. This gives the result. □

Remark 3.7.2. It may be noticed that γ does not stabilize the graded algebras $\widetilde{\mathfrak{g}}_j$ for $j = 1, \dots, k$ since $\gamma(V_j^+) \neq V_j^-$ (see in Corollary 1.8.4 the decomposition of $V_j^{\pm}$).

3.8. The orbits of P in V^- and the polynomials ∇_j

Definition 3.8.1. *For $j = 0, \dots, k$ we denote by ∇_j the polynomial defined on V^- by*

$$\nabla_j(Y) = \Delta_j(\gamma(Y)) \text{ for } Y \in V^- \ .$$

Theorem 3.8.2. *The polynomials ∇_j are irreducible polynomials of degree $\kappa(k+1-j)$.*
(1) ∇_0 is a relative invariant on V^- under the action of G.
(2) $\nabla_j (j = 0, \dots, k)$ is a relative invariant on V^- under the action of the parabolic group P. More precisely we have

$$\nabla_j(p.Y) = \chi_j^-(p)\nabla_j(Y) \text{ for } p \in P \ ,$$

where χ_j^- is the character of $P = MAN$ described below. For $n \in N$, $H \in \mathfrak{a}$ and $m \in M$ we have

- $\chi_j^-(n) = 1$;
- $\chi_j^-(e^{\operatorname{ad} H}) = e^{-\kappa(\lambda_0(H)+\cdots+\lambda_{k-j}(H))}$;
- $\chi_j^-(m) = \begin{cases} 1 & \text{in Type I and Type III cases ;} \\ m_0 \cdots m_{k-j} & \text{if } m.X_j = m_j X_j \ (m_j = \pm 1) \text{ in Type II case} \ . \end{cases}$

Proof. — The relative invariance is a direct consequence of the relative invariance of the polynomials Δ_j (Theorem 1.14.2 and Theorem 3.6.1) and of the fact that γ (Theorem 3.7.1) normalizes G and P.

To describe the character χ_j^- we first remark that the choice of the normalization of Δ_j implies that $\nabla_j(I^-) = 1$ since $\gamma(I^-) = I^+$. Then we have for $p \in P$

$$\chi_j^-(p) = \nabla_j(p.I^-) = \Delta_j(\gamma p \gamma^{-1}.I^+) = \chi_j(\gamma p \gamma^{-1}) \ .$$

Using the description of the character χ_j given in Theorem 3.6.1 and the fact that γ normalizes M, A and N we obtain the value of the character χ_j^-. More precisely χ_j^- is then trivial on N and the action of γ on $\mathfrak{a}^0$ gives the value of χ_j^- on A. Moreover χ_j^- is trivial on M in Type I and Type III cases. To obtain the result in Type II case we consider an element $m \in M$. Then we have $m.X_j = m_j X_j$ $(m_j = \pm 1)$ and we obtain

$$\begin{aligned} \nabla_j(m.I^-) &= \Delta_j\Big(\sum_{s=0}^{k} m_s \gamma.\theta(X_s)\Big) = \Delta_j\Big(\sum_{s=0}^{k} m_s X_{k-s}\Big) \\ &= \prod_{s=j}^{k} m_{k-s} = m_0 \cdots m_{k-j} \ , \end{aligned}$$

which gives the result.

□

Remark 3.8.3. The proof of the previous Theorem relies only on the properties of γ described in Theorem 3.7.1 and gives the value of ∇_j on the P–orbit of I^- in V^-. This P–orbit is equal to $\gamma(P.I^+)$ hence open by Theorem 3.6.3. Therefore the polynomials ∇_j do not depend on the choice of the element γ satisfying the properties of Theorem 3.7.1. They are the fundamental relative invariants of the prehomogeneous space (P, V^-).

We deduce from the previous Theorem a relation between Δ_j and ∇_j on the open orbits of G respectively in V^+ and V^-. For this purpose let us define for $\varepsilon = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_k)$ where $\varepsilon_j = \pm 1$:

$$I_\varepsilon^+ = \sum_{j=0}^{k} \varepsilon_j X_j \quad \text{and} \quad I_\varepsilon^- = \theta(I_\varepsilon^+) = \sum_{j=0}^{k} \varepsilon_j \theta(X_j) \ .$$

The sets $G.I_\varepsilon^+$ (*resp.* $G.I_\varepsilon^-$) are open in V^+ (*resp.* V^-) and their union is an open dense set. Nevertheless some of these sets are the same (see the classification of the open orbits of G in chapter 2).

Corollary 3.8.4. *For $\varepsilon = (\varepsilon_0, \ldots, \varepsilon_k)$ with $\varepsilon_j = \pm 1$ and $g \in G$ we have*

$$\nabla_0(g.I_\varepsilon^-) = \frac{1}{\Delta_0(g.I_\varepsilon^+)} \ ;$$

$$\nabla_j(g.I_\varepsilon^-) = \frac{\Delta_{k+1-j}(g.I_\varepsilon^+)}{\Delta_0(g.I_\varepsilon^+)} \ \textit{for } j = 1, \ldots, k \ .$$

Proof. —It is sufficient to prove these equalities of analytic functions on an open subset of G and we will do it on the open subset PH_ε where $H_\varepsilon = Z_G(I_\varepsilon^+) = Z_G(I_\varepsilon^-)$. For $p \in P$ we have

$$\nabla_j(p.I_\varepsilon^-) = \chi_j^-(p)\Delta_j(\gamma.I_\varepsilon^-) = \chi_j^-(p)\varepsilon_0 \cdots \varepsilon_{k-j} \ ;$$

$$\Delta_j(p.I_\varepsilon^+) = \chi_j(p)\Delta_j(I_\varepsilon^+) = \chi_j(p)\varepsilon_j \cdots \varepsilon_k \ .$$

But we deduce from Theorems 3.6.1 and Theorem 3.8.2 that

$$\chi_0^-(p) = \frac{1}{\chi_0(p)} \ ,$$

$$\chi_j^-(p) = \frac{\chi_{k+1-j}(p)}{\chi_0(p)} \ ,$$

which implies the result.

□

We deduce from this result that the relation between χ_0 and χ_0^- is in fact true on G and not only on P. We summarize the relations between χ_j and χ_j^- as follows.

Corollary 3.8.5. *For $g \in G$, $p \in P$ and $j = 1, \ldots, k$ we have*

$$\chi_0^-(g) = \frac{1}{\chi_0(g)} \ \textit{for } g \in G \ ;$$

$$\chi_j^-(p) = \frac{\chi_{k+1-j}(p)}{\chi_0(p)} \ \textit{for } p \in P \ .$$

Let us denote by $\mathcal{O}^-$ the open dense subset of V^- given by

$$\mathcal{O}^- = \{Y \in V^- \mid \nabla_0(Y)\nabla_1(Y)\cdots\nabla_k(Y) \neq 0\} .$$

Using again the involution γ we deduce from Theorem 3.6.3 a description of the open P–orbits in V^-.

Theorem 3.8.6. *The open dense subset $\mathcal{O}^-$ of V^- is the union of the open P–orbits in V^-.*

Type I. — *The number of open P–orbits in V^- is 2^{k+1}. Representatives of these orbits are given by the elements $I_\varepsilon^- = \sum_{j=0}^k \varepsilon_j\theta(X_j)\,(\varepsilon_j = \pm 1)$. Moreover we have*

$$P.I_\varepsilon^- = \{Y \in V^- \mid \nabla_j(Y)\varepsilon_0\cdots\varepsilon_{k-j} > 0 \text{ for } j = 0,\dots,k\} .$$

Type II. — *There are two open P–orbits in V^-. Representatives of these orbits are given by the elements $I^- = \sum_{j=0}^k \theta(X_j)$ and $I_1^- = \left(\sum_{j=0}^{k-1}\theta(X_j)\right) - \theta(X_k)$. Moreover we have*

$$P.I^- = \{Y \in \mathcal{O}^- \mid \nabla_0(Y) > 0\} \text{ and } P.I_1^- = \{X \in \mathcal{O}^- \mid \nabla_0(Y) < 0\} .$$

Type III. — *There is only one open P–orbit in V^- . Moreover we have*

$$Y \in \mathcal{O}^- \Longleftrightarrow \nabla_j(Y) > 0 \text{ for } j = 0,\dots,k .$$

As in Corollary 3.6.5 we can also describe the open AN–orbits in V^- in Type I and Type II cases.

Corollary 3.8.7 (Type I and Type II)**.**
There are 2^{k+1} open AN–orbits in V^-. Representatives of these orbits are given by the elements $I_\varepsilon^- = \sum_{j=0}^k \varepsilon_j\theta(X_j)$ $(\varepsilon_j = \pm 1)$. The corresponding orbits are described as follows :

$$\mathcal{O}_\varepsilon^- = AN.I_\varepsilon^- = \{Y \in V^- \mid \nabla_j(Y)\varepsilon_0\cdots\varepsilon_{k-j} > 0 \text{ for } j = 0,\dots k\} .$$

Proof. — The Corollary is an easy consequence of the identity

$$\nabla_j(anI_\varepsilon^-) = \chi_j^-(an)\varepsilon_0\cdots\varepsilon_{k-j} ,$$

given in the proof of Corollary 3.8.4.

□

A way to write more globally the relations between Δ_j and ∇_j is given below using the following definitions :

Definition 3.8.8. *Let $s = (s_0, s_1, \dots, s_k) \in \mathbb{C}^{k+1}$. We denote by $|\Delta|^s$ and $|\nabla|^s$ the functions defined respectively on $\mathcal{O}^+$ and $\mathcal{O}^-$ by*

$$|\Delta|^s(X) = |\Delta_0(X)|^{s_0}\cdots|\Delta_k(X)|^{s_k} \text{ for } X \in \mathcal{O}^+ ;$$
$$|\nabla|^s(Y) = |\nabla_0(Y)|^{s_0}\cdots|\nabla_k(Y)|^{s_k} \text{ for } Y \in \mathcal{O}^- .$$

Definition 3.8.9. *We will denote by t the involution on $\mathbb{C}^{k+1}$ given by*

$$t(s) = (-s_0 - s_1 - \cdots - s_k, s_k, s_{k-1}, \dots, s_1)$$
$$\text{for } s = (s_0,\dots,s_k) \in \mathbb{C}^{k+1} .$$

Corollary 3.8.4 implies the relation

$$|\nabla|^s(p.I_\varepsilon^-) = |\Delta|^{t(s)}(p.I_\varepsilon^+) \text{ for } p \in P, s \in \mathbb{C}^{k+1} .$$

Therefore the functions $|\Delta|^s$ and $|\nabla|^{s'}$ have the same A–character if and only if $s' = t(s)$.

3.9. A G–equivariant map from Ω^+ onto Ω^-

Until now we have defined several maps which interchange V^+ and V^-, namely the involutions θ, σ and γ of the Lie algebra $\widetilde{\mathfrak{g}}$. None of them is G–equivariant. It is possible to define a G–equivariant map but only on the set of generic elements of V^+. Using Theorem 1.11.2 we set the following Definition.

Definition 3.9.1. *We denote by Ω^+ and Ω^- the sets of generic elements respectively in V^+ and V^- which are given by*

$$\Omega^+ = \{X \in V^+ \mid \Delta_0(X) \neq 0\} \quad \text{and} \quad \Omega^- = \{Y \in V^- \mid \nabla_0(Y) \neq 0\} .$$

Recall from Definition 1.7.1 that Ω^+ (*resp.* Ω^-) is the union of the open G–orbits in V^+ (*resp.* V^-).

From Proposition 1.7.9 we know that, if X is a generic element in V^+, there exists an unique element $Y \in V^-$ such that $\{Y, H_0, X\}$ is a $\mathfrak{sl}_2$–triple. It follows that the map $X \longmapsto Y$ is G–invariant since $\{gY, H_0, gX\}$ is also a $\mathfrak{sl}_2$–triple for $g \in G$. We will give an equivalent definition of this map which shows that this is the "grad log" map from SATO and KIMURA ([Sa–K]–1977, p. 62).

For $X \in V^+$ let $\Delta_0'(X)$ be the differential of the polynomial function Δ_0 at the point X which is an element of the dual of V^+. Using the identification between V^- and the dual of V^+ given by the normalized Killing form b, this differential defines an element of V^-.

Definition 3.9.2. *For $X \in \Omega^+$, let $\psi(X) \in V^-$ be defined by the relation*

$$b\big(\psi(X), X'\big) = \frac{\langle \Delta_0'(X), X' \rangle}{\kappa\, \Delta_0(X)} \quad \textit{for all } X' \in V^+ .$$

Proposition 3.9.3. *For $X \in \Omega^+$, $\{\psi(X), H_0, X\}$ is a $\mathfrak{sl}_2$–triple and the map $X \longmapsto \psi(X)$ is G–equivariant on Ω^+.*

Proof. — Let $X \in \Omega^+$ and $g \in G$. As $\Delta_0(gX) = \chi_0(g)\Delta_0(X)$ we easily get

$$\langle \Delta_0'(gX), gX' \rangle = \chi_0(g) \langle \Delta_0'(X), X' \rangle .$$

It follows that

$$b\big(\psi(gX), X'\big) = \frac{\chi_0(g)\langle \Delta_0'(X), g^{-1}X' \rangle}{\kappa\, \Delta_0(gX)} = b\big(\psi(X), g^{-1}X'\big) ,$$

which implies that ψ is G–equivariant *i.e.*

$$\psi(gX) = g\psi(X) \text{ for } X \in \Omega^+, \ g \in G .$$

Let $A \in \mathfrak{g}$. Derivation of the identity $\Delta_0(e^{t\,\mathrm{ad}\,A}X) = \chi_0(\exp tA)\Delta_0(X)$ with respect to t at the point 0 implies

$$\langle \Delta_0'(X), [A,X]\rangle = \langle \chi_0'(e), A\rangle \Delta_0(X) \ ,$$

where e is the neutral element of G. It follows that

$$b\big(A,[X,\psi(X)]\big) = b\big(\psi(X),[A,X]\big) = \frac{1}{\kappa}\langle \chi_0'(e), A\rangle \ .$$

From Theorem 1.8.1 and Proposition 3.5.1 we get the following orthogonal decomposition of $\mathfrak{g}$.

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{a} \oplus \big(\oplus_{i\neq j} E_{i,j}(1,-1)\big) \ .$$

From Proposition 3.1.4 we have $\mathfrak{a}^{\mathfrak{p}} \cap \mathfrak{q} = \mathfrak{a}^0$. Thus the inclusion $\mathfrak{a}^0 \subset \mathfrak{a} \subset \mathfrak{a}^{\mathfrak{p}}$ and the invariance of $\mathfrak{a}$ under σ imply that

$$\mathfrak{a} = \mathfrak{a} \cap \mathfrak{h} \oplus \mathfrak{a}^0 \ .$$

The character χ_0 is trivial on H and on the connected component of M (by Theorem 3.6.1). Hence for $A \in \mathfrak{m} \oplus \mathfrak{a} \cap \mathfrak{h}$ we have $\langle \chi_0'(e), A\rangle = 0$. Moreover for $A \in E_{i,j}(1,-1)$ we have also $\langle \chi_0'(e), A\rangle = 0$ since $E_{i,j}(1,-1) \subset [\mathfrak{g},\mathfrak{g}]$. It follows that $[X,\psi(X)]$ is an element of $\mathfrak{a}^0$.

From Theorem 3.6.1 we get $\langle \chi_0'(e), H_{\lambda_j}\rangle = 2\kappa$ and, therefore, we have

$$b\left(H_{\lambda_j},[X,\psi(X)]\right) = 2 \ .$$

On the other hand, as $b\left(H_{\lambda_j},H_0\right) = b\left(H_{\lambda_j},H_{\lambda_j}\right) = -2$ from Lemma 1.10.2, we get

$$[X,\psi(X)] = -H_0 \ .$$

Hence $\{\psi(X), H_0, X\}$ is a $\mathfrak{sl}_2$–triple.

□

Remark 3.9.4. From Definition 3.9.2 it follows that there exists a polynomial map $Q : V^+ \longrightarrow V^-$ of degree $\kappa(k+1)-1$ such that

$$\psi(X) = \frac{Q(X)}{\Delta_0(X)} \quad \text{for } X \in \Omega^+ \ .$$

Proposition 3.9.5. *For $X \in \Omega^+$ we have*

$$\nabla_0\big(\psi(X)\big) = \frac{1}{\Delta_0(X)} \ ;$$

$$\nabla_j\big(\psi(X)\big) = \frac{\Delta_{k+1-j}(X)}{\Delta_0(X)} \ \textit{for } j = 1,\dots,k \ .$$

Then for $s \in \mathbb{C}^{k+1}$ we have

$$|\nabla|^s\big(\psi(X)\big) = |\Delta|^{t(s)}(X) \ .$$

Proof. — Since ψ is P–equivariant the fonction $X \longmapsto \nabla_j(\psi(X))$ is relatively invariant under the action of P with character χ_j^-. Therefore, up to scalar multiplication, these relations are implied by Corollary 3.8.5. Since $\{I_\varepsilon^- = \theta(I_\varepsilon^+), H_0, I_\varepsilon^+\}$ is a $\mathfrak{sl}_2$–triple, we have

$$\psi(I_\varepsilon^+) = I_\varepsilon^- = \theta(I_\varepsilon^+) \ .$$

Therefore

$$\nabla_j\big(\psi(I_\varepsilon^+)\big) = \nabla_j(I_\varepsilon^-) = \varepsilon_0 \cdots \varepsilon_{k-j} \ .$$

As $\Delta_{k+1-j}(I_\varepsilon^+) = \varepsilon_{k-j+1} \cdots \varepsilon_k$, we obtain the result.

□

3.10. Intersection of N–orbits with the diagonal

From Lemma 3.6.2 we know that each element X of $\mathcal{O}^+$ is N–conjugate to an element of the diagonal $\oplus_{j=0}^k \widetilde{\mathfrak{g}}^{\lambda_j}$. It is easy to see that X is conjugate to an unique element $f(X)$ of the diagonal (see Lemma 3.10.1 below). The aim of this section is to describe more precisely this map f in terms of the map ψ defined in the previous section.

Lemma 3.10.1. *Each element of $\mathcal{O}^+$ is N–conjugate to an unique element of $\oplus_{j=0}^k \widetilde{\mathfrak{g}}^{\lambda_j}$.*

Proof. — It remains to prove that two distinct points of $(\oplus_{j=0}^k \widetilde{\mathfrak{g}}^{\lambda_j}) \cap \mathcal{O}^+$ are not N–conjugate.

Let X and X' be N–conjugate elements of $\mathcal{O}^+ \cap (\oplus_{j=0}^k \widetilde{\mathfrak{g}}^{\lambda_j})$. Then we have

$$\Delta_j(X) = \Delta_j(X') \neq 0 \quad \text{for } j = 0, \ldots, k \ .$$

Let $I^+ = \sum_{j=0}^k X_j$ be the generic element of $\oplus_{j=0}^k \widetilde{\mathfrak{g}}^{\lambda_j}$ (choosen in section 3.6) such that $\Delta_j(I^+) = 1$ for all $j \in \{0, \ldots, k\}$.

Type I and II : In these cases we have

$$X = \sum_{j=0}^k x_j X_j \ \text{ and } \ X' = \sum_{j=0}^k x'_j X_j \ \text{ with } x_j \in \mathbb{R}\backslash\{0\},\ x'_j \in \mathbb{R}\backslash\{0\} \ .$$

It follows from Theorem 1.14.2 (3) that for all $j \in \{0, \ldots, k\}$ we have

$$x_j \cdots x_k = x'_j \cdots x'_k \ ,$$

which implies that $X = X'$.

Type III : In this case there exist (Theorem 3.5.4) elements m and m' in M such that

$$X = m \sum_{j=0}^k x_j X_j \ \text{ and } \ X' = m' \sum_{j=0}^k x'_j X_j \ \text{ with } x_j > 0,\ x'_j > 0 \ .$$

It follows from Theorem 1.14.2 (3) and Theorem 3.6.1 that for all $j \in \{0, \ldots, k\}$ we have

$$x_j^2 \cdots x_k^2 = {x'_j}^2 \cdots {x'_k}^2 \ ,$$

which implies that $X = X'$.

□

We may now define the map

$$\begin{aligned} f \ : \mathcal{O}^+ &\longrightarrow \oplus_{j=0}^k \widetilde{\mathfrak{g}}^{\lambda_j} \\ X \ &\longmapsto f(X) \ , \end{aligned}$$

where $f(X)$ is the intersection of the N–orbit of X with $\oplus_{j=0}^k \widetilde{\mathfrak{g}}^{\lambda_j}$.

In order to describe the map f, we will first give a more precise version of Proposition 2.2.1 (2). Let us recall that $\widetilde{\mathfrak{g}}_1 = V_1^- \oplus \mathfrak{g}_1 \oplus V_1^+$ is a regular

graded algebra whose gradation is given by $H_{\lambda_1} + \cdots H_{\lambda_k} = H_0 - H_{\lambda_0}$. Let ψ_1 be the map defined as in Definition 3.9.2 on the set of generic elements in V_1^+ *i.e.* the set of $X \in V_1^+$ such that $\Delta_1(X) \neq 0$.

Lemma 3.10.2. *Let X be an element of V^+ such that $\Delta_1(X) \neq 0$. Let*

$$X = X_0 + X_1 + X_2 \quad \textit{where} \quad [H_0 - H_{\lambda_0}, X_s] = sX_s \ (s = 0, 1, 2) \ ,$$

be the eigenspaces decomposition of X for $\mathrm{ad}(H_0 - H_{\lambda_0})$ and let Z be defined by

$$Z = [X_1, \psi_1(X_2)] \ .$$

Then Z belongs to $\mathfrak{n}$ and $e^{\mathrm{ad}\, Z} X = f_0(X) + X_2$ where $f_0(X) \in \widetilde{\mathfrak{g}}^{\lambda_0}$ and $X_2 \in V_1^+$. More precisely we have

$$\begin{aligned} &Z \in \oplus_{j=1}^k E_{0,j}(1,-1) \ , \\ &f_0(X) = X_0 - \frac{1}{2}\big(\mathrm{ad}\, X_1\big)^2 \psi_1(X_2) \ , \\ &\Delta_1(X_2) = \Delta_1(X) \neq 0 \ . \end{aligned}$$

Proof. — Let us first note that X_2 belongs to V_1^+ which is by Corollary 1.8.4 the set $\{X \in V^+ \mid [H_{\lambda_0}, X] = 0\}$. Moreover it follows from the definition of Δ_1 (Definition 1.14.1) that $\Delta_1(X_2) = \Delta_1(X)$ which implies that X_2 is generic in V_1^+ and that $\psi_1(X_2)$ is well defined.

It follows easily from the definition of X_1 and the fact that $\psi_1(X_2)$ belongs to V_1^- that

$$Z \in \mathfrak{g} \quad \text{and} \quad [H_{\lambda_0}, Z] = Z \ .$$

Hence we have

$$Z \in \oplus_{j=1}^k E_{0,j}(1,-1) \subset \mathfrak{n} \ .$$

From the relation $[\psi_1(X_2), X_2] = H_0 - H_{\lambda_0}$ we deduce that

$$[X_2, Z] = X_1 \ .$$

Then, writing the decomposition of $e^{\mathrm{ad}\, Z} X$ with respect to the eigenspaces of $\mathrm{ad}(H_0 - H_{\lambda_0})$, we obtain, as in the proof of Proposition 2.2.1, that

$$e^{\mathrm{ad}\, Z} X = X_0 + [Z, X_1] + \frac{1}{2}(\mathrm{ad}\, Z)^2 X_2 + X_2 \ .$$

Using again the relation $[X_2, Z] = X_1$ we obtain

$$\begin{aligned} f_0(X) &= X_0 + [Z, X_1] + \frac{1}{2}(\mathrm{ad}\, Z)^2 X_2 \\ &= X_0 + \frac{1}{2}[Z, X_1] \\ &= X_0 - \frac{1}{2}(\mathrm{ad}\, X_1)^2 \psi_1(X_2) \in \widetilde{\mathfrak{g}}^{\lambda_0} \ . \end{aligned}$$

□

Remark 3.10.3. It follows from Remark 3.9.4 that the map

$$\begin{aligned} f_0 \ : \{X \in V^+ \mid \Delta_1(X) \neq 0\} &\longrightarrow \quad \widetilde{\mathfrak{g}}^{\lambda_0} \\ X \qquad &\longmapsto f_0(X) = X_0 - \frac{1}{2}(\mathrm{ad}\, X_1)^2 \psi_1(X_2) \ , \end{aligned}$$

is of the form $f_0 = \dfrac{P_0}{\Delta_1}$ where P_0 is a polynomial map from V^+ into $\widetilde{\mathfrak{g}}^{\lambda_0}$.

Theorem 3.10.4. *The N–orbit of an element $X \in \mathcal{O}^+$ intersects the diagonal $\widetilde{\mathfrak{g}}^{\lambda_0} \oplus \cdots \oplus \widetilde{\mathfrak{g}}^{\lambda_k}$ in an unique point*

$$f(X) = f_0(X) + \cdots + f_k(X) \quad \text{where } f_j(X) \in \widetilde{\mathfrak{g}}^{\lambda_j} \ .$$

More precisely $f_k(X)$ is the orthogonal projection of X on $\widetilde{\mathfrak{g}}^{\lambda_k}$ and, for $j = 0, \ldots, k-1$, we have

$$f_j(X) = \frac{P_j(X)}{\Delta_{j+1}(X)} \ ,$$

where P_j is a polynomial map from V^+ into $\widetilde{\mathfrak{g}}^{\lambda_j}$.

Proof. — The maps f_j are defined by induction using Lemma 3.10.2 on the graded algebras $\widetilde{\mathfrak{g}}_j$ (see Corollary 1.8.4). To be more precise, let us introduce some notations. Recall that the decomposition

$$\begin{aligned} V^+ &= \oplus_{0 \le i \le j \le k} E_{i,j}(1,1) \\ &= \left(\oplus_{j=0}^{k} \widetilde{\mathfrak{g}}^{\lambda_j}\right) \oplus \left(\oplus_{0 \le i < j \le k} E_{i,j}(1,1)\right) \ . \end{aligned}$$

is orthogonal for the bilinear form b. For $j \in \{0, \ldots, k-1\}$ and $X \in V^+$ we denote by

- X_2^j the orthogonal projection of X on $V_{j+1}^+ = \oplus_{j+1 \le r \le s \le k} E_{r,s}(1,1)$,
- X_1^j the orthogonal projection of X on $\oplus_{j<s} E_{j,s}(1,1)$,
- X_0^j the orthogonal projection of X on $\widetilde{\mathfrak{g}}^{\lambda_j}$.

Then we have $X = X_0^0 + X_1^0 + X_2^0$ (which is the decomposition used in the previous lemma) and for $j = 0, \ldots, k-2$,

$$X_2^j = X_0^{j+1} + X_1^{j+1} + X_2^{j+1} \ .$$

From the definition of Δ_j we obtain

$$\Delta_{j+1}(X) = \Delta_{j+1}(X_2^j) \ .$$

Let ψ_{j+1} be the map defined on the set of generic points in V_{j+1}^+ as in Definition 3.9.2. If X belongs to $\mathcal{O}^+$, then X_2^j is a generic element of V_{j+1}^+ and $\psi_{j+1}(X_2^j)$ is well defined.

Lemma 3.10.2 implies the existence of an element $Z^0 \in \oplus_{s=1}^{k} E_{0,s}(1,-1)$ such that

$$e^{\operatorname{ad} Z^0} X = f_0(X) + X_2^0 \quad \text{where } f_0(X) = X_0^0 - \frac{1}{2}(\operatorname{ad} X_1^0)^2 \psi_1(X_2^0) \in \widetilde{\mathfrak{g}}^{\lambda_0} \ .$$

Applying now the same Lemma to the element $X_2^0 \in V_1^+$ we get the existence of an element $Z^1 \in \oplus_{s=2}^{k} E_{1,s}(1,-1)$ such that

$$e^{\operatorname{ad} Z^1} X_2^0 = f_1(X) + X_2^1$$

$$\text{where } f_1(X) = f_1(X_2^0) = X_0^1 - \frac{1}{2}(\operatorname{ad} X_1^1)^2 \psi_2(X_2^1) \in \widetilde{\mathfrak{g}}^{\lambda_1} \ .$$

The relation $[Z^1, \widetilde{\mathfrak{g}}^{\lambda_0}] = 0$ implies that

$$e^{\operatorname{ad} Z^1} e^{\operatorname{ad} Z^0} X = f_0(X) + f_1(X) + X_2^1 \ .$$

By induction on j we obtain the existence of elements $Z^j \in \mathfrak{n}$ such that

$$e^{\operatorname{ad} Z^{k-1}} \cdots e^{\operatorname{ad} Z^0} X = f_0(X) + \cdots + f_{k-1}(X) + X_2^{k-1} ,$$

where X_2^{k-1} belongs to $V_k^+ = \widetilde{\mathfrak{g}}^{\lambda_k}$ and

$$f_j(X) = X_0^j - \frac{1}{2}(\operatorname{ad} X_1^j)^2 \psi_{j+1}(X_2^j) \in \widetilde{\mathfrak{g}}^{\lambda_j} .$$

The Theorem is proved since it follows from Remark 3.9.4 that

$$\psi_{j+1}(X_2^j) = \frac{Q_{j+1}(X_2^j)}{\Delta_{j+1}(X_2^j)} = \frac{Q_{j+1}(X_2^j)}{\Delta_{j+1}(X)} ,$$

where Q_{j+1} is a polynomial map from V^{j+1} into $\widetilde{\mathfrak{g}}^{\lambda_j}$.

□

CHAPTER 4

Integral formulas

The purpose of this chapter is the computation of some integrals on V^+ which will be used in Chapter 5 and Chapter 6.

This computation is done by using a so called method of descent. This means that, starting with a graded regular Lie algebra of rank $k+1$ (Definition 1.6.2), we obtain a relation between the Lebesgue integral on V^+ and the Lebesgue integrals on the subspaces of weight 2 of two graded regular Lie algebras of rank strictly less than $k+1$. The aim of sections 4.4, 4.5 and 4.6 is to obtain a good normalization of the Lebesgue measures on these vector spaces such that there is no constant in the integral formulas. In fact all the measures satisfying the normalizing conditions will be uniquely determined.

Moreover the integral formula contains an orbital integral for the action of some nilpotent subgroup of G on V^+ (Definition 4.6.3). We obtain in section 4.8 a relation between the orbital integral of the Fourier transform of some function and the Fourier transform of its orbital integral. This relation will be essential in the sequel.

Let us recall that we choose in section 3.6 a sequence $(X_j)_{j=0,\dots,k}$ satisfying condition (D). In this chapter we will denote by σ the involution defined by $I^+ = X_0 + X_1 + \cdots + X_k$ and by σ_ε the involution defined by $I^+_\varepsilon = \varepsilon_0 X_0 + \cdots + \varepsilon_k X_k$ where $\varepsilon_j = \pm 1$. We recall that the polynomials Δ_j (*resp.* ∇_j) are normalized by $\Delta_j(I^+) = 1$ (*resp.* $\nabla_j(\theta(I^+)) = 1$).

4.1. Integrals on V^+ and V^-

We consider Lebesgue measures on V^+ and V^- respectively denoted by dX and dY. Each of these measure is defined up to scalar multiplication. Using the fact that the Cartan involution θ interchanges V^+ and V^- we will normalize the pair of measures (dX, dY) such that

$$(N_1) \qquad \int_{V^+} f(X)\, dX = \int_{V^-} f(\theta(Y))\, dY \quad \text{for } f \in L^1(V^+)\ .$$

Proposition 4.1.1. *If the pair of measures (dX, dY) is normalized by (N_1) and if τ is an involution of $\widetilde{\mathfrak{g}}$ commuting with θ such that $\tau(V^-) = V^+$ then we have*

$$\int_{V^+} f(X)\ dX = \int_{V^-} f(\tau(Y))\ dY \quad \textit{for } f \in L^1(V^+)\ .$$

Proof. — The unicity, up to scalar multiplication, of the Lebesgue measure on V^+ implies the existence of a constant $C > 0$ such that for

$f \in L^1(V^+)$ we have

$$\begin{aligned}\int_{V^+} f(X)\, dX &= C \int_{V^-} f(\tau(Y))\, dY \\ &= C \int_{V^+} f(\tau(\theta(X)))\, dX \quad \text{(using } N_1\text{)} \ .\end{aligned}$$

But the hypothesis on τ implies that the restriction of $(\tau\theta)^2$ to V^+ is the identity, hence the constant C is equal to 1.

□

Remark 4.1.2. This Proposition can be applied to any involution defined by a generic element J^+ in V^+ such that $\{\theta(J^+), H_0, J^+\}$ is a $\mathfrak{sl}_2$–triple (see Theorem 3.1.1 (5)). In particular it can be applied to the involutions σ_ε.

The involution γ defined in Theorem 3.7.1 is given by an element of $\widetilde{K}$. Therefore γ commutes with θ and Proposition 4.1.1 can also be applied to the involution γ.

It may also be noticed that the relation $\gamma(I^+) = \theta(I^+)$ implies, as in Theorem 3.1.1 (5), that γ and σ commute.

After the first normalization (N_1) the pair of measures (dX, dY) on $V^+ \times V^-$ is defined up to scalar multiplication. They will be uniquely determined after a second normalization using the Fourier transform.

We will once and for all make the choice of the character c of $\mathbb{R}$ given by

$$c(x) = e^{2i\pi x} \quad \text{for } x \in \mathbb{R} \ .$$

Using the duality between V^+ and V^- given by the normalized Killing form b (Definition 1.10.1), we define the Fourier transform as follows.

Definition 4.1.3. *The Fourier transform $\mathcal{F}$ of a function f in the Schwartz space $\mathcal{S}(V^+)$ of V^+ is the function $\mathcal{F}f \in \mathcal{S}(V^-)$ given by*

$$\big(\mathcal{F}f\big)(Y) = \int_{V^+} f(X)\, c\big(b(X,Y)\big)\, dX \ , \ Y \in V^- \ .$$

The inverse Fourier transform of a function $g \in \mathcal{S}(V^-)$ is given by

$$\big(\overline{\mathcal{F}}g\big)(X) = \int_{V^-} g(Y)\, \overline{c\big(b(X,Y)\big)}\, dY \ , \ X \in V^+ \ .$$

We will also normalize the pair of measures (dX, dY) by the condition

(N_2) $\qquad \overline{\mathcal{F}} \circ \mathcal{F}$ is the identity on $\mathcal{S}(V^+)$.

In this case the measures are called dual for the Fourier transform.

We will once and for all normalize the pair of measures (dX, dY) by the conditions (N_1) and (N_2).

We will now determine G–invariant measures on V^+ and V^-. Let us first introduce some notation.

Notation 4.1.4. *We will use in the sequel the following parameter*

$$m = \frac{\dim V^+}{\kappa(k+1)} = \frac{1}{\kappa}(\ell + \frac{kd}{2}) \quad \text{(using Proposition 1.9.5)} \ .$$

Lemma 4.1.5. *If g is an element of G we have for $f \in L^1(V^+)$ and $h \in L^1(V^-)$*

$$\int_{V^+} f(g.X)\, dX = \chi_0(g)^{-m} \int_{V^+} f(X)\, dX \ ,$$
$$\int_{V^-} h(g.Y)\, dY = \chi_0(g)^{m} \int_{V^-} h(Y)\, dY \ .$$

Proof. — We have to prove that the determinant of the adjoint action of $g \in G$ on V^+ (*resp.* V^-) is given by $\chi_0(g)^{-m}$ (*resp.* $\chi_0(g)^m$). Since the Killing form induces a duality betwen V^+ and V^- and is invariant under the action of G we have

$$\det{}_{V^+}(\mathrm{Ad}(g)) = \det{}_{V^-}(\mathrm{Ad}(g)^{-1}) \ .$$

Moreover we know (see the proof of Theorem 1.11.2) that the function

$$V^+ \ni X \mapsto P(X) = \det{}_{(V^+,V^-)}(\mathrm{ad}\, X)^2 \ ,$$

is a relative invariant under the action of G for any choice of basis in V^+ and V^-, hence a power of Δ_0. Therefore the character of this invariant which is given by $g \mapsto \det_{V^+}(g)^2$ is a power of the character χ_0 of Δ_0. Taking $g = e^{t\,\mathrm{ad}\, H_0}$, an explicit computation leads to the result.

□

Recall from Definition 3.9.1 that Ω^+ and Ω^- denote the set of generic elements in V^+ and V^- *i.e.*

$$\Omega^+ = \{X \in V^+ \mid \Delta_0(X) \neq 0\} \ \text{ and } \ \Omega^- = \{X \in V^- \mid \nabla_0(X) \neq 0\} \ .$$

Definition 4.1.6. *Let d^*X and d^*Y be the measures defined respectively on Ω^+ and Ω^- by*

$$d^*X = \frac{dX}{|\Delta_0(X)|^m} \quad \textit{and} \quad d^*Y = \frac{dY}{|\nabla_0(Y)|^m} \ .$$

It follows from Lemma 4.1.5 that these measures are G-invariant.

Proposition 4.1.7. *Let τ be an involution of $\widetilde{\mathfrak{g}}$ commuting with θ such that $\tau(V^-) = V^+$ and $\tau(I^-) = I^+$. Then for a continuous function f with compact support in Ω^+ i.e. $f \in \mathcal{C}_c(\Omega^+)$ we have*

$$\int_{V^+} f(X)\, d^*X = \int_{V^-} f(\tau(Y))\, d^*Y \ .$$

Proof. — From the normalization (N_1) of the pair (dX, dY) and from Proposition 4.1.1 we have for $f \in \mathcal{C}_c(\Omega^+)$

$$\int_{V^+} f(X)dX = \int_{V^-} f(\tau(Y))dY \ .$$

Then we also have

$$\int_{V^+} f(X)d^*X = \int_{V^-} f(\tau(Y))|\Delta_0(\tau(Y))|^{-m}dY \ .$$

The map $Y \mapsto \Delta_0(\tau(Y))$ is a relative invariant polynomial under the action of G on V^-, has the same degree as ∇_0 and takes the value 1 on I^- since $\tau(I^-) = I^+$. Thus we have

$$\Delta_0(\tau(Y)) = \nabla_0(Y) \quad \text{for } Y \in V^- \ ,$$

which implies the result.

□

For $\varepsilon \in \{\pm 1\}^{k+1}$, let H_ε be the centralizer in G of I_ε^+ (Definition 3.4.1) which is also the centralizer in G of $I_\varepsilon^- = \theta(I_\varepsilon^+)$. Then $G.I_\varepsilon^+$ (*resp.* $G.I_\varepsilon^-$), which is a connected component of Ω^+ (*resp.* Ω^-), is an open G–orbit in V^+ (*resp.* V^-). It is a realization of the symmetric space G/H_ε :

$$G.I_\varepsilon^+ \simeq G/H_\varepsilon \simeq G.I_\varepsilon^- \ .$$

Proposition 4.1.8. *The restriction of d^*X to $G.I_\varepsilon^+$ and the restriction of d^*Y to $G.I_\varepsilon^-$ induce the same G-invariant measure $d\dot{g}$ on G/H_ε. More precisely we have*

$$\int_{V^+} f(X)\, d^*X = \int_{G/H_\varepsilon} f(g.I_\varepsilon^+)\, d\dot{g} \quad \textit{for} \quad f \in \mathcal{C}_c(G.I_\varepsilon^+) \ ,$$
$$\int_{V^-} h(Y)\, d^*Y = \int_{G/H_\varepsilon} f(g.I_\varepsilon^-)\, d\dot{g} \quad \textit{for} \quad h \in \mathcal{C}_c(G.I_\varepsilon^-) \ .$$

Proof. —Let $d^+\dot{g}$ (*resp.* $d^-\dot{g}$) be the G–invariant measures on G/H_ε defined by d^*X (*resp.* d^*Y). This means that for $f \in \mathcal{C}_c(G.I_\varepsilon^+)$ and for $h \in \mathcal{C}_c(G.I_\varepsilon^-)$ we have

$$\int_{G/H\varepsilon} f(gI_\varepsilon^+) d^+\dot{g} = \int_{V^+} f(X) d^*X \ \text{ and } \ \int_{G/H_\varepsilon} h(gI_\varepsilon^-) d^-\dot{g} = \int_{V^-} h(Y) d^*Y \ .$$

Since $\sigma_\varepsilon(I_\varepsilon^+) = I_\varepsilon^-$, the function $f \circ \sigma_\varepsilon$ belongs to $\mathcal{C}_c(G.I_\varepsilon^-)$ if f belongs to $\mathcal{C}_c(G.I_\varepsilon^+)$. Moreover the involution σ_ε satisfies the hypothesis of Proposition 4.1.7. Therefore we get

$$\int_{G/H_\varepsilon} f(gI_\varepsilon^+) d^+\dot{g} = \int_{G/H_\varepsilon} f(\sigma_\varepsilon(gI_\varepsilon^-)) d^-\dot{g} = \int_{G/H_\varepsilon} f(\sigma_\varepsilon(g)I_\varepsilon^+) d^-\dot{g} \ .$$

As σ_ε is an involution on G, we finally obtain

$$\int_{G/H_\varepsilon} f(gI_\varepsilon^+) d^+\dot{g} = \int_{G/H_\varepsilon} f(gI_\varepsilon^+) d^-\dot{g} \ ,$$

which implies that $d^+\dot{g} = d^-\dot{g}$.

□

Let us consider the action of the G–equivariant map ψ defined in section 3.9 on the G–invariant measures d^*X and d^*Y.

Proposition 4.1.9. *For a function $h \in \mathcal{C}_c(\Omega^-)$ we have*

$$\int_{V^+} h(\psi(X))\, d^*X = \int_{V^-} h(Y)\, d^*Y \ .$$

Proof. — The map

$$h \longmapsto \int_{V^+} h(\psi(X))\, d^*X \ ,$$

is a G–invariant measure on Ω^-. Thus there exists a positive constant C such that for $h \in \mathcal{C}_c(\Omega^-)$ we have

$$\int_{V^+} h(\psi(X))\, d^*X = C \int_{V^-} h(Y)\, d^*Y \ .$$

It follows from Proposition 4.1.7 applied to the involution σ that for a function $f \in \mathcal{C}_c(\Omega^+)$ we have

$$\int_{V^+} f(\sigma \circ \psi(X))\, d^*X = C \int_{V^+} f(X)\, d^*X \ .$$

In order to prove that the constant C is 1, we need an explicit computation of the differential of $\sigma \circ \psi$ at a point $X \in \Omega^+$. Derivation of the identity $[\psi(X), X] = H_0$ implies the following equality between linear maps from V^+ to $\mathfrak{g}$:

$$\operatorname{ad} X \circ \psi'(X) = \operatorname{ad} \psi(X) \quad \text{for } X \in \Omega^+ \ .$$

Since $(\operatorname{ad}\psi(X) \circ \operatorname{ad} X).Y = -2Y$ for $Y \in V^-$ we obtain

$$\psi'(X) = -\frac{1}{2}(\operatorname{ad}\psi(X))^2_{|V^+} \quad \text{for } X \in \Omega^+ \ .$$

But Ω^+ is the union of the open G–orbits in V^+, which are of the form $G.I_\varepsilon^+$. Then we may chose $X = g.I_\varepsilon^+$ where $g \in G$. As $\{I_\varepsilon^-, H_0, I_\varepsilon^+\}$ is a $\mathfrak{sl}_2$–triple, we have $\psi(I_\varepsilon^+) = I_\varepsilon^-$. Moreover it follows from Theorem 3.1.1 that $\sigma_\varepsilon(U) = \frac{1}{2}(\operatorname{ad} I_\varepsilon^-)^2(U)$ for $U \in V^+$. Thus we get

$$\psi'(g.I_\varepsilon^+) = -\frac{1}{2}\operatorname{Ad}(g) \circ (\operatorname{ad} I_\varepsilon^-)^2 \circ \operatorname{Ad}(g)^{-1} = -\operatorname{Ad}(g) \circ \sigma_\varepsilon \circ \operatorname{Ad}(g)^{-1} \ ,$$

which implies that

$$(\sigma \circ \psi)'(g.I_\varepsilon^+) = -\sigma \circ \operatorname{Ad}(g) \circ \sigma_\varepsilon \circ \operatorname{Ad}(g)^{-1}{}_{|V^+} \ .$$

We have seen during the proof of Lemma 4.1.5 that

$$\det{}_{V^-} \operatorname{Ad}(g) = \chi_0(g)^{-m} \text{ and } \det{}_{V^+} \operatorname{Ad}(g) = \chi_0(g)^m \ .$$

Since σ and σ_ε are involutions which commute (Proposition 3.1.3) we have also $|\det(\sigma \circ \sigma_\varepsilon)| = 1$. It follows that

$$|\det{}_{V^+}\big((\sigma \circ \psi)'(g.I_\varepsilon^+)\big)| = \chi_0(g)^{-2m} \ ,$$

which implies that for $X \in \Omega^+$

$$|\det{}_{V^+}\big((\sigma \circ \psi)'(X)\big)| = |\Delta_0(X)|^{-2m} \ .$$

The map $Y \longmapsto \Delta_0(\sigma(Y))$ is a relative invariant on V^- under the action of G, has the same degree as ∇_0 and takes the value 1 on $I^- = \sigma(I^+)$. Thus we have $\Delta_0\big(\sigma(Y)\big) = \nabla_0(Y)$ for $Y \in V^-$ and it follows from Proposition 3.9.5 that for $X \in \Omega^+$ we have

$$\Delta_0\big(\sigma \circ \psi(X)\big) = \frac{1}{\Delta_0(X)} \ .$$

Finally we obtain

$$\begin{aligned}\int_{V^+} f\big(\sigma\circ\psi(X)\big)\,d^*X &= \int_{V^+} f\big(\sigma\circ\psi(X)\big)|\Delta_0\big(\sigma\circ\psi(X)\big)|^m\,dX\\ &= \int_{V^+} f(\xi)\frac{|\Delta_0(\xi)|^m}{|\Delta_0(\xi)|^{2m}}\,d\xi\\ &= \int_{V^+} f(\xi)\,d^*\xi\ ,\end{aligned}$$

which gives the result. □

4.2. Two diffeomorphisms

In this section we consider two graded regular subalgebras of $\widetilde{\mathfrak{g}}$ which are of the form $\widetilde{\mathfrak{g}}_A$ with $A\subset\{\lambda_0,\lambda_1,\dots,\lambda_k\}$ (see Proposition 1.8.5). The first one is defined for $A=\{\lambda_k\}$, is hence of rank 1 and is denoted by $\widetilde{\mathfrak{g}}_{\{\lambda_k\}}$. The second one is defined for $A=\{\lambda_0,\lambda_1,\dots,\lambda_{k-1}\}$, is hence of rank $k=\operatorname{rank}(\widetilde{\mathfrak{g}})-1$ and is denoted by $\underline{\widetilde{\mathfrak{g}}}$. Recall from Remark 1.8.6 that the two graded algebras $\widetilde{\mathfrak{g}}_{\{\lambda_k\}}$ and $\widetilde{\mathfrak{g}}_k$ may be distinct.

A way to describe these algebras is to use the decomposition of $\widetilde{\mathfrak{g}}$ given by the eigenspaces of $\operatorname{ad}H_{\lambda_k}$.

Definition 4.2.1. *We denote for* $q\in\mathbb{Z}$

$$\begin{aligned}V^+(q)&=\{X\in V^+\mid[H_{\lambda_k},X]=qX\}\ ;\\ \mathfrak{g}(q)&=\{X\in\mathfrak{g}\mid[H_{\lambda_k},X]=qX\}\ ;\\ V^-(q)&=\{X\in V^-\mid[H_{\lambda_k},X]=qX\}\ .\end{aligned}$$

From the definition of $\widetilde{\mathfrak{g}}_A$ we get

$$\begin{aligned}\widetilde{\mathfrak{g}}_{\{\lambda_k\}}&=V^-(-2)\oplus\mathfrak{g}(0)\oplus V^+(2)\ ;\\ \underline{\widetilde{\mathfrak{g}}}&=V^-(0)\oplus\mathfrak{g}(0)\oplus V^+(0)\ .\end{aligned}$$

Moreover the decomposition of $\widetilde{\mathfrak{g}}$ in joint eigenspaces for $\operatorname{ad}H_{\lambda_k}$ and for $\operatorname{ad}H_{\lambda_j}(j\neq k)$ (Theorem 1.8.1) implies that

$$\begin{aligned}V^+&=V^+(2)\oplus V^+(1)\oplus V^+(0)\quad\text{where } V^+(2)=\widetilde{\mathfrak{g}}^{\lambda_k}\ ;\\ \mathfrak{g}&=\mathfrak{g}(1)\oplus\mathfrak{g}(0)\oplus\mathfrak{g}(-1)\quad\text{where }\mathfrak{g}(-1)\subset\mathfrak{n};\\ V^-&=V^-(-2)\oplus V^-(-1)\oplus V^-(0)\quad\text{where } V^-(-2)=\widetilde{\mathfrak{g}}^{-\lambda_k}\ .\end{aligned}$$

Let $G(0)$ be the subgroup of G generated by the set $\{e^{\operatorname{ad}X}\mid X\in\mathfrak{g}(0)\}$. Since the graded Lie algebra $\widetilde{\mathfrak{g}}_{\{\lambda_k\}}$ (*resp.* $\underline{\widetilde{\mathfrak{g}}}$) is regular (Proposition 1.8.5) there exists a unique (up to scalar multiplication) irreducible polynomial on $V^+(2)$ (*resp.* $V^-(0)$) which is relatively invariant under the action of the group $G(0)$.

Lemma 4.2.2.
(1) *The set of generic elements in* $V^+(2)$ *under the action of* $G(0)$ *is*

$$V^+(2)'=\{v\in V^+(2)\mid\Delta_k(v)\neq0\}\ .$$

(2) *The set of generic elements in* $V^-(0)$ *under the action of* $G(0)$ *is*

$$V^-(0)' = \{u' \in V^-(0) \mid \nabla_1(u') \neq 0\} \ .$$

Proof. —
(1) From Definition 1.14.1 the restriction of Δ_k to $V^+(2)$ is the unique (up to scalar multiplication) irreducible relative invariant on $V^+(2)$ under the action of the group G_k. Since G_k is a subgroup of $G(0)$ (see Remark 1.8.6) it follows that the restriction of Δ_k to $V^+(2)$ is also relatively invariant under the action of $G(0)$. And the result follows from Theorem 1.11.2 applied to the graded regular algebra $\widetilde{\mathfrak{g}}_{\{\lambda_k\}}$.

(2) Similarly the restriction of Δ_1 to V_1^+ is the unique (up to scalar multiplication) irreducible relative invariant on V_1^+ under the action of the group G_1 (Definition 1.14.1). The involution γ (see Theorem 3.7.1) is an isomorphism from $V^-(0)$, eigenspace in V^- of $\operatorname{ad} H_{\lambda_k}$ for the eigenvalue 0, onto the eigenspace in V^+ of $\operatorname{ad}(-H_{\lambda_0})$ for the eigenvalue 0, which is V_1^+. From the definition of $\mathfrak{g}_1$ ($\mathfrak{g}_1 = \mathfrak{z}_{\mathfrak{g}}(\widetilde{\mathfrak{l}}_0) \subset \mathfrak{z}_{\mathfrak{g}}(H_{\lambda_0})$) we deduce that $\gamma(\mathfrak{g}_1)$ is a subalgebra of $\mathfrak{g}(0)$ hence $\gamma G_1 \gamma^{-1}$ a subgroup of $G(0)$. Therefore the restriction of $\nabla_1 = \Delta_1 \circ \gamma$ to $V^-(0)$ is also relatively invariant under the action of $G(0)$. And the result follows from Theorem 1.11.2 applied to the graded regular algebra $\widetilde{\underline{\mathfrak{g}}}$.

□

Proposition 4.2.3. *Let* $V^+(2)'$ *(resp.* $V^-(0)'$*) be the set of generic elements in* $V^+(2)$ *(resp.* $V^-(0)$*) under the action of* $G(0)$.
(1) *The map* Φ *defined by*

$$\begin{aligned} \Phi : \ V^+(0) \times \mathfrak{g}(-1) \times V^+(2)' &\longrightarrow V^+ \\ (u, \ A, \ v) &\longmapsto e^{\operatorname{ad} A}(u+v) \end{aligned}$$

is a diffeomorphism onto the set

$$O_k = \{X \in V^+ \mid \Delta_k(X) \neq 0\} \ .$$

(2) *The map* Ψ *defined by*

$$\begin{aligned} \Psi : \ V^-(0)' \times \mathfrak{g}(-1) \times V^-(-2) &\longrightarrow V^- \\ (u', \ A, \ v') &\longmapsto e^{\operatorname{ad} A}(u'+v') \end{aligned}$$

is a diffeomorphism onto the set

$$\{Y \in V^- \mid \nabla_1(Y) \neq 0\} \ .$$

Proof. —
(1) Using the gradations given by the eigenspaces of $\operatorname{ad} H_0$ and $\operatorname{ad} H_{\lambda_k}$ we obtain

$$(*) \quad \Phi(u, A, v) = u + \frac{1}{2}(\operatorname{ad} A)^2 v + \operatorname{ad} A.v + v \ ,$$

where $u + \frac{1}{2}(\operatorname{ad} A)^2 v \in V^+(0)$, $\operatorname{ad} A.v \in V^+(1)$ and $v \in V^+(2)$.

If v is a generic element of $V^+(2)$ *i.e.* if $\Delta_k(v) \neq 0$, there exists (Proposition 1.7.9) an element v' in $V^-(-2)$ such that $\{v', H_{\lambda_k}, v\}$ is a $\mathfrak{sl}_2$-triple.

Then $\operatorname{ad} v$ is an isomorphism from $V^-(-1) \oplus \mathfrak{g}(-1)$ onto $\mathfrak{g}(1) \oplus V^+(1)$, hence from $\mathfrak{g}(-1)$ onto $V^+(1)$ since $v \in V^+$. We have

$$\Phi(u, A, v) = \Phi(u', A', v') \Longrightarrow v' = v \text{ and } \operatorname{ad} v.A = \operatorname{ad} v'.A' \ .$$

It follows that $A' = A$ and, therefore, that $u' = u$. Hence Φ is one to one.

To prove that Φ is onto, we decompose an element $X \in O_k$ in

$$X = X^{(0)} + X^{(1)} + X^{(2)} \text{ where } X^{(q)} \in V^+(q) \ .$$

The definition of Δ_k (Definition 1.14.1) implies the equality

$$\Delta_k(X) = \Delta_k(X^{(2)}) \ .$$

Since X belongs to O_k, $X^{(2)}$ is generic in $V^+(2)$. By the argument used above, it follows that $\operatorname{ad} X^{(2)}$ is an isomorphism from $\mathfrak{g}(-1)$ onto $V^+(1)$. Therefore there exists an element $A \in \mathfrak{g}(-1)$ such that $\operatorname{ad} X^{(2)}.A = X^{(1)}$. If we define $u \in V^+(0)$ by

$$u = X^{(0)} - \frac{1}{2}(\operatorname{ad} A)^2 X^{(2)} \ ,$$

we obtain the equality $\Phi(u, A, X^{(2)}) = X$.

It follows from the equality $(*)$, that the differential of Φ at the point (u, A, v) as a linear morphism from $V^+(0) \oplus \mathfrak{g}(-1) \oplus V^+(2)$ on $V^+(0) \oplus V^+(1) \oplus V^+(2)$ has the following matrix (according to theses decompositions)

$$\begin{pmatrix} \mathrm{Id} & * & * \\ 0 & -\operatorname{ad} v & * \\ 0 & 0 & \mathrm{Id} \end{pmatrix}$$

Since $\operatorname{ad} v$ is an isomorphism from $\mathfrak{g}(-1)$ onto $V^+(1)$, Φ is a diffeomorphism.

(2) The proof is the same modulo the following remarks.

If the decomposition of an element $Y \in V^-$ is written as $Y = Y^{(0)} + Y^{(1)} + Y^{(2)}$ with $Y^{(p)} \in V^-(-p)$, then $\nabla_1(Y) = \nabla_1(Y^{(0)})$. This is due to the fact that $\nabla_1(Y) = \Delta_1(\gamma.Y)$ and $\gamma(V^-(0)) = V_1^+$. The result obtained in 1) can be rewritten using the decomposition in eigenspaces of $\operatorname{ad} H_{\lambda_0}$ instead of $\operatorname{ad} H_{\lambda_k}$ and the properties of γ imply the result.

□

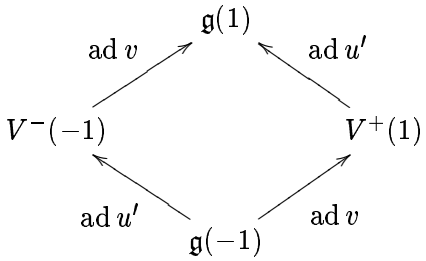
Remark 4.2.4. The formula $(*)$ and the definition of Δ_k imply that

$$\Delta_k(\Phi(u, A, v)) = \Delta_k(v) \ .$$

4.3. Isomorphisms between $\mathfrak{g}(1)$, $\mathfrak{g}(-1)$, $V^+(1)$ and $V^-(-1)$

The aim of the next sections is to show that it is possible to normalize the measures such that there is no constant in the integral formula using the change of variables Φ (or Ψ). The final result is given in section 4.6.

Let v (*resp.* u') be a generic element in $V^+(2)$ (*resp.* $V^-(0)$). As $[v, u'] = 0$, the following diagram commutes and it follows from the proof of Proposition 4.2.3 that the arrows $\operatorname{ad} v$ and $\operatorname{ad} u'$ are isomorphisms.



Proposition 4.3.1. *There exist positive constants α, β, γ and δ which depend on the choice of the Lebesgue measures dA on $\mathfrak{g}(-1)$, dB on $\mathfrak{g}(1)$, dw on $V^+(1)$ and dw' on $V^-(-1)$ such that the following assertions hold.*
(1) *Let v be a generic element in $V^+(2)$ i.e. such that $\Delta_k(v) \neq 0$. Then we have for $f \in L^1(V^+(1))$ and $g \in L^1(\mathfrak{g}(1))$*

$$\int_{V^+(1)} f(w)\, dw = \alpha\, |\Delta_k(v)|^{\frac{dk}{\kappa}} \int_{\mathfrak{g}(-1)} f([A,v])\, dA\ ;$$
$$\int_{\mathfrak{g}(1)} g(B)\, dB = \beta\, |\Delta_k(v)|^{\frac{dk}{\kappa}} \int_{V^-(-1)} g([w',v])\, dw'\ .$$

(2) *Let u' be a generic element in $V^-(0)$ i.e. such that $\nabla_1(u') \neq 0$. Then we have for $f \in L^1(V^-(-1))$ and $g \in L^1(\mathfrak{g}(1))$*

$$\int_{V^-(-1)} f(w')\, dw' = \gamma\, |\nabla_1(u')|^{\frac{d}{\kappa}} \int_{\mathfrak{g}(-1)} f([A,u'])\, dA\ ;$$
$$\int_{\mathfrak{g}(1)} g(B)\, dB = \delta\, |\nabla_1(u')|^{\frac{d}{\kappa}} \int_{V^+(1)} g([w,u'])\, dw\ .$$

Proof. — We only give the proof of the first formula.
If v is a generic element in $V^+(2)$, then $\operatorname{ad} v$ is an isomorphism from $\mathfrak{g}(-1)$ onto $V^+(1)$ and, therefore, there exists a nonzero function α defined on $V^+(2)'$ such that for $f \in L^1(V^+(1))$ we have

$$(*) \qquad \int_{V^+(1)} f(w)\, dw = \alpha(v) \int_{\mathfrak{g}(-1)} f([A,v])\, dA\ .$$

This function is in fact the absolute value of a polynomial function on $V^+(2)$.

The subspace $\mathfrak{g}(-1)$ is invariant under the action of $G(0)$ and there exists a character χ_1 of $G(0)$ such that for $F \in L^1(\mathfrak{g}(-1))$ and $g \in G(0)$ we have

$$\int_{\mathfrak{g}(-1)} F(gA)\, dA = \chi_1(g) \int_{\mathfrak{g}(-1)} F(A)\, dA\ .$$

On the other hand $V^+(2)'$ and $V^+(1)$ are also invariant under the action of $G(0)$ and we obtain

$$\begin{aligned}\int_{V^+(1)} f(w)\, dw &= \alpha(gv) \int_{\mathfrak{g}(-1)} f([A,gv])\, dA \\ &= \alpha(gv)\chi_1(g^{-1}) \int_{\mathfrak{g}(-1)} f(g[A,v])\, dA\ .\end{aligned}$$

Using again formula $(*)$ we obtain

$$\alpha(v) \int_{V^+(1)} f(w)\, dw = \alpha(gv)\chi_1(g^{-1}) \int_{V^+(1)} f(g.w)\, dw \ .$$

But there exists a character χ_2 of $G(0)$ such that

$$\int_{V^+(1)} f(g.w)\, dw = \chi_2(g) \int_{V^+(1)} f(w)\, dw \ .$$

Consequently we obtain that for $g \in G(0)$ and $v \in V^+(2)'$

$$\alpha(gv) = \chi_1(g)\chi_2(g)^{-1}\alpha(v) \ .$$

In other words the function α is a relative invariant under the action of $G(0)$. Every relative invariant polynomial on $V^+(2)$ under the action of $G(0)$ is a power of Δ_k hence there exists a positive constant α and an integer m such that

$$\alpha(v) = \alpha\, |\Delta_k(v)|^m \ .$$

To determine m, we take $g = e^{t \operatorname{ad} H_{\lambda_k}}$ with $t \in \mathbb{R}$. Since $g.v = e^{2t}v$ for $v \in V^+(2)$ we deduce from formula $(*)$ that

$$\alpha(v) \int_{\mathfrak{g}(-1)} f([A, v])\, dA = \alpha(e^{2t}v) \int_{\mathfrak{g}(-1)} f([e^{2t}A, v])\, dA \ .$$

Therefore for every $t \in \mathbb{R}$ we have

$$\alpha(e^{2t}v) = e^{2t \dim \mathfrak{g}(-1)}\ \alpha(v) \ .$$

The dimension of $\mathfrak{g}(-1)$ is the sum of the dimensions of the spaces $E_{j,k}(1,-1)$ for $j = 0, 1, \ldots, k-1$, hence we have $\dim \mathfrak{g}(-1) = kd$. The polynomial function Δ_k has degree κ and, therefore, we have proved that

$$\alpha(v) = \alpha|\Delta_k(v)|^{\frac{dk}{\kappa}} \ .$$

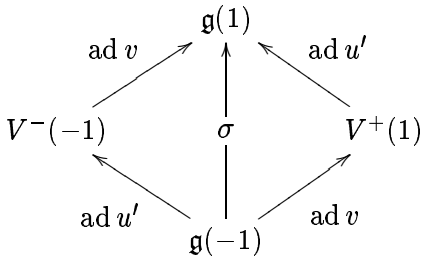
The formulas of the second part of the Proposition are due to the fact that the relative invariant polynomials on $V^-(0)$ under the action of $G(0)$ are the powers of ∇_1 which is a polynomial of degree κk.

□

4.4. A first normalization and its consequence

We consider now $v = X_k$ which is a generic element in $V^+(2)$ and $u' = \theta(X_0 + \cdots + X_{k-1})$ which is a generic element in $V^-(0)$.

Lemma 4.4.1. *For such a choice of v and u', the following diagram is commutative :*



In other words, we have

$$\forall A \in \mathfrak{g}(-1),\ \mathrm{ad}\, u' \,\mathrm{ad}\, v.A = \mathrm{ad}\, v \,\mathrm{ad}\, u'.A = \sigma(A)\ .$$

Moreover we have

$$\forall w \in V^+(1),\ (\mathrm{ad}\, u' \circ \sigma \circ \mathrm{ad}\, u').w = -\sigma(w)\ ,$$
$$\forall w' \in V^-(-1),\ (\mathrm{ad}\, v \circ \sigma \circ \mathrm{ad}\, v).w' = -\sigma(w')\ .$$

Proof. —
1) Since $[u', v] = 0$, the operators $\mathrm{ad}\, u'$ and $\mathrm{ad}\, v$ commute. On the other hand since $I^+ - v = X_0 + \cdots X_{k-1}$ belongs to $V^+(0)$, $[I^+ - v, \mathfrak{g}(-1)]$ is a subset of $V^+(-1) = \{0\}$. Therefore

$$\forall A \in \mathfrak{g}(-1), \quad \mathrm{ad}\, u' \,\mathrm{ad}\, v.A = \mathrm{ad}\, u' \,\mathrm{ad}\, I^+.A\ .$$

But $[u', I^+]$ is equal to $H_{\lambda_0} + \cdots + H_{\lambda_{k-1}}$, so that

$$\forall A \in \mathfrak{g}(-1), \quad \mathrm{ad}\, u' \,\mathrm{ad}\, v.A = A + \mathrm{ad}\, I^+ \,\mathrm{ad}\, u'.A\ .$$

Since $I^- - u' = \theta(X_k)$ belongs to $V^-(-2)$, $[I^- - u', \mathfrak{g}(-1)]$ is a subset of $V^-(-3) = \{0\}$. Therefore

$$\forall A \in \mathfrak{g}(-1), \quad \mathrm{ad}\, u' \,\mathrm{ad}\, v.A = A + \mathrm{ad}\, I^+ \,\mathrm{ad}\, I^-.A\ .$$

From Theorem 3.1.1 (1), we obtain

$$\forall A \in \mathfrak{g}(-1), \quad A\,\mathrm{ad}\, u' \,\mathrm{ad}\, v.A = A + \mathrm{ad}\, I^- \,\mathrm{ad}\, I^+.A = \sigma(A)\ .$$

2) The definition of σ implies that $\sigma(X_j) = \theta(X_j)$ for $j \in \{0, \ldots, k\}$, hence $\sigma(u') = X_0 + \cdots + X_{k-1}$. Then the relation $[\sigma(u'), u'] = -(H_{\lambda_0} + \cdots + H_{\lambda_{k-1}})$ implies that

$$\forall w \in V^+(1), \quad \mathrm{ad}\, \sigma(u') \,\mathrm{ad}\, u'.w = -w + \mathrm{ad}\, u' \,\mathrm{ad}\, \sigma(u')w = -w\ .$$

Then we obtain

$$\forall w \in V^+(1), \quad (\mathrm{ad}\, u' \circ \sigma \circ \mathrm{ad}\, u').w = (\sigma \circ \mathrm{ad}\, \sigma(u') \circ \mathrm{ad}\, u').w = -\sigma(w)\ .$$

The last result is proved with the same method.

□

The involution θ interchanges $V^-(-1)$ and $V^+(1)$, $\mathfrak{g}(-1)$ and $\mathfrak{g}(1)$. It is then possible to normalize the pairs of measures (dw, dw') on $V^+(1) \times$

$V^-(-1)$ and the pair of measures (dA, dB) on $\mathfrak{g}(-1) \times \mathfrak{g}(1)$ such that

$$\text{(4-4-1)} \quad \begin{cases} \displaystyle\int_{V^+(1)} f(w)\, dw = \int_{V^-(-1)} f(\theta(w'))\, dw' \quad \text{for } f \in L^1(V^+(1)) , \\ \displaystyle\int_{\mathfrak{g}(1)} g(B)\, dB = \int_{\mathfrak{g}(-1)} g(\theta(A))\, dA \quad \text{for } g \in L^1(\mathfrak{g}(1)) . \end{cases}$$

Since σ is also an involution interchanging $V^-(-1)$ and $V^+(1)$, $\mathfrak{g}(-1)$ and $\mathfrak{g}(1)$ which commutes with θ, we have also (see Proposition 4.1.1)

$$\text{(4-4-2)} \quad \begin{cases} \displaystyle\int_{V^+(1)} f(w)\, dw = \int_{V^-(-1)} f(\sigma(w'))\, dw' \quad \text{for } f \in L^1(V^+(1)) , \\ \displaystyle\int_{\mathfrak{g}(1)} g(B)\, dB = \int_{\mathfrak{g}(-1)} g(\sigma(A))\, dA \quad \text{for } g \in L^1(\mathfrak{g}(1)) . \end{cases}$$

Proposition 4.4.2. *If the measures are normalised by* (4-4-1)*, then the strictly positive constants obtained in Proposition* 4.3.1 *satisfy the relations*

$$\alpha\delta = \beta\gamma = \alpha\beta = \gamma\delta = 1 .$$

Proof. — The choice of the normalizations of the polynomials Δ_k and ∇_1 implies that

$$\Delta_k(I^+) = \Delta_k(v) = 1 \text{ and } \nabla_1(I^-) = \nabla_1(u') = 1 .$$

Using Proposition 4.3.1, we obtain for $g \in L^1(\mathfrak{g}(1))$

$$\begin{aligned} \int_{\mathfrak{g}(1)} g(B)\, dB &= \delta \int_{V^+(1)} g(-\operatorname{ad} u'.w)\, dw \\ &= \alpha\delta \int_{\mathfrak{g}(-1)} g(\operatorname{ad} u' \operatorname{ad} v.A)\, dA . \end{aligned}$$

Then we deduce from Lemma 4.4.1 that

$$\int_{\mathfrak{g}(1)} g(B)\, dB = \alpha\delta \int_{\mathfrak{g}(-1)} g(\sigma(A))\, dA .$$

Then (4-4-2) implies the equality $\alpha\delta = 1$.

We prove with the same argument that

- the relation $\operatorname{ad} v \circ \operatorname{ad} u'|_{\mathfrak{g}(-1)} = \sigma$ implies the equality $\beta\gamma = 1$;
- the relation $\operatorname{ad} v \circ \sigma \circ \operatorname{ad} v|_{V^-(-1)} = -\sigma$ implies the equality $\alpha\beta = 1$;
- the relation $\operatorname{ad} u' \circ \sigma \circ \operatorname{ad} u'|_{V^+(1)} = -\sigma$ implies the equality $\gamma\delta = 1$.

□

4.5. A second normalization and its consequence

The form b induces a duality between the spaces $V^+(1)$ and $V^-(-1)$, and between the spaces $\mathfrak{g}(-1)$ and $\mathfrak{g}(1)$. We define as in Definition 4.1.3 the Fourier transform and its inverse on the Schwarz functions on these spaces. Unless a specific notation is needed these Fourier transforms will also be denoted by $\mathcal{F}$ and $\overline{\mathcal{F}}$.

After the first normalization (4-4-1) the pairs of measures (dw, dw') on $V^+(1) \times V^-(-1)$ and (dA, dB) on $\mathfrak{g}(-1) \times \mathfrak{g}(1)$ are defined up to scalar multiplication. We normalize moreover these pairs of measures by the conditions

$$(4\text{-}5\text{-}1) \quad \begin{cases} \overline{\mathcal{F}} \circ \mathcal{F} \text{ is the identity on } \mathcal{S}(V^+(1)) \ ; \\ \\ \overline{\mathcal{F}} \circ \mathcal{F} \text{ is the identity on } \mathcal{S}(\mathfrak{g}(-1)) \ . \end{cases}$$

These measures are now uniquely determined by (4-4-1) and (4-5-1) and we obtain the following result.

Proposition 4.5.1. *If the pairs of measures (dw, dw') on $V^+(1) \times V^-(-1)$ and (dA, dB) on $\mathfrak{g}(-1) \times \mathfrak{g}(1)$ are normalized by the two conditions* (4-4-1) *and* (4-5-1)*, then the constant α, β, γ and δ obtained in Proposition* 4.3.1 *are equal to 1.*

Proof. — We will prove that $\alpha\gamma = 1$. Then Proposition 4.4.2 will imply the result.

Let f be a Schwartz function on $V^+(1)$. We recall that $u' = \theta(X_0 + \cdots + X_{k-1})$ satisfies $\nabla_1(u') = 1$. From (4-5-1) and Proposition 4.3.1 (2) we obtain for $w \in V^+(1)$

$$(*) \quad f(w) = \big(\overline{\mathcal{F}} \circ \mathcal{F} f\big)(w) = \gamma \int_{\mathfrak{g}(-1)} \big(\mathcal{F} f\big)([A, u']) \ \overline{c\Big(b(w, [A, u'])\Big)} \ dA \ .$$

On the other hand, the element $v = X_k$ satisfies $\Delta_k(v) = 1$. From (4-5-1) and Proposition 4.3.1 (1) we obtain for $w' \in V^-(-1)$

$$\big(\mathcal{F} f\big)(w') = \alpha \int_{\mathfrak{g}(-1)} f([A, v]) \ c\Big(b([A, v], w')\Big) \ dA \ .$$

If we define the function φ on $\mathfrak{g}(-1)$ by $\varphi(A) = f([A, v])$, we obtain that

$$\big(\mathcal{F} f\big)(w') = \alpha (F\varphi)([v, w']) \ ,$$

where, here, F denotes the Fourier transform from $\mathcal{S}(\mathfrak{g}(-1))$ onto $\mathcal{S}(\mathfrak{g}(1))$. Then we have

$$\big(\mathcal{F} f\big)([A, u']) = \alpha \big(F\varphi\big)(-\operatorname{ad} v \circ \operatorname{ad} u'.A) \ .$$

Lemma 4.4.1 implies that for $A \in \mathfrak{g}(-1)$ we have $\operatorname{ad} v \circ \operatorname{ad} u'.A = \sigma(A)$. Then equality $(*)$ may be written as

$$f(w) = \alpha\gamma \int_{\mathfrak{g}(-1)} \big(F\varphi\big)(-\sigma(A)) \ \overline{c\Big(b(\operatorname{ad} u'.w, A)\Big)} \ dA \ .$$

Using now the normalizations (4-4-2) and (4-5-1) for the measures (dA, dB) we obtain

$$f(w) = \alpha\gamma \ \varphi\big(-\sigma(\operatorname{ad} u'.w)\big) = \alpha\gamma \ f\big(\operatorname{ad} v \circ \sigma \circ \operatorname{ad} u'.w\big) \ .$$

Writing $w = \operatorname{ad} v.A$ with $A \in \mathfrak{g}(-1)$, which is possible since $\operatorname{ad} v$ is an isomorphism from $\mathfrak{g}(-1)$ onto $V^+(1)$, we deduce from Lemma 4.4.1 that

$$\operatorname{ad} v \circ \sigma \circ \operatorname{ad} u'.w = \operatorname{ad} v \circ \sigma \circ \operatorname{ad} u' \circ \operatorname{ad} v.A = \operatorname{ad} v \circ \sigma^2.A = w \ .$$

This proves that for every $f \in \mathcal{S}(V^+(1))$ we have $f = \alpha\gamma f$, and therefore $\alpha\gamma = 1$.

□

4.6. Integral formulas on V^+ and V^-

Recall that the pair of measures (dX, dY) on $V^+ \times V^-$ is determined by conditions (N_1) and (N_2). We will also normalize the pairs of measures (du, du') on $V^+(0) \times V^-(0)$, (dw, dw') on $V^+(1) \times V^-(-1)$, and (dv, dv') on $V^+(2) \times V^-(-2)$ by conditions similar to (N_1) and (N_2). Then we get

$$dX = du\, dw\, dv \quad \text{and} \quad dY = du'\, dw'\, dv' \ .$$

In particular (dw, dw') is now normalized by (4-4-1) and (4-5-1).

Moreover we also choose a pair of measures (dA, dB) on $\mathfrak{g}(-1) \times \mathfrak{g}(1)$ normalized by conditions (4-4-1) and (4-5-1).

Remark 4.6.1. The pair of measures (du, du') *(resp.* (dv, dv')*)* is then normalized by conditions (N_1) and (N_2) written for the regular graded Lie algebra $\widetilde{\mathfrak{g}}$ *(resp.* $\widetilde{\mathfrak{g}}_{\{\lambda_k\}}$*)*.

Theorem 4.6.2. *These normalizations of the measures imply that for* $f \in L^1(V^+)$ *and* $g \in L^1(V^-)$ *we have*

$$\int_{V^+} f(X)\, dX = \int_{u \in V^+(0)} \int_{A \in \mathfrak{g}(-1)} \int_{v \in V^+(2)} f\Big(e^{\operatorname{ad} A}(u+v)\Big) |\Delta_k(v)|^{\frac{dk}{\kappa}}\, du\, dA\, dv \ ;$$

$$\int_{V^-} g(Y)\, dY = \int_{u' \in V^-(0)} \int_{A \in \mathfrak{g}(-1)} \int_{v' \in V^-(-2)} g\Big(e^{\operatorname{ad} A}(u'+v')\Big) |\nabla_1(u')|^{\frac{d}{\kappa}} du' dA\, dv' \ .$$

Proof. — We will prove the first equality (the proof of the second one is similar). Let us start with the right hand side of the formula which is equal to

$$\int_{u \in V^+(0)} \int_{A \in \mathfrak{g}(-1)} \int_{v \in V^+(2)} f\Big(u + \frac{1}{2}(\operatorname{ad} A)^2.v + \operatorname{ad} A.v + v\Big)\ |\Delta_k(v)|^{\frac{dk}{\kappa}}\, du\ dA\ dv \ .$$

Using the change of variable $u + \frac{1}{2}(\operatorname{ad} A)^2.v \mapsto u$ we get

$$\int_{u \in V^+(0)} \int_{A \in \mathfrak{g}(-1)} \int_{v \in V^+(2)} f\Big(u + [A, v] + v\Big)\ |\Delta_k(v)|^{\frac{dk}{\kappa}}\, du\ dA\ dv \ .$$

Using Proposition 4.3.1 (1) and the equality $\alpha = 1$ from Proposition 4.5.1, we obtain that this last integral is equal to

$$\int_{u \in V^+(0)} \int_{w \in V^+(1)} \int_{v \in V^+(2)} f(u + w + v)\ du\ dw\ dv \ .$$

Since $dX = du\ dw\ dv$, this gives the result.

□

Let us now introduce the orbital integral of a function f for the action of the unipotent subgroup of G with Lie algebra $\mathfrak{g}(-1)$.

Definition 4.6.3.

(1) *For* $f \in L^1(V^+)$*, the function* T_f^+ *is defined on* $V^+(0) \times V^+(2)$ *by*

$$T_f^+(u, v) = |\Delta_k(v)|^{\frac{kd}{2\kappa}} \int_{A \in \mathfrak{g}(-1)} f\Big(e^{\operatorname{ad} A}(u+v)\Big)\ dA \ .$$

(2) *For $g \in L^1(V^-)$, the function T_g^- is defined on $V^-(0) \times V^-(-2)$ by*

$$T_g^-(u',v') = |\nabla_1(u')|^{\frac{d}{2\kappa}} \int_{A \in \mathfrak{g}(-1)} g\Big(e^{\operatorname{ad} A}(u'+v')\Big)\, dA\ .$$

The integral formulas of Theorem 4.6.2 may rewritten as follows.

Proposition 4.6.4. *With the previous normalizations of the measures we have for $f \in L^1(V^+)$ and $g \in L^1(V^-)$*

$$\int_{V^+} f(X)\, dX = \int_{u \in V^+(0)} \int_{v \in V^+(2)} T_f^+(u,v)\, |\Delta_k(v)|^{\frac{dk}{2\kappa}} du\ dv\ ;$$
$$\int_{V^-} g(Y)\, dY = \int_{u' \in V^-(0)} \int_{v' \in V^-(-2)} T_g^-(u',v')\, |\nabla_1(u')|^{\frac{d}{2\kappa}} du'\ dv'\ .$$

We list now some properties of the orbital integral T_f^+.

Proposition 4.6.5.

(1) *For $f \in L^1(V^+)$ the orbital integral T_f^+ is defined almost everywhere and the product of $T_f^+(u,v)$ by $|\Delta_k(v)|^{\frac{kd}{2\kappa}}$ is integrable over $V^+(0) \times V^+(2)$.*

(2) *For $f \in \mathcal{S}(V^+)$, $u \in V^+(0)$ and $v \in V^+(2)'$ (i.e. $\Delta_k(v) \neq 0$) the function $A \mapsto f\Big(e^{\operatorname{ad} A}(u+v)\Big)$ belongs to $\mathcal{S}(\mathfrak{g}(-1))$ and $T_f^+(u,v)$ is everywhere defined on $V^+(0) \times V^+(2)'$.*

(3) *If f is a $\mathcal{C}^\infty$ function with compact support in $O_k = \{X \in V^+ \mid \Delta_k(X) \neq 0\}$, then T_f^+ is a $\mathcal{C}^\infty$ function with compact support in $V^+(0) \times V^+(2)'$.*

Proof. —

Assertion (1) is an immediate consequence, via Fubini's Theorem, of the integration formula of Theorem 4.6.2.

For (2), let us consider the map φ from $\mathfrak{g}(-1)$ into V^+ given by

$$\varphi(A) = e^{\operatorname{ad} A}(u+v) = u + \frac{1}{2}(\operatorname{ad} A)^2 v + \operatorname{ad} A.v + v \quad (A \in \mathfrak{g}(-1))\ ,$$

where $u \in V^+(0)$ and v generic in $V^+(2)'$. The map $A \mapsto \operatorname{ad} A.v$ is a linear isomorphism from $\mathfrak{g}(-1)$ onto $V^+(1)$. Hence for some choice of norms on $\mathfrak{g}(-1)$ and on V^+ there exists a constant a such that

$$||\varphi(A)|| \geq ||\operatorname{ad} A.v|| \geq a||A||\ .$$

Moreover the map $A \mapsto e^{\operatorname{ad} A}(u+v)$ is polynomial. Therefore, for $f \in \mathcal{S}(V^+)$, the function $A \mapsto (f \circ \varphi)(A)$ belongs to the Schwartz space of $\mathfrak{g}(-1)$ and the result follows.

Assertion (3) is an immediate consequence of Proposition 4.2.3.

□

Similar results for the orbital integral T_g^- are given in the following Proposition.

Proposition 4.6.6.
(1) *For $g \in L^1(V^-)$ the orbital integral T_g^- is defined almost everywhere and the product of $T_g^-(u',v')$ by $|\nabla_1(u')|^{\frac{d}{2\kappa}}$ is integrable over $V^-(0) \times V^-(-2)$.*

(2) *For $g \in \mathcal{S}(V^-)$, $v' \in V^-(-2)$ and $u' \in V^-(0)'$ (i.e. $\nabla_1(u') \neq 0$) the function $A \mapsto f\Big(e^{\operatorname{ad} A}(u'+v')\Big)$ belongs to $\mathcal{S}(\mathfrak{g}(-1))$ and $T_g^-(u',v')$ is everywhere defined on $V^-(0)' \times V^-(-2)$.*

(3) *If g is a $\mathcal{C}^\infty$ function with compact support in $\{Y \in V^- \mid \nabla_1(Y) \neq 0\}$, then T_g^- is a $\mathcal{C}^\infty$ function with compact support in $V^-(0)' \times V^-(-2)$.*

4.7. Fourier transform of a quadratic character

We give in this section the so-called Weil formula on $\mathbb{R}$ ([We]–1964) for the Fourier transform of a quadratic character using the details given by Rallis and Schiffmann in ([Ra-S]–1975).

Let E be a real vector space and E^* his dual. Using the character $c : x \mapsto e^{2i\pi x}$ of $\mathbb{R}$, we define the Fourier transform of a function f in the Schwartz space $\mathcal{S}(E)$ by the formula

$$\widehat{f}(y^*) = \int_E f(x)\, c(< y^*, x >)\, dx \quad \text{for } y^* \in E^* .$$

We take dual measures dx and dy^* on E and E^* *i.e.* measures such that

$$f(x) = \int_{E^*} \widehat{f}(y^*)\, \overline{c(< y^*, x >)}\, dy^* \quad \text{for } f \in \mathcal{S}(E) .$$

On the other hand, let Q be a non degenerate quadratic form on E and β be the bilinear symmetric form defined by

$$\beta(x,x') = Q(x+x') - Q(x) - Q(x') \quad \text{for } x, x' \in E .$$

This form induces a linear isomorphism ψ_β between E and E^* given by

$$< \psi_\beta(x), x' >= \beta(x,x'), \quad \text{for } x, x' \in E .$$

Thus there exists a constant $C > 0$ such that for every $f^* \in \mathcal{S}(E^*)$

$$\int_{E^*} f^*(y^*)\, dy^* = C \int_E f^*(\psi_\beta(x))\, dx .$$

Remark 4.7.1. If dx is replaced by λdx, then dy^* is replaced by $\frac{1}{\lambda} dy^*$ and C by $\frac{1}{\lambda^2} C$.

Theorem 4.7.2 (A. Weil)**.**
If the signature of the quadratic form Q is (p,q) then, for $f \in \mathcal{S}(E)$, we have

$$\int_E \widehat{f}\big(\psi_\beta(x)\big)\, c\big(Q(x)\big)\, dx = \frac{1}{\sqrt{C}} e^{i\frac{\pi}{4}(p-q)} \int_E f(x)\, \overline{c(Q(x))}\, dx.$$

This theorem means that the Fourier transform of the quadratic character $c \circ Q$ is equal, as a tempered distribution on E (identified to E^* by ψ_β), to the product of $\overline{c \circ Q}$ by $\frac{1}{\sqrt{C}} e^{i\frac{\pi}{4}(p-q)}$. This is exactly a rewriting (with our choice of measures) of the formula (1–4) in [Ra-S]–1975 p. 500. The coefficient $e^{i\frac{\pi}{4}(p-q)}$ is given in [Ra-S]–1975 p. 504 (case of the field $\mathbb{R}$).

We will apply this formula to the following situation :
We take $E = \mathfrak{g}(-1)$ and we identify E^* with $\mathfrak{g}(+1)$ using the bilinear form b. The Fourier transform $\mathcal{F}$ defined in section 4.5 and the Fourier transform $f \mapsto \widehat{f}$ defined here are the same. More precisely, if $y^* \in E^*$ is given for all $A \in \mathfrak{g}(-1)$ by

$$< y^*, A > = b(B, A) \text{ where } B \in \mathfrak{g}(1) \ ,$$

then we have for $f \in \mathcal{S}(E)$

$$\widehat{f}(y^*) = \int_{\mathfrak{g}(-1)} f(A)\ c\Big(b(A,B)\Big)\ dA = (\mathcal{F}f)(B) \ .$$

Recall that the measures dA and dB were choosen to be dual for this Fourier transform (Condition (4-5-1)).

Let u' and v be elements of respectively $V^-(0)$ and $V^+(2)$ which are generic and let $Q_{u',v}$ be the quadratic form defined on $\mathfrak{g}(-1)$ by

$$Q_{u',v}(A) = \frac{1}{2}\ b\Big((\operatorname{ad} A)^2 u', v\Big) \quad \text{for } A \in E = \mathfrak{g}(-1) \ .$$

The fact that $\operatorname{ad} u' \operatorname{ad} v = \operatorname{ad} v \operatorname{ad} u'$ implies that the associated symmetric bilinear form is given by

$$\beta_{u',v}(A, A') = b(\operatorname{ad} u' \operatorname{ad} v.A, A') \ .$$

Thus we obtain

$$\psi_{\beta_{u',v}}(A) = \operatorname{ad} u' \operatorname{ad} v.A \ .$$

Moreover the fact that u' and v are generic implies that the quadratic form $Q_{u',v}$ is non degenerate.

To calculate the constant C we use Proposition 4.3.1 (where $\alpha = 1$ and $\delta = 1$ due to the choice of measures). For $f \in \mathcal{S}(\mathfrak{g}(1))$we have

$$\begin{aligned}
\int_E f\Big(\psi_{\beta_{u',v}}(A)\Big)\ dA &= \int_{\mathfrak{g}(-1)} f(\operatorname{ad} u' \operatorname{ad} v.A)\ dA \\
&= |\Delta_k(v)|^{-\frac{kd}{\kappa}} \int_{V^+(1)} f(\operatorname{ad} u'.w)\ dw \\
&= |\Delta_k(v)|^{-\frac{kd}{\kappa}} |\nabla_1(u')|^{-\frac{d}{\kappa}} \int_{\mathfrak{g}(1)} f(B)\, dB \ .
\end{aligned}$$

Therefore the constant C is given by

$$C = |\Delta_k(v)|^{\frac{kd}{\kappa}} |\nabla_1(u')|^{\frac{d}{\kappa}} \ .$$

Definition 4.7.3. *Let (p, q) be the signature of the quadratic form*

$$Q_{u',v} : A \in \mathfrak{g}(-1) \mapsto \frac{1}{2} b\big((\operatorname{ad} A)^2 u', v\big) \ .$$

We denote by γ the function defined on $V^-(0) \times V^+(2)$ by

$$\gamma(u', v) = e^{i\frac{\pi}{4}(p-q)} \quad \text{for } u' \in V^-(0), v \in V^+(2) \ .$$

Theorem 4.7.2 can now be written as follows.

Corollary 4.7.4. *Let (u', v) be an element of $V^-(0)' \times V^+(2)'$. Then, for $f \in \mathcal{S}(\mathfrak{g}(-1))$, we have*

$$\int_{\mathfrak{g}(-1)} \Big(\mathcal{F}f\Big)(\operatorname{ad} u' \operatorname{ad} v.A)\ c\big(Q_{u',v}(A)\big)\ dA$$
$$= \gamma(u', v)|\Delta_k(v)|^{-\frac{kd}{2\kappa}}|\nabla_1(u')|^{-\frac{d}{2\kappa}} \int_{\mathfrak{g}(-1)} f(A)\ \overline{c\big(Q_{u',v}(A)\big)}\ dA \ .$$

4.8. A relation between $T^-_{\mathcal{F}f}$ and T^+_f

In this section we give the proof of a formula given by I. MULLER for the case $\ell = 1$ in ([Mu]–1986 - Théorème de décomposition (3)).

We normalize the Lebesgue measures by (N_1) and (N_2) and we denote by $\mathcal{F}$ (*resp.* $\mathcal{F}_u$, *resp.* $\mathcal{F}_v$) the Fourier transform on V^+ (*resp.* $V^+(0)$, *resp.* $V^+(2)$).

Theorem 4.8.1 (I. MULLER for $\ell = 1$)**.**
If f is a Schwartz function on V^+ then, for $(u', v') \in V^-(0)' \times V^-(-2)$, we have the equality

$$\Big(T^-_{\mathcal{F}f}\Big)(u', v') = \mathcal{F}_v\Big(\gamma(u', v)\big(\mathcal{F}_u T^+_f\big)(u', v)\Big)(v') \ .$$

Remark 4.8.2. We know from Proposition 4.6.6 (2) that $T^-_{\mathcal{F}f}(u', v')$ is everywhere defined for v' in $V^-(-2)$ and u' generic in $V^-(0)$ since $\mathcal{F}f$ is a Schwartz function on V^-.

Similarly we know from Proposition 4.6.5 (2) that $T^+_f(u, v)$ is everywhere defined for u in $V^+(0)$ and v generic in $V^+(2)$. Moreover we know from Proposition 4.6.5 (1) that $T^+_f(u, v)$ is almost everywhere defined on $V^+(0) \times V^+(2)$ and that the function $u \mapsto T^+_f(u, v)$ is integrable on $V^+(0)$ for almost all $v \in V^+(2)$. We will prove that this function is in fact integrable for v generic in $V^+(2)$ which implies that $\Big(\mathcal{F}_u T^+_f\Big)(u', v)$ is everywhere defined on $V^-(0) \times V^+(2)'$. The key point of the proof of the Theorem will be the fact that, for u' generic in $V^-(0)$, the Fourier transform of the function $v \mapsto \gamma(u', v)\Big(\mathcal{F}_u T^+_f\Big)(u', v)$ is well defined.

Proof. — During this proof we will omit the spaces where the integrations are performed, making the following convention : the integrations in the variables A and A' will be on $\mathfrak{g}(-1)$, in u on $V^+(0)$, in v on $V^+(2)$, in u' on $V^-(0)$, in v' on $V^-(-2)$ and in w on $V^+(1)$.

Let v be a fixed generic element in $V^+(2)$.

We recall (Definition 4.6.3) that

$$T^+_f(u, v) = |\Delta_k(v)|^{\frac{kd}{2\kappa}} \int_A f\big(u + \frac{1}{2}(\operatorname{ad} A)^2 v + [A, v] + v\big)\ dA \ .$$

Let us make an arbitrary choice of norms $||\ ||$ on each spaces under consideration. Since f is a Schwartz function on V^+, for any integers m and n there exists a positive constant $C_{m,n}$ such that, for $u \in V^+(0)$ and $w \in V^+(1)$, we have

$$(1+||u||^2)^m(1+||w||^2)^n\big|f(u+w+v)\big| \leq C_{m,n}\ .$$

And we obtain

$$(1)\quad \int_{u,A}\Big|f\big(u+\frac{1}{2}(\operatorname{ad}A)^2v+[A,v]+v\big)\Big|\,dA\,du \leq$$
$$C_{m,n}\int_{u,A}\frac{1}{(1+||u+\frac{1}{2}(\operatorname{ad}A)^2v||^2)^m}\times\frac{1}{(1+||[A,v]||^2)^n}\,du\,dA\ .$$

By the change of variable $u+\frac{1}{2}(\operatorname{ad}A)^2v \to u$, the integral in u is finite for m large enough. Moreover, since v is generic in $V^+(2)$, the application $A \mapsto [A,v]$ is an isomorphism from $\mathfrak{g}(-1)$ to $V^+(1)$. It follows that there exists a constant C_v such that $||[A,v]|| \geq C_v||A||$. Thus the integral in A is finite for n large enough. From Fubini's Theorem we deduce that

$$u \mapsto T^+_f(u,v) \in L^1\Big(V^+(0)\Big) \qquad \text{for } v \in V^+(2)'\ .$$

Let us now compute the Fourier transform in u of T^+_f which is given by

$$(2)\quad \big(\mathcal{F}_uT^+_f\big)(u',v) =$$
$$|\Delta_k(v)|^{\frac{kd}{2\kappa}}\int_{u,A}f\big(u+\frac{1}{2}(\operatorname{ad}A)^2v+[A,v]+v\big)\ c\big(b(u,u')\big)dA\,du\ .$$

Due to (1) the order of integration does not matter.
Making the change of variable $u+\frac{1}{2}(\operatorname{ad}A)^2v \mapsto u$ we obtain

$$(3)\quad \big(\mathcal{F}_uT^+_f\big)(u',v) =$$
$$|\Delta_k(v)|^{\frac{kd}{2\kappa}}\int_{u,A}f\big(u+[A,v]+v\big)\ c\Big(b(u-\frac{1}{2}(\operatorname{ad}A)^2v,u')\Big)dA\,du\ .$$

Let us define the function J_1 by

$$J_1(u',A,v) = \int_u f\big(u+[A,v]+v\big)\ c\Big(b(u,u')\Big)\,du$$
$$= \left(\mathcal{F}_uf\right)(u',[A,v],v)\ .$$

Then, using the definition of $Q_{u',v}$ (section 4.7), formula (3) may be written as

$$(4)\qquad \big(\mathcal{F}_uT^+_f\big)(u',v) = |\Delta_k(v)|^{\frac{kd}{2\kappa}}\int_A J_1(u',A,v)\ \overline{c\big(Q_{u',v}(A)\big)}\ dA\ .$$

We suppose now that u' is generic in $V^-(0)$.

As v generic in $V^+(2)$, the function $(u,A) \mapsto f(u+[A,v]+v)$ belongs to the Schwartz space of $V^+(0)\times\mathfrak{g}(-1)$ and the function $A \mapsto J_1(u',A,v)$ belongs to the Schwartz space of $\mathfrak{g}(-1)$. As u' is generic too, the quadratic

form $Q_{u',v}$ is non degenerate. Hence we may apply the WEIL formula as in Corollary 4.7.4 and we obtain

$$(5)\quad \int_A J_1(u',A,v)\,\overline{c\big(Q_{u',v}(A)\big)}\,dA = \gamma(u',v)^{-1}|\Delta_k(v)|^{\frac{kd}{2\kappa}}\,|\nabla_1(u')|^{\frac{d}{2\kappa}} \times \int_{A'}\Big(\mathcal{F}_A J_1\Big)(u',\operatorname{ad}u'\operatorname{ad}vA,v')\,c\big(Q_{u',v}(A')\big)\,dA'\ ,$$

where $\mathcal{F}_A$ denotes the Fourier transform on $\mathfrak{g}(-1)$. More precisely we have

$$\Big(\mathcal{F}_A J_1\Big)(u',\operatorname{ad}u'\operatorname{ad}vA',v) = \int_A (\mathcal{F}_u f)\,(u',[A,v],v)\,c\Big(b(A,\operatorname{ad}u'\operatorname{ad}vA')\Big)\,dA\ .$$

Now we make the change of variable $w = [A,v]$. Using the equality $b(A,\operatorname{ad}u'\operatorname{ad}vA') = b([A,v],[u',A'])$, the integral formula in Proposition 4.3.1 (and the fact that the constant appearing there is equal to 1, according to Proposition 4.5.1) we obtain

$$(6)\quad \Big(\mathcal{F}_A J_1\Big)(u',\operatorname{ad}u'\operatorname{ad}vA',v) = |\Delta_k(v)|^{-\frac{kd}{\kappa}}\,(\mathcal{F}_w\mathcal{F}_u f)\,(u',[u',A'],v)\ .$$

Then it follows from (4), (5) and (6) that, for u' generic in $V^-(0)$ and v generic in $V^+(2)$, we have

$$(7)\quad \gamma(u',v)\big(\mathcal{F}_u T_f^+\big)(u',v) = |\nabla_1(u')|^{\frac{d}{2\kappa}}\ \int_{A'}\Big(\mathcal{F}_w\mathcal{F}_u f\Big)(u',[u',A'],v))\,c\big(Q_{u',v}(A')\big)\,dA'\ .$$

From now on we suppose only u' generic in $V^-(0)$.

The Fourier transform $\mathcal{F}_w\mathcal{F}_u f$ is a Schwartz function on the space $V^-(0)\times V^-(-1)\times V^+(2)$. Then we can prove, as before, that since u' is generic, we have

$$\int_{v,A'}\Big|\Big(\mathcal{F}_w\mathcal{F}_u f\Big)(u',[u',A'],v))\Big|\,dA'\,dv < +\infty\ .$$

Using (7), it follows that the function $v\mapsto\gamma(u',v)\big(\mathcal{F}_u T_f^+\big)(u',v)$ belongs to $L^1(V^+(2))$ for u' generic and this allows, by Fubini's Theorem, to exchange the integrations in A' and in v for the computation of the Fourier transform of this function. Thus we have

$$\mathcal{F}_v\Big(\gamma(u',v)\big(\mathcal{F}_u T_f^+\big)(u',v)\Big)(v')$$

$$= |\nabla_1(u')|^{\frac{d}{2\kappa}}\int_{v,A'}\Big(\mathcal{F}_w\mathcal{F}_u f\Big)\big(u',[u',A'],v\big)c\Big(b(\frac{1}{2}(\operatorname{ad}A')^2u',v)+b(v',v)\Big)dA'dv$$

$$= |\nabla_1(u')|^{\frac{d}{2\kappa}}\int_{A'}\Big(\mathcal{F}_v\mathcal{F}_w\mathcal{F}_u f\Big)\Big(u',[u',A'],v'+\frac{1}{2}(\operatorname{ad}A')^2u'\Big)\,dA'\ .$$

Since we have $u'+[u',A']+v'+\frac{1}{2}(\operatorname{ad}A')^2u' = e^{\operatorname{ad}A'}(u'+v')$, we recognize the orbital integral T^- (Definition 4.6.3). Moreover $\mathcal{F}_v\mathcal{F}_w\mathcal{F}_u f$ is the Fourier transform of f on V^+ and we conclude that

$$\mathcal{F}_v\Big(\gamma(u',v)\big(\mathcal{F}_u T_f^+\big)(u',v)\Big)(v') = T^-_{\mathcal{F}f}(u',v')\ .$$

for u' generic in $V^-(0)$.

□

CHAPTER 5

Functional equation of the Zeta function for Type I and II

Throughout this chapter we suppose that $\widetilde{\mathfrak{g}}$ is of Type I or of Type II (except for Lemma 4.1.5, 5.3.6 and 5.3.7 which are general results). The aim of the chapter is to give an explicit functional equation for the local Zeta function of the prehomogeneous space (P, V^+) which is defined in section 5.5.

5.1. Definition of the local Zeta functions

Let (X_j) $(j = 0, 1, \ldots, k)$ be the sequence satisfying condition (D) chosen in section 3.6. For each $\varepsilon = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_k) \in \{\pm 1\}^{k+1}$ we consider the elements

$$I_\varepsilon^+ = \sum_{j=0}^{k} \varepsilon_j X_j \in V^+ ,$$

$$I_\varepsilon^- = \sum_{j=0}^{k} \varepsilon_j Y_j \in V^- \quad \text{where } Y_j = \theta(X_j) .$$

Recall from Corollary 3.6.5 and Corollary 3.8.7 that for Type I and Type II the representations (AN, V^+) and (AN, V^-) are prehomogeneous vector spaces. More precisely there are 2^{k+1} open orbits in V^+ and in V^- under the action of the connected group AN. These orbits are parametrized by the elements $\varepsilon = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_k) \in \{\pm 1\}^{k+1}$ and are given respectively by

$$\mathcal{O}_\varepsilon^+ = AN.I_\varepsilon^+ = \{X \in V^+ \mid \Delta_j(X)\varepsilon_j \cdots \varepsilon_k > 0 \quad \text{for } j = 0, 1, \ldots, k\}$$
$$\text{and } \mathcal{O}_\varepsilon^- = AN.I_\varepsilon^- = \{Y \in V^- \mid \nabla_j(Y)\varepsilon_0 \cdots \varepsilon_{k-j} > 0 \quad \text{for } j = 0, 1, \ldots, k\} .$$

The fundamental relative invariants of the prehomogeneous vector space (AN, V^+) are the polynomials Δ_j $(j = 0, \ldots, k)$ and the fundamental relative invariants of (AN, V^-) are the polynomials ∇_j $(j = 0, \ldots, k)$. Therefore, according to F. Sato ([Sat]–1982), the definition of the local Zeta functions of these prehomogeneous vector spaces is as follows:

Definition 5.1.1. *Let $s = (s_0, \ldots, s_k) \in \mathbb{C}^{k+1}$. For $f \in \mathcal{S}(V^+)$ and $g \in \mathcal{S}(V^-)$, the local Zeta functions of the prehomogeneous vector spaces*

(AN, V^+) and (AN, V^-) are defined by

$$Z_\varepsilon^+(f,s) = \int_{\mathcal{O}_\varepsilon^+} f(X)|\Delta(X)|^s \, dX \ ,$$

$$Z_\varepsilon^-(g,s) = \int_{\mathcal{O}_\varepsilon^-} g(Y)|\nabla(Y)|^s \, dY \ .$$

The notations $|\Delta|^s$ and $|\nabla|^s$ are defined in 3.8.8. The two previous integrals defining the Zeta functions converge for $\mathrm{Re}(s) > 0$ (*i.e.* $\mathrm{Re}\, s_j > 0$ for $j = 0, \ldots, k$) and define an holomorphic function in this open subset of $\mathbb{C}^{k+1}$. By the result of BERNSHTEIN and GELFAND ([B–G]–1969) these integrals have meromorphic continuations in $\mathbb{C}^{k+1}$ which are tempered distributions.

5.2. Existence of a functional equation for (AN, V^+)

Recall from Definition 3.8.9 that the involution t on the parameter $s \in \mathbb{C}^{k+1}$ is defined by

$$t(s) = (-s_0 - s_1 - \cdots - s_k, s_k, s_{k-1}, \ldots, s_1).$$

We will also make the convention that in $\mathbb{C}^{k+1}$

$$(z_0, z_1, \ldots, z_k) - m = (z_0 - m, z_1, \ldots, z_k) \quad \text{for } m \in \mathbb{Z} \ .$$

If $f \in \mathcal{S}(V^+)$ (resp. $g \in \mathcal{S}(V^-)$) the Fourier transform $\mathcal{F}f$ (resp. $\overline{\mathcal{F}}g$) is defined as in 4.1.3.

We first prove the existence of a relation between the Zeta functions $Z_\varepsilon^+(f, .)$ and $Z_\eta^-(\mathcal{F}f, .)$ in the case where f is a $\mathcal{C}^\infty$ function with compact support in $\mathcal{O}^+$, the union of the open AN–orbits in V^+ :

$$\mathcal{O}^+ = \{X \in V^+ \mid \Delta_0(X)\Delta_1(X)\cdots\Delta_k(X) \neq 0\} = \bigcup_{\varepsilon \in \{\pm 1\}^{k+1}} \mathcal{O}_\varepsilon^+ \ .$$

It should be noticed that, if $f \in \mathcal{C}_c^\infty(\mathcal{O}^+)$, then $s \mapsto Z_\varepsilon^+(f,s)$ is an entire function for each ε.

Proposition 5.2.1. *For each $\eta \in \{\pm 1\}^{k+1}$ there exist meromorphic functions $s \mapsto \nu(s, \eta, \varepsilon)$ on $\mathbb{C}^{k+1}$ such that for any $f \in \mathcal{C}_c^\infty(\mathcal{O}^+)$ we have*

$$Z_\eta^-(\mathcal{F}f, s) = \sum_{\varepsilon \in \{\pm 1\}^{k+1}} \nu(s, \eta, \varepsilon) Z_\varepsilon^+(f, t(s) - m) \quad .$$

Proof. —It will be enough to prove the Proposition on an open subset of $\mathbb{C}^{k+1}$ where $Z_\eta^-(\mathcal{F}f, s)$ and the various $Z_\varepsilon^+(f, t(s) - m)$ are holomorphic. Since f has compact support in $\mathcal{O}^+$ and $\mathcal{F}f$ is a Schwartz function on V^-, this is the case on the set $\mathrm{Re}\, s > 0$.

We first determine how the distributions $f \mapsto Z_\eta^-(\mathcal{F}f, s)$ and $f \mapsto Z_\varepsilon^+(f, t(s) - m)$ transform under the action of the group AN on f. Let us suppose that $\mathrm{Re}\, s > 0$.

Define for $p \in AN$ and $f \in \mathcal{C}_c^\infty(\mathcal{O}^+)$:

$$f_p(X) = f(p^{-1}X) \quad (X \in V^+) \ .$$

Since the characters χ_j of the relative invariant Δ_j under the action of P are positive on the subgroup AN (Theorem 3.6.1) we deduce from Lemma 4.1.5 that for $p \in AN$ we have

$$Z_\varepsilon^+(f_p, t(s) - m) = \chi^{t(s)}(p) Z_\varepsilon^+(f, t(s) - m) \ , \tag{5-2-1}$$

where the character $\chi^{t(s)}$ of P is given by

$$\chi^{t(s)} = \chi_0^{-s_0 - s_1 - \cdots - s_k} \chi_1^{s_k} \chi_2^{s_{k-1}} \cdots \chi_k^{s_1} \ .$$

On the other hand Lemma 4.1.5 implies also that

$$\mathcal{F}(f_p) = \chi_0(p)^m (\mathcal{F}f)_p \, . \tag{5-2-2}$$

An easy computation using (5-2-2) and the relations

$$\chi_0^- = \frac{1}{\chi_0} \quad , \qquad \chi_j^- = \frac{\chi_{k+1-j}}{\chi_0} \quad (j = 1, \ldots, k) \ ,$$

which were proved in Corollary 3.8.5 imply that for $p \in AN$ we have

$$Z_\eta^-(\mathcal{F}(f_p), s) = \chi^{t(s)}(p) Z_\eta^-(\mathcal{F}f, s) \ . \tag{5-2-3}$$

If $f \in \mathcal{C}_c^\infty(\mathcal{O}^+)$, let us write

$$f = \sum_\varepsilon f \mathbf{1}_{\mathcal{O}_\varepsilon^+}$$

where $\mathbf{1}_{\mathcal{O}_\varepsilon^+}$ denotes the characteristic function of $\mathcal{O}_\varepsilon^+$. We get

$$Z_\eta^-(\mathcal{F}f, s) = \sum_\varepsilon Z_\eta^-(\mathcal{F}(f \mathbf{1}_{\mathcal{O}_\varepsilon^+}), s) \ . \tag{5-2-4}$$

A well known result by Bruhat ([Bru]–1956, Th 3.1 p. 124) implies that there exists at most one distribution (up to scalar multiplication) on each $\mathcal{O}_\varepsilon^+$ which transforms under the action of AN as in (5-2-1) or in (5-2-3). The fact that the maps $f \mapsto Z_\varepsilon^+(f, s)$ and $g \mapsto Z_\eta^-(g, s)$ are distributions was first proved by Bernshtein and Gelfand ([B–G]–1969). This is also proved below (see Lemma 5.2.5). Therefore there exists, for $\operatorname{Re} s > 0$, a constant $\nu(s, \eta, \varepsilon)$ such that

$$Z_\eta^-(\mathcal{F}(f \mathbf{1}_{\mathcal{O}_\varepsilon^+}), s) = \nu(s, \eta, \varepsilon) Z_\varepsilon^+(f, t(s) - m) \, .$$

Then from (5-2-4) we deduce that, for $\operatorname{Re} s > 0$,

$$Z_\eta^-(\mathcal{F}f, s) = \sum_\varepsilon \nu(s, \eta, \varepsilon) Z_\varepsilon^+(f, t(s) - m) \, . \tag{5-2-5}$$

As the functions $Z_\eta^-(\mathcal{F}f, s)$ and $Z_\varepsilon^+(f, t(s) - m)$ are meromorphic, the functions $\nu(s, \eta, \varepsilon)$ have a meromorphic continuation in $\mathbb{C}^{k+1}$ such that relation (5-2-5) is satisfied.

□

The end of this section will be devoted to prove that (5-2-5) is also satisfied for functions f in the Schwartz space of V^+. For this purpose we need a more precise study of the distributions $f \mapsto Z_\varepsilon^+(f, s)$ and we will use the existence of generalized Bernstein operators and polynomials in several variables which is due to C. Sabbah ([Sab]–1987, Prop. 1.2).

For $m \in \mathbb{N}$ and $i \in \{0, \dots, k\}$, let $m[i] \in \mathbb{N}^{k+1}$ be defined by

$$m[i] = (0, \dots, 0, m, 0, \dots, 0) \ ,$$

where m is at the i–th place. Then for $s = (s_0, \dots, s_k)$ we have

$$s - 1[i] = (s_0, \dots, s_{i-1}, s_i - 1, , s_{i+1}, \dots, s_k) \ .$$

Proposition 5.2.2 (C. SABBAH)**.**
For $i = 0, \dots, k$, there exists a differential operator $P_i(X, s, \partial_X)$ on V^+, with polynomial coefficients in $X \in V^+$ and in $s = (s_0, s_1, \dots, s_k)$ such that

$$P_i(X, s, \partial_X)\Delta^s = B_i(s)\Delta^{s-1[i]} \ , \tag{5-2-6}$$

where:

- $B_i(s) = \prod_{\ell \in I_i} \alpha_{i,\ell}(s)^{\gamma_{i,\ell}}$,
- I_i *is a finite set,*
- $\alpha_{i,\ell}(s) = a^0_{i,\ell}s_0 + a^1_{i,\ell}s_1 + \ldots + a^k_{i,\ell}s_k + b_{i,\ell}$, *with* $a^j_{i,\ell} \in \mathbb{N}, b_{i,\ell} \in \mathbb{Q}^+$,
- $\gamma_{i,\ell} \in \mathbb{N}$.

More precisely relation (5-2-6) *is true locally for any determination of* $\Delta^s = \Delta_0^{s_0}\Delta_1^{s_1} \dots \Delta_k^{s_k}$.

Let $\delta_j = \varepsilon_j \dots \varepsilon_k$ be the sign of Δ_j on $\mathcal{O}_\varepsilon^+$ As an easy consequence of (5-2-6) the following identity is true on $\mathcal{O}_\varepsilon^+$:

$$P_i(X, s, \partial_X)|\Delta|^s = \delta_i B_i(s)|\Delta|^{s-1[i]} \ . \tag{5-2-7}$$

For $q \in \mathbb{N}$ and $i \in \{0, 1, \dots, k\}$, we set

$$D^i_q(\partial, s) = P_i(X, s - (q-1)[i], \partial_X) \cdots P_i(X, s - 1[i], \partial_X)P_i(X, s, \partial_X) \ .$$

By iteration of formula (5-2-7) we get on $\mathcal{O}_\varepsilon^+$

$$D^i_q(\partial, s)\ |\Delta|^s = \delta_i^q \Big(\prod_{m=0}^{q-1} B_i\big(s - m[i]\big)\Big)|\Delta|^{s-q[i]} \ . \tag{5-2-8}$$

Performing integration by parts, the proof of the following Lemma is then straightforward.

Lemma 5.2.3. *Let $f \in \mathcal{S}(V^+)$. The function $s \mapsto Z^+_\varepsilon(f, s)$ has a meromorphic continuation in $\mathbb{C}^{k+1}$ which is holomorphic in the open set*

$$U = \{s \in \mathbb{C}^{k+1} \mid B_i\big(s - m[i]\big) \neq 0, \forall i = 0, \dots, k, \ \forall m \in \mathbb{N}\}.$$

Moreover if ${}^tD^i_q(\partial, s)$ denotes the transpose of the operator $D^i_q(\partial, s)$, then for $i \in \{0, \dots, k\}$, and $m \in \mathbb{N}$, the Zeta function $Z^+_\varepsilon(f, s)$ satisfies the following functional equation

$$Z^+_\varepsilon({}^tD^i_q(\partial, s)f, s) = \delta_i^q \Big(\prod_{m=0}^{q-1} B_i\big(s - m[i]\big)\Big) Z^+_\varepsilon(f, s - q[i]) \ . \tag{5-2-9}$$

Let us choose a basis $(e_i)_{i=1,\dots,r}$ of V^+, and let $||\ ||$ be a norm on V^+, for example we can take $||X|| = -b(X, \theta X)$. For $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_r) \in \mathbb{N}^r$, let D_α be the differential operator on V^+ given by

$$D_\alpha = \frac{\partial^{\alpha_1}}{\partial X_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_r}}{\partial X_r^{\alpha_r}} \ ,$$

where X_i are the coordinates in the basis (e_i). For $M, N \in \mathbb{N}$, let $\nu_{M,N}$ be the classical semi-norms on $\mathcal{S}(V^+)$ given by

$$\nu_{M,N}(f) = \max_{|\alpha| \leq N} \Big(\sup_{X \in V^+} (1 + \|X\|^2)^M |D_\alpha f(X)| \Big) \quad ,$$

where $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_r$.

As the operator $D_q^i(\partial, s)$ has polynomial coefficients in the V^+ variables and in s, we get easily the following Lemma.

Lemma 5.2.4. *Let K be a compact subset of $\mathbb{C}^{k+1}$. There exist three constants $C \in \mathbb{R}_+^*$, $M_0 \in \mathbb{N}$, $N_0 \in \mathbb{N}$ (each one depending on K) such that*

$$\forall f \in \mathcal{S}(V^+) \ , \qquad \sup_{X \in V^+,\ s \in K} |{}^t D_q^i(\partial, s) f(X)| \leq C\, \nu_{M_0,N_0}(f).$$

Lemma 5.2.5. *Let $\widetilde{U}_\varepsilon$ be the open subset of $\mathbb{C}^{k+1}$ on which $Z_\varepsilon^+(f, s)$ is holomorphic for all $f \in \mathcal{S}(V^+)$. Let K be a compact subset of $\widetilde{U}_\varepsilon$. There exist three constants $C \in \mathbb{R}_+^*$, $M \in \mathbb{N}$, $N \in \mathbb{N}$ (each one depending on K) such that, for $s \in K$*

$$\forall f \in \mathcal{S}(V^+) \ , \quad |Z_\varepsilon^+(f, s)| \leq C\, \nu_{M,N}(f). \tag{5-2-10}$$

This implies in particular that $f \longmapsto Z_\varepsilon^+(f, s)$ is a tempered distribution of order $\leq N$ for $s \in K$.

Proof. — We first prove the Lemma on the open set U defined in Lemma 5.2.3.

For $\mathbf{q} = (q_0, q_1, \ldots, q_k) \in \mathbb{N}^{k+1}$, let $D_{\mathbf{q}}(\partial, s)$ be the differential operator defined by

$$D_{\mathbf{q}}(\partial, s) = D_{q_k}^k(\partial, s) D_{q_{k-1}}^{k-1}(\partial, s) \cdots D_{q_0}^0(\partial, s) \ .$$

Note that the various $D_{q_j}^j(\partial, s)$ commute if they are applied to $|\Delta|^s$. Let $B_\varepsilon(s, \mathbf{q})$ be the polynomial defined by

$$D_{\mathbf{q}}(\partial, s) |\Delta|^s = B_\varepsilon(s, \mathbf{q})\, |\Delta|^{s-\mathbf{q}} \ \text{ on } \mathcal{O}_\varepsilon^+ \ .$$

Note that the explicit computation of $B_\varepsilon(s, \mathbf{q})$ as product of B_i's could be easily derived from (5-2-8) but we only need the fact that this polynomial is nonzero on U.

Let ${}^t D_{\mathbf{q}}(\partial, s)$ be the transpose of $D_{\mathbf{q}}(\partial, s)$. We get the following formula, similar to (5-2-9),

$$Z_\varepsilon^+(f, s) = \frac{1}{B_\varepsilon(s + \mathbf{q}, \mathbf{q})} Z_\varepsilon^+\big({}^t D_{\mathbf{q}}(\partial, s + \mathbf{q}) f, s + \mathbf{q}\big) \ . \tag{5-2-11}$$

Let $K \subset U$ be a compact set and let $\mathbf{q}$ be choosen such that, for $s \in K$, we have $\mathrm{Re}(s + \mathbf{q}) > 0$. If $s \in K$ then $Z_\varepsilon^+\big({}^t D_{\mathbf{q}}(\partial, s + \mathbf{q}) f, s + \mathbf{q}\big)$ is defined by the converging integral

$$\int_{\mathcal{O}_\varepsilon^+} {}^t D_{\mathbf{q}}(\partial, s + \mathbf{q}) f(X) |\Delta(X)|^{s+\mathbf{q}}\, dX \ ,$$

and therefore we obtain

$$\big| Z_\varepsilon^+\big({}^t D_{\mathbf{q}}(\partial, s + \mathbf{q}) f, s + \mathbf{q}\big) \big| \leq \int_{\mathcal{O}_\varepsilon^+} |{}^t D_{\mathbf{q}}(\partial, s + \mathbf{q}) f(X)|\, |\Delta(X)|^{\mathrm{Re}\, s + \mathbf{q}}\, dX \quad .$$

Since each Δ_i is a polynomial of degree less or equal to $k+1$, there exists a constant $C_1 > 0$ such that

$$|\Delta_i(X)| \leq C_1(1+\|X\|^2)^{k+1} \quad \text{for } i = 0, 1, \ldots, k \ .$$

Let us define $\mathbf{R}(s) = \sum \mathrm{Re}\,(s_j)$ and $\mathbf{R}(\mathbf{q}) = \sum q_j$. Then we have

$$\left|Z_\varepsilon^+\left({}^tD_{\mathbf{q}}(\partial, s+\mathbf{q})f, s+\mathbf{q}\right)\right| \leq$$
$$\int_{\mathcal{O}_\varepsilon^+} |{}^tD_{\mathbf{q}}(\partial, s+\mathbf{q})f(X)|\ C_1^{\mathbf{R}(s)+\mathbf{R}(\mathbf{q})}(1+\|X\|^2)^{(k+1)(\mathbf{R}(s)+\mathbf{R}(\mathbf{q}))}\ dX \ .$$

This implies that

$$\left|Z_\varepsilon^+\left({}^tD_{\mathbf{q}}(\partial, s+\mathbf{q})f, s+\mathbf{q}\right)\right| \leq C_1^{\mathbf{R}(s)+\mathbf{R}(\mathbf{q})} \int_{\mathcal{O}_\varepsilon^+} (1+\|X\|^2)^{-\alpha}\ dX$$
$$\times \sup_{X\in V^+} \left((1+\|X\|^2)^{(k+1)(\mathbf{R}(s)+\mathbf{R}(\mathbf{q}))+\alpha} \left|{}^tD_{\mathbf{q}}(\partial, s+\mathbf{q})f(X)\right|\right) \ ,$$

where α is an integer sufficiently large such that $\int_{V^+}(1+\|X\|^2)^{-\alpha}dX < +\infty$. Therefore, for $s \in K$, there are constants $C_2 \in \mathbb{R}_+$ and $M_0 \in \mathbb{N}$ such that

$$\left|Z_\varepsilon^+\left({}^tD_{\mathbf{q}}(\partial, s+\mathbf{q})f, s+\mathbf{q}\right)\right| \leq C_2\ \nu_{M_0,0}\left({}^tD_{\mathbf{q}}(\partial, s+\mathbf{q})f\right).$$

Lemma 5.2.4 and the fact that in (5-2-11) the fonction $\dfrac{1}{B_\varepsilon(s+\mathbf{q}, \mathbf{q})}$ is bounded on K imply the existence of C, M and N such that the inequality (5-2-10) is true on U.

It remains to prove that the same inequality is true if $K \subset \widetilde{U}_\varepsilon$. Therefore it is sufficient to prove (5-2-10) on a compact neighbourhood of a point s^0 where $Z_\varepsilon^+(f, s)$ is holomorphic and which does not belong to U. Then s^0 belongs to a finite union of hyperplanes given by $\alpha_{i,\ell}(s+\mathbf{q}) = 0$ ($\mathbf{q} \in \mathbb{N}^{k+1}$). As compact neighbourhood, we can choose a polydisc of center s^0 such that its boundary does not intersect the previous hyperplanes. Classical estimates, using the Cauchy formula in several variables relatively to this polydisc, lead to the result.

□

In order to extend the functional equation of Proposition 5.2.1 to functions $f \in \mathcal{S}(V^+)$, we define the distribution T_s^η by

$$(5\text{-}2\text{-}12) \qquad < T_s^\eta, f >= Z_\eta^-(\mathcal{F}f, s) - \sum_\varepsilon \nu(s, \eta, \varepsilon) Z_\varepsilon^+(f, t(s)-m) \ .$$

The aim is to prove that this is the zero distribution. We first prove the following result.

Lemma 5.2.6. *There exists an integer $p \in \mathbb{N}$ such that*

$$\forall s \in \mathbb{C}^{k+1} \ , \quad \Delta_0^p \Delta_1^p \ldots \Delta_k^p T_s^\eta = 0$$

Proof. —Proposition 5.2.1 implies that the support of the distribution T_s^η is inside $V^+ \setminus \mathcal{O}^+ = \{X \in V^+ \,|\, \Delta_0(X)\Delta_1(X)\ldots\Delta_k(X) = 0\}$. A well known result (see for example [Sh]–1972, Lemma 1.3) implies that, if this distribution is of finite order r, then, for $p = r+1$, we have

$$(5\text{-}2\text{-}13) \qquad \Delta_0^p \Delta_1^p \ldots \Delta_k^p T_s^\eta = 0 \ .$$

Let K be a compact set (with $\overset{\circ}{K} \neq \emptyset$) such that $Z_\eta^-(\mathcal{F}f, s)$ and each $Z_\varepsilon^+(f, t(s) - m)$ $(\varepsilon \in \{\pm 1\}^{k+1})$ are holomorphic in a neighborhood of K. From Lemma 5.2.5 there exists an integer r_1 such that the distributions $Z_\varepsilon^+(f, t(s) - m)$ are of order $\leq r_1$ for $s \in K$ and all $\varepsilon \in \{\pm 1\}^{k+1}$. Again Lemma 5.2.5 and the fact that $f \mapsto \mathcal{F}f$ is a topological isomorphism from $\mathcal{S}(V^+)$ onto $\mathcal{S}(V^-)$ imply that there exists an integer r_2 such that the distribution $Z_\eta^-(\mathcal{F}f, s)$ is of order $\leq r_2$ for $s \in K$. Then, if $s \in K$, the distribution T_s^η is of order $\leq r = \max(r_1, r_2)$.

It follows that (5-2-13) is satisfied for $s \in K$. By analytic continuation the same equality is true everywhere since the distribution T_η^s has a meromorphic continuation in $\mathbb{C}^{k+1}$ by Lemma 5.2.3.

□

For $i = 0, 1, \ldots, k$, let $\Delta_i(\partial)$ be the differential operator (with constant coefficients) on V^- defined by

$$\Delta_i(\partial) e^{b(X,Y)} = \Delta_i(X) e^{b(X,Y)}, \quad X \in V^+,\ Y \in V^- \ . \tag{5-2-14}$$

Lemma 5.2.7. *Let $\eta \in \{\pm 1\}^{k+1}$. There exist polynomials $b_0^\eta, b_1^\eta, \ldots, b_k^\eta$ in the variable $s = (s_0, s_1, \ldots, s_k)$ such that on $\mathcal{O}_\eta^-$ the following identities are satisfied.*

$$\text{(5-2-15)} \quad \begin{cases} \Delta_0(\partial)|\nabla|^s = b_0^\eta(s)\,|\nabla_0|^{s_0 - 1}|\nabla_1|^{s_1} \ldots |\nabla_k|^{s_k} \ , \\ \quad \textit{and for } i \neq 0 \\ \Delta_i(\partial)|\nabla|^s = b_i^\eta(s)\,|\nabla_0|^{s_0 - 1}|\nabla_1|^{s_1} \ldots |\nabla_{k-i+1}|^{s_{k-i+1}+1} \ldots |\nabla_k|^{s_k}. \end{cases}$$

Proof. — Let ℓ be the regular representation of AN on the space $\mathcal{C}^\infty(\mathcal{O}_\eta^-)$ given by

$$(\ell(p)f)(X) = f(p^{-1}X) \quad \text{where } f \in \mathcal{C}^\infty(\mathcal{O}_\eta^-),\ p \in AN,\ X \in \mathcal{O}_\eta^- \ .$$

For $i = 0, 1, \ldots, k$, we have for $p \in P$

$$\ell(p) \circ \Delta_i(\partial) = \chi_i(p)^{-1} \Delta_i(\partial) \circ \ell(p) \ . \tag{5-2-16}$$

These relations are easily proved by restricting the operators to the functions $Y \mapsto e^{b(X,Y)}$.

Then, using (5-2-16) and the relations (Corollary 3.8.5)

$$\chi_0(p) = \frac{1}{\chi_0^-(p)} \text{ and } \chi_i(p) = \frac{\chi_{k+i-1}^-(p)}{\chi_0^-(p)} \ (i \neq 0),$$

we show that the functions on both sides of (5-2-15) are relatively invariant under the action of AN, with the same character. As the action of AN is transitive on $\mathcal{O}_\eta^-$ these two functions differ only by a constant denoted $b_i^\eta(s)$. The value of $|\nabla|^s$ at the point I_η^- is equal to 1 and the value of $\Delta_i(\partial)|\nabla|^s$ at the same point is polynomial in s. Thus $b_i^\eta(s)$ is polynomial in s.

□

For an integer $p \in \mathbb{N}$ we define

$$\Delta^p = \Delta_0^p \Delta_1^p \ldots \Delta_k^p \text{ and } \Delta(\partial)^p = \Delta_0(\partial)^p \Delta_1(\partial)^p \ldots \Delta_k(\partial)^p \ .$$

Using the fact that the degree of the polynomial Δ_j is $k+1-j$ (Theorem 1.14.2) we deduce from (5-2-14) that

$$\Delta(\partial)^p e^{2i\pi b(X,Y)} = (2i\pi)^{\frac{p(k+1)(k+2)}{2}} \Delta^p(X) e^{2i\pi b(X,Y)} \quad (X \in V^+,\ Y \in V^-) .$$

This leads to the computation of the following Fourier transform.

Lemma 5.2.8. *For $f \in \mathcal{S}(V^+)$*

$$\mathcal{F}(\Delta^p f) = (2i\pi)^{-\frac{p(k+1)(k+2)}{2}} \Delta(\partial)^p \mathcal{F} f .$$

Now we have everything we need to prove the functional equation which extends Proposition 5.2.1 to functions in the Schwartz space of V^+ and V^-.

Theorem 5.2.9 (Abstract functional equation for Type I and II)**.** *For $f \in \mathcal{S}(V^+)$, $g \in \mathcal{S}(V^-)$, $\varepsilon \in \{\pm 1\}^{k+1}$ and $\eta \in \{\pm 1\}^{k+1}$ the integrals defining $Z_\varepsilon^+(f,s)$ and $Z_\eta^-(g,s)$ have meromorphic continuations in $\mathbb{C}^{k+1}$ and there exist meromorphic functions $\nu(s,\eta,\varepsilon)$ and $\delta(s,\varepsilon,\eta)$ such that*

$$Z_\eta^-(\mathcal{F}f,s) = \sum_{\varepsilon \in \{\pm 1\}^{k+1}} \nu(s,\eta,\varepsilon) Z_\varepsilon^+(f, t(s)-m) \tag{5-2-17}$$

$$Z_\varepsilon^+(f,s) = \sum_{\eta \in \{\pm 1\}^{k+1}} \delta(s,\varepsilon,\eta) Z_\eta^-(\mathcal{F}f, t(s)-m) \tag{5-2-18}$$

Proof. —First of all notice that (5-2-18) is obtained from (5-2-17) by exchanging the roles of V^+ and V^-. Therefore we only prove (5-2-17).

It remains to prove that $T_s^\eta = 0$ where T_η^s is the tempered distribution given by (5-2-12). Recall from Lemma 5.2.6 that there exists an integer p (independant of s) such that

$$\Delta^p T_\eta^s = 0 .$$

For $f \in \mathcal{S}(V^+)$, $< \Delta^p T_\eta^s, f >$ is a linear combination of terms of the form $Z_\eta^-(\mathcal{F}(\Delta^p f), s)$ and $Z_\varepsilon^+(\Delta^p f, t(s)-m)$.

From Lemma 5.2.8 we get

$$Z_\eta^-(\mathcal{F}(\Delta^p f), s) = (2i\pi)^{-\frac{p(k+1)(k+2)}{2}} Z_\eta^-(\Delta(\partial)^p \mathcal{F}(f), s) .$$

For $\mathrm{Re}(s) > 0$ this Zeta function is an integral and, using the fact that the differential operator $\Delta(\partial)^p$ is a homogeneous differential operator of degree $\frac{1}{2}p(k+1)(k+2)$ we obtain by integration by parts that

$$Z_\eta^-(\mathcal{F}(\Delta^p f), s) = (-2i\pi)^{-\frac{p(k+1)(k+2)}{2}} \int_{\mathcal{O}_\eta^-} \big(\mathcal{F}f\big)(Y) \big(\Delta(\partial)^p |\nabla|^s\big)(Y)\, dY .$$

Then Lemma 5.2.7 implies the existence of a polynomial $P_\eta(s)$ such that

$$Z_\eta^-(\mathcal{F}(\Delta^p f), s) = P_\eta(s) \int_{\mathcal{O}_\eta^-} \big(\mathcal{F}f\big)(Y) |\nabla(Y)|^{\alpha(s)} dY ,$$

where

$$\alpha(s_0, s_1, \ldots, s_k) = (s_0 - (k+1)p, s_1 + p, \ldots, s_k + p) .$$

On the other hand we have

$$Z_\varepsilon^+(\Delta^p f, t(s)-m) = \delta_0^p \delta_1^p \ldots \delta_k^p Z_\varepsilon^+\big(f, (t(s)-m) + (p, \ldots, p)\big) ,$$

where δ_j is the sign of Δ_j on $\mathcal{O}_\varepsilon^+$.

Therefore we obtain

$$0 =< \Delta^p T_s^\eta, f >= P_\eta(s) Z_\eta^-(\mathcal{F}f, \alpha(s)) - \sum_{\varepsilon\in\{\pm1\}^{k+1}} \nu(s,\eta,\varepsilon)\delta_0^p\delta_1^p\dots\delta_k^p Z_\varepsilon^+(f,(t(s)-m)+(p,\dots,p)).$$

As $t(\alpha(s))-m=(t(s)-m)+(p,\dots,p)$, we obtain meromorphic functions which are, for obvious reasons, equal to the functions $\nu(\alpha(s),\eta,\varepsilon)$ given in Proposition 5.2.1 such that for $f\in\mathcal{S}(V^+)$ and $\operatorname{Re} s>0$ we have

$$Z_\eta^-(\mathcal{F}f,\alpha(s)) = \sum_{\varepsilon\in(\pm1)^{k+1}} \nu(\alpha(s),\eta,\varepsilon) Z_\varepsilon^+(f, t\big(\alpha(s)\big)-m) \quad .$$

Theorem 5.2.9 is therefore proved by meromorphic continuation.

□

Remark 5.2.10. The previous Theorem is essentially Theorem 1 in F. SATO's paper ([Sat]–1982). Howewer the proof by Sato relies on the result of BERNSHTEIN and GELFAND ([B–G]–1969]) and on the fact the distributions $f\mapsto Z_\varepsilon^+(f,s)$ are of bounded order for s in a compact set (which is not proved in [B–G]). Our proof is an extension to the case of several variables of the proof by SHINTANI ([Sh]–1972). It relies on a result of C. SABBAH (see Proposition 5.2.2) which implies the result of BERNSHTEIN and GELFAND and the local boundedness of the order of the Zeta distributions (see Lemma 5.2.5).

5.3. Computation of the coefficients $\delta(s,\varepsilon,\eta)$

In this section we will compute the coefficients $\delta(s,\varepsilon,\eta)$ defined in Theorem 5.2.9. The proof is done by induction on the rank (Definition 1.6.2) of the graded algebra $(\widetilde{\mathfrak{g}}, H_0)$.

Let us start with the case where $k=0$ (rank one case).

We will first recall the functional equation of J.T. TATE in the real case ([Ta]–1967).

Let $\mathbf{1}$ (*resp.* $-\mathbf{1}$) be the character χ of $\mathbb{R}^*$ given by

$$\chi(x) = \begin{cases} 1 & \text{if } \chi=\mathbf{1}\ , \\ \operatorname{sg}(x) & \text{if } \chi=-\mathbf{1}\ , \end{cases}$$

where $\operatorname{sg}(x)=+1$ if $x>0$ and -1 if $x<0$.

Definition 5.3.1 (TATE's rho factors)**.**
The TATE rho factors are the meromorphic functions $z \mapsto \rho(z,\chi)$ on $\mathbb{C}$ defined by

$$\rho(z,\chi) = \begin{cases} \pi^{\frac{1}{2}-z}\dfrac{\Gamma\left(\frac{z}{2}\right)}{\Gamma\left(\frac{1-z}{2}\right)} = 2(2\pi)^{-z}\Gamma(z)\cos\frac{\pi z}{2} & \text{if } \chi = \mathbf{1}\ ; \\ -i\pi^{\frac{1}{2}-z}\dfrac{\Gamma\left(\frac{z+1}{2}\right)}{\Gamma\left(1-\frac{z}{2}\right)} = -2i(2\pi)^{-z}\Gamma(z)\sin\frac{\pi z}{2} & \text{if } \chi = -\mathbf{1}\ . \end{cases}$$

Proposition 5.3.2 (TATE's functional equation)**.**
For $\varphi \in \mathcal{S}(\mathbb{R})$ and χ a character of $\mathbb{R}^*$, the function defined for $\operatorname{Re} z > 0$ by the integral $Z(\varphi,z,\chi) = \int_{\mathbb{R}} \varphi(x)\chi(x)|x|^z \frac{dx}{|x|}$ has a meromorphic continuation to $\mathbb{C}$ and satisfies the following relation

$$Z(\varphi,z,\chi) = \rho(z,\chi)Z(\widehat{\varphi},1-z,\chi)\ ,$$

where $\widehat{\varphi}(y) = \int_{\mathbb{R}} \varphi(x)e^{2i\pi xy}dx$.

In the case $k = 0$ and $\ell = 1$ (Type I and II) we have $V^+ = \mathbb{R}X_0$ and $V^- = \mathbb{R}Y_0$ where $Y_0 = \theta(X_0)$. If dx denotes the Lebesgue measure on $\mathbb{R}$, there exists positive constants λ and μ such that

$$\int_{V^+} f(X)\,dX = \lambda\int_{\mathbb{R}} f(xX_0)\,dx \text{ and } \int_{V^-} g(Y)\,dY = \mu\int_{\mathbb{R}} g(xY_0)\,dx\ .$$

The normalization of the measures dX and dY done in section 4.1 implies that $\lambda = \mu = 1$: the equality $\theta(X_0) = \sigma(X_0)$ implies by (N_1) that $\lambda = \mu$. The equality $b(xX_0, yY_0) = xy$ (Lemma 1.10.2) and the choice of the character c of $\mathbb{R}$ implies by (N_2) that $\lambda\mu = 1$. Then we have for ε and η in $\{-1,+1\}$

$$Z_\varepsilon^+(f,z) = \int_{\varepsilon x>0} f(xX_0)\,|x|^z\,dx \text{ and } Z_\eta^-(g,z) = \int_{\eta x>0} g(xY_0)\,|x|^z\,dx\ .$$

Proposition 5.3.3. *In the case where $k = 0$, the functions $Z_\varepsilon^+(f,z)$ and $Z_\eta^-(g,z)$ have meromorphic continuations to $\mathbb{C}$ for $f \in \mathcal{S}(V^+)$ and $g \in \mathcal{S}(V^-)$ and satisfy the following relation*

$$Z_\varepsilon^+(f,z) = \sum_{\eta=\pm1} \delta_0(z,\varepsilon,\eta)Z_\eta^-(\mathcal{F}f,-z-1)\ ,$$

where

$$\begin{aligned}\delta_0(z,\varepsilon,\eta) &= \frac{1}{2}\big(\rho(z+1,\mathbf{1}) + \varepsilon\eta\rho(z+1,-\mathbf{1})\big) \\ &= (2\pi)^{-(z+1)}e^{-i\frac{\pi}{2}\varepsilon\eta(z+1)}\Gamma(z+1)\ .\end{aligned}$$

Remark 5.3.4. It may be noted that this is an explicit expression of the relation (5-2-18) since, in this case ($k = 0$ and $\kappa = 1$), we have $t(s) = -s$ and $m = 1$.

Remark 5.3.5. The function $\delta_0(z,\varepsilon,\eta)$ depends on z of course but only on the sign of the product $\varepsilon\eta$.

Proof of Proposition 5.3.3. — The result is a direct consequence of TATE's functional equation. If φ is the function $x \mapsto f(xX_0)$, the relations

$$Z(\varphi, z+1, \mathbf{1}) = Z_1^+(f,z) + Z_{-1}^+(f,z)$$
$$\text{and } Z(\varphi, z+1, -\mathbf{1}) = Z_1^+(f,z) - Z_{-1}^+(f,z)$$

give the result.

□

We precise now the procedure of induction which will lead to the general result

Recall from Chapter 4 that if the regular graded algebra $\widetilde{\mathfrak{g}}$ is of rank $k+1$, then the regular graded algebra $\underline{\widetilde{\mathfrak{g}}} = V^-(0) \oplus \mathfrak{g}(0) \oplus V^+(0)$ is of rank k.

We will denote by underlined letters the elements of the algebra $\underline{\widetilde{\mathfrak{g}}}$ which is graded by $\underline{H}_0 = \sum_{j=0}^{k-1} H_{\lambda_j}$. For example $\underline{G} = G(0)$ denotes the analytic subgroup of G with Lie algebra $\underline{\mathfrak{g}} = \mathfrak{g}(0)$, $\underline{\mathfrak{a}}^0$ denotes the sum of $\mathbb{R}H_{\lambda_j}$ for $j = 0, \ldots, k-1$, $\underline{P}$ denotes the parabolic subgroup of $\underline{G}$ defined as in section 3.5 and $\underline{\Delta}_j$ $(j = 0, \ldots, k-1)$ the relative invariants for the action of $\underline{P}$ on $\underline{V}^+ = V^+(0)$ normalized by $\underline{\Delta}_j(\sum_{j=0}^{k-1} X_j) = 1$ and so on...

Lemma 5.3.6 (Type I, II and III)**.**
(1) *The Lie algebra $\underline{\mathfrak{P}}$ of the parabolic subgroup $\underline{P}$ is given by*

$$\underline{\mathfrak{P}} = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}^0) \oplus \sum_{i<j<k} E_{i,j}(1,-1) \ .$$

(2) *The* LANGLANDS *decomposition of $\underline{P}$ is given by $\underline{P} = \underline{M}\ A\ \underline{N}$ where $\underline{N} = N \cap \underline{G}$ and $\underline{M} = M \cap \underline{G}$. Moreover M and $\underline{M}$ have the same Lie algebra.*

Proof. —
(1) We have only to prove that $\mathfrak{z}_{\underline{\mathfrak{g}}}(\underline{\mathfrak{a}}^0) = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}^0)$. Since $\underline{\mathfrak{g}}$ is the centralizer of H_0 and of H_{λ_k}, an element X belongs to $\mathfrak{z}_{\underline{\mathfrak{g}}}(\underline{\mathfrak{a}}^0)$ if and only if it centralizes H_{λ_k} and each H_{λ_j} for $j = 0, \ldots, k-1$ *i.e.* if it centralizes $\mathfrak{a}^0$.
(2) By Proposition 3.5.1 and (1) the Levi parts $\mathfrak{m} \oplus \mathfrak{a} = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}^0)$ and $\underline{\mathfrak{m}} \oplus \underline{\mathfrak{a}} = \mathfrak{z}_{\underline{\mathfrak{g}}}(\underline{\mathfrak{a}}^0)$ of $\mathfrak{P}$ and $\underline{\mathfrak{P}}$ respectively are equal. It follows, also from Proposition 3.5.1, that $\underline{\mathfrak{a}} = \mathfrak{a}$ and consequently that $\underline{\mathfrak{m}} = \mathfrak{m}$. In the case where M is disconnected it may happen that $\underline{M}$ is strictly contained in M. But in any case we have $Z_{\underline{K}}(\underline{\mathfrak{a}}^0) = Z_K(\mathfrak{a}^0) \cap \underline{G}$ hence $\underline{M} = M \cap \underline{G}$.

□

Lemma 5.3.7 (Type I, II and III)**.**
(1) *If u, v and A are elements of $V^+(0)$, $V^+(2)$ and $\mathfrak{g}(-1)$ respectively, then*

$$\Delta_j\big(e^{\operatorname{ad} A}(u+v)\big) = \Delta_j(u+v) = \begin{cases} \Delta_k(v)\underline{\Delta}_j(u) \ \textit{if } j = 0, \ldots, k-1 \ , \\ \Delta_k(v) \ \textit{if } j = k \ . \end{cases}$$

(2) *If u', v' and A are elements of $V^-(0)$, $V^-(-2)$ and $\mathfrak{g}(-1)$ respectively, then*

$$\nabla_j\left(e^{\mathrm{ad}\,A}(u'+v')\right) = \nabla_j(u'+v') = \begin{cases} \underline{\nabla}_0(u')\nabla_k(v') & \text{if } j=0\ , \\ \underline{\nabla}_{j-1}(u') & \text{if } j=1,\dots,k\ . \end{cases}$$

Proof of (1). — The polynomials Δ_j are invariant under the action of N, hence, in particular, under the action of $\exp(\mathfrak{g}(-1))$ and the first equality follows.

Let Q_j be the function defined for $u \in V^+(0)$ and $v \in V^+(2)$ by

$$Q_j(u+v) = \begin{cases} \Delta_k(v)\underline{\Delta}_j(u) & \text{if } j=0,\dots,k-1\ , \\ \Delta_k(v) & \text{if } j=k\ . \end{cases}$$

The functions Δ_j and Q_j are polynomials functions on $V^+(0)\oplus V^+(2)$. The inclusion $\underline{N} \subset N$ and the fact that $\underline{N}$ centralize $V^+(2) = \widetilde{\mathfrak{g}}^{\lambda_k}$ imply that these functions are invariant under the action of $\underline{N}$. We know (Lemma 3.6.2) that if $\underline{\Delta}_j(u) \neq 0$ for all $j \in \{0,\dots,k-1\}$ there exists an element $n \in \underline{N}$ such that $n.u$ belongs to $\oplus_{j=0}^{k-1}\widetilde{\mathfrak{g}}^{\lambda_j}$. It is therefore sufficient to show the equality of Δ_j and Q_j on $\oplus_{j=0}^{k}\widetilde{\mathfrak{g}}^{\lambda_j}$.

In the case $\ell = 1$ (Type I and II), the result follows from the following relations (see Theorem 1.14.2(3)) :

$$\Delta_j\left(\sum_{s=0}^{k} x_s X_s\right) = \prod_{s=j}^{k} x_s^{\kappa} \quad \text{and} \quad \underline{\Delta}_j\left(\sum_{s=0}^{k-1} x_s X_s\right) = \prod_{s=j}^{k-1} x_s^{\kappa}\ . \tag{5-3-1}$$

In the case $\ell > 1$ (Type III) we know from Theorem 3.5.4 that M acts transitively on the spheres of $\widetilde{\mathfrak{g}}^{\lambda_j}$. From the proof of this Theorem, we know moreover that if $X \in \oplus_{j=0}^{k}\widetilde{\mathfrak{g}}^{\lambda_j}$ there exists an element $m = m_0 m_1 \cdots m_k$ with $m_j \in L_j \cap K$ such that

$$m.X \in \oplus_{j=0}^{k}\mathbb{R}X_j\ .$$

Since $L_j \cap K$ is the analytic subgroup of G with Lie algebra $[\widetilde{\mathfrak{g}}^{-\lambda_j}, \widetilde{\mathfrak{g}}^{\lambda_j}] \cap \mathfrak{k}$ contained in $\mathfrak{m} = \underline{\mathfrak{m}}$, it follows from Lemma 5.3.6 that

$$m \in \underline{M}\ .$$

Since, in Type III case, Δ_j and Q_j are invariant under the action of $\underline{M}$ (Theorem 3.6.1), it is now sufficient to prove the equality of Δ_j and Q_j on $\oplus_{j=0}^{k}\mathbb{R}X_j$. The result , as in the case $\ell = 1$, is a consequence of the relation (5-3-1) above.

□

As a direct consequence of the previous Lemma and of the characterization of the AN–orbits given in Corollary 3.6.5 and 3.8.7 , we obtain, in Type I and II cases, the following relations between AN and $A\underline{N}$ orbits.

For $\varepsilon \in \{\pm 1\}^{k+1}$, let $\underline{I_\varepsilon}^+$ be the element of $\underline{V}^+ = V^+(0)$ given by $\sum_{j=0}^{k-1}\varepsilon_j X_j$ and $\underline{\mathcal{O}_\varepsilon}^+$ the orbit under $A\ \underline{N}$ of $\underline{I_\varepsilon}^+$. For $\eta \in \{\pm 1\}^{k+1}$, let $\underline{I_\eta}^-$ be the element of $\underline{V}^- = V^-(0)$ given by $\sum_{j=0}^{k-1}\eta_j Y_j$ and $\underline{\mathcal{O}_\eta}^-$ the orbit under $A\ \underline{N}$ of $\underline{I_\eta}^-$.

Proposition 5.3.8 (Type I and II).
(1) *If u, v and A are elements of $V^+(0)$, $V^+(2)$ and $\mathfrak{g}(-1)$ respectively, then*

$$e^{\operatorname{ad} A}(u+v) \in \mathcal{O}_\varepsilon^+ \iff \varepsilon_k \Delta_k(v) > 0 \quad \text{and} \quad u \in \underline{\mathcal{O}_\varepsilon}^+ \ ,$$

where $u \in \underline{\mathcal{O}_\varepsilon}^+ \iff \underline{\Delta}_j(u)\varepsilon_j \cdots \varepsilon_{k-1} > 0$ for $j = 0, \ldots, k-1$.
(2) *If u', v' and A are elements of $V^-(0)$, $V^-(-2)$ and $\mathfrak{g}(-1)$ respectively, then*

$$e^{\operatorname{ad} A}(u'+v') \in \mathcal{O}_\eta^- \iff \eta_k \nabla_k(v') > 0 \quad \text{and} \quad u' \in \underline{\mathcal{O}_\eta}^- \ ,$$

where $u' \in \underline{\mathcal{O}_\eta}^- \iff \underline{\nabla}_j(u')\eta_0 \cdots \eta_{k-1-j} > 0$ for $j = 0, \ldots, k-1$.

In order to use Theorem 4.8.1 to compute by induction the coefficients of the functional equation, we first prove that $\gamma(u', v)$ (see Definition 4.7.3). depends only on the orbit under $A\underline{N}$ of $u' \in V^-(0)$ and $v \in V^+(2)$.

Let us recall that e denotes the dimension of the root spaces $\widetilde{\mathfrak{g}}^{\frac{\pm\lambda_i \pm \lambda_j}{2}}$, which is equal to d in Type I case.

Lemma 5.3.9. *Let u' be an element of $\underline{\mathcal{O}_\eta}^-$ with $\eta \in \{\pm 1\}^{k+1}$ and v an element of $V^+(2)$ such that $\varepsilon_k \Delta_k(v) > 0$ with $\varepsilon_k = \pm 1$. Then we have*

$$\gamma(u', v) = e^{i\frac{\pi}{4} e\, \varepsilon_k (\sum_{j=0}^{k-1} \eta_j)} \ .$$

Proof. —From Definition 4.7.3 we have $\gamma(u', v) = e^{i\frac{\pi}{4}(p-q)}$ where (p, q) is the signature of the quadratic form $Q_{u',v}$ defined on $\mathfrak{g}(-1)$ by $Q_{u',v}(A) = \frac{1}{2} b((\operatorname{ad} A)^2 u', v)$.

By hypothesis u' belongs to the $A\underline{N}$–orbit of $\underline{I_\eta}^-$. Thus there exist an element $n \in \underline{N}$ and strictly positive real numbers u_j $(j = 0, \ldots, k-1)$ such that

$$n.u' = u'_0 \quad \text{where} \quad u'_0 = \sum_{j=0}^{k-1} u_j \eta_j Y_j \ .$$

Moreover $\underline{N}$ centralizes $V^+(2) = \widetilde{\mathfrak{g}}^{\lambda_k}$ and there exists a strictly positive real number v_k such that

$$n.v = v_0 \text{ where } v_0 = \varepsilon_k v_k X_k \ .$$

As $\underline{N}$ stabilizes $\mathfrak{g}(-1)$, the quadratic form $Q_{u',v}$ is equivalent to the quadratic form $Q_{u'_0,v_0}$.

The space $\mathfrak{g}(-1)$ is the direct sum of the subspaces $E_{j,k}(1,-1)$ for $j = 0, \ldots k-1$ which are orthogonal for the quadratic form $Q_{u'_0,v_0}$. Moreover if A belongs to $E_{j,k}(1,-1)$ then $b((\operatorname{ad} A)^2 Y_i, X_k) = 0$ for $i \neq j$. Thus we have the following equivalence of quadratic forms :

$$Q_{u'_0,v_0} \sim \oplus_{j=0}^{k-1} \eta_j \varepsilon_k Q_{j,k} \ ,$$

where $Q_{j,k}$ is the quadratic form defined on $E_{j,k}(1,-1)$ by

$$Q_{j,k}(A) = \frac{1}{2} b\big((\operatorname{ad} A)^2 Y_j, X_k\big) \text{ for } A \in E_{j,k}(1,-1) \ .$$

In section 2.5 we have defined quadratic forms q_{X_j,X_k} on $E_{j,k}(-1,-1)$ by

$$q_{X_j,X_k}(Y) = -\frac{1}{2} b(\operatorname{ad} X_j.Y, \operatorname{ad} X_k.Y) \ .$$

A simple calculation shows that the forms $Q_{j,k}$ and q_{X_j,X_k} are related by the formula

$$Q_{j,k}(\operatorname{ad} X_j.Y) = q_{X_j,X_k}(Y) \quad \text{for } Y \in E_{j,k}(-1,-1) \ .$$

Since $\operatorname{ad} X_j$ is an isomorphism from $E_{j,k}(-1,-1)$ onto $E_{j,k}(1,-1)$ the quadratic forms $Q_{j,k}$ and q_{X_j,X_k} are equivalent.

We know from Theorem 2.8.12 that for Type I and II the signature of the quadratic form q_{X_j,X_k} is equal to $(\frac{d+e}{2}, \frac{d-e}{2})$. Then if (p,q) is the signature of $Q_{u',v}$, we obtain

$$p - q = e\,\varepsilon_k \sum_{j=0}^{k-1} \eta_j \ ,$$

and the result follows from the definition of $\gamma(u',v)$.

□

Theorem 5.3.10 (Type I and II)**.**
Let $f \in \mathcal{S}(V^+)$ and $\varepsilon \in \{\pm 1\}^{k+1}$. In the functional equation (5-2-18) *proved in Theorem* 5.2.9

$$Z_\varepsilon^+(f,s) = \sum_{\eta \in \{\pm 1\}^{k+1}} \delta_k(s,\varepsilon,\eta) Z_\eta^-(\mathcal{F}f, t(s) - m) \ ,$$

the coefficients $s \mapsto \delta_k(s,\varepsilon,\eta)$ are the meromorphic functions given by

$$\delta_k(s,\varepsilon,\eta) = \prod_{j=0}^{k} \delta_0(s_0 + \cdots + s_j + \frac{jd}{2}, \varepsilon_j, \eta_j) \prod_{j=1}^{k} e^{-i\frac{e\pi}{4}\varepsilon_j(\eta_0+\cdots+\eta_{j-1})} \ .$$

Remark 5.3.11. The coefficient $\delta(s,\varepsilon,\eta)$ defined in Theorem 5.2.9 depends of course on the graded algebra $\widetilde{\mathfrak{g}}$. We prove here that for Type I and II it depends only on the rank $k+1$ of this graded algebra and on the constant d. Henceforth we will denote it by δ_k to emphasize the dependance on k.

Remark 5.3.12. The formula given for δ_0 in Proposition 5.3.3 implies that the function δ_k is a product of an analytic function of s with the product $\prod_{j=0}^{k} \Gamma(s_0 + \cdots + s_j + \frac{jd}{2} + 1)$.

Proof. — To obtain the explicit expression of the coefficients δ_k it is sufficient to use $\mathcal{C}^\infty$ functions f with compact support in $\mathcal{O}^+$. In that case the Zeta function $s \mapsto Z_\varepsilon^+(f,s)$ is analytic on $\mathbb{C}^{k+1}$.

We will prove the formula by induction on k. For $k = 0$ the formula reduces to TATE's result (see Proposition 5.3.3). We suppose now that this formula is true on $\widetilde{\underline{\mathfrak{g}}}$ where it can be written for $\varphi \in \mathcal{C}_c^\infty(\underline{\mathcal{O}}^+)$

$$Z_{\underline{\varepsilon}}(\varphi, \underline{s}) = \sum_{\underline{\eta} \in \{\pm 1\}^k} \delta_{k-1}(\underline{s}, \underline{\varepsilon}, \underline{\eta}) Z_{\underline{\eta}}^-(\mathcal{F}\varphi, t(\underline{s}) - \underline{m}) \ .$$

Here we have $\underline{\varepsilon} = (\varepsilon_0, \ldots, \varepsilon_{k-1})$, $\underline{s} = (s_0, \ldots, s_{k-1})$ and

$$\underline{m} = \frac{\dim \underline{V}^+}{\kappa\, k} = 1 + \frac{(k-1)d}{2} \quad \text{since } \kappa = 1 \ .$$

Let f be a function in $\mathcal{C}_c^\infty(\mathcal{O}^+)$. Using the integration formula proved in Proposition 4.6.4, Lemma 5.3.7(1) and Proposition 5.3.8(1), $Z_\varepsilon^+(f,s)$ can be written as

$$Z_\varepsilon^+(f,s) = \int_{\varepsilon_k\Delta_k(v)>0}\int_{u\in\underline{\mathcal{O}_\varepsilon}^+} T_f^+(u,v)\,|\underline{\Delta}(u)|^{\underline{s}}\,|\Delta_k(v)|^{s_0+\cdots+s_k+\frac{kd}{2}}\,du\,dv\ .$$

We know from Proposition 4.2.3 that the map $\Phi : (u,A,v)\mapsto e^{\mathrm{ad}\,A}(u+v)$ is a diffeomorphism from $V^+(0)\times\mathfrak{g}(-1)\times V^+(2)'$ onto the set $O_k=\{X\in V^+ \mid \Delta_k(X)\neq 0\}$, where $V^+(2)'=\{v\in V^+(2)\mid \Delta_k(v)\neq 0\}$. Moreover, by Lemma 5.3.7, the inverse image of $\mathcal{O}^+$ by this diffeomorphism is the open set $\underline{\mathcal{O}}^+\times\mathfrak{g}(-1)\times V^+(2)'$. It follows that the function $(u,v)\mapsto T_f^+(u,v)$ is a $\mathcal{C}^\infty$ function with compact support in $\underline{\mathcal{O}}^+\times V^+(2)'$. Thus the integral above is absolutely convergent for each value of $s\in\mathbb{C}^{k+1}$. Moreover the induction hypothesis can be applied to the function $u\mapsto\varphi(u)=T_f^+(u,v)$ and we obtain

$$\int_{u\in\underline{\mathcal{O}_\varepsilon}^+} T_f^+(u,v)|\underline{\Delta}(u)|^{\underline{s}}\,du = \sum_{\underline{\eta}\in\{\pm1\}^k}\delta_{k-1}(\underline{s},\underline{\varepsilon},\underline{\eta})\int_{u'\in\underline{\mathcal{O}_\eta}^-}(\mathcal{F}_uT_f^+)(u',v)|\underline{\nabla}(u')|^{t(\underline{s})-\underline{m}}\,du'\ .$$

Let us suppose now that $t(\underline{s})-\underline{m}$ is such that the function $u'\mapsto|\underline{\nabla}(u')|^{t(\underline{s})-\underline{m}}$ is continuous on $V^-(0)$. Then each integral in the previous sum is absolutely convergent and compactly supported in v. Therefore we may exchange the order of integration in the computation of $Z_\varepsilon^+(f,s)$ which gives

$$Z_\varepsilon^+(f,s) = \sum_{\underline{\eta}\in\{\pm1\}^k}\delta_{k-1}(\underline{s},\underline{\varepsilon},\underline{\eta})\int_{u'\in\underline{\mathcal{O}_\eta}^-}|\underline{\nabla}(u')|^{t(\underline{s})-\underline{m}} \times\int_{\varepsilon_k\Delta_k(v)>0}(\mathcal{F}_uT_f^+)(u',v)|\Delta_k(v)|^{s_0+\cdots+s_k+\frac{kd}{2}}\,dv\,du'\,.$$

In order to apply Theorem 4.8.1 we wish to introduce $\gamma(u',v)$ which is, by Lemma 5.3.9, independent of the choice of $u'\in\underline{\mathcal{O}_\eta}^-$. Thus we introduce a function $v\in V^+(2)\mapsto G_{u'}(v)$, defined for a fixed element $u'\in\underline{\mathcal{O}_\eta}^-$, by

$$G_{u'}(v)=\gamma(u',v)\Big(\mathcal{F}_uT_f^+\Big)(u',v)=e^{i\frac{\pi}{4}e\,\varepsilon_k(\sum_{j=0}^{k-1}\eta_j)}\Big(\mathcal{F}_uT_f^+\Big)(u',v)\ .$$

This function belongs to $\mathcal{C}_c^\infty(V^+(2)')$ and TATE's functional equation (Proposition 5.3.3) applied to this function gives

$$\int_{\varepsilon_k\Delta_k(v)>0}G_{u'}(v)|\Delta_k(v)|^{s_0+\cdots+s_k+\frac{dk}{2}}dv=\sum_{\eta_k=\pm1}\delta_0(s_0+\cdots+s_k+\frac{dk}{2},\varepsilon_k,\eta_k) \times\int_{\eta_k\nabla_k(v')>0}\Big(\mathcal{F}_vG_{u'}\Big)(v')\,|\nabla_k(v')|^{-1-s_0-\cdots-s_k-\frac{dk}{2}}dv'\,.$$

From Theorem 4.8.1 we obtain

$$\Big(\mathcal{F}_vG_{u'}\Big)(v')=\Big(\mathcal{F}_v\big(\gamma(u',v)\mathcal{F}_uT_f^+\big)\Big)(v')=T_{\mathcal{F}f}^-(u',v')\ .$$

The formula giving $Z_\varepsilon^+(f,s)$ can now be written as

$$Z_\varepsilon^+(f,s) = \sum_{\underline{\eta}\in\{\pm1\}^k} \delta_{k-1}(\underline{s},\underline{\varepsilon},\underline{\eta}) e^{-i\frac{\pi}{4}e\varepsilon_k(\sum_{j=o}^{k-1}\eta_j)} \times \sum_{\eta_k=\pm1} \delta_0(,s_0+\cdots+s_k+\frac{dk}{2},\varepsilon_k,\eta_k) \times \int_{u'\in\underline{\mathcal{O}_\eta}^-}\int_{\eta_k\nabla_k(v')>0} T_{\overline{\mathcal{F}}f}^-(u',v')\,|D(u',v')|du'dv' \ ,$$

where

$$|D(u',v')| = |\underline{\nabla}(u')|^{t(\underline{s})-\underline{m}}|\nabla_k(v')|^{-1-s_0-\cdots-s_k-\frac{dk}{2}} \ .$$

Using the relations $\underline{m} = m - \frac{d}{2}$ and $m = 1 + \frac{kd}{2}$ we deduce from Lemma 5.3.7(2) that

$$|D(u',v')| = |\nabla(u'+v')|^{t(s)-m}|\nabla_1(u')|^{\frac{d}{2}} \ .$$

Using again the integral formula proved in Proposition 4.6.4 (but this time on V^-) and Proposition 5.3.8(2) we obtain

$$Z_\varepsilon^+(f,s) = \sum_{\eta\in\{\pm1\}^{k+1}} \delta_k(s,\varepsilon,\eta) Z_\eta^-(\mathcal{F}f, t(s)-m) \ ,$$

where

$$\delta_k(s,\varepsilon,\eta) = \delta_{k-1}(\underline{s},\underline{\varepsilon},\underline{\eta})\,\delta_0(s_0+\cdots+s_k+\frac{dk}{2},\varepsilon_k,\eta_k)\,e^{-i\frac{e\pi}{4}\varepsilon_k(\sum_{j=0}^{k-1}\eta_j)} \ .$$

By analytic continuation we obtain a relation between δ_k and δ_{k-1} which are meromorphic functions by Theorem 5.2.9. A proof by induction on k gives the announced formula for δ_k.

□

5.4. Computation of the coefficients $\nu(s,\eta,\varepsilon)$

The matrix $\big(\nu(s,\eta,\varepsilon)\big)_{\varepsilon,\eta\in\{\pm1\}^{k+1}}$ defined in Theorem 5.2.9 is the inverse of the matrix $\big(\delta(t(s)-m,\varepsilon,\eta)\big)_{\varepsilon,\eta\in\{\pm1\}^{k+1}}$ since we have

$$t(t(s)-m)-m = s \ .$$

In fact, using the relation $\overline{\mathcal{F}}\circ\mathcal{F} = \mathrm{Id}_{\mathcal{S}(V^+)}$, we can express the coefficients ν in terms of the coefficients δ.

Theorem 5.4.1 (Type I and II)**.**
Let $f\in\mathcal{S}(V^+)$ and $\eta\in\{\pm1\}^{k+1}$. In the functional equation (5-2-17) *proved in Theorem* 5.2.9

$$Z_\eta^-(\mathcal{F}f,s) = \sum_{\varepsilon\in\{\pm1\}^{k+1}} \nu_k(s,\eta,\varepsilon) Z_\varepsilon^+(f,t(s)-m) \ ,$$

the coefficients $s\mapsto\nu_k(s,\eta,\varepsilon)$ are the meromorphic functions given by

$$\nu_k(s,\eta,\varepsilon) = \delta_k(s,\widetilde{\eta},-\widetilde{\varepsilon}) \ ,$$

where $-\varepsilon = (-\varepsilon_0,\ldots,-\varepsilon_k)$ and $\widetilde{\varepsilon} = (\varepsilon_k,\varepsilon_{k-1},\ldots,\varepsilon_1,\varepsilon_0)$.

Proof. —In order to exchange the role of V^+ and V^- we will use the involution γ of $\widetilde{\mathfrak{g}}$ defined in section 3.7. It follows from Remark 4.1.2 that

$$\int_{V^-} g(Y)\,dY = \int_{V^+} g(\gamma X)\,dX \quad \text{for } g \in L^1(V^-)\ .$$

Moreover we have from Definition 3.8.1

$$\nabla_j(Y) = \Delta_j(\gamma Y) \quad \text{for } j = 0,\dots,k\ .$$

It follows from Corollary 3.6.5 and 3.8.7 that

$$X \in \mathcal{O}_\eta^+ \Longleftrightarrow \gamma X \in \mathcal{O}_{\widetilde{\eta}}^-\ .$$

Let us start with a function $f \in \mathcal{C}_c^\infty(\mathcal{O}^+)$. Then we have

$$Z_\eta^-(\mathcal{F}f,s) = \int_{\mathcal{O}_{\widetilde{\eta}}^+} \big(\mathcal{F}f\big)(\gamma X)\,|\Delta(X)|^s\,dX = Z_{\widetilde{\eta}}^+(\varphi,s)\ ,$$

where φ is the function $V^+ \ni X \mapsto \big(\mathcal{F}f\big)(\gamma X)$. Then the functional equation (5-2-18) gives

$$Z_\eta^-(\mathcal{F}f,s) = \sum_{\varepsilon\in\{\pm1\}^{k+1}} \delta_k(s,\widetilde{\eta},\varepsilon)\,Z_\varepsilon^-(\mathcal{F}\varphi,t(s)-m)\ .$$

Since γ is an involution we deduce from the relation $b(X',\gamma X) = b(\gamma X',X)$ that

$$\varphi(X) = \Big(\overline{\mathcal{F}}(f\circ\gamma)\Big)(-X) \quad \text{for } X \in V^+\ .$$

And the relation $\mathcal{F}\circ\overline{\mathcal{F}} = \mathrm{Id}_{\mathcal{S}(V^-)}$ implies that

$$\Big(\mathcal{F}\varphi\Big)(Y) = \big(f\circ\gamma\big)(-Y) \quad \text{for } Y \in V^-\ .$$

Moreover, if $-\varepsilon$ denotes $(-\varepsilon_0,\dots,-\varepsilon_k)$ for $\varepsilon = (\varepsilon_0,\dots,\varepsilon_k) \in \{\pm1\}^{k+1}$, we have

$$Y \in \mathcal{O}_\varepsilon^- \Longleftrightarrow -Y \in \mathcal{O}_{-\varepsilon}^-\ .$$

Therefore we obtain

$$\begin{aligned} Z_\varepsilon^-(\mathcal{F}\varphi,t(s)-m) &= \int_{\mathcal{O}_\varepsilon^-} \big(f\circ\gamma\big)(-Y)\,|\nabla(Y)|^{t(s)-m}\,dY \\ &= \int_{\mathcal{O}_{-\varepsilon}^-} \big(f\circ\gamma\big)(Y)\,|\nabla(Y)|^{t(s)-m}\,dY \\ &= \int_{\mathcal{O}_{-\widetilde{\varepsilon}}^+} f(X)\,|\Delta(X)|^{t(s)-m}\,dX \\ &= Z_{-\widetilde{\varepsilon}}^+(f,t(s)-m)\ . \end{aligned}$$

And consequently we have

$$\begin{aligned} Z_\eta^-(\mathcal{F}f,s) &= \sum_{\varepsilon\in\{\pm1\}^{k+1}} \delta_k(s,\widetilde{\eta},\varepsilon)\,Z_{-\widetilde{\varepsilon}}^+(f,t(s)-m) \\ &= \sum_{\varepsilon\in\{\pm1\}^{k+1}} \delta_k(s,\widetilde{\eta},-\widetilde{\varepsilon})\,Z_\varepsilon^+(f,t(s)-m)\ . \end{aligned}$$

Taking a function $f \in \mathcal{C}_c^\infty(\mathcal{O}_\varepsilon^+)$ such that $Z_\varepsilon^+(f,t(s)-m) \neq 0$ we obtain the result.

□

Now the matrix $\big(\delta_k(t(s)-m,\varepsilon,\eta)\big)_{\varepsilon,\eta\in\{\pm1\}^{k+1}}$ is explicitely inverted and we obtain the following Corollary, which would be difficult to deduce from the explicit formulas given in Theorem 5.3.10.

Corollary 5.4.2. *If η and η' are two elements of $\{\pm1\}^{k+1}$ then we have*

$$\sum_{\varepsilon\in\{\pm1\}^{k+1}} \delta_k\big(s,\widetilde{\eta},-\widetilde{\varepsilon}\big)\delta_k\big(t(s)-m,\varepsilon,\eta'\big) = \begin{cases} 0 \text{ if } \eta\neq\eta' \ , \\ 1 \text{ if } \eta=\eta' \ . \end{cases}$$

5.5. Functional equation for (P,V^+)

We will now consider the local Zeta functions of the prehomogeneous vector spaces (P,V^+) and (P,V^-).

In Type I case the open orbits of P and of AN are the same on V^+ and V^- since the action of M is trivial on $\oplus_{j=0}^k\widetilde{\mathfrak{g}}^{\lambda_j}$. The local Zeta functions of these prehomogeneous spaces are the functions Z_ε^+ and Z_η^- defined in 5.1.1.

But in Type II case there are only two open P–orbits in V^+ (Theorem 3.6.3) which are given by

$$\mathcal{O}_0^+ = P.I^+ = \{X\in\mathcal{O}^+ \mid \Delta_0(X)>0\} \text{ where } I^+ = \sum_{j=0}^k X_j \ ,$$

$$\mathcal{O}_1^+ = P.I_1^+ = \{X\in\mathcal{O}^+ \mid \Delta_0(X)<0\} \text{ where } I_1^+ = \big(\sum_{j=0}^{k-1} X_j\big) - X_k \ .$$

Similarly there are also only two open P–orbits in V^- (Theorem 3.8.6) which are given by

$$\mathcal{O}_0^- = P.I^- = \{Y\in\mathcal{O}^- \mid \nabla_0(X)>0\} \text{ where } I^- = \sum_{j=0}^k \theta(X_j) \ ,$$

$$\mathcal{O}_1^- = P.I_1^- = \{Y\in\mathcal{O}^- \mid \nabla_0(X)<0\} \text{ where } I_1^- = \big(\sum_{j=0}^{k-1} \theta(X_j)\big) - \theta(X_k) \ .$$

Each open P–orbit is a finite union of AN–orbits and for simplicity we will introduce the following notation.

Notation 5.5.1 (Type II). *Let $p\in\{0,1\}$. We denote by $\underline{p}$ the set*

$$\underline{p} = \{\varepsilon\in\{\pm1\}^{k+1} \mid \mathcal{O}_\varepsilon^+ \subset \mathcal{O}_p^+\} \ .$$

From Theorem 3.6.3 and Corollary 3.6.5 we deduce that

$$\varepsilon\in\underline{0} \Longleftrightarrow \varepsilon_0\cdots\varepsilon_k = 1 \ ,$$
$$\varepsilon\in\underline{1} \Longleftrightarrow \varepsilon_0\cdots\varepsilon_k = -1 \ .$$

And from Theorem 3.8.6 and Corollary 3.8.7 we obtain also

$$\varepsilon\in\underline{p} \Longleftrightarrow \mathcal{O}_\varepsilon^- \subset \mathcal{O}_p^- \ .$$

The Zeta functions of (P,V^+) and (P,V^-) are then defined as follows (in the next section we will generalize this Definition).

Definition 5.5.2 (Type II).
Let $s = (s_0, \dots, s_k) \in \mathbb{C}^{k+1}$. For $f \in \mathcal{S}(V^+)$ and $h \in \mathcal{S}(V^-)$, the local Zeta functions of (P, V^+) and (P, V^-) are defined for $p \in \{0, 1\}$ by

$$Z_p^+(f, s) = \int_{\mathcal{O}_p^+} f(X)\, |\Delta(X)|^s\, dX = \sum_{\varepsilon \in \underline{p}} Z_\varepsilon^+(f, s) \ ,$$

$$Z_p^-(h, s) = \int_{\mathcal{O}_p^-} h(Y)\, |\nabla(Y)|^s\, dY = \sum_{\eta \in \underline{p}} Z_\eta^-(h, s) \ .$$

We will deduce the functional equation for these Zeta functions from the previous functional equation (Theorem 5.2.9). The proof relies on the following Lemma.

Lemma 5.5.3. *Let $q \in \{0, 1\}$ and η and η' be elements of $\underline{q}$. Then, for $p \in \{0, 1\}$, there exists a permutation $\varepsilon \mapsto \varepsilon'$ of the set $\underline{p}$ such that we have*

$$\delta_k(s, \varepsilon, \eta) = \delta_k(s, \varepsilon', \eta') \quad \text{and} \quad \nu_k(s, \varepsilon, \eta) = \nu_k(s, \varepsilon', \eta') \ .$$

(equality of meromorphic functions)

Proof. — Let us recall that from Theorem 5.3.10 and Proposition 5.3.3 we have

$$\delta_k(s, \varepsilon, \eta) = \prod_{j=0}^{k} \delta_0(s_0 + \cdots + s_j + \frac{jd}{2}, \varepsilon_j, \eta_j) \prod_{j=1}^{k} e^{-i\frac{e\pi}{4}\varepsilon_j(\eta_0 + \cdots + \eta_{j-1})} \ ,$$

where

$$\delta_0(s_0 + \cdots + s_j + \frac{jd}{2}, \varepsilon_j, \eta_j)$$
$$= (2\pi)^{-(1+s_0+\cdots+s_k+\frac{jd}{2})} \Gamma(1 + s_0 + \cdots + s_k + \frac{jd}{2}) \times e^{-i\frac{\pi}{2}\varepsilon_j\eta_j(1+s_0+\cdots+s_k+\frac{jd}{2})} \ .$$

Let us define ε' by

$$\varepsilon_j'\eta_j' = \varepsilon_j\eta_j \quad \text{for } j = 0, \dots, k \ .$$

As $\eta_0 \cdots \eta_k = \eta_0' \cdots \eta_k'$, the map $\varepsilon \mapsto \varepsilon'$ is a permutation of $\underline{p}$ for $p \in \{0, 1\}$.

In the cases where $e = 0$, $\delta_k(s, \varepsilon, \eta)$ depends only on the products $\varepsilon_j\eta_j$ and the result follows.

In the case where $e \neq 0$ we have necessarily (in Type II case) $k = 0$ or $k = 1$ by Corollary 2.8.7. If $k = 0$ there is nothing to prove since there are only two open AN–orbits which are also the P–orbits. If $k = 1$ it remains to prove that

$$e^{-i\frac{e\pi}{4}\varepsilon_1\eta_0} = e^{-i\frac{e\pi}{4}\varepsilon_1'\eta_0'} \ .$$

But the relations $\varepsilon_1'\eta_1' = \varepsilon_1\eta_1$ (definition of ε') and $\eta_1'\eta_0' = \eta_1\eta_0$ (hypothesis on η and η') imply that

$$\varepsilon_1'\eta_0' = \varepsilon_1\eta_0 \ ,$$

and the first equality is proved.

In order to prove the second equality we use the relation proved in Theorem 5.4.1 :

$$\nu_k(s, \varepsilon, \eta) = \delta_k(s, \widetilde{\varepsilon}, -\widetilde{\eta}) \ .$$

As η and η' are elements of $\underline{q}$, there exists $r \in \{0,1\}$ such that $-\widetilde{\eta}$ and $-\widetilde{\eta'}$ are elements of $\underline{r}$. In fact $r = q$ if $k+1$ is even and $r = 1-q$ if $k+1$ is odd. As ε' is defined by $\varepsilon'_j \eta'_j = \varepsilon_j \eta_j$ for $j \in \{0, \ldots, k\}$, the relation $\widetilde{\varepsilon}_j = \varepsilon_{k-j}$ implies

$$(\widetilde{\varepsilon})_j (-\widetilde{\eta})_j = (\widetilde{\varepsilon'})_j (-\widetilde{\eta'})_j \ .$$

Thus the second equality is a direct consequence of the first one.

□

It follows from the previous Lemma that $\sum_{\varepsilon \in \underline{p}} \delta_k(s, \varepsilon, \eta)$ does not depend on η but only on the open P–orbit in which $\mathcal{O}_\eta^-$ is contained. Similarly $\sum_{\eta \in \underline{q}} \nu_k(s, \eta, \varepsilon)$ does not depend on ε but only on the open P–orbit in which $\mathcal{O}_\varepsilon^+$ is contained. Therefore the following Definition makes sense.

Definition 5.5.4. *For q and q elements of $\{0,1\}$ we set*

$$\delta_k(s,p,q) = \sum_{\varepsilon \in \underline{p}} \delta_k(s,\varepsilon,\eta) \quad \text{if } \eta \in \underline{q} \ ,$$

$$\nu_k(s,q,p) = \sum_{\eta \in \underline{q}} \nu_k(s,\eta,\varepsilon) \quad \text{if } \varepsilon \in \underline{p} \ .$$

For $f \in \mathcal{S}(V^+)$ and $p \in \{0,1\}$ we deduce from Theorem 5.2.9 that

$$\begin{aligned} Z_p^+(f,s) &= \sum_{\varepsilon \in \underline{p}} \sum_{q \in \{0,1\}} \sum_{\eta \in \underline{q}} \delta_k(s,\varepsilon,\eta) Z_\eta^-(\mathcal{F}f, t(s)-m) \\ &= \sum_{q \in \{0,1\}} \delta_k(s,p,q) \sum_{\eta \in \underline{q}} Z_\eta^-(\mathcal{F}f, t(s)-m) \ . \end{aligned}$$

Similarly we have for $q \in \{0,1\}$

$$Z_q^-(\mathcal{F}f,s) = \sum_{p \in \{0,1\}} \nu_k(s,q,p) \sum_{\varepsilon \in \underline{p}} Z_\varepsilon^+(f, t(s)-m) \ .$$

Therefore we obtain the following functional equations for the Zeta function of (P, V^+).

Theorem 5.5.5 (Type II, functional equation for (P, V^+))**.** *For $f \in \mathcal{S}(V^+)$, $p, q \in \{0,1\}$ the integrals defining $Z_p^+(f,s)$ and $Z_q^-(\mathcal{F}f,s)$ have meromorphic continuations in $\mathbb{C}^{k+1}$ and satisfy the following functional equations*

$$Z_q^-(\mathcal{F}f,s) = \sum_{p \in \{0,1\}} \nu_k(s,q,p) Z_p^+(f, t(s)-m),$$

$$Z_p^+(f,s) = \sum_{q \in \{0,1\}} \delta_k(s,p,q) Z_q^-(\mathcal{F}f, t(s)-m) \ .$$

5.6. A generalized functional equation for (P, V^+) in Type II

Throughout this section we suppose that $\widetilde{\mathfrak{g}}$ is of Type II.

In fact, in order to prove our main theorem on Zeta functions related to the minimal spherical series in the Type II case (see chapter 7), we need now to introduce a slight generalization of the Zeta functions $Z_p^+(f,s)$ and $Z_p^-(h,s)$ which were introduced in Definition 5.5.2.

The polynomials Δ_j (*resp.* ∇_j) are nonzero on the open P–orbits. The signs of these polynomials are constant on the open AN–orbits but are nonconstant on the P–orbits. Let us denote by $\mathrm{sg}(x)$ the sign of a nonzero real number x. For $\tau = (\tau_0, \ldots, \tau_k) \in \mathbb{Z}^{k+1}$ and for $X \in \mathcal{O}^+$, the sign of $\Delta(X)^\tau = \Delta_0(X)^{\tau_0} \cdots \Delta_k(X)^{\tau_k}$ depends only on the class of each τ_j in $\mathbb{Z}/2\mathbb{Z}$. Thus we can define for $\tau = (\tau_0, \ldots, \tau_k) \in (\mathbb{Z}/2\mathbb{Z})^{k+1}$

$$\mathrm{sg}\Big(\Delta(X)^\tau\Big) = \mathrm{sg}\Big(\Delta_0(X)^{\tau_0} \cdots \Delta_k(X)^{\tau_k}\Big) \text{ for } X \in \mathcal{O}^+ \ ,$$
$$\mathrm{sg}\Big(\nabla(Y)^\tau\Big) = \mathrm{sg}\Big(\nabla_0(Y)^{\tau_0} \cdots \nabla_k(Y)^{\tau_k}\Big) \text{ for } Y \in \mathcal{O}^- \ .$$

The following Lemma is a direct consequence of Corollary 3.6.5 and Corollary 3.8.7.

Lemma 5.6.1. *For $\tau = (\tau_0, \tau_1, \ldots, \tau_k) \in (\mathbb{Z}/2\mathbb{Z})^{k+1}$ we have*

$$X \in \mathcal{O}_\varepsilon^+ \Longrightarrow \mathrm{sg}\big(\Delta(X)^\tau\big) = (\varepsilon_0 \cdots \varepsilon_k)^{\tau_0} (\varepsilon_1 \cdots \varepsilon_k)^{\tau_1} \cdots \varepsilon_k^{\tau_k} \ ,$$
$$Y \in \mathcal{O}_\varepsilon^- \Longrightarrow \mathrm{sg}\big(\nabla(Y)^\tau\big) = (\varepsilon_0 \cdots \varepsilon_k)^{\tau_0 + \cdots + \tau_k} (\varepsilon_1 \cdots \varepsilon_k)^{\tau_k} \cdots \varepsilon_k^{\tau_1} \ .$$

The first expression will be denoted by $\tau(\varepsilon)$ and the second expression by $t(\tau)(\varepsilon)$ since, from Definition 3.8.9, *we have* $t(\tau) = (-\tau_0 - \cdots - \tau_k, \tau_k, \ldots \tau_1)$.

Definition 5.6.2 (Type II). *Let $s = (s_0, s_1, \ldots, s_k) \in \mathbb{C}^{k+1}$ and let $\tau = (\tau_0, \tau_1, \ldots, \tau_k) \in (\mathbb{Z}/2\mathbb{Z})^{k+1}$. For $p \in \{0,1\}, f \in \mathcal{S}(V^+)$ and $h \in \mathcal{S}(V^-)$ we define*

$$Z_p^+(f, s, \tau) = \int_{\mathcal{O}_p^+} f(X) |\Delta(X)|^s \, \mathrm{sg}\big(\Delta(X)^\tau\big) \, dX \ ,$$
$$Z_p^-(h, s, \tau) = \int_{\mathcal{O}_p^-} h(Y) |\nabla(Y)|^s \, \mathrm{sg}\big(\nabla(Y)^{t(\tau)}\big) \, dY \ .$$

Remark 5.6.3. The parameter τ occurring in the previous definition will essentially parametrize the $(M \cap H)$–spherical representations of M as we shall see in chapter 7.

From Lemma 5.6.1 we obtain

$$Z_p^+(f, s, \tau) = \sum_{\varepsilon \in \underline{p}} \tau(\varepsilon) \, Z_\varepsilon^+(f, s) \ ,$$
$$Z_p^-(h, s, \tau) = \sum_{\varepsilon \in \underline{p}} \tau(\varepsilon) \, Z_\varepsilon^-(h, s) \ .$$

As in section 5.5 the functional equation for these Zeta functions is a consequence of Theorem 5.2.9, using the following Lemma.

Lemma 5.6.4 (Type II). *Let $\tau \in (\mathbb{Z}/2\mathbb{Z})^{k+1}$.*
(1) *For $\eta \in \underline{q}$, the meromorphic function $\sum_{\varepsilon \in \underline{p}} \tau(\eta)\, \tau(\varepsilon)\, \delta_k(s, \varepsilon, \eta)$ does not depend on the choice of $\eta \in \underline{q}$ but depends only on q (and on p).*
(2) *For $\varepsilon \in \underline{p}$, the meromorphic function $\sum_{\eta \in \underline{q}} \tau(\eta)\, \tau(\varepsilon)\, \nu_k(s, \eta, \varepsilon)$ does not depend on the choice of $\varepsilon \in \underline{p}$ but depends only on p (and on q).*

Proof of (1). —Let $\eta,\ \eta' \in \underline{q}$. We know from the proof of Lemma 5.5.3 that if the permutation $\varepsilon \mapsto \varepsilon'$ of $\underline{p}$ is defined by

$$\varepsilon'_j \eta'_j = \varepsilon_j \eta_j \quad \text{for } j = 0, \dots, k\ ,$$

then

$$\delta_k(s, \varepsilon, \eta) = \delta_k(s, \varepsilon', \eta').$$

Moreover for such a permutation, we have for any $\tau \in (\mathbb{Z}/2\mathbb{Z})^{k+1}$

$$\tau(\eta)\tau(\varepsilon) = \tau(\eta')\tau(\varepsilon').$$

This yields the result. The proof of (2) is similar.

□

Then, from previous Lemma, the following Definition makes sense.

Definition 5.6.5. *For q and q elements of $\{0,1\}$ we set*

$$\delta_k(s, \tau, p, q) = \tau(\eta) \sum_{\varepsilon \in \underline{p}} \tau(\varepsilon) \delta_k(s, \varepsilon, \eta) \quad \text{if } \eta \in \underline{q}\ ,$$

$$\nu_k(s, \tau, q, p) = \tau(\varepsilon) \sum_{\eta \in \underline{q}} \tau(\eta) \nu_k(s, \eta, \varepsilon) \quad \text{if } \varepsilon \in \underline{p}\ .$$

Theorem 5.6.6 (Type II). *For $f \in \mathcal{S}(V^+)$, $h \in \mathcal{S}(V^-)$, $p, q \in \{0,1\}$, the integrals defining $Z_p^+(f, s, \tau)$ and $Z_q^-(h, s, \tau)$ have meromorphic continuations to $\mathbb{C}^{k+1}$ which satisfy the following functional equations*

$$Z_p^+(f, s, \tau) = \sum_{q=0,1} \delta_k(s, \tau, p, q) Z_q^-(\mathcal{F}f, t(s) - m, \tau)\ ,$$

$$Z_q^-(\mathcal{F}f, s, \tau) = \sum_{p=0,1} \nu_k(s, \tau, q, p) Z_p^+\big(f, t(s) - m, \tau\big)$$

Proof. —The fact that $Z_p^+(f, s, \tau)$ and $Z_q^-(h, s, \tau)$ have meromorphic continuations is a direct consequence of Theorem 5.2.9. Let us prove the second relation (the proof of the first one is similar). Using the definition of $Z_q^-(h, s, \tau)$ and Theorem 5.4.1 we get

$$\begin{aligned} Z_q^-(\mathcal{F}f, s, \tau) &= \sum_{\eta \in \underline{q}} \tau(\eta) Z_\eta^-(\mathcal{F}f, s) \\ &= \sum_{\eta \in \underline{q}} \tau(\eta) \sum_{\varepsilon \in \{\pm 1\}^{k+1}} \nu_k(s, \eta, \varepsilon) Z_\varepsilon^+(f, t(s) - m)\ . \end{aligned}$$

In order to introduce

$$Z_p^+(f, t(s) - m, \tau) = \sum_{\varepsilon \in \underline{p}} \tau(\varepsilon)\ Z_\varepsilon^+(f, t(s) - m)$$

we remark that $\tau(\varepsilon)^2 = 1$ and then write

$$Z_q^-(\mathcal{F}f, s, \tau) = \sum_{p=0}^{1} \sum_{\varepsilon \in \underline{p}} \tau(\varepsilon) \Big(\sum_{\eta \in \underline{q}} \tau(\varepsilon)\ \tau(\eta)\ \nu_k(s, \eta, \varepsilon) \Big) Z_\varepsilon^+(f, t(s) - m)\ .$$

Then we get from Definition 5.6.5

$$\begin{aligned} Z_q^-(\mathcal{F}f, s, \tau) &= \sum_{p=0}^{1} \nu_k(s, \tau, q, p) \sum_{\varepsilon \in \underline{p}} \tau(\varepsilon)\ Z_\varepsilon^+(f, t(s) - m) \\ &= \sum_{p=0}^{1} \nu_k(s, \tau, q, p)\ Z_p^+\big(f, t(s) - m, \tau\big)\ . \end{aligned}$$

□

5.7. Functional equation for (G, V^+)

We will now consider the local Zeta functions of the prehomogeneous vector spaces (G, V^+) and (G, V^-). As before we will obtain the corresponding functional equation from Theorem 5.2.9.

In **Type II case** we know from Theorem 2.7.3 that there are two open G–orbits in V^+, as well as in V^-. Let us call $\Omega_0^\pm$ and $\Omega_1^\pm$ these open orbits where

$$\begin{aligned} \Omega_0^+ &= G.I^+ = \{X \in V^+ \mid \Delta_0(X) > 0\} \subset V^+ \text{ where } I^+ = \sum_{j=0}^{k} X_j\ , \\ \Omega_1^+ &= G.I_1^+ = \{X \in V^+ \mid \Delta_0(X) < 0\} \subset V^+ \text{ where } I_1^+ = -X_k + \sum_{j=0}^{k-1} X_j\ , \\ \Omega_0^- &= G.I^- = \{X \in V^- \mid \nabla_0(X) > 0\} \subset V^- \text{ where } I^- = \theta(I^+)\ , \\ \Omega_1^- &= G.I_1^- = \{X \in V^- \mid \nabla_0(X) < 0\} \subset V^- \text{ where } I_1^- = \theta(I_1^+)\ . \end{aligned}$$

Each open P–orbit is dense in an open G–orbit. More precisely we have $\mathcal{O}_p^+ \subset \Omega_p^+$ and $\mathcal{O}_p^- \subset \Omega_p^-$ for $p \in \{0, 1\}$. Moreover the unique irreducible relative G–invariant on V^+ is Δ_0 and on V^- is ∇_0. Therefore the corresponding Zeta functions are defined for $f \in \mathcal{S}(V^+)$, $h \in \mathcal{S}(V^-)$, $z \in \mathbb{C}$ and $p \in \{0, 1\}$ by

$$\begin{aligned} Z_p^+(f, z) &= \int_{\Omega_p^+} f(X)\, |\Delta_0(X)|^z\ dX = \int_{\mathcal{O}_p^+} f(X)\, |\Delta_0(X)|^z\ dX\ , \\ Z_p^-(h, z) &= \int_{\Omega_p^-} h(Y)\, |\nabla_0(Y)|^z\, dY = \int_{\mathcal{O}_p^-} h(Y)\, |\nabla_0(Y)|^z\, dY\ . \end{aligned}$$

Hence the corresponding functional equation is a direct consequence of Theorem 5.5.5.

Theorem 5.7.1 (Type II, functional equation for (G, V^+))**.**

For $f \in \mathcal{S}(V^+)$, $p, q \in \{0,1\}$, the integrals defining $Z_p^+(f,z)$ and $Z_q^-(\mathcal{F}f,z)$ have meromorphic continuation in $\mathbb{C}$ and satisfy the following functional equation

$$Z_p^+(f,z) = \sum_{q\in\{0,1\}} \delta_k((z,0,\dots,0),p,q) Z_q^-(\mathcal{F}f,-z-m) \ .$$

In **Type I case** there are only $k+2$ open G–orbits in V^+(Theorem 2.10.1) which are given by

$$\Omega_p^+ = G.I_p^+ \ \text{ where } I_p^+ = \sum_{j=0}^{k-p} X_j - \sum_{j=k-p+1}^{k} X_j \quad (p = 0,1,\dots,k+1) \ .$$

Similarly there are also $k+2$ open G–orbits in V^- which are given by

$$\Omega_p^- = G.I_p^- \ \text{ where } I_p^- = \sum_{j=0}^{k-p} \theta(X_j) - \sum_{j=k-p+1}^{k} \theta(X_j) \quad (p = 0,1,\dots,k+1) \ .$$

Each open G–orbit is, up to a set of measure 0, a finite union of AN–orbits and for simplicity we will introduce the following notation.

Notation 5.7.2 (Type I)**.** *Let $p \in \{0,1,\dots,k+1\}$. We denote by $\underline{p}$ the set*

$$\underline{p} = \{\varepsilon \in \{\pm 1\}^{k+1} \mid \mathcal{O}_\varepsilon^+ \subset \Omega_p^+\} \ .$$

From Theorem 2.10.1 and Corollary 3.6.5 we deduce that

$$\varepsilon \in \underline{p} \Longleftrightarrow \#\{j \mid \varepsilon_j = -1\} = p \ ,$$

where $\#A$ denotes the number of elements of the set A. From Corollary 3.8.7 we also obtain

$$\varepsilon \in \underline{p} \Longleftrightarrow \mathcal{O}_\varepsilon^- \subset \Omega_p^- \ .$$

Definition 5.7.3 (Type I)**.**
Let $z \in \mathbb{C}$. For $f \in \mathcal{S}(V^+)$ and $h \in \mathcal{S}(V^-)$, the local Zeta functions of (G, V^+) and (G, V^-) are defined for $p \in \{0,\dots,k+1\}$ by

$$Z_p^+(f,z) = \int_{\Omega_p^+} f(X)\,|\Delta_0(X)|^z \, dX = \sum_{\varepsilon\in\underline{p}} Z_\varepsilon^+(f,(z,0,\dots,0)) \ ,$$

$$Z_p^-(h,z) = \int_{\Omega_p^-} h(Y)\,|\nabla_0(Y)|^z \, dY = \sum_{\eta\in\underline{p}} Z_\eta^-(h,(z,0,\dots,0)) \ .$$

The existence of the functional equation for these Zeta functions as well as some details concerning the coefficients was obtained by Sato and Shintani ([Sa–Sh]–1974 Theorem 1). The explicit computation of the coefficients in Type I case is also known (see [S–F]–1984 Theorem 1 and [Mu]–1986 Proposition 3). But we will show that it may be obtained from the previous functional equation (Theorem 5.2.9 and Theorem 5.3.10). The proof relies on the following Lemma.

Lemma 5.7.4 (Type I). *Let Ω_q^- be an open G–orbit in V^-, η and η' be elements of $\underline{q}$. Then, for $p \in \{0, \dots, k+1\}$, there exists a permutation $\varepsilon \mapsto \varepsilon'$ of the set $\underline{p}$ such that we have*

$$\delta_k((z,0,\dots,0),\varepsilon,\eta) = \delta_k((z,0,\dots,0),\varepsilon',\eta') \ .$$

Proof. —In Type I case we have $e = d$. Then, from Theorem 5.3.10 and Proposition 5.3.3, we may write for $s = (z, 0, \dots, 0)$

$$\begin{aligned}\delta_k(s,\varepsilon,\eta) &= \prod_{j=0}^{k} \delta_0(z + \frac{jd}{2}, \varepsilon_j, \eta_j) \prod_{j=1}^{k} e^{-i\frac{d\pi}{4}\varepsilon_j(\eta_0+\cdots+\eta_{j-1})} \ , \\ &= \omega_k(z) \times e^{-i\frac{\pi}{2}(1+z)(\sum_{j=0}^{k}\varepsilon_j\eta_j)} \times e^{-i\frac{\pi}{4}dA} \ ,\end{aligned}$$

where

$$\omega_k(z) = \prod_{j=0}^{k} (2\pi)^{-(1+z+j\frac{d}{2})}\, \Gamma(1+z+j\frac{d}{2}) \ ,$$

and

$$A = \sum_{j=0}^{k} j\varepsilon_j\eta_j + \sum_{j=1}^{k} \varepsilon_j(\eta_0+\cdots+\eta_{j-1}) = \sum_{j=1}^{k} \varepsilon_j(\eta_0+\eta_1+\cdots+\eta_{j-1}+j\eta_j) \ .$$

Let us consider the map $\eta \in \underline{q} \mapsto \{a_1, \dots, a_k\}$ where

$$a_j = \eta_0 + \eta_1 + \cdots + \eta_{j-1} + j\eta_j \ .$$

As $\eta_j \equiv 1 \bmod 2$ the a_j's are even numbers. Let us show that the a_j's are distinct. If $a_i = a_j$ with $1 \le i < j$ then

$$i\eta_i = \eta_i + \eta_{i+1} + \ldots + \eta_{j-1} + j\eta_j \ ,$$

which implies

$$|j\eta_j - (i-1)\eta_i| = |\eta_{i+1} + \cdots + \eta_{j-1}| \le j - i - 1 \ .$$

But if $\eta_i = \eta_j$ we have

$$|j\eta_j - (i-1)\eta_i| = j - i + 1 > j - i - 1 \ ,$$

and if $\eta_i = -\eta_j$ we have

$$|j\eta_j - (i-1)\eta_i| = j + i - 1 > j - i - 1 \ .$$

Hence we obtain a contradiction. Therefore the a_j's are distinct. Moreover the relation $\eta \in \underline{q}$ implies that

$$-2(q-1) \le a_j \le 2(k-q) \quad \text{for } j = 1, \dots, k \ .$$

Therefore the image of the map $\eta \in \underline{q} \mapsto \{a_1, \dots, a_k\}$ is the set

$$\{-2(q-1), -2(q-2), \dots, 2(k-q-1), 2(k-q)\} \ .$$

It follows that, if η and η' are two elements in $\underline{q}$, there exists a permutation τ of $\{1, \dots, k\}$ such that

$$a'_j = a_{\tau(j)} \ ,$$

where $\{a_1, \dots, a_k\}$ and $\{a'_1, \dots, a'_k\}$ are the elements associated respectively to η and η' by the previous map. Moreover we have

$$\begin{aligned} a_j > 0 &\Longrightarrow \eta_j > 0 \quad i.e.\ \ \eta_j = 1 \ , \\ a_j < 0 &\Longrightarrow \eta_j < 0 \quad i.e.\ \ \eta_j = -1 \ , \\ a_j = 0 &\Longrightarrow \eta_0\eta_j < 0 \quad i.e.\ \ \eta_0\eta_j = -1 \ . \end{aligned}$$

These relations imply that for $j = 1, \dots, k$ we have

$$\begin{aligned} a'_j \neq 0 &\Longrightarrow \eta'_j = \eta_{\tau(j)} \ , \\ a'_j = 0 &\Longrightarrow \eta'_0\eta'_j = \eta_0\eta_{\tau(j)} < 0 \ . \end{aligned}$$

Now we define a permutation σ of $\{0, \dots, k\}$ as follows.
If $\eta'_0 = \eta_0$ then, for j_0 such that $a'_{j_0} = 0$, we have $\eta'_{j_0} = \eta_{\tau(j_0)}$, and we set

$$\sigma(0) = 0 \ \text{ and } \ \sigma(j) = \tau(j) \text{ for } j = 1, \dots, k \ .$$

If $\eta'_0 \neq \eta_0$ then $1 \leq q \leq k$, and therefore, there exists j_0 such that $a'_{j_0} = 0$. We have $\eta'_{j_0} = -\eta'_0 = \eta_0$ and $\eta_{\tau(j_0)} = -\eta_0 = \eta'_0$. In this case we set

$$\sigma(j) = \begin{cases} \tau(j_0) & \text{if } j = 0 \ , \\ 0 & \text{if } j = j_0 \ , \\ \tau(j) & \text{if } j \neq 0, j_0 \ . \end{cases}$$

In the two cases we obtain

$$a'_j = a_{\sigma(j)} \text{ for } j = 1, \dots, k \ \text{ and } \eta'_j = \eta_{\sigma(j)} \text{ for } j = 0, \dots k \ .$$

Therefore if, for $\varepsilon \in \underline{p}$, we define ε' by

$$\varepsilon'_j = \varepsilon_{\sigma(j)} \ \text{ for } j = 0, \dots, k \ ,$$

then $\varepsilon' \in \underline{p}$ and we obtain for both cases

$$\sum_{j=0}^{k} \varepsilon'_j\eta'_j = \sum_{j=0}^{k} \varepsilon_j\eta_j \ \text{ and } \ \sum_{j=1}^{k} \varepsilon'_j a'_j = \sum_{j=1}^{k} \varepsilon_j a_j \ .$$

It follows that

$$\delta_k(s, \varepsilon', \eta') = \delta_k(s, \varepsilon, \eta) \quad \text{ for } s = (z, 0, \dots, 0) \ .$$

□

The previous Lemma implies that $\sum_{\varepsilon \in \underline{p}} \delta_k((z, 0, \dots, 0), \varepsilon, \eta)$ does not depend on η but only on the open G–orbit in which $\mathcal{O}_\eta^-$ is contained. Then we may define $\widetilde{\delta}_k(z, p, q)$ for $p, q \in \{0, \dots, k+1\}$ by

$$\widetilde{\delta}_k(z, p, q) = \sum_{\varepsilon \in \underline{p}} \delta_k((z, 0, \dots, 0), \varepsilon, \eta) \ \text{ if } \eta \in \underline{q} \ .$$

For $f \in \mathcal{S}(V^+)$ and $p \in \{0, \ldots, k+1\}$ we deduce from Theorem 5.2.9 that

$$Z_p^+(f,z) = \sum_{\varepsilon \in \underline{p}} \sum_{q} \sum_{\eta \in \underline{q}} \delta_k\big((z,0,\ldots,0),\varepsilon,\eta\big)\, Z_\eta^-\big(\mathcal{F}f,(-z-m,0,\ldots,0)\big)$$
$$= \sum_{q \in \{0,\ldots,k+1\}} \widetilde{\delta}_k(z,p,q) \sum_{\eta \in \underline{q}} Z_\eta^-\big(\mathcal{F}f,(-z-m,0,\ldots,0)\big) \ .$$

Therefore we obtain the following functional equation for the Zeta function related to the G–orbits.

Theorem 5.7.5 (Type I, functional equation for (G, V^+))**.**
For $f \in \mathcal{S}(V^+)$, $p,q \in \{0,\ldots,k+1\}$ the integrals defining $Z_p^+(f,z)$ and $Z_q^-(\mathcal{F}f,z)$ have meromorphic continuation in $\mathbb{C}$ and satisfy the following functional equation

$$Z_p^+(f,z) = \sum_{q \in \{0,\ldots,k+1\}} \widetilde{\delta}_k(z,p,q) Z_q^-(\mathcal{F}f,-z-m) \ .$$

Remark 5.7.6. Using Theorem 5.4.1 we also obtain with the same hypothesis

$$Z_q^-(\mathcal{F}f,z-m) = \sum_{p \in \{0,\ldots,k+1\}} \widetilde{\nu}_k(z-m,q,p) Z_p^+(f,-z) \ ,$$

where

$$\widetilde{\nu}_k(z-m,q,p) = \sum_{\eta \in \underline{q}} \delta_k((z-m,0,\ldots,0),\widetilde{\eta},-\widetilde{\varepsilon}) \quad \text{for } \varepsilon \in \underline{p} \ .$$

This is exactly Theorem 1 of [S–F]–1984. Using the fact that $\varepsilon \in \underline{p} \Longleftrightarrow \widetilde{\varepsilon} \in \underline{p}$ we may also write

$$\widetilde{\nu}_k(z-m,q,p,) = \sum_{\eta \in \underline{q}} \delta_k((z-m,0,\ldots,0),\eta,-\varepsilon) \quad \text{for } \varepsilon \in \underline{p} \ .$$

More precisely, using the equality $m = 1 + \frac{kd}{2}$, it may be verified that we have

$$\widetilde{\nu}_k(z-m,q,p) = (2\pi)^{-(k+1)z} \times e^{i\frac{\pi}{2}(k+1)z} \times (2\pi)^{\frac{k(k+1)d}{4}} \prod_{j=0}^{k} \Gamma(z - j\frac{d}{2})$$
$$\times \sum_{\eta \in \underline{q}} e^{-i\pi z B_{\varepsilon,\eta}} \times e^{i\frac{\pi}{4} d A_{\varepsilon,\eta}} \ ,$$

where

- ε is any element of $\underline{p}$,
- $B_{\varepsilon,\eta} = \frac{1}{2}\Big((k+1) - \sum_{j=0}^{k} \varepsilon_j \eta_j\Big) = \#\{j \mid \varepsilon_j \eta_j = -1\}$,
- $A_{\varepsilon,\eta} = \sum_{j=1}^{k} \eta_j(\varepsilon_0 + \cdots + \varepsilon_{j-1}) - \sum_{j=0}^{k} \varepsilon_j \eta_j (k-j)$.

This is exactly SATAKE–FARAUT formula.

CHAPTER 6

Functional equation of the Zeta function for Type III

Throughout this chapter we suppose that $\widetilde{\mathfrak{g}}$ is of type III.

There is an unique open G–orbit in V^+ (*resp.* V^-) which will be denoted by Ω^+ (*resp.* Ω^-). Recall also that there is an unique open P–orbit in V^+ (*resp.* V^-) denoted by $\mathcal{O}^+$ (*resp.* $\mathcal{O}^-$).

Let $(X_j)_{j=0,\dots,k}$ be the sequence satisfying condition (D) chosen in section 3.6 and define $I^+ = X_0 + \cdots + X_k$. The sets Ω^+ and Ω^- (*resp.* $\mathcal{O}^+$ and $\mathcal{O}^-$) are the G–orbits (*resp.* P–orbit) of I^+ and $I^- = \theta(I^+)$.

In this chapter we will consider representations of the group M (defined by the LANGLANDS decomposition of P as in section 3.5) with an $(M \cap H)$–invariant vector where H is the centralizer in G of I^+ (and of I^-). The key point here is the fact that $M/(M \cap H)$ is compact (see Theorem 3.5.4).

6.1. $(M \cap H)$–spherical representations of M

Let $(\rho, \mathcal{H}_\rho)$ be a unitary representation of M and $\mathcal{H}_\rho^\infty$ be the Fréchet space of C^∞ vectors in $\mathcal{H}_\rho$. Then its topological anti-dual $\mathcal{H}_\rho^{-\infty}$ is called the space of distribution vectors of ρ. The group M acts naturally both on $\mathcal{H}_\rho^\infty$ and on $\mathcal{H}_\rho^{-\infty}$. The space of $(M \cap H)$–invariant distributions vectors is denoted by $(\mathcal{H}_\rho^{-\infty})^{M\cap H}$.

Of course, if $\mathcal{H}_\rho$ is finite dimensional, then $\mathcal{H}_\rho^\infty = \mathcal{H}_\rho$.

Proposition 6.1.1. *Let $(\rho, \mathcal{H}_\rho)$ be a irreducible unitary representation of M having a nonzero $(M \cap H)$–invariant distribution vector. Then $\dim \mathcal{H}_\rho < +\infty$ and $\mathcal{H}_\rho$ is (equivalent to) a closed subspace of $L^2(M/M \cap H)$ (i.e. $(\rho, \mathcal{H}_\rho)$ is a discrete series representation of $M/M \cap H$).*

Proof. — This Proposition is essentially Lemma 3.5. of [vdB]–1988. For the convenience of the reader and also because we will need later on some details and notations, we nevertheless give the proof.

Recall from the proof of Theorem 3.5.4 that the inclusion $M \cap K \hookrightarrow M$ induces a homeomorphism $\varphi : M \cap K/M \cap K \cap H \longrightarrow M/M \cap H$. This is due to the fact that $M = (M \cap K)\exp(\mathfrak{m} \cap \mathfrak{p})$ where $\mathfrak{m} \cap \mathfrak{p} \subset \mathfrak{h}$. If the invariant measures are normalized appropriately, this gives rise by pull back to an isometric isomorphism

$$\varphi^* : L^2(M/M \cap H) \longrightarrow L^2(M \cap K/M \cap K \cap H) \ .$$

This map φ^* is $M \cap K$-equivariant. As $\exp(\mathfrak{m} \cap \mathfrak{p})$ acts trivially on the space $L^2(M/M \cap H)$, the map φ^* sets up in fact a one-to-one correspondence

between M-invariant subspaces of $L^2(M/M\cap H)$ and $M\cap K$-invariant subspaces of $L^2(M\cap K/M\cap K\cap H)$.

As $M\cap K$ is compact, the irreducible M–invariant closed subspaces of $L^2(M/M\cap H)$ are therefore finite-dimensional.

Let now $(\rho,\mathcal{H}_\rho)$ be an irreducible unitary representation of M, having a nonzero $(M\cap H)$–invariant distribution vector $v_\rho\in(\mathcal{H}_\rho^{-\infty})^{M\cap H}$. Then the map $j:\mathcal{H}_\rho^\infty\longrightarrow C^\infty(M/M\cap H)$ defined for $v\in\mathcal{H}_\rho^\infty$ and $m\in M$ by

$$j(v)(m)=\overline{\langle v_\rho,\rho(m^{-1})v\rangle}$$

is a M–equivariant embedding, which is continuous for the Fréchet topologies.

For any irreducible unitary representation π of M occurring in the space $L^2(M/M\cap H)$ (π is necessarily finite dimensional from above), denote by U_π the isotypic component of π in $L^2(M/M\cap H)$. Again, as $M/M\cap H$ is compact, the space U_π is finite dimensional. Let P_π be the orthogonal projection on U_π in $L^2(M/M\cap H)$. As $L^2(M/M\cap H)$ is the direct sum of the isotypic components we have

$$0\neq j=\sum_\pi(P_\pi\circ j)$$

and therefore there exists π_0 such that $P_{\pi_0}\circ j\neq 0$. Consider

$$P_{\pi_0}\circ j:\mathcal{H}_\rho^\infty\longrightarrow U_{\pi_0}\ .$$

This map is M–equivariant and continuous. Recall now that $(M,\mathcal{H}_\rho)$ is topologically irreducible if and only if $(M,\mathcal{H}_\rho^\infty)$ is topologically irreducible ([Bru]–1956, Prop. 2.6). Therefore the kernel of $P_{\pi_0}\circ j$ must be $\{0\}$ and $P_{\pi_0}\circ j$ is injective. Hence $\dim\mathcal{H}_\rho^\infty<+\infty$. It follows that $\mathcal{H}_\rho$ is also finite dimensional and that $\mathcal{H}_\rho=\mathcal{H}_\rho^\infty$. Moreover j is an M–equivariant embeddind of $\mathcal{H}_\rho$ into $L^2(M/M\cap H)$.

□

Let us now recall some basic facts from the representation theory of the orthogonal group $O(Q)$ where Q is a positive definite quadratic form on $\mathbb{R}^\ell$ (see for example [C]–1983). Denote by S the unit sphere

$$S=\{x\in\mathbb{R}^\ell\mid Q(x)=1\}\ .$$

The group $O(Q)$ acts transitively on S and if $x_0\in S$, the centralizer of x_0 in $O(Q)$ (denoted by $\widetilde{H}$) is isomorphic to an orthogonal group in $\ell-1$ variables.

The Laplacian $Q(\partial)$ is defined to be the differential operator with constant coefficients such that

$$Q(\partial)_y\,\mathrm{e}^{B_Q(x,y)}=Q(x)\,\mathrm{e}^{B_Q(x,y)}$$

where $B_Q(x,y)=\frac{1}{2}(Q(x+y)-Q(x)-Q(y))$ is the scalar product defined by Q. A function f on $\mathbb{R}^\ell$ is said to be *harmonic* if

$$Q(\partial)f=0\ .$$

The homeomorphism $O(Q)/\widetilde{H}\simeq S$ gives rise to an isometry from $L^2(O(Q)/\widetilde{H})$ onto $L^2(S)$ (with appropriate invariant measures on $O(Q)/\widetilde{H}$

and S). The irreducible (necessarily finite dimensional) representations of $O(Q)$ having a $\widetilde{H}$–invariant vector are exactly the so–called spherical harmonics $\mathcal{H}_\ell^t$ $(t \in \mathbb{N})$ where $\mathcal{H}_\ell^t$ is the space of complex valued harmonic polynomials of degree t on $\mathbb{R}^\ell$ on which the group $O(Q)$ acts by left translations. These representations are non equivalent for $t \neq t'$. Moreover

$$L^2(S) = \oplus_{t=0}^{\infty} \mathcal{H}_\ell^t \ ,$$

where the embedding of $\mathcal{H}_\ell^t$ into $L^2(S)$ is given by restriction to S.

Suppose now we are given a family $Q_0, Q_1, \ldots, Q_k$ of positive definite quadratic forms on $\mathbb{R}^{\ell_0}, \mathbb{R}^{\ell_1}, \ldots, \mathbb{R}^{\ell_k}$ respectively. For $j \in \{0, 1, \ldots, k\}$ denote by $\widetilde{H_j}$ the isotropy subgroup in $O(Q_j)$ of a fixed point on the Q_j–sphere S_j. Put $\Gamma = O(Q_0) \times O(Q_1) \times \ldots \times O(Q_k)$ and $\widetilde{\Gamma} = \widetilde{H_0} \times \widetilde{H_1} \times \cdots \times \widetilde{H_k}$. Suppose also we are looking for irreducible representations τ of Γ having a $\widetilde{\Gamma}$–invariant vector. Then elementary representation theory tells us that there exists $(\tau_0, \tau_1, \ldots, \tau_k) \in \mathbb{N}^{k+1}$ such that

$$\tau \simeq \mathcal{H}_{\ell_0}^{\tau_0} \otimes \mathcal{H}_{\ell_1}^{\tau_1} \otimes \cdots \otimes \mathcal{H}_{\ell_k}^{\tau_k} \ .$$

It should be noticed that if $(\tau_0, \ldots, \tau_k) \neq (\tau_0', \ldots, \tau_k')$, then the corresponding representations are non equivalent.

In our context, a scalar product on $\oplus_{j=0}^k \widetilde{\mathfrak{g}}^{\lambda_j}$ is defined by

$$(X, Y) \longmapsto b(X, \theta Y) \quad \text{ for } X, Y \in \oplus_{j=0}^k \widetilde{\mathfrak{g}}^{\lambda_j} \ .$$

For this scalar product the spaces $\widetilde{\mathfrak{g}}^{\lambda_j}$ are mutually orthogonal. Denote by Q_j the corresponding quadratic form on $\widetilde{\mathfrak{g}}^{\lambda_j}$ *i.e.* $Q_j(X) = b(X, \theta(X))$ for $X \in \widetilde{\mathfrak{g}}^{\lambda_j}$. Denote by S_j the corresponding unit sphere in $\widetilde{\mathfrak{g}}^{\lambda_j}$ and by $O(Q_j)$ the corresponding orthogonal group of $\widetilde{\mathfrak{g}}^{\lambda_j}$. Denote by $\widetilde{\Gamma}$ the isotropy subgroup of I^+ inside $\Gamma = O(Q_0) \times O(Q_1) \times \cdots \times O(Q_k)$. As X_j belongs to S_j by Lemma 1.10.2, $\widetilde{\Gamma}$ is the product of the isotropy subgroups of X_j in $O(Q_j)$. According to the preceding remarks the irreducible representations of Γ having a $\widetilde{\Gamma}$–invariant vector are of the form $\mathcal{H}_\ell^{\tau_0} \otimes \mathcal{H}_\ell^{\tau_1} \otimes \cdots \otimes \mathcal{H}_\ell^{\tau_k}$, where $\ell = \dim \widetilde{\mathfrak{g}}^{\lambda_j}$.

Notation 6.1.2. *For $\tau = (\tau_0, \tau_1, \ldots, \tau_k) \in \mathbb{N}^{k+1}$ we set*

$$\mathcal{H}_\tau = \mathcal{H}_\ell^{\tau_0} \otimes \mathcal{H}_\ell^{\tau_1} \otimes \cdots \otimes \mathcal{H}_\ell^{\tau_k} \ ,$$

where $\mathcal{H}_\ell^{\tau_j}$ is the space of $Q_j(\partial)$–harmonic polynomials of degree τ_j on $\widetilde{\mathfrak{g}}^{\lambda_j}$.

As the quadratic forms Q_j are invariant under M (see the proof of Theorem 3.5.4), the group M acts on the space $\mathcal{H}_\tau$.

From now on, we will also denote by τ the corresponding representation of M on $\mathcal{H}_\tau$.

As $M \cap H$ centralizes I^+, this representation has a $(M \cap H)$–invariant vector.

Theorem 6.1.3 (Type III)**.** *The representation $(\tau, \mathcal{H}_\tau)$ of M is irreducible and conversely any irreducible unitary representation of M having*

a $(M \cap H)$–invariant distribution vector is equivalent to $(\tau, \mathcal{H}_\tau)$ for some parameter $\tau \in \mathbb{N}^{k+1}$. Moreover we have the multiplicity free decomposition :

$$L^2(M/M \cap H) = \oplus_{\tau \in \mathbb{N}^{k+1}} \mathcal{H}_\tau.$$

Proof. —Let L_j be the analytic subgroup of G with Lie algebra $\mathfrak{l}_j = [\widetilde{\mathfrak{g}}^{-\lambda_j}, \widetilde{\mathfrak{g}}^{\lambda_j}]$. As L_j centralizes $\mathfrak{a}^0$, we have $L_j \subset MA$. On the other hand the representation of L_j on $\widetilde{\mathfrak{g}}^{\lambda_j}$ gives rise to an irreducible prehomogeneous vector space of commutative parabolic type, whose relative invariant is the quadratic form Q_j. From the classification of real prehomogeneous vector spaces of parabolic type (see [Ru]– 86), we know that $\mathfrak{l}_j \supset \mathfrak{o}(Q_j) \times$ center, where $\mathfrak{o}(Q_j)$ is the Lie algebra of $O(Q_j)$. Therefore the Lie algebra of $L_j \cap K$ contains $\mathfrak{o}(Q_j)$. This implies that the representation of $L_j \cap K$ on $\mathcal{H}_\ell^{\tau_j}$ is irreducible for all $\tau_j \in \mathbb{N}$. As the subgroups L_j commute, the product $\prod_{j=0}^k (L_j \cap K)$ is contained in M, and therefore the infinitesimal action of M on $\mathcal{H}_\tau$ contains the infinitesimal action of $\prod_{j=0}^k \mathfrak{o}(Q_j)$. Hence $(\tau, \mathcal{H}_\tau)$ is an irreducible representation of M. Moreover, if $\tau \neq \tau'$, the representations $(\tau, \mathcal{H}_\tau)$ and $(\tau', \mathcal{H}_{\tau'})$ are non equivalent.

Conversely let ρ be an irreducible unitary representation of M having a $(M \cap H)$–invariant distribution vector. From Proposition 6.1.1 we know that ρ is a finite dimensional subrepresentation of

$$L^2(M/M \cap H) \simeq L^2(\prod_{j=0}^k S_j) \simeq L^2(S_0) \otimes L^2(S_1) \otimes \cdots \otimes L^2(S_k) \ .$$

As $L^2(S_j) = \oplus_{t=0}^\infty \mathcal{H}_\ell^t$, we necessarily have $\rho = \tau$ for some $\tau \in \mathbb{N}^{k+1}$. The multiplicity free decomposition is then also clear.

□

Corollary 6.1.4. *Let $\mathcal{H}_\tau^{M \cap H}$ be the subspace of $(M \cap H)$–invariant vectors in $\mathcal{H}_\tau$. Then $\mathcal{H}_\tau^{M \cap H} = (\mathcal{H}_\tau^{-\infty})^{M \cap H}$ and $\dim \mathcal{H}_\tau^{M \cap H} = 1$.*

Proof. —Since τ is a finite–dimensional representation of M, we have $\mathcal{H}_\tau = \mathcal{H}_\tau^\infty = \mathcal{H}_\tau^{-\infty}$ and , therefore, $\mathcal{H}_\tau^{M \cap H} = (\mathcal{H}_\tau^{-\infty})^{M \cap H}$. As the regular representation of M on $L^2(M/M \cap H)$ is the representation induced from the trivial representation of $M \cap H$, and as τ occurs wilt multiplicity one in $L^2(M/M \cap H)$, the result follows from the Frobenius reciprocity theorem.

□

Definition 6.1.5. *The irreducible unitary representations $(\tau, \mathcal{H}_\tau)$ defined as before will be called $(M \cap H)$–spherical representations of M.*

The contragredient representation $(\check{\tau}, \mathcal{H}_{\check{\tau}})$, where $\mathcal{H}_{\check{\tau}}$ is the dual space of $\mathcal{H}_\tau$, is also an $(M \cap H)$–spherical representations of M. It has a very natural $(M \cap H)$–invariant vector, which is the evaluation map $ev_\tau \in \mathcal{H}_{\check{\tau}}$ defined by

$$\begin{aligned} ev_\tau \ &: \mathcal{H}_\tau \longrightarrow \mathbb{C} \\ &R \longmapsto R(I^+) \ . \end{aligned}$$

This map is well defined since each element of $\mathcal{H}_\tau$ may be considered as a polynomial function on the space $\oplus_{j=0}^k \widetilde{\mathfrak{g}}^{\lambda_j}$, homogeneous of degree τ_j on $\widetilde{\mathfrak{g}}^{\lambda_j}$.

Definition 6.1.6. *We will denote by v_τ the $(M \cap H)$–invariant vector of $\mathcal{H}_\tau$ defined by*

$$(R \mid v_\tau)_\tau = R(I^+) \quad \text{for all } R \in \mathcal{H}_\tau \ ,$$

where $(\ \mid\ )_\tau$ stands for the Hilbert product in $\mathcal{H}_\tau$.

6.2. Equivariant mappings from $\mathcal{O}^{\pm}$ into $\mathcal{H}_\tau$

Theorem 6.2.1. *Let $(\tau, \mathcal{H}_\tau)$ be a $(M \cap H)$–spherical representation of M and v_τ be the $(M \cap H)$–invariant vector defined above. Then there exists a unique map $b_\tau^+ : \mathcal{O}^+ \longrightarrow \mathcal{H}_\tau$ (resp. $b_\tau^- : \mathcal{O}^- \longrightarrow \mathcal{H}_\tau$) such that for all $X \in \mathcal{O}^+$, (resp. $Y \in \mathcal{O}^-$) $m \in M$, $a \in A$, $n \in N$ we respectively have*

$$b_\tau^+(manX) = \tau(m) b_\tau^+(X) \quad \text{and} \quad b_\tau^+(I^+) = v_\tau \ ,$$
$$b_\tau^-(manY) = \tau(m) b_\tau^-(Y) \quad \text{and} \quad b_\tau^-(I^-) = v_\tau \ .$$

Proof. — As AN is a normal subgroup of $MAN = P$ we first extend τ into a representation of P in $\mathcal{H}_\tau$ by setting:

$$\tau(man) = \tau(m) \quad \text{for all } m \in M, a \in A, n \in N \ .$$

The mapping $P \ni p \mapsto \tau(p) v_\tau \in \mathcal{H}_\tau$ is right invariant under $P \cap H = M \cap H$. Therefore the mapping b_τ^+ defined on $\mathcal{O}^+ = P.I^+$ by

$$b_\tau^+(pI^+) = \tau(p) v_\tau \text{ for } p \in P \ ,$$

is well defined and satisfies the relations given in the Theorem..

Suppose now that a map $b_\tau^+ : \mathcal{O}^+ \longrightarrow \mathcal{H}_\tau$ is given which satisfies $b_\tau^+(manX) = \tau(m) b_\tau^+(X)$ for $m \in M, a \in A, n \in N$ and $X \in \mathcal{O}^+$. Then it is clear that $b_\tau^+(I^+)$ is an $M \cap H$-invariant vector in $\mathcal{H}_\tau$. From Corollary 6.1.4 we know that there exists a constant c such that $b_\tau^+(I^+) = c v_\tau$. If moreover the map b_τ^+ is required to verify $b_\tau^+(I^+) = v_\tau$, the uniqueness therefore follows.

The proof for b_τ^- is similar.

□

Again let τ be a $(M \cap H)$–spherical representation of M which is extended into a representation of P as above. Then the same proof as Theorem 6.2.1 leads to the following Proposition.

Proposition 6.2.2. *Let $s \in \mathbb{C}^{k+1}$, let χ^s be the character of the P–relative invariant Δ^s and $(\chi^-)^s$ be the character of the P–relative invariant ∇^s. Define, for $X \in \mathcal{O}^+$ and $Y \in \mathcal{O}^-$,*

$$b_{\tau,s}^+(X) = \Delta^s(X) b_\tau^+(X) \quad \text{and} \quad b_{\tau,s}^-(Y) = \nabla^s(Y) b_\tau^-(Y).$$

Then $b_{\tau,s}^+$ (resp. $b_{\tau,s}^-$) is the unique map from $\mathcal{O}^+$ (resp. $\mathcal{O}^-$) into $\mathcal{H}_\tau$ which satisfies for $p \in P$ and $X \in \mathcal{O}^+$ (resp. $Y \in \mathcal{O}^-$)

$$b_{\tau,s}^+(pX) = (\chi)^s(p) \tau(p) b_{\tau,s}^+(X) \quad \text{and} \quad b_{\tau,s}^+(I^+) = v_\tau \ ,$$
$$b_{\tau,s}^-(pY) = (\chi^-)^s(p) \tau(p) b_{\tau,s}^-(Y) \quad \text{and} \quad b_{\tau,s}^-(I^-) = v_\tau \ .$$

The next results give some information about the behaviour of b_τ^+ near the boundary of $\mathcal{O}^+$ and of b_τ^- near the boundary of $\mathcal{O}^-$.

Lemma 6.2.3. *Let R be an element of $\mathcal{H}_\tau$.*
Then for $X \in (\widetilde{\mathfrak{g}}^{\lambda_0} \oplus \widetilde{\mathfrak{g}}^{\lambda_1} \oplus \cdots \oplus \widetilde{\mathfrak{g}}^{\lambda_k}) \cap \mathcal{O}^+$ we have

$$(R \,|\, b_\tau^+(X))_\tau = \frac{R(X)}{\Delta_0(X)^{\frac{\tau_0}{2}} \Delta_1(X)^{\frac{\tau_1-\tau_0}{2}} \cdots \Delta_k(X)^{\frac{\tau_k-\tau_{k-1}}{2}}},$$

and for $Y \in (\widetilde{\mathfrak{g}}^{-\lambda_0} \oplus \widetilde{\mathfrak{g}}^{-\lambda_1} \oplus \cdots \oplus \widetilde{\mathfrak{g}}^{-\lambda_k}) \cap \mathcal{O}^-$ we have

$$(R \,|\, b_\tau^-(Y))_\tau = \frac{R(\theta(Y))}{\nabla_0(Y)^{\frac{\tau_k}{2}} \nabla_1(Y)^{\frac{\tau_{k-1}-\tau_k}{2}} \cdots \nabla_k(Y)^{\frac{\tau_0-\tau_1}{2}}},$$

It may be noticed that, as $\Delta_j(X) \geq 0$ in Type III case, the square root $\Delta_j(X)^{\frac{1}{2}}$ is well defined for $j \in \{0, \ldots, k\}$.

Proof. — For an element X of $(\widetilde{\mathfrak{g}}^{\lambda_0} \oplus \widetilde{\mathfrak{g}}^{\lambda_1} \oplus \cdots \oplus \widetilde{\mathfrak{g}}^{\lambda_k}) \cap \mathcal{O}^+$, there exist $m \in M$ and $a \in A$ such that $X = maI^+$. Hence it follows from Definition 6.1.6 that

$$(R \,|\, b_\tau^+(X))_\tau = (R \,|\, \tau(m)v_\tau)_\tau = \big(\tau(m)^{-1}R\big)(I^+) = R(mI^+) \ .$$

Moreover, we have $mI^+ = Y_0 + Y_1 + \cdots + Y_k$ with $Y_j \in \widetilde{\mathfrak{g}}^{\lambda_j}$. Then for $a = \exp(H)$ with $H \in \mathfrak{a}$ we get

$$maI^+ = amI^+ = a.Y_0 + a.Y_1 + \cdots + a.Y_k = a^{\lambda_0}Y_0 + a^{\lambda_1}Y_1 + \cdots + a^{\lambda_k}Y_k \ ,$$

where $a^{\lambda_j} = e^{\lambda_j(H)}$. As R is homogeneous of degree τ_j on $\widetilde{\mathfrak{g}}^{\lambda_j}$, we obtain

$$R(maI^+) = a^{\tau_0\lambda_0 + \cdots + \tau_k\lambda_k} R(mI^+).$$

On the other hand, in Type III case, Theorem 3.6.1 implies that we have for $u = (u_0, \ldots, u_k) \in \mathbb{Z}^{k+1}$

$$\Delta(maI^+)^{\frac{u}{2}} = \prod_{j=0}^{k} a^{\kappa\frac{u_j}{2}(\lambda_j + \cdots + \lambda_k)} = a^{u_0\lambda_0 + (u_0+u_1)\lambda_1 + \cdots + (u_0+\cdots+u_k)\lambda_k} \ .$$

Hence, for $u = (\tau_0, \tau_1 - \tau_0, \ldots, \tau_k - \tau_{k-1})$, we get

$$R(maI^+) = \Delta(maI^+)^{\frac{u}{2}} R(mI^+) \ ,$$

and this yields the first result.

Let $Y = maI^-$ be an element of $(\widetilde{\mathfrak{g}}^{-\lambda_0} \oplus \widetilde{\mathfrak{g}}^{-\lambda_1} \oplus \cdots \oplus \widetilde{\mathfrak{g}}^{-\lambda_k}) \cap \mathcal{O}^-$. We have as above

$$(R \,|\, b_\tau^-(Y))_\tau = R(mI^+) \ .$$

From the proof of Theorem 3.5.4 we may write $m = m_K m_H$ with $m_K \in M \cap K$ and $m_H \in M \cap H$. As m_H centralizes I^+ and I^- we have

$$\theta(maI^-) = \theta(a)mI^+ = a^{-1}mI^+ \ .$$

Then we obtain

$$R(\theta(Y)) = R(a^{-1}mI^+) = a^{-\tau_0\lambda_0 - \cdots - \tau_k\lambda_k} R(mI^+) \ .$$

On the other hand, using Theorem 3.8.2, we have for $v = (v_0, \ldots, v_k) \in \mathbb{Z}^{k+1}$

$$\nabla(maI^-)^{\frac{v}{2}} = a^{-(v_0+\cdots+v_k)\lambda_0 - (v_0+\cdots+v_{k-1})\lambda_1 - \cdots - v_0\lambda_k} \ .$$

Hence, for $v = (\tau_k, \tau_{k-1} - \tau_k, \ldots, \tau_0 - \tau_1)$, we get

$$R(\theta(maI^-)) = \nabla(maI^-)^{\frac{v}{2}} R(mI^+) \ ,$$

and this yields the second result.

□

Let us denote by $\widetilde{\tau}$ the $(M \cap H)$–spherical representation associated to the parameter $\widetilde{\tau} = (\tau_k, \tau_{k-1}, \dots, \tau_0)$. The involution $\gamma \circ \theta = \theta \circ \gamma$ (where γ is the involution defined in Theorem 3.7.1) interchanges $\widetilde{g}^{\lambda_j}$ and $\widetilde{g}^{\lambda_{k-j}}$. Therefore, if R is an element of $\mathcal{H}_\tau$, the polynomial $R \circ \gamma \circ \theta$ belongs to $\mathcal{H}_{\widetilde{\tau}}$ and we deduce from Lemma 6.2.3 the following result.

Lemma 6.2.4. *Let Y be an element of $(\widetilde{\mathfrak{g}}^{-\lambda_0} \oplus \widetilde{\mathfrak{g}}^{-\lambda_1} \oplus \cdots \oplus \widetilde{\mathfrak{g}}^{-\lambda_k}) \cap \mathcal{O}^-$. Then we have for $R \in \mathcal{H}_\tau$*

$$(R \,|\, b_\tau^-(Y))_\tau = (\widetilde{R} \,|\, b_{\widetilde{\tau}}^+(\gamma Y))_{\widetilde{\tau}} \ ,$$

where $\widetilde{R} = R \circ \gamma \circ \theta$.

Proposition 6.2.5. *There exist polynomial maps $U^+ : V^+ \to \mathcal{H}_\tau$ and $U^- : V^- \to \mathcal{H}_\tau$, such that for $X \in \mathcal{O}^+$ and $Y \in \mathcal{O}^-$ we have*

$$b_\tau^+(X) = \frac{U^+(X)}{\Delta(X)^{\frac{\delta}{2}}} \quad \text{and} \quad b_\tau^-(Y) = \frac{U^-(Y)}{\nabla(Y)^{\frac{\widetilde{\delta}}{2}}} \ ,$$

where $\delta = (\delta_0, \delta_1, \dots, \delta_k)$ is defined by

$$\delta_0 = \tau_0, \ \ \delta_1 = \tau_1 + \tau_0, \dots, \delta_k = \tau_k + \tau_{k-1} \ ,$$

and $\widetilde{\delta} = (\widetilde{\delta}_0, \widetilde{\delta}_1, \dots, \widetilde{\delta}_k)$ is defined by

$$\widetilde{\delta}_0 = \tau_k, \ \ \widetilde{\delta}_1 = \tau_{k-1} + \tau_k, \dots, \widetilde{\delta}_k = \tau_0 + \tau_1 \ .$$

Proof. — Recall from Theorem 3.10.4 that every $X \in \mathcal{O}^+$ is N-conjugate to a unique element $f(X) \in \widetilde{\mathfrak{g}}^{\lambda_0} \oplus \cdots \oplus \widetilde{\mathfrak{g}}^{\lambda_k}$. More precisely there exists $n \in N$ such that

$$nX = f(X) = f_0(X) + f_1(X) + \cdots + f_k(X),$$

where $f_j(X) \in \widetilde{\mathfrak{g}}^{\lambda_j}$. The map $X \mapsto f_k(X)$ is the orthogonal projection of X on $\widetilde{\mathfrak{g}}^{\lambda_k}$, hence a polynomial map from V^+ into $\widetilde{\mathfrak{g}}^{\lambda_k}$ and for $j = 0, \dots, k-1$, we have $f_j(X) = \dfrac{P_j(X)}{\Delta_{j+1}(X)}$, where P_j is a polynomial map from V^+ into $\widetilde{\mathfrak{g}}^{\lambda_j}$.

As $b_\tau^+(X) = b_\tau^+(f(X))$, it follows from the previous Lemma 6.2.3 that for $R \in \mathcal{H}_\tau$ we have

$$(R \mid b_\tau^+(X))_\tau = \frac{R(f(X))}{\Delta(f(X))^{\frac{u}{2}}} = \frac{R(f(X))}{\Delta(X)^{\frac{u}{2}}} \ ,$$

where $u = (\tau_0, \tau_1 - \tau_0, \dots, \tau_k - \tau_{k-1})$. As R is homogeneous of degree τ_j on $\widetilde{\mathfrak{g}}^{\lambda_j}$, we obtain

$$\begin{aligned} R \circ f &= R\Big(\frac{P_0}{\Delta_1} + \cdots + \frac{P_{k-1}}{\Delta_k} + f_k\Big) \\ &= \frac{1}{\Delta_1^{\tau_0}\Delta_2^{\tau_1} \dots \Delta_k^{\tau_{k-1}}} R(P_0 + P_1 + \cdots + P_{k-1} + f_k) \ . \end{aligned}$$

Setting $\mathbf{R}(X) = R\big(P_0(X) + P_1(X) + \cdots + P_{k-1}(X) + f_k(X)\big)$, we get a polynomial function $\mathbf{R}$ on V^+ such that

$$\begin{aligned}(R \mid b_\tau^+(X))_\tau &= \frac{\mathbf{R}(X)}{\Delta(X)^{\frac{u}{2}}\Delta_1(X)^{\tau_0}\dots\Delta_k(X)^{\tau_{k-1}}} \\ &= \frac{\mathbf{R}(X)}{\Delta_0(X)^{\frac{\tau_0}{2}}\Delta_1(X)^{\frac{\tau_0+\tau_1}{2}}\dots\Delta_k(X)^{\frac{\tau_{k-1}+\tau_k}{2}}} \\ &= \frac{\mathbf{R}(X)}{\Delta(X)^{\frac{\delta}{2}}} \ .\end{aligned}$$

Taking now an orthonormal basis R_i of $\mathcal{H}_\tau$, we have

$$b_\tau^+(X) = \sum_i (b_\tau^+(X) \mid R_i)_\tau R_i = \sum_i \frac{\overline{\mathbf{R}_i}(X)}{\Delta(X)^{\frac{\delta}{2}}} R_i \ .$$

By setting

$$U^+(X) = \sum_i \overline{\mathbf{R}_i}(X)\, R_i,$$

we obtain the first assertion.

In order to prove the second assertion, we use Lemma 6.2.4. For $Y \in \mathcal{O}^-$ we have $\gamma Y \in \mathcal{O}^+$ and $\Delta_j(\gamma Y) = \nabla_j(Y)$ for $j \in \{0, \dots, k\}$. Therefore we get

$$(R \mid b_\tau^-(Y))_\tau = (\widetilde{R} \mid b_{\widetilde{\tau}}^+(\gamma Y))_{\widetilde{\tau}} = \frac{\widetilde{\mathbf{R}}(\gamma Y)}{\Delta(\gamma Y)^{\frac{\widetilde{\delta}}{2}}} = \frac{\widetilde{\mathbf{R}}(\gamma Y)}{\nabla(Y)^{\frac{\widetilde{\delta}}{2}}} \ ,$$

where $\widetilde{\delta} = (\tau_k, \tau_{k-1} + \tau_k, \dots, \tau_0 + \tau_1)$ and $\widetilde{\mathbf{R}}$ is the polynomial function on V^+ associated to $\widetilde{R} = R \circ \gamma \circ \theta$ by setting

$$\widetilde{\mathbf{R}}(X) = \widetilde{R}(P_0(X) + \cdots + P_{k-1}(X) + f_k(X)) \ .$$

Taking again an orthonormal basis R_i of $\mathcal{H}_\tau$ we obtain

$$b_\tau^-(Y) = \sum_i (b_\tau^-(Y) \mid R_i)_\tau R_i = \sum_i \frac{\overline{\widetilde{\mathbf{R}}_i}(\gamma Y)}{\nabla(Y)^{\frac{\widetilde{\delta}}{2}}} R_i \ .$$

By setting

$$U^-(Y) = \sum_i \overline{\widetilde{\mathbf{R}}_i}(\gamma Y)\, R_i,$$

we obtain the second assertion.

□

6.3. Definition of the local Zeta functions

Recall from Theorem 3.6.3 and Theorem 3.8.6 that (P, V^+) and (P, V^-) are prehomogeneous vector spaces with a single open orbit given respectively by

$$\begin{aligned}\mathcal{O}^+ = P.I^+ &= \{X \in V^+ \mid \Delta_j(X) > 0 \ \text{ for } j = 0, 1, \dots, k\} \ , \\ \mathcal{O}^- = P.I^- &= \{Y \in V^- \mid \nabla_j(Y) > 0 \ \text{for } j = 0, 1, \dots, k\} \ .\end{aligned}$$

The fundamental relative invariants of the prehomogeneous space (P, V^+) are the polynomials Δ_j $(j = 0, 1, \ldots, k)$ and the fundamental relative invariants of the prehomogeneous space (P, V^-) are the polynomials ∇_j $(j = 0, 1, \ldots, k)$.

The following definition of local Zeta functions of (P, V^+) and (P, V^-) are not the usual one because they involve b_τ^+ and b_τ^- and are $\mathcal{H}_\tau$–valued.

Definition 6.3.1. *For $s = (s_0, s_1, \ldots, s_k) \in \mathbb{C}^{k+1}$, τ a $(M \cap H)$–spherical representation of M, $f \in \mathcal{S}(V^+)$ and $h \in \mathcal{S}(V^-)$, the local Zeta functions for Type III are defined by*

$$Z^+(f, s, \tau) = \int_{\mathcal{O}^+} f(X)\Delta^s(X)b_\tau^+(X)\, dX \ ,$$

$$Z^-(h, s, \tau) = \int_{\mathcal{O}^-} h(Y)\nabla^s(Y)b_\tau^-(Y)\, dY \ .$$

Recall that all the polynomials Δ_j and ∇_j take only non negative values. Therefore there is no absolute value needed in the definition of Δ^s and ∇^s (see Definition 3.8.8).

Remark 6.3.2. Using Proposition 6.2.5, it is easy to see that there exists $u_0 \in \mathbb{R}$ such that the integrals defining the Zeta functions converge for $\operatorname{Re} s = \max \operatorname{Re} s_j > u_0$ and define holomorphic functions on this open subset of $\mathbb{C}^{k+1}$. We know also from [B-G]–1969 that these Zeta functions have a meromorphic continuation on $\mathbb{C}^{k+1}$.

6.4. Existence of a functional equation for (P, V^+)

Recall from chapter 4 (Notation 4.1.4) that

$$m = \frac{\dim V^+}{\kappa(k+1)} = \frac{\dim V^+}{2(k+1)} = \frac{\ell}{2} + \frac{kd}{4} \ .$$

The involution t on $\mathbb{C}^{k+1}$ is still defined by

$$t(s) = (-s_0 - s_1 - \cdots - s_k, s_k, s_{k-1}, \ldots, s_1)$$

and we also make the convention that in $\mathbb{C}^{k+1}$

$$(z_0, z_1, \ldots, z_k) - m = (z_0 - m, z_1, \ldots, z_k) \ .$$

For $f \in \mathcal{S}(V^+)$ (*resp.* $h \in \mathcal{S}(V^-)$ the Fourier transform $\mathcal{F}f$ (*resp.* $\overline{\mathcal{F}}h$) is defined as in Definition 4.1.3.

Let $u_0 \in \mathbb{R}$ be given such that the integrals defining $Z^+(f, s, \tau)$ and $Z^-(h, s, \tau)$ converge for $\operatorname{Re}(s) > u_0$ and any $f \in \mathcal{S}(V^+)$ and $h \in \mathcal{S}(V^-)$. Then, for such a parameter s, the mappings $f \longmapsto Z^+(f, s, \tau)$ and $h \longmapsto Z^-(h, s, \tau)$ define tempered $\mathcal{H}_\tau$–valued distributions on V^+ and V^-.

If now $f \in \mathcal{C}_c^\infty(\mathcal{O}^+)$, then $Z^+(f, s, \tau)$ is an entire function on $\mathbb{C}^{k+1}$ and $f \longmapsto Z^+(f, s, \tau)$ is a distribution on $\mathcal{O}^+$ for any s. As in Type I and II cases, we will first prove the existence of the functional equation for $f \in \mathcal{C}_c^\infty(\mathcal{O}^+)$.

Define for $p \in P$ and φ any function on V^+

$$\varphi_p(X) = \varphi(p^{-1}X) \quad (X \in V^+) \ .$$

Recall that $(M \cap H)$–spherical representations of M in $\mathcal{H}_\tau$ are extented to representations of P which are trivial on AN.

Lemma 6.4.1. *Let τ be a $(M \cap H)$–spherical representation of M. Then for any $p \in P$ and $f \in \mathcal{C}_c^\infty(\mathcal{O}^+)$ we have*

$$Z^+(f_p, s, \tau) = \chi^{s+m}(p)\tau(p)Z^+(f, s, \tau) \ . \tag{6-4-1}$$

Conversely if T is a H_τ–valued distribution on $\mathcal{O}^+$ which satisfies

$$\forall p \in P, \forall f \in \mathcal{C}_c^\infty(\mathcal{O}^+), \quad \langle T, f_p\rangle = \chi^{s+m}(p)\tau(p)\langle T, f\rangle \ ,$$

then T is a multiple of $f \longmapsto Z^+(f, s, \tau)$.

Recall that $\chi^s = \chi_0^{s_0} \cdots \chi_k^{s_k}$ is the character of the P–relative invariant Δ^s.

Proof. — As the characters χ_j of Δ_j are positive on P, an easy computation using Lemma 4.1.5 and Proposition 6.2.1 implies (6-4-1).

Let now T be a H_τ–valued distribution on $\mathcal{O}^+$ which transforms under P as in (6-4-1). Let $(\check{\tau}, \mathcal{H}_{\check{\tau}})$ be the contragredient representation of $(\tau, \mathcal{H}_\tau)$. We transform first T into a so-called $\mathcal{H}_{\check{\tau}}$–distribution (see [Bru]–1956 p. 105), that is a continuous linear form $\widetilde{T}$ on $\mathcal{C}_c^\infty(\mathcal{O}^+, \mathcal{H}_{\check{\tau}}) = \mathcal{C}_c^\infty(\mathcal{O}^+) \otimes \mathcal{H}_{\check{\tau}}$, by setting

$$\forall Q \in \mathcal{H}_\tau, \ \forall f \in \mathcal{C}_c^\infty(\mathcal{O}^+), \quad \langle \widetilde{T}, f \otimes Q\rangle = \langle\langle T, f\rangle, Q\rangle.$$

The correspondence between $\mathcal{H}_\tau$–valued distributions and $\mathcal{H}_{\check{\tau}}$–distributions is one to one (see [Bru]–1956, Prop. 1.1.b. p. 105).

If $\varphi \in \mathcal{C}_c^\infty(\mathcal{O}^+, \mathcal{H}_{\check{\tau}})$, we may define $\check{\tau}(p)\varphi$ for $p \in P$ by

$$(\check{\tau}(p)\varphi)(X) = \check{\tau}(p)\varphi(X) \quad (X \in \mathcal{O}^+) \ .$$

Then we have

$$\forall p \in P, \ \langle \widetilde{T}, \varphi_p\rangle = \langle \widetilde{T}, \chi^{s+m}(p)\check{\tau}(p)^{-1}\varphi\rangle \ .$$

This means that $\widetilde{T}$ is a $\mathcal{H}_{\check{\tau}}$–distribution which is quasi-invariant under P with multiplier $\chi^{-(s+m)}\check{\tau}$ (see [Bru]–1959, Definition 3.3. p. 123). By Bruhat's Theorem 3.1 ([Bru]–1959 p. 124), there exists a unique vector $v \in \mathcal{H}_\tau$ such that

$$\chi(h)^{s+m}\tau(h)v = v \quad \text{for } h \in P \cap H \ ,$$

$$\text{and } \langle \widetilde{T}, \varphi\rangle = \int_{P/P\cap H} \langle \chi(p)^{s+m}\tau(p)v, \varphi(\dot{p})\rangle \, d\dot{p} \quad \text{for } \varphi \in \mathcal{C}_c^\infty(\mathcal{O}^+, \mathcal{H}_{\check{\tau}}) \ .$$

Here $d\dot{p}$ denotes an invariant measure on $P/P \cap H$. Using the identification of $P/P \cap H$ with $\mathcal{O}^+$, it follows from Lemma 4.1.5 that this measure is given by $\Delta_0(X)^{-m}dX$ which implies that $\chi_0(p)^m d\dot{p}$ corresponds to dX. On the other hand χ^s is trivial on $P \cap H$ hence v is $(P \cap H)$–invariant which implies, by Corollary 6.1.4, that v is a scalar multiple of v_τ and $\tau(p)v$ a scalar multiple of the function $p \mapsto b_\tau^+(pI^+)$. Therefore there exists $c \in \mathbb{C}$ such that

$$\langle \widetilde{T}, \varphi\rangle = c \int_{\mathcal{O}^+} \Delta(X)^s \langle b_\tau^+(X), \varphi(X)\rangle \, dX \ .$$

It follows that for $f \in \mathcal{C}_c^\infty(\mathcal{O}^+)$ we obtain

$$\langle T, f\rangle = c \, Z^+(f, s, \tau) \ .$$

□

Proposition 6.4.2. *Let τ be a $(M \cap H)$–spherical representation of M and $s \in \mathbb{C}^{k+1}$. There exists a meromorphic function $\nu(s,\tau)$ such that for any $f \in \mathcal{C}_c^\infty(\mathcal{O}^+)$ we have*

$$Z^-(\mathcal{F}f, s, \tau) = \nu(s,\tau) Z^+(f, t(s) - m, \tau)$$

(equality of meromorphic functions).

Proof. —It is enough to prove the Proposition on an open subset of $\mathbb{C}^{k+1}$ where both $Z^+(f, t(s) - m, \tau)$ and $Z^-(\mathcal{F}f, s, \tau)$ are holomorphic. As $f \in \mathcal{C}_c^\infty(\mathcal{O}^+)$ this is the case for $\mathrm{Re}(s) > u_0$. Moreover, in this case both $Z^+(f, t(s) - m, \tau)$ and $Z^-(\mathcal{F}f, s, \tau)$ are defined by integrals. In order to use Lemma 6.4.1 let us determine how $Z^-(\mathcal{F}f, s, \tau)$ transforms under the natural action of P on f.

Lemma 4.1.5 implies that

$$\mathcal{F}(f_p) = \chi_0^m(p)(\mathcal{F}f)_p \ .$$

An easy computation using again Lemma 4.1.5 and the relations

$$\chi_0^- = \frac{1}{\chi_0}, \quad \chi_j^- = \frac{\chi_{k+1-j}}{\chi_0} \ (j = 1, \ldots, k) \ ,$$

which were proved in Corollary 3.8.5 imply that for $p \in P$ we have:

$$Z^-(\mathcal{F}(f_p), s, \tau) = \chi^{t(s)}(p)\tau(p) Z^-(\mathcal{F}(f), s, \tau) \ . \tag{6-4-2}$$

It follows from (6-4-1) and (6-4-2) that $Z^-(\mathcal{F}(f), s, \tau)$ transforms under P as $Z^+(f, t(s) - m, \tau)$ and Lemma 6.4.1 yields the result.

□

Proposition 6.4.3. *Let τ be a $M \cap H$-spherical representation of M. Let $||\ ||$ be a fixed norm on $\mathcal{H}_\tau$. For any $f \in \mathcal{S}(V^+)$ and any $h \in \mathcal{S}(V^-)$ the analytic functions $s \longmapsto Z^+(f, s, \tau)$ and $s \longmapsto Z^-(h, s, \tau)$, defined for $\mathrm{Re}(s) > u_0$, have a meromorphic continuation on an open subset $\widetilde{U} \subset \mathbb{C}^{k+1}$ which contains the complementary set of a countable union of hyperplanes. Moreover if K is a compact subset of $\widetilde{U}$, there exist three constants $C \in \mathbb{R}^+$, $M \in \mathbb{N}$ and $N \in \mathbb{N}$ (each depending on K), such that for $s \in K$*

$$\forall f \in \mathcal{S}(V^+), \ \ ||Z^+(f, s, \tau)|| \leq C\nu_{M,N}(f) \tag{6-4-3}$$

$$\forall h \in \mathcal{S}(V^-), \ \ ||Z^-(h, s, \tau)|| \leq C\nu_{M,N}(h) \tag{6-4-4}$$

(where $\nu_{M,N}$ is the usual semi-norm on $\mathcal{S}(V^+)$ or $\mathcal{S}(V^-)$ defined in section 5.2.). This implies in particular that $f \longmapsto Z^+(f, s, \tau)$ and $h \longmapsto Z^-(h, s, \tau)$ are $\mathcal{H}_\tau$–valued tempered distributions of order $\leq N$ for $s \in K$.

Proof. —As said in Remark 6.3.2 the fact that $s \longmapsto Z^+(f, s, \tau)$ and $s \longmapsto Z^-(h, s, \tau)$ have meromorphic continuation to $\mathbb{C}^{k+1}$ is due to Bernshtein and Gel'fand ([B-G]–1969) using Hironaka's Theorem. We prefer here use directly Sabbah's result (see Proposition 5.2.2) as for Type I and Type II.

If $(e_i)_{i=1,\ldots,p}$ is a basis of $\mathcal{H}_\tau$, let us denote by $Z_i^+(f, s, \tau)$ and $Z_i^-(h, s, \tau)$ the i-th component of $Z^+(f, s, \tau)$ and $Z^-(h, s, \tau)$ respectively. In order to

prove the assertion concerning the holomorphic extension to $\widetilde{U}$, it is enough to prove it for each component $Z_i^+(f,s,\tau)$ and $Z_i^-(h,s,\tau)$. But from Proposition 6.2.5 we know that, for example,

$$Z_i^+(f,s,\tau) = \int_{V^+} f(X)\Delta^{s-\left(\frac{\tau_0}{2},\frac{\tau_0+\tau_1}{2},\dots,\frac{\tau_{k-1}+\tau_k}{2}\right)} U_i^+(X)\,dX \ ,$$

where U_i^+ is a polynomial function. Therefore Proposition 5.2.2 applied as in Lemma 5.2.3 gives the result. The same way, in order to prove the inequalities (6-4-3) and (6-4-4), it is enough to prove the same inequality for $|Z_i^+(f,s,\tau)|$ and $|Z_i^-(h,s,\tau)|$. This again is proved as Lemma 5.2.5.

□

In order to extend the functional equation of Proposition 6.4.2 to functions $f \in \mathcal{S}(V^+)$, we define the distribution $T_{s,\tau}$ for $f \in \mathcal{S}(V^+)$ by

$$\langle T_{s,\tau}, f\rangle = Z^-(\mathcal{F}f,s,\tau) - \nu(s,\tau)Z^+(f,t(s)-m,\tau) \ . \tag{6-4-5}$$

The aim is to prove that this is the zero distribution.

For $i \in \{0,1,\dots,k\}$ let $\Delta_i(\partial)$ be the differential operator with constant coefficients on V^- defined as in (5-2-14) by

$$\Delta_i(\partial)e^{b(X,Y)} = \Delta_i(X)e^{b(X,Y)} \quad (X \in V^+, Y \in V^-) \ .$$

Lemma 6.4.4. *There exist polynomials $b_{0,\tau}, b_{1,\tau}\dots, b_{k,\tau}$ in the variable $s = (s_0,s_1,\dots,s_k)$ such that*

$$\Delta_0(\partial)\nabla_0^{s_0}\dots\nabla_k^{s_k}b_\tau^- = b_{0,\tau}(s)\nabla_0^{s_0-1}\nabla_1^{s_1}\dots\nabla_k^{s_k}b_\tau^- \ ,$$
$$\Delta_i(\partial)\nabla_0^{s_0}\dots\nabla_k^{s_k}b_\tau^- = b_{i,\tau}(s)\nabla_0^{s_0-1}\nabla_1^{s_1}\dots\nabla_{k-i+1}^{s_{k-i+1}+1}\dots\nabla_k^{s_k}b_\tau^- \quad (i\neq 0) \ .$$

Proof. —In order to apply Proposition 6.2.2 let us determine how the function $\Delta_i(\partial)\nabla^s b_\tau^-$ transforms under the action of P. Let π be the regular representation of P on functions on $\mathcal{O}^-$ *i.e.* for a function Ψ on $\mathcal{O}^-$, π is defined by

$$(\pi(p)\Psi)(Y) = \Psi(p^{-1}Y) \quad (p\in P,\ Y\in\mathcal{O}^-) \ .$$

Recall from (5-2-16) that for $i = 0,1,\dots,k$ we have

$$\pi(p)\circ\Delta_i(\partial) = \chi_i(p)^{-1}\Delta_i(\partial)\circ\pi(p) \ .$$

On the other hand for any function $\Psi \in \mathcal{C}^\infty(\mathcal{O}^-,\mathcal{H}_\tau)$ we have

$$\Delta_i(\partial)\tau(p)\Psi = \tau(p)\Delta_i(\partial)\Psi \quad (p\in P) \ .$$

Using Theorem 6.2.1 it follows that

$$\pi(p)^{-1}\Big(\Delta_i(\partial)\Delta^s b_\tau^-\Big) = \chi_i(p)(\chi^-)^s(p)\tau(p)\Delta_i(\partial)b_\tau^- \ .$$

Suppose for example that $i\neq 0$. From Corollary 3.8.5 we have

$$\chi_i(p) = \frac{\chi_{k+1-i}^-(p)}{\chi_0^-(p)} \ ,$$

which implies that

$$\chi_i(p)(\chi^-)^s(p) = (\chi^-)^{s'}(p) \text{ where } s' = (s_0-1,s_1,\dots,s_{k+1-i}+1,\dots,s_k) \ .$$

Therefore the function $\Delta_i(\partial)\Delta^s b_\tau^-$ transforms under P as the function $\Delta^{s'} b_\tau^-$. Then, Proposition 6.2.2 implies the existence of a constant $b_{i,\tau}(s)$ such that

$$\Delta_i(\partial)\nabla^s b_\tau^- = b_{i,\tau}(s)\nabla_0^{s_0-1}\nabla_1^{s_1}\dots\nabla_{k-i+1}^{s_{k-i+1}+1}\dots\nabla_k^{s_k} b_\tau^- .$$

As in the proof of Lemma 5.2.7, it is easy to see that $b_{i,\tau}$ is a polynomial. □

Theorem 6.4.5. (Abstract functional equation for type III)
Let τ be a $(M \cap H)$–spherical representation of M. For $f \in \mathcal{S}(V^+)$ and $h \in \mathcal{S}(V^-)$ the integrals defining $Z^+(f,s,\tau)$ and $Z^-(h,s,\tau)$ have meromorphic continuation to $\mathbb{C}^{k+1}$ and there exist meromorphic functions $\nu(s,\tau)$ and $\delta(\tau,s)$ such that

$$Z^-(\mathcal{F}f,s,\tau) = \nu(s,\tau)Z^+(f,t(s)-m,\tau) , \tag{6-4-6}$$

$$Z^+(f,s,\tau) = \delta(s,\tau)Z^-(\mathcal{F}f,t(s)-m,\tau) . \tag{6-4-7}$$

Proof. — First of all notice that $\delta(s,\tau)$ is the inverse of $\nu(t(s)-m,\tau)$ since $t(t(s)-m)-m = s$. Therefore it remains to prove that $T_{s,\tau} = 0$, *i.e.* that for all $f \in \mathcal{S}(V^+)$,

$$Z^-(\mathcal{F}(f),s,\tau) - \nu(s,\tau)Z^+(f,t(s)-m,\tau) = 0.$$

Recall from Proposition 6.2.5 that

$$b_\tau^+ = \frac{U^+}{\Delta_0^{\frac{\tau_0}{2}}\Delta_1^{\frac{\tau_0+\tau_1}{2}}\dots\Delta_k^{\frac{\tau_{k-1}+\tau_k}{2}}}$$

and

$$b_\tau^- = \frac{U^-}{\nabla_0^{\frac{\tau_k}{2}}\nabla_1^{\frac{\tau_{k-1}+\tau_k}{2}}\dots\nabla_k^{\frac{\tau_0+\tau_1}{2}}}$$

where $U^\pm$ is a $\mathcal{H}_\tau$-valued polynomial. Therefore, once a basis of $\mathcal{H}_\tau$ is chosen, the same proof as Lemma 5.2.6 (which needs Proposition 6.4.3) can be applied to each component of $T_{s,\tau}$. Hence there exists an integer $p \in \mathbb{N}$ such that

$$\Delta_0^p\Delta_1^p\dots\Delta_k^p T_{s,\tau} = 0 .$$

Define $\Delta^p = \Delta_0^p\Delta_1^p\dots\Delta_k^p$ and $\Delta(\partial)^p = \Delta_0(\partial)^p\Delta_1(\partial)^p\dots\Delta_k(\partial)^p$. We will now compute explicitly $\Delta^p T_{s,\tau}$. Using Lemma 5.2.8, which is still true in the type III case, we get

$$Z^-(\mathcal{F}(\Delta^p f),s,\tau) = (2i\pi)^{-p\frac{(k+1)(k+2)}{2}} Z^-(\Delta(\partial)^p\mathcal{F}(f),s,\tau) .$$

For $\mathrm{Re}(s) > u_0$ this Zeta function is an integral. Integration by parts using Lemma 6.4.4 leads to the existence of a polynomial $P_\tau(s)$ such that

$$\begin{aligned} Z^-(\mathcal{F}(\Delta^p f),s,\tau) &= P_\tau(s)\int_{V^-}\mathcal{F}(f)(Y)\nabla(Y)^{\alpha(s)}b_\tau^-(Y)\,dY \\ &= P_\tau(s)Z^-(\mathcal{F}(f),\alpha(s),\tau) \end{aligned}$$

where $\alpha(s_0,s_1,\dots,s_k) = (s_0-(k+1)p, s_1+p,\dots,s_k+p)$. On the other hand we have

$$Z^+(\Delta^p f,t(s)-m) = Z^+(f,t(s)-m+(p,\dots,p)) .$$

Hence we obtain

$$0 = \langle \Delta^p T_{s,\tau}, f \rangle = P_\tau(s) Z^-(\mathcal{F}f, \alpha(s), \tau) - \nu(s,\tau) Z^+(f, t(s) - m + (p,\dots,p)) .$$

As $t(\alpha(s)) - m = t(s) - m + (p,\dots,p)$, there exists a meromorphic function, which for obvious reasons is equal to the function $\nu(s,\tau)$ from Proposition 6.4.2, such that for $f \in \mathcal{S}(V^+)$ and $\mathrm{Re}(s) > u_0$ we have:

$$Z^-(\mathcal{F}(f), \alpha(s), \tau) = \nu(s,\tau) Z^+(f, t(\alpha(s)) - m, \tau).$$

The result follows by meromorphic continuation.

□

6.5. Explicit functional equation for $k = 0$

The explicit functional equation for the local Zeta function $Z(f,s,\tau)$ will be obtained in the next section. As for Type I and Type II, the proof will be done by induction on the rank k. One of the key ingredients will be the proof in the case where $k = 0$. We have seen in chapter 5 that for Type I and Type II, the case $k = 0$ is essentially TATE's functional equation of the MELLIN transform.

For Type III the case $k = 0$ is somewhat more involved. In fact it relies on the functional equation of a Zeta function attached to a positive definite quadratic form and a harmonic polynomial, which was obtained by RALLIS and SCHIFFMANN ([Ra-S]–1975). We first explain their result.

Let Q be a positive definite quadratic form on $\mathbb{R}^\ell$. Define a bilinear form Φ on $\mathbb{R}^\ell$ by

$$\forall x, y \in \mathbb{R}^\ell, \quad \Phi(x,y) = Q(x+y) - Q(x) - Q(y) \tag{6-5-1}$$

and define the Fourier transform on the Schwartz space $\mathcal{S}(\mathbb{R}^\ell)$ by

$$\widetilde{f}(x') = \int_{\mathbb{R}^\ell} f(x) e^{2i\pi\Phi(x,x')}\, d_s x \ ,$$

where $d_s x$ is a self-dual Lebesgue measure for this Fourier transform. This means that the inversion formula is given by

$$f(x) = \int_{\mathbb{R}^\ell} \widetilde{f}(x') e^{-2i\pi\Phi(x,x')}\, d_s x' \ ,$$

where $d_s x'$ is the same measure.

Let $Q(\partial)$ be the differential operator with constant coefficients defined by the formula $(4i\pi)^2 \widetilde{(Qf)} = Q(\partial)\widetilde{f}$ for $f \in \mathcal{S}(\mathbb{R}^\ell)$. A polynomial R on $\mathbb{R}^\ell$ is called *harmonic* if $Q(\partial)R = 0$. Let $\mathcal{H}_\ell^t$ be the space of harmonic homogeneous polynomials of degree t. For $f \in \mathcal{S}(\mathbb{R}^\ell)$, $R \in \mathcal{H}_\ell^t$ and $z \in \mathbb{C}$, $\mathrm{Re}(z) > \frac{\ell}{2}$ we define the following Zeta function

$$\mathbf{Z}(f,z,R) = \int_{\mathbb{R}^\ell} f(x) Q(x)^{z-\frac{\ell}{2}} R(x)\, d_s x \ . \tag{6-5-2}$$

Theorem 6.5.1 (RALLIS–SCHIFFMANN).
Let $R \in \mathcal{H}_\ell^t$ and $f \in \mathcal{S}(\mathbb{R}^\ell)$. The function $z \longmapsto \mathbf{Z}(f, z, R)$ is holomorphic for $\mathrm{Re}(z) > \frac{\ell}{2}$ and has a meromorphic continuation to the whole complex plane which satisfies the following functional equation

$$\mathbf{Z}(f, z, R) = \gamma(z, t)\mathbf{Z}(\widetilde{f}, \frac{\ell}{2} - z - t, R) \ ,$$

where

$$\gamma(z,t) = \rho(z - \frac{\ell}{2} + 1, \mathbf{1})\Big(\rho(t+z, (-\mathbf{1})^t)\cos(\frac{\pi}{4}\ell) + i\rho(t+z, (-\mathbf{1})^{t+1})\sin(\frac{\pi}{4}\ell)\Big)$$

and where $\rho(z, \pm\mathbf{1})$ are TATE's rho factors defined in section 5.3.

Proof. — This is a particular case of a functional equation proved by RALLIS and SCHIFFMANN (they deal more generally with arbitrary non-singular quadratic forms). More precisely it is a particular case of formula (5-14) p. 551 of [Ra-S]–1975 (see also formula (5-20) p. 554 in *loc.cit.*). □

Let us now consider a graded Lie algebra $\widetilde{\mathfrak{g}} = V^- \oplus \mathfrak{g} \oplus V^+$ which is of Type III and where $k = 0$. The condition $k = 0$ implies that $V^+ = \widetilde{\mathfrak{g}}^{\lambda_0}$, $V^- = \widetilde{\mathfrak{g}}^{-\lambda_0}$ and $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{a}$. We know from Theorem 1.12.1 that the fundamental relative invariant Δ_0 is then of degree 2 and is, up to a multiplicative constant, equal to $b(X, \theta X)$. More precisely, as $b(X_0, \theta X_0) = 1$ by Lemma 1.10.2, we have $\Delta_0(X) = b(X, \theta X)$ for any $X \in V^+ = \widetilde{\mathfrak{g}}^{\lambda_0}$. Hence Δ_0 is a positive definite quadratic form on $\widetilde{\mathfrak{g}}^{\lambda_0}$.

In order to use Theorem 6.5.1 in our context we have to proceed to a careful comparison between several Fourier transforms. If $\Phi(X, X')$ is the bilinear form defined as in (6-5-1) by $\Phi(X, X') = \Delta_0(X + X') - \Delta_0(X) - \Delta_0(X')$, then $\Phi(X, X') = 2b(X, \theta X')$ $(X, X' \in \widetilde{\mathfrak{g}}^{\lambda_0})$. We define first two Fourier transforms on $\mathcal{S}(\widetilde{\mathfrak{g}}^{\lambda_0})$ by

$$\widetilde{f}(X') = \int_{\widetilde{\mathfrak{g}}^{\lambda_0}} f(X) e^{2i\pi\Phi(X,X')}\, d_s X \ , \tag{6-5-3}$$

$$\hat{f}(X') = \int_{\widetilde{\mathfrak{g}}^{\lambda_0}} f(X) e^{2i\pi b(X,\theta X')}\, dX \ , \tag{6-5-4}$$

where $d_s X$ and dX are self-dual Lebesgue measures on $\widetilde{\mathfrak{g}}^{\lambda_0}$ for $\widetilde{f}$ and $\hat{f}$ respectively. It may be proved that if (dX, dY) is a pair of measures respectively on $V^+ = \widetilde{\mathfrak{g}}^{\lambda_0}$ and $V^- = \widetilde{\mathfrak{g}}^{-\lambda_0}$ normalized as in section 4.1, then dX is self–dual for $\hat{f}$. Recall that ℓ denotes the dimension of $\widetilde{\mathfrak{g}}^{\lambda_0}$. An elementary computation shows that

$$d_s X = \sqrt{2}^\ell dX \quad \text{and} \quad \widetilde{f}(X') = \sqrt{2}^\ell \hat{f}(2X') \ (X' \in \widetilde{\mathfrak{g}}^{\lambda_0}) \ .$$

Moreover if the Fourier transform $\mathcal{F}f$ between $\mathcal{S}(\widetilde{\mathfrak{g}}^{\lambda_0})$ and $\mathcal{S}(\widetilde{\mathfrak{g}}^{-\lambda_0})$ is defined as in Definition 4.1.3 by

$$\mathcal{F}f(Y) = \int_{\widetilde{\mathfrak{g}}^{\lambda_0}} f(X) e^{2i\pi b(X,Y)}\, dX \quad (Y \in \widetilde{\mathfrak{g}}^{-\lambda_0}) \ , \tag{6-5-5}$$

then we have

$$\mathcal{F}f(\theta X') = \hat{f}(X') = \frac{1}{\sqrt{2}^{\ell}}\widetilde{f}(\frac{X'}{2}) \quad (X' \in \widetilde{\mathfrak{g}}^{\lambda_0}) \ . \tag{6-5-6}$$

As $k = 0$ the $(M \cap H)$–spherical representations of M are just the representation of M on the spaces $\mathcal{H}_{\ell}^{\tau_0}$ of $\Delta_0(\partial)$–harmonic polynomials of degree τ_0 on $\widetilde{\mathfrak{g}}^{\lambda_0}$. As in section 6.1, we also denote by τ_0 the corresponding representation of M and by v_{τ_0} the $(M \cap H)$–invariant vector in $\mathcal{H}_{\ell}^{\tau_0}$ defined in 6.1.6.

Let us remark that, in the case $k = 0$, we have $I^+ = X_0$, $P = MA$ and $\mathcal{O}^+ = MA.I^+ = \widetilde{\mathfrak{g}}^{\lambda_0} \backslash \{0\}$.

We are now able to give the explicit functional equation for Type III and $k = 0$. For the $(M \cap H)$–spherical representation on $\mathcal{H}_{\ell}^{\tau_0}$ $(\tau_0 \in \mathbb{N})$, the Zeta functions defined in 6.3.1 can be written in this case as

$$Z^+(f, z, \tau_0) = \int_{\mathcal{O}^+} f(X)\Delta_0^z(X)b_{\tau_0}^+(X)\, dX \quad (f \in \mathcal{S}(\widetilde{\mathfrak{g}}^{\lambda_0}), z \in \mathbb{C}) \ ,$$

and

$$Z^-(h, z, \tau_0) = \int_{\mathcal{O}^-} h(Y)\nabla_0^z(Y)b_{\tau_0}^-(Y)\, dY \quad (h \in \mathcal{S}(\widetilde{\mathfrak{g}}^{-\lambda_0}), z \in \mathbb{C}) \ .$$

Theorem 6.5.2 (Type III and $k = 0$)**.**
Let $\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{g}}^{-\lambda_0} \oplus \mathfrak{g} \oplus \widetilde{\mathfrak{g}}^{\lambda_0}$ be a graded Lie algebra of Type III *and of rank 1 ($k = 0$). For $\tau_0 \in \mathbb{N}$, the Zeta functions $Z^+(f, z, \tau_0)$ and $Z^-(h, z, \tau_0)$, which are holomorphic for* $\mathrm{Re}(z)$ *large enough, admit a meromorphic continuation to the whole complex plane, and there exists a meromorphic function $\delta_0(z, \tau_0)$ such that*

$$Z^+(f, z, \tau_0) = \delta_0(z, \tau_0)Z^-(\mathcal{F}f, -z - \frac{\ell}{2}, \tau_0) \ .$$

Moreover we have

$$\delta_0(z, \tau_0) = \begin{cases} -\pi^{-2z-\frac{\ell}{2}-1}\Gamma(z - \frac{\tau_0}{2} + 1)\Gamma(z + \frac{\tau_0}{2} + \frac{\ell}{2}) \sin \pi z & \text{if } \tau_0 \text{ is even} \ , \\ -i\,\pi^{-2z-\frac{\ell}{2}-1}\Gamma(z - \frac{\tau_0}{2} + 1)\Gamma(z + \frac{\tau_0}{2} + \frac{\ell}{2}) \cos \pi z & \text{if } \tau_0 \text{ is odd} \ . \end{cases}$$

Proof. — It is enough to prove that for any $R \in \mathcal{H}_{\ell}^{\tau_0}$ we have

$$\big(R \mid Z^+(f, z, \tau_0)\big)_{\tau_0} = \delta_0(z, \tau_0)\big(R \mid Z^-(\mathcal{F}f, -z - \frac{\ell}{2}, \tau_0)\big)_{\tau_0}$$

with the given factor $\delta_0(z, \tau_0)$.

From Lemma 6.2.3 and the relation $d_sX = \sqrt{2}^{\ell}dX$ we get

$$\begin{aligned} \big(R \mid Z^+(f, z, \tau_0)\big)_{\tau_0} &= \int_{\mathcal{O}^+} f(X)\Delta_0(X)^z \big(R \mid b_{\tau_0}^+(X)\big)_{\tau_0}\, dX \\ &= \frac{1}{\sqrt{2}^{\ell}} \int_{\mathcal{O}^+} f(X)\Delta_0(X)^{z-\frac{\tau_0}{2}} R(X)\, d_sX \ . \end{aligned}$$

It follows that

$$\big(R \mid Z^+(f, z, \tau_0)\big)_{\tau_0} = \frac{1}{\sqrt{2}^{\ell}}\mathbf{Z}(f, z - \frac{\tau_0}{2} + \frac{\ell}{2}, R) \ . \tag{6-5-6}$$

On the other hand we have

$$\big(R \mid Z^-(\mathcal{F}f, -z - \frac{\ell}{2}, \tau_0)\big)_{\tau_0} = \int_{\mathcal{O}^-} \mathcal{F}f(Y)\nabla_0(Y)^{-z-\frac{\ell}{2}} \big(R \mid b^-_{\tau_0}(Y)\big)_{\tau_0}\, dY \ .$$

In the previous integral we make the change of variable $Y = \theta X$ and use Proposition 4.1.1. From the relations $\nabla_0(\theta X) = b(\theta X, X) = \Delta_0(X)$, $\mathcal{F}f(\theta X) = \dfrac{1}{\sqrt{2}^\ell}\widetilde{f}(\dfrac{X}{2})$, $dX = \dfrac{1}{\sqrt{2}^\ell} d_s X$ and Lemma 6.2.3 we get

$$\begin{aligned}\big(R \mid Z^-(\mathcal{F}f, -z - \frac{\ell}{2}, \tau_0)\big)_{\tau_0} &= \frac{1}{2^\ell}\int_{\mathcal{O}^+} \widetilde{f}(\frac{X}{2})\Delta_0(X)^{-z-\frac{\ell}{2}-\frac{\tau_0}{2}} R(X)\, d_s X \\ &= \int_{\mathcal{O}^+} \widetilde{f}(X)\Delta_0(2X)^{-z-\frac{\ell}{2}-\frac{\tau_0}{2}} R(2X)\, d_s X \ .\end{aligned}$$

Since $\Delta_0(2X) = 2^2\Delta_0(X)$ and $R(2X) = 2^{\tau_0}R(X)$ we obtain

$$\big(R \mid Z^-(\mathcal{F}f, -z - \frac{\ell}{2}, \tau_0)\big)_{\tau_0} = 2^{-2z-\ell}\mathbf{Z}(\widetilde{f}, -z - \frac{\tau_0}{2}, R) \ . \tag{6-5-7}$$

Using (6-5-6) and (6-5-7) together with Theorem 6.5.1 we obtain

$$Z^+(f, z, \tau_0) = 2^{2z+\frac{\ell}{2}}\gamma(z - \frac{\tau_0}{2} + \frac{\ell}{2}, \tau_0) Z^-(\mathcal{F}f, -z - \frac{\ell}{2}, \tau_0) \ .$$

Hence

$$\delta_0(z, \tau_0) = 2^{2z+\frac{\ell}{2}}\gamma(z - \frac{\tau_0}{2} + \frac{\ell}{2}, \tau_0).$$

From the explicit value of $\gamma(z, \tau_0)$ given in Theorem 6.5.1 we get

$$\begin{aligned}\delta_0(z, \tau_0) &= 2^{2z+\frac{\ell}{2}}\rho(z - \frac{\tau_0}{2} + 1, \mathbf{1}) \\ &\times \Big(\rho(z + \frac{\tau_0}{2} + \frac{\ell}{2}, (-\mathbf{1})^{\tau_0})\cos(\frac{\pi}{4}\ell) + i\rho(z + \frac{\tau_0}{2} + \frac{\ell}{2}, (-\mathbf{1})^{\tau_0+1})\sin(\frac{\pi}{4}\ell)\Big) \ .\end{aligned}$$

Using the value of ρ (Definition 5.3.1) we obtain

$$\delta_0(z, \tau_0) = 2\pi^{-2z-\frac{\ell}{2}-1}\Gamma(z - \frac{\tau_0}{2} + 1)\Gamma(z + \frac{\tau_0}{2} + \frac{\ell}{2})\alpha(z) \ ,$$

where $z \mapsto \alpha(z)$ depends on $(-\mathbf{1})^{\tau_0}$. For example if τ_0 is even we get

$$\begin{aligned}\alpha(z) &= \cos\big(\frac{\pi}{2}(z - \frac{\tau_0}{2} + 1)\big) \\ &\times \Big(\cos\big(\frac{\pi}{2}(z + \frac{\tau_0}{2} + \frac{\ell}{2})\big)\cos(\frac{\pi}{4}\ell) + \sin\big(\frac{\pi}{2}(z + \frac{\tau_0}{2} + \frac{\ell}{2})\big)\sin(\frac{\pi}{4}\ell)\Big) \ .\end{aligned}$$

The result follows from elementary trigonometric formulas.

□

6.6. Computation of the coefficients $\delta(s,\tau)$

In this section we will compute explicitely the coefficients $\delta(s,\tau)$ defined in Theorem 6.4.5. This will be done by induction on the rank. Recall from Chapter 4 that if the regular graded algebra $\widetilde{\mathfrak{g}}$ is of rank $k+1$, then the regular graded algebra $\underline{\widetilde{\mathfrak{g}}} = V^-(0) \oplus \mathfrak{g}(0) \oplus V^+(0)$ is of rank k.

As for Type I and Type II in section 5.3 we will denote by underlined letters all objects attached to the algebra $\underline{\widetilde{\mathfrak{g}}}$ which is graded by the element $\underline{H}_0 = \sum_{j=0}^{k-1} H_{\lambda_j}$. For example $\underline{G} = G(0)$ denotes the analytic subgroup of G with Lie algebra $\underline{\mathfrak{g}} = \mathfrak{g}(0)$ and $\underline{P}$ denotes the parabolic subgroup of $\underline{G}$ defined as in section 3.5. The polynomials $\underline{\Delta}_j$ $(j = 0, \ldots, k-1)$ are the fundamental relative invariants for the action of $\underline{P}$ on $\underline{V}^+ = V^+(0)$, normalized by the condition $\underline{\Delta}_j(\underline{I}^+) = 1$ where $\underline{I}^+ = \sum_{j=0}^{k-1} X_j$.

From Lemma 5.3.6 we know that the LANGLANDS decomposition of $\underline{P}$ is given by

$$\underline{P} = \underline{M}A\underline{N} \ ,$$

where $\underline{M} = M \cap \underline{G}$ and $\underline{N} = N \cap \underline{G}$. Moreover M and $\underline{M}$ have the same Lie algebra.

Let L_j be the analytic subgroup of G with Lie algebra $\mathfrak{l}_j$ $(j = 0, \ldots, k)$. The Cartan decomposition of L_j is given by

$$L_j = M_j A_j \quad \text{where } M_j = L_j \cap K \text{ and } A_j = e^{\mathbb{R}H_{\lambda_j}} \ .$$

It follows that the group M_j is connected and therefore a subgroup of $\underline{M}$.

Lemma 6.6.1. *The set $M_0 M_1 \cdots M_{k-1} A_0 A_1 \cdots A_{k-1} \underline{N}$ is a subgroup of $\underline{P}$. Moreover the action of this subgroup on $\underline{V}^+$ is prehomogeneous with $\underline{\mathcal{O}}^+ = \underline{P}\,\underline{I}^+$ as single open orbit.*

Proof. — The subgroups L_j commute. Hence $M_0 \cdots M_{k-1} A_0 \cdots A_{k-1} \underline{N}$ is a subgroup of $\underline{P}$. The proof of Theorem 3.5.4 shows that not only the action of $\underline{M}$ but also the action of $M_0 M_1 \cdots M_{k-1}$ is transitive on the product of the spheres $\prod_{j=0}^{k-1} S_j$. Using this result, the same proof as in Theorem 3.6.3 (Type III) yields the result.

□

Let $\underline{H}$ be the centralizer of $\underline{I}^+$ in $\underline{G}$.

Remark 6.6.2. It may be noticed that $\underline{H}$ is not a subset of H and that, moreover, $\underline{M} \cap \underline{H}$ is not a subset of $M \cap H$. To prove this consider a element $m \in M_k$ which does not centralize X_k. Such an element exists since the orbit of X_k under the action of M_k is a sphere. This element belongs to $\underline{M} \subset \underline{G}$ and centralizes $\underline{I}^+$, hence belongs to $\underline{H}$. But m does not centralize $I^+ = \underline{I}^+ + X_k$ and, therefore, does not belong to H.

Let $(\tau, \mathcal{H}_\tau)$ be a $(M \cap H)$–sperical representation of M where $\mathcal{H}_\tau$ is the space $\mathcal{H}_0^{\tau_0} \otimes \mathcal{H}_1^{\tau_1} \otimes \cdots \otimes \mathcal{H}_k^{\tau_k}$. The restriction of this representation to the space

$$\mathcal{H}_{\underline{\tau}} = \mathcal{H}_0^{\tau_0} \otimes \mathcal{H}_1^{\tau_1} \otimes \cdots \otimes \mathcal{H}_{k-1}^{\tau_{k-1}}$$

defines a representation of the subgroup $\underline{M}$ of M which, by Theorem 6.1.3 is a $(\underline{M} \cap \underline{H})$–spherical representation of $\underline{M}$ denoted $\underline{\tau}$. Similarly the restriction

to the space $\mathcal{H}_{\tau_k} = \mathcal{H}_k^{\tau_k}$ defines a $(M_k \cap H_k)$–spherical representation τ_k of M_k where $H_k = Z_{L_k}(X_k)$.

A $(M \cap H)$–invariant vector $v_\tau \in \mathcal{H}_\tau$ of the representation τ has been defined in Definition 6.1.6. In a similar manner we define a $\underline{M} \cap \underline{H}$–invariant vector in $\mathcal{H}_{\underline{\tau}}$ for $\underline{\tau}$ and a $M_k \cap H_k$–invariant vector in $\mathcal{H}_{\tau_k}$ for τ_k.

Lemma 6.6.3. *If we identify canonically $\mathcal{H}_\tau$ whith $H_{\underline{\tau}} \otimes \mathcal{H}_{\tau_k}$, then we have*

$$v_\tau = v_{\underline{\tau}} \otimes v_{\tau_k} \ .$$

Proof. —Using the canonical identification between the Hilbert spaces $\mathcal{H}_\tau$ and $H_{\underline{\tau}} \otimes \mathcal{H}_{\tau_k}$ we get, for $P \in \mathcal{H}_{\underline{\tau}}$ and $Q \in \mathcal{H}_{\tau_k}$

$$\begin{aligned}\big(v_{\underline{\tau}} \otimes v_{\tau_k} \mid P \otimes Q\big)_\tau &= \big(v_{\underline{\tau}} \mid P\big)_{\underline{\tau}} \big(v_{\tau_k} \mid Q\big)_{\tau_k} \\ &= P(\underline{I}^+)Q(X_k) = (P \otimes Q)(I^+) = \big(v_\tau \mid P \otimes Q\big)_\tau \ .\end{aligned}$$

□

Proposition 6.6.4.

(1) *For $u \in \underline{\mathcal{O}}^+$, $v \in \mathcal{O}_k^+ = \widetilde{\mathfrak{g}}^{\lambda_k} \backslash \{0\}$ and $A \in \mathfrak{g}(-1)$ we have*

$$b_\tau^+(e^{\operatorname{ad} A}(u+v)) = b_\tau^+(u+v) = b_{\underline{\tau}}^+(u) \otimes b_{\tau_k}^+(v) \ .$$

(2) *For $u' \in \underline{\mathcal{O}}^-$, $v \in \mathcal{O}_k^- = \widetilde{\mathfrak{g}}^{-\lambda_k} \backslash \{0\}$ and $A \in \mathfrak{g}(-1)$ we have*

$$b_\tau^-(e^{\operatorname{ad} A}(u+v)) = b_\tau^-(u+v) = b_{\underline{\tau}}^-(u) \otimes b_{\tau_k}^-(v) \ .$$

Proof. — We only give the proof for (1).

It follows from Lemma 5.3.7 that for $u \in \underline{\mathcal{O}}^+$, $v \in \mathcal{O}_k^+$ and $A \in \mathfrak{g}(-1)$, $e^{\operatorname{ad} A}(u+v)$ is an element of $\mathcal{O}^+$. Hence $b_\tau^+(e^{\operatorname{ad} A}(u+v))$ is well defined and we obtain from Theorem 6.2.1

$$b_\tau^+(e^{\operatorname{ad} A}(u+v)) = b_\tau^+(u+v) \ .$$

Since the action of M_k is transitive on the spheres of $\widetilde{\mathfrak{g}}^{\lambda_k}$ there exist $m_k \in M_k$ and $a_k \in A_k = \exp(\mathbb{R}H_{\lambda_k})$ such that

$$v = m_k a_k X_k \ .$$

Lemma 6.6.1 implies that there exist $\underline{m} \in M_0 \cdots M_{k-1}$, $\underline{a} \in A_0 \cdots A_{k-1}$ and $\underline{n} \in \underline{N}$ such that

$$u = \underline{m}\,\underline{a}\,\underline{n}\,\underline{I}^+ \ .$$

Moreover $m_k a_k$ commutes with $\underline{m}\,\underline{a}\,\underline{n}$. Hence if $m = \underline{m} m_k$, $a = \underline{a} a_k$ we have $u = ma\underline{n}\,\underline{I}^+$ since $m_k a_k \underline{I}^+ = I^+$ and $v = ma\underline{n}\,I_k^+$ since $\underline{m}\,\underline{a}\,\underline{n}\,X_k = X_k$. It follows from Theorem 6.2.1 that

$$b_\tau^+(u+v) = b_\tau^+(ma\underline{n}.I^+) = \tau(\underline{m}\,m_k)v_\tau \ .$$

Using Lemma 6.6.3 and again Theorem 6.2.1 applied to the regular graded algebras $(\widetilde{\underline{\mathfrak{g}}}, \underline{H_0})$ and $(\widetilde{\mathfrak{l}}_k, H_{\lambda_k})$ we obtain

$$\begin{aligned}b_\tau^+(u+v) &= \tau(\underline{m})v_{\underline{\tau}} \otimes \tau(m_k)v_{\tau_k} \\ &= b_{\underline{\tau}}^+(\underline{m}\,\underline{a}\,\underline{n}\,\underline{I}^+) \otimes b_{\tau_k}^+(m_k a_k X_k) \\ &= b_{\underline{\tau}}^+(u) \otimes b_{\tau_k}^+(v) \ .\end{aligned}$$

□

As in Type I and II cases we need an explicit computation of $\gamma(u', v)$, but here the result is very simple.

Lemma 6.6.5 (Type III).
For $u' \in \underline{\mathcal{O}}^-$ and $v \in \mathcal{O}_k^+$ we have $\gamma(u', v) = 1$.

Proof. — By Definition 4.7.3 we have $\gamma(u', v) = e^{i\frac{\pi}{4}(p-q)}$ where (p, q) is the signature of the quadratic form $Q_{u',v}$ which is defined on $\mathfrak{g}(-1)$ by $Q_{u',v}(A) = \frac{1}{2}b\big((\operatorname{ad} A)^2 u', v\big)$.

As u' belongs to $\underline{\mathcal{O}}^-$ we know from Lemma 3.6.2 that there exists an element $n \in \underline{N}$ such that

$$n.u' = u'_0 \quad \text{where} \quad u'_0 = \sum_{j=0}^{k-1} Z_j \quad (Z_j \in \widetilde{\mathfrak{g}}^{-\lambda_j}) \ .$$

Moreover nv still belongs to $\mathcal{O}_k$ and we note $nv = Z_k$.

As $\underline{N}$ stabilizes $\mathfrak{g}(-1)$, the quadratic form $Q_{u',v}$ is equivalent to the quadratic form $Q_{u'_0,Z_k}$. The decomposition $\mathfrak{g}(-1) = \oplus_{j=0}^{k-1} E_{j,k}(1,-1)$ is an orthogonal decomposition for $Q_{u'_0,Z_k}$. Moreover if A belongs to $E_{j,k}(1,-1)$ then $b\big((\operatorname{ad} A)^2 Z_i, Z_k\big) = 0$ for $i \neq j$. Thus we have the following equivalence of quadratic forms :

$$Q_{u'_0,Z_k} \sim \oplus_{j=0}^{k-1} Q_{j,k} \ ,$$

where $Q_{j,k}$ is the quadratic form defined on $E_{j,k}(1,-1)$ by

$$Q_{j,k}(A) = \frac{1}{2} b\big((\operatorname{ad} A)^2 Z_j, Z_k\big) \quad \text{for } A \in E_{j,k}(1,-1) \ .$$

The quadratic forms q_{Z_j,Z_k} on $E_{j,k}(-1,-1)$ have been defined in section 2.5 by

$$q_{Z_j,Z_k}(Y) = -\frac{1}{2} b(\operatorname{ad} Z_j.Y, \operatorname{ad} Z_k.Y) \quad \text{for } Y \in E_{j,k}(-1,-1) \ .$$

A simple calculation shows that

$$Q_{j,k}(\operatorname{ad} Z_j.Y) = q_{Z_j,Z_k}(Y) \ \text{ for } Y \in E_{j,k}(-1,-1) \ .$$

As $\operatorname{ad} Z_j$ is an isomorphism from $E_{j,k}(-1,-1)$ onto $E_{j,k}(1,-1)$ the quadratic forms $Q_{j,k}$ and q_{Z_j,Z_k} are equivalent.

We know from Theorem 2.8.12 that for Type III the signature of the quadratic form q_{Z_j,Z_k} is equal to $(\frac{d}{2}, \frac{d}{2})$.This yields the result.

□

Theorem 6.6.6 (Type III).
Let $f \in \mathcal{S}(V^+)$ and $\tau \in \mathbb{N}^{k+1}$. In the functional equation (6-4-7) *proved in Theorem* 6.4.5

$$Z^+(f, s, \tau) = \delta_k(s, \tau) Z^-(\mathcal{F}f, t(s) - m, \tau) \ ,$$

the meromorphic functions $s \mapsto \delta_k(s, \tau)$ are given by

$$\delta_k(s, \tau) = \prod_{j=0}^{k} \delta_0(s_0 + \cdots + s_j + \frac{jd}{4}, \tau_j) \ .$$

Remark 6.6.7. The coefficient $\delta(s,\tau)$ defined in Theorem 6.4.5 depends of course on the graded algebra $\widetilde{\mathfrak{g}}$. We prove here that for Type III it depends only on the rank k of this graded algebra and on the constants d and ℓ. Henceforth we will denote it by δ_k to emphasize the dependance on k.

Proof of Theorem 6.6.6. — To obtain the explicit expression of the coefficients δ_k it is sufficient to use $\mathcal{C}^\infty$ functions f with compact support in $\mathcal{O}^+$. In that case the Zeta function $s \mapsto Z^+(f,s,\tau)$ is analytic on $\mathbb{C}^{k+1}$.

We will prove the formula by induction on k. For $k=0$ the formula reduces to Theorem 6.5.2. We suppose now that this formula is true on $\underline{\widetilde{\mathfrak{g}}}$ where it can be written for $\varphi \in \mathcal{C}_c^\infty(\underline{\mathcal{O}}^+)$

$$Z(\varphi,\underline{s},\underline{\tau}) = \delta_{k-1}(\underline{s},\underline{\tau}) Z^-(\mathcal{F}\varphi, t(\underline{s}) - \underline{m}) \ .$$

Here we have $\underline{s} = (s_0,\dots,s_{k-1})$ and

$$\underline{m} = \frac{\dim \underline{V}^+}{\kappa\, k} = \frac{\ell}{2} + \frac{(k-1)d}{4} = m - \frac{d}{4} \quad \text{since } \kappa = 2 \ .$$

Let f be a function in $\mathcal{C}_c^\infty(\mathcal{O}^+)$. Notice that if $u \in \underline{V}^+$ and $v \in V^+(2)$, then from Lemma 5.3.7 we have

$$u+v \in \mathcal{O}^+ \iff u \in \underline{\mathcal{O}}^+ \text{ and } v \in \mathcal{O}_k^+ \ .$$

Then, from the integration formula proved in Proposition 4.6.4, Lemma 5.3.7 and Proposition 6.6.4 we get

$$Z^+(f,s,\tau) = \int_{v\in\mathcal{O}_k}\int_{u\in\underline{\mathcal{O}}^+} T_f^+(u,v)\ \underline{\Delta}(u)^{\underline{s}}\,\Delta_k(v)^{s_0+\dots+s_k+\frac{kd}{4}} \times b_{\underline{\tau}}^+(u)\otimes b_{\tau_k}^+(v)\ du\ dv \ .$$

As the map $\Phi : (u,A,v) \mapsto e^{\operatorname{ad} A}(u+v)$ is a diffeomorphism from the open set $\underline{\mathcal{O}}^+ \times \mathfrak{g}(-1) \times \mathcal{O}_k^+$ onto $\mathcal{O}^+$, the function $(u,v) \mapsto T_f^+(u,v)$ is a $\mathcal{C}^\infty$ function with compact support in $\underline{\mathcal{O}}^+ \times \mathcal{O}_k^+$. Thus the previous integral is absolutely convergent for each value of $s \in \mathbb{C}^{k+1}$ and $\tau \in \mathbb{N}^{k+1}$. If we apply now the induction hypothesis to the function $u \mapsto T_f^+(u,v)$ we obtain

$$\int_{u\in\underline{\mathcal{O}}^+} T_f^+(u,v)\underline{\Delta}(u)^{\underline{s}} b_{\underline{\tau}}^+(u)\ du = \delta_{k-1}(\underline{s},\underline{\tau}) \int_{u'\in\underline{\mathcal{O}}^-} (\mathcal{F}_u T_f^+)(u',v)\underline{\nabla}(u')^{t(\underline{s})-\underline{m}}\, b_{\underline{\tau}}^-(u')\ du' \ .$$

Suppose now that s is chosen such that the function $u' \mapsto \nabla(u')^{t(\underline{s})-\underline{m}}$ is continuous on $\underline{V}^- = V^-(0)$. Then the last integral is absolutely convergent and compactly supported in v. Hence we may exchange the order of integration in the computation of $Z^+(f,s,\tau)$ and we get

$$Z^+(f,s,\tau) = \delta_{k-1}(\underline{s},\underline{\tau}) \int_{u'\in\underline{\mathcal{O}}^-} \underline{\nabla}(u')^{t(\underline{s})-\underline{m}}\, b_{\underline{\tau}}^-(u') \otimes \int_{v\in\mathcal{O}_k^+} \big(\mathcal{F}_u T_f^+\big)(u',v)\Delta_k(v)^{s_0+\dots+s_k+\frac{kd}{4}}\, b_{\tau_k}^+(v)\ dv\ du' \ .$$

From Theorem 6.5.2 applied to the graded algebra $\widetilde{\mathfrak{g}}_k = \widetilde{\mathfrak{g}}^{-\lambda_k} \oplus \mathfrak{m}_k \oplus \mathfrak{a}_k \oplus \widetilde{\mathfrak{g}}^{-\lambda_k}$ we get

$$\int_{\mathcal{O}_k^+} \left(\mathcal{F}_u T_f^+\right)(u',v) \Delta_k(v)^{s_0+\cdots+s_k+\frac{dk}{4}} b_{\tau_k}^+(v)\, dv = \delta_0(s_0+\cdots+s_k+\frac{dk}{4},\tau_k)$$
$$\times \int_{v'\in\mathcal{O}_k^-} \left(\mathcal{F}_v\mathcal{F}_u T_f^+\right)(u',v') \nabla_k(v')^{-s_0-\cdots-s_k-\frac{dk}{4}-\frac{\ell}{2}} b_{\tau_k}^-(v')\, dv' .$$

From Lemma 6.6.5 and Theorem 4.8.1 we get

$$\mathcal{F}_v\mathcal{F}_u T_f^+ = \mathcal{F} T_f^+ = T_{\mathcal{F}f}^- .$$

The formula giving $Z^+(f,s,\tau)$ can now be written as

$$Z^+(f,s,\tau) = \delta_{k-1}(\underline{s},\underline{\tau})\delta_0(s_0+\cdots+s_k+\frac{dk}{4},\tau_k)$$
$$\times \int_{\underline{\mathcal{O}}^-}\int_{\mathcal{O}_k^-} T_{\mathcal{F}f}^-(u',v')\, D(u',v')\, du'dv' ,$$

where

$$D(u',v') = \underline{\nabla}(u')^{t(\underline{s})-\underline{m}} \nabla_k(v')^{-s_0-\cdots-s_k-\frac{dk}{4}-\frac{\ell}{2}} b_{\underline{\tau}}^-(u') \otimes b_{\tau_k}^-(v') .$$

Using the relations $\underline{m} = m - \frac{d}{4}$ and $m = \frac{\ell}{2} + \frac{kd}{4}$ we deduce from Lemma 5.3.7(2) and Proposition 6.6.4(2) that

$$D(u',v') = \nabla(u'+v')^{t(s)-m} \nabla_1(u')^{\frac{d}{4}} b_\tau^-(u'+v') .$$

From the integral formula proved in Proposition 4.6.4 (but this time on V^-) we obtain

$$Z^+(f,s,\tau) = \delta_{k-1}(\underline{s},\underline{\tau})\delta_0(s_0+\cdots+s_k+\frac{dk}{4},\tau_k) Z^-(\mathcal{F}f, t(s)-m,\tau) .$$

An induction on k gives the announced formula for δ_k.

□

6.7. Computation of the coefficients $\nu(s,\tau)$

The coefficient $\nu(s,\tau)$ defined in Theorem 6.4.5 is the inverse of the coefficient $\delta(t(s)-m,\tau)$ since we have $t(t(s)-m)-m = s$.

Theorem 6.7.1 (Type III)**.**
In the functional equation (6-4-6) *proved in Theorem* 6.4.5

$$Z^-(\mathcal{F}f,s,\tau) = \nu_k(s,\tau) Z^+(f,t(s)-m,\tau) \quad \left(f\in\mathcal{S}(V^+)\right) ,$$

the coefficient $s\mapsto\nu_k(s,\tau)$ *is the meromorphic function given by*

$$\nu_k(s,\tau) = (-1)^{|\tau|}\delta_k(s,\widetilde{\tau}) ,$$

where $\widetilde{\tau} = (\tau_k,\tau_{k-1},\ldots,\tau_1,\tau_0)$ *and* $|\tau| = \tau_0+\tau_1+\cdots+\tau_k$.

Remark 6.7.2. A proof similar to the proof given in Theorem 5.4.1 namely a proof using the involution γ which interchanges V^+ and V^- and the fact that $\overline{\mathcal{F}}\circ\mathcal{F} = \mathrm{Id}_{V^+}$ may be done. But here a direct computation using the explicit value of δ_k obtained in section 6.6 gives the result.

Proof. — We first give the value of $\nu_0(z,\tau_0)$ which is the inverse of $\delta_0(-z-\frac{\ell}{2},\tau_0)$, since we have $m=\frac{\ell}{2}$ if $k=0$ (Type III). From Theorem 6.5.2 we get

$$\delta_0(-z-\frac{\ell}{2},\tau_0)=\pi^{2z+\frac{\ell}{2}-1}\Gamma(-z-\frac{\tau_0}{2}-\frac{\ell}{2}+1)\Gamma(-z+\frac{\tau_0}{2})\alpha(z)\ ,$$

where

$$\alpha(z)=\begin{cases}\sin\pi(z+\frac{\ell}{2}) & \text{if } \tau_0 \text{ even}\ ,\\ -i\cos\pi(z+\frac{\ell}{2}) & \text{if } \tau_0 \text{ odd}\ .\end{cases}$$

Using the formula $\Gamma(z)\Gamma(1-z)=\dfrac{\pi}{\sin\pi z}$ we obtain

$$\delta_0(-z-\frac{\ell}{2},\tau_0)^{-1}=\pi^{-2z-\frac{\ell}{2}-1}\Gamma(z+\frac{\tau_0}{2}+\frac{\ell}{2})\Gamma(z-\frac{\tau_0}{2}+1)\beta(z)\ ,$$

where

$$\beta(z)=\begin{cases}\dfrac{\sin\pi(z+\frac{\tau_0}{2}+\frac{\ell}{2})\sin\pi(z-\frac{\tau_0}{2}+1)}{\sin\pi(z+\frac{\ell}{2})}=-\sin\pi z & \text{if } \tau_0 \text{ even}\ ,\\ \dfrac{\sin\pi(z+\frac{\tau_0}{2}+\frac{\ell}{2})\sin\pi(z-\frac{\tau_0}{2}+1)}{-i\cos\pi(z+\frac{\ell}{2})}=i\cos\pi z & \text{if } \tau_0 \text{ odd}\ .\end{cases}$$

It follows that

$$\nu_0(z,\tau_0)=\delta_0(-z-\frac{\ell}{2},\tau_0)^{-1}=(-1)^{\tau_0}\delta_0(z,\tau_0)\ . \tag{6-7-1}$$

We know from Theorem 6.6.6 that

$$\delta_k(s,\tau)=\prod_{j=0}^{k}\delta_0(s_0+\cdots+s_j+j\frac{d}{4},\tau_j)\ .$$

Recall that $m=\frac{\ell}{2}+k\frac{d}{4}$. Then if $s'=(s'_0,\dots,s'_k)$ denotes the element $t(s)-m$ we get

$$s'_0+\cdots s'_j+j\frac{d}{4}=-s_0-s_1-\cdots-s_{k-j}-(k-j)\frac{d}{4}-\frac{\ell}{2}\ .$$

Thus we have

$$\delta_k(t(s)-m,\tau)=\prod_{j=0}^{k}\delta_0\big(-(s_0+\cdots+s_j+j\frac{d}{4})-\frac{\ell}{2},\tau_{k-j}\big)\ .$$

Since $\nu_k(s,\tau)=\delta_k(t(s)-m,\tau)^{-1}$, the result is obtained from formula (6-7-1). □

CHAPTER 7

Zeta function attached to a representation in the minimal spherical principal series

In this chapter, we will define the main object of this article, namely the Zeta function attached to a representation in the minimal spherical principal series. We will also prove a functional equation for such a Zeta function (see Theorem below). This functional equation is the main result of the paper.

Let $\widetilde{\mathfrak{g}} = V^- \oplus \mathfrak{g} \oplus V^+$ be a regular graded Lie algebra. We will denote by $(\Omega_p^+)_{p=0,\dots,r}$ the collection of open G–orbits in V^+ (the number of open G–orbits is $r+1$). Let $(X_j)_{j=0,\dots,k}$ be the sequence satisfying condition (D) chosen in section 3.6. For each p we choose once and for all an element $I_p^+ = \sum_{j=0}^k \varepsilon_j^p X_j$ $(\varepsilon_j^p = \pm 1)$ belonging to Ω_p^+. Let us recall how this choice may be done for each Type.

- Type I (see Theorem 2.10.1 and section 5.6):

$$r = k+1 \quad \text{and} \quad I_p^+ = \Big(\sum_{j=0}^{k-p} X_j\Big) - \Big(\sum_{j=k-p+1}^{k} X_j\Big) \ .$$

- Type II (see Theorem 2.7.3 and section 5.6):

$$r = 1, \quad I_0^+ = I^+ = \sum_{j=0}^{k} X_j \quad \text{and} \quad I_1^+ = \Big(\sum_{j=0}^{k-1} X_j\Big) - X_k \ .$$

- Type III (see Theorem 2.6.2) :

$$r = 0 \quad \text{and} \quad I_0^+ = I^+ = \sum_{j=0}^{k} X_j \ .$$

In any case the index p in I_p^+ denotes the number of coefficients ε_j^p in the decomposition $I_p^+ = \sum_{j=0}^k \varepsilon_j^p X_j$ which are equal to -1.

We also define $I_p^- = \theta(I_p^+)$ $(p = 0, \dots, r)$ and the open G-orbit $G.I_p^-$ will be denoted by Ω_p^-. The collection of open G-orbits in V^- is given by $(\Omega_p^-)_{p=0,\dots,r}$.

We know from Chapter 3, that if $H_p = Z_G(I_p^+) = Z_G(I_p^-)$, then

$$\Omega_p^+ \simeq G/H_p \simeq \Omega_p^-$$

is a symmetric space defined by the involution $\sigma_p = e^{\operatorname{ad} I_p^+} e^{\operatorname{ad} I_p^-} e^{\operatorname{ad} I_p^+}$.

7.1. The minimal spherical principal series

For a general symmetric space G/H, van den BAN has defined in ([vdB]–1988) the so-called minimal spherical principal series of representations of G. The representations in this series do really have (generically) nonzero H–invariant distribution vectors, and those which are unitary, give the most continuous part of the Plancherel formula for G/H.

In this section we will describe these representations in some details for the symmetric spaces G/H_p. These representations are a subset of the principal series of representations induced from the minimal $\sigma_p\theta$-stable parabolic subgroup P introduced in section 3.5.

Remark 7.1.1. The first key fact to be noticed in our context is that the parabolic subgroup P is minimal $\sigma_p\theta$-stable for all $p = 0, \ldots, r$ (see Proposition 3.5.1). It follows that the principal series induced from the minimal $\sigma_p\theta$-stable parabolic subgroup for the various σ_p will be induced from the same P.

Therefore we will in this section and in section 7.2 denote by H one of the subgroup H_p and by σ the corresponding involution σ_p.

Recall from Theorem 3.5.3 and the remarks before it, that there is a finite number of open P–orbits in G/H which are parametrised by $W^0/W^{0\sigma}$. We will denote by $\mathcal{W}$ a set of representatives of $W^0/W^{0\sigma}$ in $N_K(\mathfrak{a}^0)$.

Let us now recall the definition of the principal series of representations induced from $P = MAN$. Recall that $\mathfrak{m}$, $\mathfrak{a}$ and $\mathfrak{n}$ are the Lie algebras of M, A and N respectively. As usually we define $\rho \in \mathfrak{a}^*$ by

$$\forall X \in \mathfrak{a}, \quad \rho(X) = \frac{1}{2}\operatorname{tr}(\operatorname{ad} X_{|\mathfrak{n}}).$$

Let $(\tau, \mathcal{H}_\tau)$ be a unitary representation of M and $(\ |\ )_\tau$ the Hilbert product on $\mathcal{H}\tau_\tau$. For $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ (the dual of the complexification of $\mathfrak{a}$), let $\mathcal{C}_{\tau,\lambda}(G, \mathcal{H}_\tau)$ be the space of continuous function $\Phi : G \longrightarrow \mathcal{H}_\tau$ such that

$$\Phi(mang) = a^{\lambda+\rho}\tau(m)\Phi(g) \quad (m \in M,\ a \in A,\ n \in N, g \in G)\ . \tag{7-1-1}$$

Since $G = PK$, the sesquilinear form defined on $\mathcal{C}_{\tau,\lambda}(G, \mathcal{H}_\tau)$ by

$$(\Phi_1|\Phi_2) = \int_K \big(\Phi_1(k)|\Phi_2(k)\big)_\tau\, dk\ ,$$

is positive definite. Let $\mathcal{H}_{\tau,\lambda}$ be the Hilbert space obtained by completion with respect to this scalar product. The right regular action of G on $\mathcal{C}_{\tau,\lambda}(G, \mathcal{H}_\tau)$ given by

$$\pi_{\tau,\lambda}(g)\Phi(x) = \Phi(xg), \quad \Phi \in \mathcal{C}_{\tau,\lambda}(G, \mathcal{H}_\tau)\ .$$

extends to a bounded operator on $\mathcal{H}_{\tau,\lambda}$ (also denoted $\pi_{\tau,\lambda}(g)$) which defines a continuous representation $\pi_{\tau,\lambda}$ of G on this Hilbert space.

The spherical principal series are the principal series which satisfy some additional assumptions given by Van den BAN

Definition 7.1.2 (Van den BAN's conditions).
The representation $\pi_{\tau,\lambda}$ belongs to the minimal spherical principal series if
(a) $\lambda_{|\mathfrak{a}\cap\mathfrak{h}} = 0$;
(b) *there exists $w \in \mathcal{W}$ such that the representation $w\tau$ of M (defined by $w\tau(m) = \tau(wmw^{-1})$ for $m \in M$) has a nonzero $(M \cap H)$–invariant distribution vector.*

Let us examine these conditions in our situation.

Concerning condition (a), it should be noticed that, as a consequence of Proposition 3.1.4, the set $\mathfrak{a} \cap \mathfrak{h}$ is independent from the choice of $H = H_p$. Moreover we have $\mathfrak{a} = (\mathfrak{a} \cap \mathfrak{h}) \oplus \mathfrak{a}^0$ and therefore condition (a) reduces to $\lambda \in (\mathfrak{a}^0_{\mathbb{C}})^*$ with λ extended to $\mathfrak{a}^*_{\mathbb{C}}$ by the condition $\lambda_{|\mathfrak{a}\cap\mathfrak{h}} = 0$.

Remark 7.1.3. As $\sigma\theta$ stabilizes $\mathfrak{n}$, we have

$$\mathrm{tr}\big(\mathrm{ad}(\sigma\theta X)_{|\mathfrak{n}}\big) = \mathrm{tr}\big(\mathrm{ad}(X)_{|\mathfrak{n}}\big) \text{ for } X \in \mathfrak{a} .$$

Thus we get $\rho(X) = 0$ for $X \in \mathfrak{a} \cap \mathfrak{h}$ ([vdB]–1988, Lemma 3.1). It follows that ρ may also be considered as an element of $(\mathfrak{a}^0_{\mathbb{C}})^*$.

Concerning condition (b) the situation depends on the Type of the regular graded algebra. For Type I the quotient $M/M \cap H$ is trivial (Theorem 3.5.4) and therefore $w\tau$ and τ are the same representation, namely the trivial representation. For Type II and III there is only one open P-orbit in G/H (see Theorem 3.5.3) and therefore $\mathcal{W}$ is trivial. Hence in any cases condition (b) reduces to the condition

(b') *τ is a representation of M which has a nonzero $(M \cap H)$-invariant distribution vector.*

As $M/M \cap H$ is compact such a representation τ is always finite dimensional (Proposition 6.1.1).

Representations satisfying condition (b') are described in Theorem 6.1.3 for Type III and are trivial for Type I.
Here is the description of such representations for Type II.

Lemma 7.1.4 (Type II). *In Type II case we define for $m \in M$ the elements $m_j = \pm 1$ by $m.X_j = m_j X_j$ $(j = 0, \ldots, k)$. Then any $(M \cap H)$-spherical representation τ is given by a character of M of the form*

$$\tau(m) = (m_1 \ldots m_k)^{\tau_1}(m_2 \ldots m_k)^{\tau_2} \ldots m_k^{\tau_k} \quad \textit{where } \tau_j \in \mathbb{Z}/2\mathbb{Z} .$$

Proof. — In Type II case there are only two open G-orbits in V^+, namely $\Omega_0^+ = GI^+ \simeq G/H_0$, and $\Omega_1^+ = GI_1^+ \simeq G/H_1$. We know from the proof of Theorem 3.5.4 that $m_j \in \{+1, -1\}$ and that $m_0 m_1 \ldots m_k = 1$. Let $\mathcal{M}$ be the multiplicative group of such sequences $(m_0, m_1, \ldots, m_k)$. The proof of Theorem 3.5.4 also shows that the map $m \mapsto (m_0, m_1, \ldots, m_k)$ is a surjective group homomorphism from M onto $\mathcal{M}$ whose kernel is $M \cap H$. Hence we have $M \cap H_0 = M \cap H_1$ and $M/M \cap H_0 = M/M \cap H_1$ is a commutative group isomorphic to $\mathcal{M}$. It is then a straightforward exercise,

using Proposition 6.1.1, to show that any representation τ of $\mathcal{M}$ with an $(M \cap H)$–invariant vector has the form given in the Lemma. □

Remark 7.1.5. It may be noticed that for all Types, representations τ of M having a $(M \cap H_p)$–invariant distribution vector have also, for all $q \in \{0, \dots, r\}$, a $(M \cap H_q)$–invariant vector. Therefore we will use the mention $(M \cap H)$–spherical representations to characterize these representations of M.

In our context the minimal spherical principal series can be defined as follows.

Definition 7.1.6. *The representation $(\pi_{\tau,\lambda}, \mathcal{H}_{\tau,\lambda})$ belongs to the minimal spherical principal series if*

(1) $\lambda \in (\mathfrak{a}_{\mathbb{C}}^0)^*$;

(2) $\begin{cases} \tau \text{ is the trivial representation for Type I,} \\ \tau \text{ is defined as in Lemma 7.1.4 for Type II ,} \\ \tau \text{ is one of the } (M \cap H)\text{–spherical representations determined} \\ \quad \text{in section 6.1 for Type III.} \end{cases}$

Remark 7.1.7. In Type I case, it follows from Proposition 3.2.2 that the parabolic subgroup P is minimal and that $H_0 = K$. Since τ is trivial on M, $(\pi_{\tau,\lambda}, \mathcal{H}_{\tau,\lambda})$ is a representation of class 1 (or K–spherical) of the minimal principal series. Thus there exists a K–invariant vector in the space of this representation. This will be generalized (in section 7.4) to each subgroup H_p in terms of H_p–invariant distribution vectors.

7.2. Summary with unified notations

In this section we will summarize some results obtained in the previous chapter by using unified notations for the three Types. For any Type a representation τ as defined in 7.1.6 will occur in the definition of the Zeta functions.

Let $p \in \{0, \dots r\}$. Recall from Theorem 3.5.3 that the open P-orbits in G/H_p are parametrized by a set $\mathcal{W}_p$ of representatives of $W^0/W^{0\sigma_p}$ in $N_K(\mathfrak{a}^0)$. Let us recall the description of these orbits in the different cases (see Theorem 3.6.3).

• Type I : The open P–orbits $\mathcal{O}_\varepsilon^+$ in V^+ and $\mathcal{O}_\varepsilon^-$ in V^- are parametrized by $\varepsilon \in \{\pm 1\}^{k+1}$ and given by

$$\mathcal{O}_\varepsilon^\pm = P.I_\varepsilon^\pm \quad \text{where } I_\varepsilon^+ = \sum_{j=0}^{k} \varepsilon_j X_j \text{ and } I_\varepsilon^- = \theta(I_\varepsilon^+) .$$

Moreover $\mathcal{O}_\varepsilon^\pm$ is a subset of $\Omega_p^\pm$ if and only if $\varepsilon \in \underline{p}$ where (Notation 5.7.2)

$$\underline{p} = \{\varepsilon \in (\pm 1)^{k+1} \mid \varepsilon \text{ has } p \text{ components equal to } -1\} .$$

For $\varepsilon \in \underline{p}$ we choose an element $\gamma_\varepsilon \in N_K(\mathfrak{a}^0)$ such that $\gamma_\varepsilon I_p^+ = I_\varepsilon^+$ (hence $\gamma_\varepsilon I_p^- = I_\varepsilon^-$). Then

$$\mathcal{W}_p = \{\gamma_\varepsilon \mid \varepsilon \in \underline{p}\} \quad (p = 0, \dots, k+1)$$

can be taken as a set of representatives of $W^0/W^{0\sigma_p}$.

• Type II : Any open G–orbit contains only one open P-orbit namely

$$\mathcal{O}_0^+ = PI_0^+ \subset \Omega_0^+ \quad \text{and } \mathcal{O}_1^+ = PI_1^+ \subset \Omega_1^+ \ ,$$
$$\mathcal{O}_0^- = PI_0^- \subset \Omega_0^- \quad \text{and } \mathcal{O}_1^- = PI_1^+ \subset \Omega_1^- \ .$$

Therefore $\mathcal{W}_p$ $(p = 0, 1)$ contains an unique element, the neutral element e.

• Type III : In this case $\Omega^+ = \Omega_0^+$ and $\Omega^- = \Omega_0^-$ contain only one open P– orbit, namely

$$\mathcal{O}^+ = PI^+ = PI_0^+ \subset \Omega^+ \quad \text{and} \quad \mathcal{O}^- = PI^- = PI_0^- \subset \Omega^-$$

Therefore $\mathcal{W}_p$ $(p = 0)$ contains an unique element, the neutral element e.

Definition 7.2.1. *For $\gamma \in \mathcal{W}_p$ $(p = 0, \dots, r)$ we will denote by $\mathcal{O}_{p,\gamma}^{\pm}$ the open P–orbit in $\Omega_p^{\pm}$ defined by an element $\gamma \in \mathcal{W}_p$. More precisely we have $\mathcal{O}_{p,\gamma}^{\pm} = P.I_{p,\gamma}^{\pm}$ where*

$$I_{p,\gamma}^{\pm} = I_p^{\pm} \text{ for Type II and Type III} \ ,$$
$$I_{p,\gamma}^{\pm} = I_\varepsilon^{\pm} \text{ if } \gamma.I_p^{\pm} = I_\varepsilon^{\pm} \text{ for Type I} \ .$$

Recall that, in Type III case, we have defined (Theorem 6.2.1) two maps $b_\tau^+ : \mathcal{O}^+ \longrightarrow \mathcal{H}_\tau$ and $b_\tau^- : \mathcal{O}^- \longrightarrow \mathcal{H}_\tau$ by the relations

$$\begin{cases} b_\tau^+(I^+) = v_\tau, \\ b_\tau^+(manX) = \tau(m)b_\tau^+(X) \ , \end{cases} \quad \text{and} \quad \begin{cases} b_\tau^-(I^-) = v_\tau \\ b_\tau^-(manY) = \tau(m)b_\tau^-(Y) \ , \end{cases}$$

where $m \in M$, $a \in A$, $n \in N, X \in \mathcal{O}^+$ and $Y \in \mathcal{O}^-$. This definition extends naturally to the other Types as follows.

• Type I : In this case $\mathcal{H}_\tau = \mathbb{C}$ and the representation τ is trivial. Therefore we define b_τ^+ on $\mathcal{O}^+ = \cup_{\varepsilon \in \{\pm 1\}^{k+1}} \mathcal{O}_\varepsilon^+$ and b_τ^- on $\mathcal{O}^- = \cup_{\varepsilon \in \{\pm 1\}^{k+1}} \mathcal{O}_\varepsilon^-$ by

$$b_\tau^+(X) = 1 \ \text{ for } X \in \mathcal{O}^+ \quad \text{and} \quad b_\tau^-(Y) = 1 \ \text{ for } Y \in \mathcal{O}^- \ .$$

• Type II : In this case $\mathcal{H}_\tau = \mathbb{C}$ and τ is a character (described in Lemma 7.1.4) which is trivial on $M \cap H_p$. Therefore we define b_τ^+ on $\mathcal{O}_0^+ \cup \mathcal{O}_1^+$ and b_τ^- on $\mathcal{O}_0^- \cup \mathcal{O}_1^-$ by

$$b_\tau^+(manI_p^+) = \tau(m) \quad \text{and} \quad b_\tau^-(manI_p^-) = \tau(m) \ ,$$

for $m \in M$, $a \in A$, $n \in N$ and $p \in \{0, 1\}$.

Finally we set the following unified definition.

Definition 7.2.2. *Let τ be a $(M \cap H)$–invariant spherical representation of M. The map $b_\tau^+ : \mathcal{O}^+ \longrightarrow \mathcal{H}_\tau$ and the map $b_\tau^- : \mathcal{O}^- \longrightarrow \mathcal{H}_\tau$ are charaterized as follows. For $m \in M$, $a \in A$, $n \in N, X \in \mathcal{O}^+$, $Y \in \mathcal{O}^-$, $p \in \{0, \dots, r\}$ and $\gamma \in \mathcal{W}_p$ these maps satisfy the relations*

$$\begin{cases} b_\tau^+(I_{p,\gamma}^+) = v_\tau, \\ b_\tau^+(manX) = \tau(m)b_\tau^+(X) \ , \end{cases} \quad \textit{and} \quad \begin{cases} b_\tau^-(I_{p,\gamma}^-) = v_\tau \ , \\ b_\tau^-(manY) = \tau(m)b_\tau^-(Y) \ , \end{cases}$$

where $v_\tau = 1$ in Type I and Type II cases and v_τ is the $M \cap H$–invariant vector of τ given in Definition 6.1.6 in Type III case.

This allows us to give the following definition.

Definition 7.2.3. *Let τ be an irreducible $(M \cap H)$–spherical representation of M, γ be an element of $\mathcal{W}_p$ $(p \in \{0, \dots, r\})$ and $s \in \mathbb{C}^{k+1}$. For $f \in \mathcal{S}(V^+)$, $h \in \mathcal{S}(V^-)$ and $\operatorname{Re} s$ large we define elements $Z^+_{p,\gamma}(f, s, \tau) \in \mathcal{H}_\tau$ and $Z^-_{p,\gamma}(h, s, \tau) \in \mathcal{H}_\tau$ by*

$$Z^+_{p,\gamma}(f, s, \tau) = \int_{\mathcal{O}^+_{p,\gamma}} f(X)|\Delta(X)|^s \, b^+_\tau(X)\, dX \ ,$$
$$Z^-_{p,\gamma}(h, s, \tau) = \int_{\mathcal{O}^-_{p,\gamma}} h(Y)|\nabla(Y)|^s \, b^-_\tau(Y)\, dY \ .$$

Let us make more explicit the relationship between these integrals and the Zeta functions defined in Chapter 5 and Chapter 6. This will show in particular that the integrals defining $Z^\pm_{p,\gamma}$ are convergent for $\operatorname{Re} s$ large .

• Type I : In this case $\mathcal{H}_\tau = \mathbb{C}$, τ is trivial $(\tau = 1)$ and $b^\pm_\tau = 1$. If ε is the element of $\underline{p}$ such that $\gamma I^+_p = I^+_\varepsilon$ and $\gamma I^-_p = I^-_\varepsilon$ we get

$$Z^+_{p,\gamma}(f, s, 1) = Z^+_\varepsilon(f, s) \quad \text{and} \quad Z^-_{p,\gamma}(h, s, 1) = Z^-_\varepsilon(h, s) \ .$$

• Type II : In this case $p \in \{0, 1\}$, $\gamma = e$ and $\mathcal{H}_\tau = \mathbb{C}$. From Lemma 7.1.4 we have for $m \in M$

$$\tau(m) = (m_1 \dots m_k)^{\tau_1}(m_2 \dots m_k)^{\tau_2} \dots m_k^{\tau_k} \ ,$$

where $\tau_j \in \mathbb{Z}/2\mathbb{Z}$ and the elements $m_j \in \{\pm 1\}$ are defined by $m.X_j = m_j X_j$ for $j = 0, \dots, k$. Moreover we have $m_0 m_1 \cdots m_k = 1$. As we have

$$mI^+_0 = m_0 X_0 + \cdots + m_{k-1} X_{k-1} + m_k X_k \ ,$$
$$mI^+_1 = m_0 X_0 + \cdots + m_{k-1} X_{k-1} - m_k X_k \ ,$$

it follows from Lemma 5.6.1 that for $u = (u_0, u_1, \dots, u_k) \in (\mathbb{Z}/2\mathbb{Z})^{k+1}$ we have

$$\operatorname{sg}\big(\Delta(mI^+_p)^u\big) = \operatorname{sg}\big(\nabla(mI^-_p)^{t(u)}\big) = (m_1 \cdots m_k)^{u_1}(m_2 \cdots m_k)^{u_2} \cdots m_k^{u_k} \times \begin{cases} 1 & \text{if } p = 0, \\ (-1)^{u_0+u_1+\cdots+u_k} & \text{if } p = 1 \ . \end{cases}$$

Moreover, for $m \in M$, $a \in A$ and $n \in N$ we have

$$\operatorname{sg}\big(\Delta(manI^+_p)^u\big) = \operatorname{sg}\big(\Delta(mI^+_p)^u\big) = \operatorname{sg}\big(\nabla(manI^-_p)^{t(u)}\big) \ .$$

Therefore we obtain a simple relation between $b^\pm_\tau$ and the sign of some power of the relative invariants.

Lemma 7.2.4 (Type II)**.** *Let τ be an irreductible $(M \cap H)$–spherical representation of M parametrized by $(\tau_1, \dots, \tau_k) \in (\mathbb{Z}/2\mathbb{Z})^k$. If we define $u_\tau \in (\mathbb{Z}/2\mathbb{Z})^{k+1}$ by*

$$u_\tau = (\tau_1 + \cdots + \tau_k, \tau_1, \dots, \tau_k) \ ,$$

we have for $p \in \{0,1\}$

$$X \in \mathcal{O}_0^+ \cup \mathcal{O}_1^+ \Longrightarrow b_\tau^+(X) = \mathrm{sg}\big(\Delta(X)^{u_\tau}\big) \ ,$$
$$Y \in \mathcal{O}_0^- \cup \mathcal{O}_1^- \Longrightarrow b_\tau^-(Y) = \mathrm{sg}\big(\nabla(Y)^{t(u_\tau)}\big) \ .$$

From Definition 5.6.2, this result implies that

$$Z_{p,e}^+(f,s,\tau) = Z_p^+(f,s,u_\tau) \quad \text{and} \quad Z_{p,e}^-(h,s,\tau) = Z_p^-(h,s,u_\tau) \ .$$

- Type III : In this case $p=0$ and $\gamma = e$. Then we get

$$Z_{0,e}^+(f,s,\tau) = Z^+(f,s,\tau) \quad \text{and} \quad Z_{0,e}^-(h,s,\tau) = Z^-(h,s,\tau)) \ .$$

We can now summarize the properties of the Zeta functions obtained in chapter 5 and 6 as follows.

Proposition 7.2.5. *Let* τ *be an irreducible* $(M \cap H)$*–spherical representation of* M*. For* $f \in \mathcal{S}(V^+)$ *and* $h \in \mathcal{S}(V^-)$ *the integrals defining* $Z_{p,\gamma}^+(f,s,\tau) \in \mathcal{H}_\tau$ *and* $Z_{p,\gamma}^-(h,s,\tau) \in \mathcal{H}_\tau$ *have meromorphic continuations to* $\mathbb{C}^{k+1}$ *which define* $\mathcal{H}_\tau$*–valued distributions.*

Moreover the open subset $\mathcal{U}^+$ *(resp.* $\mathcal{U}^-$*) of* $\mathbb{C}^{k+1}$ *on which* $Z_{p,\gamma}^+(f,s,\tau)$ *(resp.* $Z_{p,\gamma}^-(h,s,\tau)$*) is holomorphic for all* $f \in \mathcal{S}(V^+)$ *(resp.* $h \in \mathcal{S}(V^-)$*), for all* $p \in \{0,\dots,r\}$ *and for all* $\gamma \in \mathcal{W}_p$ *contains the complementary set of a countable union of hyperplanes.*

Proof. — For the positive side, the first part of the Proposition is a consequence of Lemma 5.2.3 and Lemma 5.2.5 in Type I and Type II cases and of Proposition 6.4.3 in Type III case.

In Type I case we take for $\mathcal{U}^+$ the intersection for $\varepsilon \in \{\pm 1\}$ of the open subsets $\widetilde{U}_\varepsilon$ defined in Lemma 5.2.5. From lemma 5.2.3 it follows that $\mathcal{U}^+$ contains the complementary set of a countable union of hyperplanes. In Type II case $\mathcal{U}^+$ is given by the same Definition since $Z_{p,e}^+(f,s,u_\tau)$ is a linear combination of $Z_\varepsilon(f,s)$ with coefficients in $\{\pm 1\}$. In Type III case we take the open subset $\widetilde{U}$ defined in Proposition 6.4.3.

Similar arguments give the result on the negative side.

□

Using the previous notations, we can now write, in a unique form, the functional equations obtained in Theorem 5.2.9 for Type I, in Theorem 5.6.6 for Type II and in Theorem 6.4.5 for Type III.

Theorem 7.2.6. *Let* τ *be an irreducible* $(M \cap H)$*–spherical representation of* M*. The Zeta functions* $Z_{p,\gamma}^+(f,s,\tau)$ *and* $Z_{p,\gamma}^-(h,s,\tau)$ *satisfy for* $f \in \mathcal{S}(V^+)$ *and* $h \in \mathcal{S}(V^-)$ *the following functional equations*

$$Z_{p,\gamma}^+(f,s,\tau) = \sum_{q=0}^{r} \sum_{\gamma' \in \mathcal{W}_q} \delta_k(s,\tau,p,\gamma,q,\gamma') Z_{q,\gamma'}^-(\mathcal{F}f, t(s)-m, \tau) \ ,$$

$$Z_{q,\gamma'}^-(\mathcal{F}f,s,\tau) = \sum_{p=0}^{r} \sum_{\gamma \in \mathcal{W}_p} \nu_k(s,\tau,q,\gamma',p,\gamma) Z_{p,\gamma}^+(f, t(s)-m, \tau) \ ,$$

where the meromorphic functions $\delta_k(s,\tau,p,\gamma,q,\gamma')$ *and* $\nu_k(s,\tau,q,\gamma',p,\gamma)$ *are defined for* $p,q \in \{0,\dots,r\}$*,* $\gamma \in \mathcal{W}_p$ *and* $\gamma' \in \mathcal{W}_q$ *as follows.*

• Type I: *here* $\tau = 1$. *If* $\gamma I_p^+ = I_\varepsilon^+$ *and* $\gamma' I_q^- = I_\eta^-$ *we have*

$$\delta_k(s,1,p,\gamma,q,\gamma') = \delta_k(s,\varepsilon,\eta) \quad \text{and} \quad \nu_k(s,1,q,\gamma',p,\gamma) = \nu_k(s,\eta,\varepsilon) \ ,$$

where $\delta_k(s,\varepsilon,\eta)$ *and* $\nu_k(s,\eta,\varepsilon)$ *have been computed in Theorem 5.3.10 and in Theorem 5.4.1 respectively.*

• Type II: *here* $\gamma = \gamma' = e$ *and we have*

$$\delta_k(s,\tau,p,e,q,e) = \delta_k(s,\tau,p,q) \quad \text{and} \quad \nu_k(s,\tau,q,e,p,e) = \nu_k(s,\tau,q,p) \ ,$$

where $\delta_k(s,\tau,p,q)$ *and* $\nu_k(s,\tau,q,p)$ *are defined in Definition 5.6.5.*

• Type III: *here* $p = q = 0$, $\gamma = \gamma' = e$ *and we have*

$$\delta_k(s,\tau,0,e,0,e) = \delta_k(s,\tau) \quad \text{and} \quad \nu_k(s,\tau,0,e,0,e) = \nu_k(s,\tau) \ ,$$

where $\delta_k(s,\tau)$ *and* $\nu_k(s,\tau)$ *have been computed in Theorem 6.6.6 and Theorem 6.7.1.*

7.3. Differentiable vectors and distribution vectors for the minimal principal spherical series

Let $\mathcal{H}_{\tau,\lambda}^\infty$ be the subspace of differentiable vectors in $\mathcal{H}_{\tau,\lambda}$ endowed with its usual Fréchet space topology.

Proposition 7.3.1. *The space* $\mathcal{H}_{\tau,\lambda}^\infty$ *of differentiable vectors for* $\pi_{\tau,\lambda}$ *is the space of* $\mathcal{C}^\infty$ *functions* $\Phi : G \longrightarrow \mathcal{H}_\tau$ *such that*

$$\Phi(mang) = a^{\lambda+\rho}\tau(m)\Phi(g) \quad (m \in M,\ a \in A,\ n \in N, g \in G) \ . \tag{7-1-1}$$

Proof. — From DIXMIER– MALLIAVIN's Theorem ([D–M]–1978) we know that the space of differentiable vectors of a continuous representation (here the representation$(\pi_{\tau,\lambda}, \mathcal{H}_{\tau,\lambda})$) is generated by elements of the form

$$\pi_{\tau,\lambda}(\varphi)u = \int_G \varphi(g)\pi_{\tau,\lambda}(g)u \ dg \quad \text{where } u \in \mathcal{H}_{\tau,\lambda},\ \varphi \in \mathcal{C}_c^\infty(G) \ .$$

Since $\mathcal{H}_\tau$ is finite–dimensional, it follows that $\mathcal{H}_{\tau,\lambda}^\infty$ is contained in the space of $\mathcal{C}^\infty$ function $\Phi : G \longrightarrow \mathcal{H}_\tau$ satisfying condition (7-1-1). And it is easy to see that we have in fact equality between these spaces.

□

Let $\mathcal{C}_c^\infty(G, \mathcal{H}_\tau)$ be the space of $\mathcal{H}_\tau$-valued $\mathcal{C}^\infty$ functions with compact support on G. We normalize the Haar measure dk on K such that $\int_K dk = 1$, and then normalize the right Haar measure on dp on P such that

$$dg = \delta(p)dpdk \text{ where } \delta(man) = a^{-2\rho} \quad (m \in M,\ a \in A,\ n \in N) \ .$$

Then the map

$$L_{\tau,\lambda} : \mathcal{C}_c^\infty(G, \mathcal{H}_\tau) \longrightarrow \mathcal{H}_{\tau,\lambda}^\infty$$

$$\Psi \longmapsto L_{\tau,\lambda}\Psi(g) = \int_P a^{-\rho-\lambda}\tau(m)^{-1}\Psi(pg)\, dp \quad (p = man)$$

is well known to be a surjective G-equivariant continuous map (G acting on the right hand side). Moreover if $p_0 = m_0 a_0 n_0$ and if $\ell(p_0)\Psi$ denotes the function $g \mapsto \Psi(p_0^{-1} g)$ then we have

$$L_{\tau,\lambda}\big(\ell(p_0)\Psi\big) = a_0^{\rho-\lambda} L_{\tau,\lambda}(\tau(m_0)^{-1}\Psi) \ . \tag{7-3-1}$$

Recall that the space $\mathcal{H}^{-\infty}_{\tau,\lambda}$ of distribution vectors for $\pi_{\tau,\lambda}$ is the anti-dual of the space $\mathcal{H}^{\infty}_{\tau,\lambda}$ of $\mathcal{C}^{\infty}$ vectors of the representation. There is a natural representation $\pi_{-\infty}$ of the group G on $\mathcal{H}^{-\infty}_{\tau,\lambda}$ given by

$$\langle \pi_{-\infty}(g)a, \Phi\rangle = \langle a, \pi(g)^{-1}\Phi\rangle \quad \text{for } a \in \mathcal{H}^{-\infty}_{\tau,\lambda},\ \Phi \in \mathcal{H}^{\infty}_{\tau,\lambda} \ .$$

When no confusion can arrive we will use the notation π instead of $\pi_{-\infty}$.

The image by the transpose of $L_{\tau,\lambda}$ of an element $a \in \mathcal{H}^{-\infty}_{\tau,\lambda}$ is an element T in the anti-dual of $\mathcal{C}^{\infty}_c(G, \mathcal{H}_\tau)$ given by

$$\langle T, \Psi\rangle = \langle a, L_{\tau,\lambda}\Psi\rangle \quad \text{for } \Psi \in \mathcal{C}^{\infty}_c(G, \mathcal{H}_\tau) \ .$$

Proposition 7.3.2. *The transpose of $L_{\tau,\lambda}$ is a G–equivariant isomorphism from $\mathcal{H}^{-\infty}_{\tau,\lambda}$ onto the subspace of elements T belonging to the anti-dual of $\mathcal{C}^{\infty}_c(G, \mathcal{H}_\tau)$ such that*

$$\langle T, \ell(man)\Psi\rangle = a^{\rho-\overline{\lambda}}\langle T, \tau(m)^{-1}\Psi\rangle \quad \textit{for } \Psi \in \mathcal{C}^{\infty}_c(G, \mathcal{H}_\tau) \ . \tag{7-3-2}$$

If we denote by $\mathcal{D}'_{\tau,-\overline{\lambda}}(G, \mathcal{H}_\tau)$ the space of antilinear forms on $\mathcal{C}^{\infty}_c(G, \mathcal{H}_\tau)$ satisfying condition (7-3-2) then we have

$$\mathcal{H}^{-\infty}_{\tau,\lambda} \simeq \mathcal{D}'_{\tau,-\overline{\lambda}}(G, \mathcal{H}_\tau) \ .$$

Proof. — The properties of $L_{\tau,\lambda}$ imply that its transpose, ${}^t L_{\tau,\lambda}$, is G–equivariant, injective and continuous into the anti–dual of $\mathcal{C}^{\infty}_c(G, \mathcal{H}_\tau)$. Moreover the relation (7-3-1) implies that the image of ${}^t L_{\tau,\lambda}$ is a subspace of $\mathcal{D}'_{\tau,-\overline{\lambda}}(G, \mathcal{H}_\tau)$.

To prove that the image is the space $\mathcal{D}'_{\tau,-\overline{\lambda}}(G, \mathcal{H}_\tau)$ we use a positive function $w \in \mathcal{C}^{\infty}_c(G)$ such that

$$\int_P w(pg)\, dp = 1 \quad \text{for all } g \in G \ .$$

(See for example [Kn] p. 140 for the existence of such a function). It may be noticed that for $\Phi \in \mathcal{H}^{\infty}_{\tau,\lambda}$ we have $w\Phi \in \mathcal{C}^{\infty}_c(G, \mathcal{H}_\tau)$ and $L_{\tau,\lambda}(w\Phi) = \Phi$ which implies in particular the surjectivity of $L_{\tau,\lambda}$.

Let T be an element of $\mathcal{D}'_{\tau,-\overline{\lambda}}(G, \mathcal{H}_\tau)$ and a the antilinear form on $\mathcal{H}^{\infty}_{\tau,\lambda}$ given by

$$\langle a, \Phi\rangle = \langle T, w\Phi\rangle \quad (\Phi \in \mathcal{H}^{\infty}_{\tau,\lambda}) \ .$$

Then we will prove that $T = {}^t L_{\tau,\lambda}(a)$ which is equivalent to prove that

$$\langle T, \Psi\rangle = \langle a, L_{\tau,\lambda}\Psi\rangle = \langle T, w.L_{\tau,\lambda}\Psi\rangle \quad \text{for } \Psi \in \mathcal{C}^{\infty}_c(G, \mathcal{H}_\tau) \ .$$

Using the continuity of T and the properties of w we have

$$\begin{aligned}\langle T, \Psi\rangle &= \langle T_g, \int_P w(pg)\, dp\ \Psi(g)\rangle \\ &= \int_P \langle T, \ell(p)^{-1}(w\,\ell(p)\Psi)\rangle\, dp \ .\end{aligned}$$

Since T satisfies relation (7-3-2) we get

$$\langle T, \Psi \rangle = \int_P a^{-\rho+\overline{\lambda}} \langle T, \tau(m)^{-1}(w\,\ell(p)\Psi)\rangle \; dp \quad (p = man)$$
$$= \langle T_g, w(g) \int_P a^{-\rho+\lambda} \tau(m)^{-1} \Psi(p^{-1}g) \; dp \rangle \; .$$

Using the change of variables $p \mapsto p^{-1}$ we obtain the result from the definition of $L_{\tau,\lambda}$.

□

There is a natural G–equivariant injection from $\mathcal{C}_{\tau,-\overline{\lambda}}(G, \mathcal{H}_\tau)$ into the space $\mathcal{D}'_{\tau,-\overline{\lambda}}(G, \mathcal{H}_\tau)$ given by the application $\Upsilon \mapsto T_\Upsilon$ where

$$\langle T_\Upsilon, \Psi \rangle = \int_G \big(\Upsilon(g) \mid \Psi(g)\big)_\tau \, dg \quad \text{for } \Psi \in \mathcal{C}_c^\infty(G, \mathcal{H}_\tau) \; .$$

Lemma 7.3.3. *For $\Upsilon \in \mathcal{C}_{\tau,-\overline{\lambda}}(G, \mathcal{H}_\tau)$ we have for any $\Psi \in \mathcal{C}_c^\infty(G, \mathcal{H}_\tau)$*

$$\int_G \big(\Upsilon(g) \mid \Psi(g)\big)_\tau \, dg = \int_K \big(\Upsilon(k) \mid (L_{\tau,\lambda}\Psi)(k)\big)_\tau \; dk \; .$$

Proof. — Using the decomposition $G = PK$ we obtain

$$\int_G \big(\Upsilon(g) \mid \Psi(g)\big)_\tau \, dg = \int_P \int_K \big(\Upsilon(pk) \mid \Psi(pk)\big)_\tau \delta(p) \, dp \, dk \; .$$

Since Υ satisfies relation (7-1-1) for the pair $(\tau, -\overline{\lambda})$ we have

$$\big(\Upsilon(pk) \mid \Psi(pk)\big)_\tau = \big(\Upsilon(k) \mid a^{\rho-\lambda}\tau(m)^{-1}\Psi(pk)\big)_\tau \quad (p = man) \; ,$$

And the result follows from the definition of $L_{\tau,\lambda}$.

□

Therefore the previous G–equivariant embedding from $\mathcal{C}_{\tau,-\overline{\lambda}}(G, \mathcal{H}_\tau)$ into the $\mathcal{D}'_{\tau,-\overline{\lambda}}(G, \mathcal{H}_\tau)$ gives rise to a G–equivariant embedding from $\mathcal{C}_{\tau,-\overline{\lambda}}(G, \mathcal{H}_\tau)$ into the space $\mathcal{H}^{-\infty}_{\tau,\lambda}$ defined as follows.

Proposition 7.3.4. *The map from $\mathcal{C}_{\tau,-\overline{\lambda}}(G, \mathcal{H}_\tau)$ into the space $\mathcal{H}^{-\infty}_{\tau,\lambda}$ given by the application $\Upsilon \mapsto a_\Upsilon$ where*

$$\langle a_\Upsilon, \Phi \rangle = \int_K \big(\Upsilon(k) \mid \Phi(k)\big)_\tau \; dk \quad \text{for } \Phi \in \; \mathcal{H}^\infty_{\tau,\lambda}$$

is a G–equivariant embedding i.e. for $g \in G$ we have

$$\pi_{\tau,\lambda}(g) a_\Upsilon = a_{r(g)\Upsilon} \; ,$$

where r denotes the right regular representation of G on $\mathcal{C}_{\tau,-\overline{\lambda}}(G, \mathcal{H}_\tau)$

In conclusion we have the following commutative diagram where the arrows are G–equivariant embedding.

$$\begin{array}{ccc} \mathcal{D}'_{\tau,-\overline{\lambda}}(G, \mathcal{H}_\tau) & \xrightarrow{\quad\simeq\quad} & \mathcal{H}^{-\infty}_{\tau,\lambda} \\ & \nwarrow \quad \nearrow & \\ & \mathcal{C}_{\tau,-\overline{\lambda}}(G, \mathcal{H}_\tau) & \end{array}$$

The space of H–invariant distribution vectors is denoted by $\left(\mathcal{H}_{\tau,\lambda}^{-\infty}\right)^H$. It follows from Lemma 3.3 in [vdB]–1988 (see also [Schl]–1994) that Van den BAN's conditions are necessary for $\pi_{\tau,\lambda}$ to have a H–invariant distribution vector which has, as an element of $\mathcal{D}'_{\tau,-\overline{\lambda}}(G,\mathcal{H}_\tau)$, a nonzero restriction to the union of open P–orbits in G/H. Moreover these conditions are sufficient for almost all λ as we will see in the next section.

7.4. H_p–invariant distribution vectors for the minimal principal spherical series

We fix in this section a $(M\cap H)$–spherical representation τ of M and consider representations $(\pi_{\tau,\lambda},\mathcal{H}_{\tau,\lambda})$ in the minimal spherical principal series. We will construct H_p–invariant distribution vectors for $p\in\{0,\dots,r\}$. Such constructions have been given in more general situations by Van den BAN ([vdB]–1988) and BRYLINSKI–DELORME ([B-D]-1992). From Proposition 7.3.2 this is equivalent to give H_p–invariant elements of $\mathcal{D}'_{\tau,-\overline{\lambda}}(G,\mathcal{H}_\tau)$ *i.e.* antilinear forms on $\mathcal{C}_c^\infty(G,\mathcal{H}_\tau)$ which transform under the left action of P as in (7-3-2) and which are invariant under the right action of H_p.

Let us recall (see Proposition 4.1.8) that the G–invariant measure d^*X on V^+ induces naturally a G–invariant measure on $G/H_p\simeq\Omega_p^+$.

We will once and for all choose a Haar measure dg on G, and then choose a Haar measure d_ph on each H_p such that

$$\int_G\varphi(g)\,dg=\int_{G/H_p}\left(\int_{H_p}\varphi(gh)\,d_ph\right)d_p\dot{g}\qquad\text{for }\varphi\in\mathcal{C}_c^\infty(G)\ .$$

Let us now start with a function $\Psi\in\mathcal{C}_c^\infty(G,\mathcal{H}_\tau)$ and define a function $\Psi_p\in\mathcal{C}_c^\infty(G/H_p,\mathcal{H}_\tau)$ by

$$\Psi_p(\dot{g})=\int_{H_p}\Psi(gh)\,d_ph\ .$$

It is well known that the mapping $\Psi\mapsto\Psi_p$ is surjective and continuous from $\mathcal{C}_c^\infty(G,\mathcal{H}_\tau)$ onto $\mathcal{C}_c^\infty(G/H_p,\mathcal{H}_\tau)$ for the usual topologies. Moreover the map $\Psi_p\longmapsto F_p$ where

$$F_p(g.I_p^+)=\Psi_p(\dot{g})\qquad\text{for }g\in G\ ,$$

is an isomorphism from $\mathcal{C}_c^\infty(G/H_p,\mathcal{H}_\tau)$ onto $\mathcal{C}_c^\infty(\Omega_p^+,\mathcal{H}_\tau)$. Therefore in order to obtain H_p–invariant distribution vectors it is sufficient to define antilinear forms on $\mathcal{C}_c^\infty(\Omega_p^+,\mathcal{H}_\tau)$ which transform under the action of P similarly to (7-3-2). This will be done in Proposition 7.4.4 below by integration on open P–orbits in Ω_p^+.

We use the notations introduced in section 7.2.

Definition 7.4.1. *We denote by* $\mathbf{U}$ *the open subset of* $\mathbb{C}^{k+1}$ *on which* $s\longmapsto Z_{p,\gamma}^+(f,s,\tau)$ *is holomorphic for all* $f\in\mathcal{C}_c^\infty(\Omega^+)$, $p\in\{0,\dots,k\}$ *and* $\gamma\in\mathcal{W}_p$.

It follows from Proposition 7.2.5 that $\mathcal{U}^+$ is a subset of $\mathbf{U}$. Therefore $\mathbf{U}$ contains the complementary set of a countable union of hyperplanes.

Remark 7.4.2. As the product by $|\Delta_0|^z (z \in \mathbb{C})$ of a function in $\mathcal{C}_c^\infty(\Omega^+)$ is also a function in $\mathcal{C}_c^\infty(\Omega^+)$ we have

$$s = (s_0, s_1, \dots, s_k) \in \mathbf{U} \Longrightarrow (s_0 + z, s_1, \dots, s_k) \in \mathbf{U} \text{ for } z \in \mathbb{C} .$$

In particular we have $s \in \mathbf{U} \Longleftrightarrow s - m \in \mathbf{U}$.

Remark 7.4.3. The open set $\mathbf{U}$ is also the subset of $\mathbb{C}^{k+1}$ on which $s \longmapsto Z^-_{p,\gamma}(h, t(s), \tau)$ is holomorphic for all $h \in \mathcal{C}_c^\infty(\Omega^-)$, $p \in \{0, \dots, k\}$ and $\gamma \in \mathcal{W}_p$. To prove this we use the G–equivariant map $\psi : \Omega^+ \longrightarrow \Omega^-$ defined in section 3.9. We deduce from Proposition 3.9.5 and Proposition 4.1.9 that for $h \in \mathcal{C}_c^\infty(\Omega^-)$ we have

$$Z^-_{p,\gamma}(h, t(s), \tau) = Z^+_{p,\gamma}(h \circ \psi, s - 2m, \tau) .$$

The result follows from Remark 7.4.2.

Proposition 7.4.4. *Let $p \in \{0, \dots, r\}$ and $\gamma \in \mathcal{W}_p$. If s is an element of the open set $\mathbf{U}$ the map*

$$\Theta^{p,\gamma}_{\tau,s} : \mathcal{C}_c^\infty(\Omega_p^+, \mathcal{H}_\tau) \longrightarrow \mathbb{C}$$
$$F \longmapsto \int_{\mathcal{O}^+_{p,\gamma}} \big(b^+_\tau(X) \,\big|\, F(X)\big)_\tau |\Delta(X)|^s \, d^*X ,$$

is a nonzero continuous antilinear form on $\mathcal{C}_c^\infty(\Omega_p^+, \mathcal{H}_\tau)$ which satisfies for $m \in M$, $a \in A$ and $n \in N$ the relation

$$\langle \Theta^{p,\gamma}_{\tau,s}, \ell(man)F\rangle = \chi^s(a)\langle \Theta^{p,\gamma}_{\tau,s}, \tau(m)^{-1}F\rangle \quad \big(F \in \mathcal{C}_c^\infty(\Omega_p^+, \mathcal{H}_\tau)\big) .$$

Moreover, for $F \in \mathcal{C}_c^\infty(\Omega_p^+, \mathcal{H}_\tau)$, the map $s \longmapsto \langle \Theta^{p,\gamma}_{\tau,s}, F\rangle$ is holomorphic on $\mathbf{U}$.

Recall that for $s = (s_0, \dots, s_k)$ we set $\chi^s(a) = \prod_{j=0}^k \chi_j(a)^{s_j}$.

Proof. — Assuming that $\Theta^{p,\gamma}_{\tau,s}$ is well defined, let us first study the action of P. For $m \in M$, $a \in A$, $n \in N$ and $X \in \mathcal{O}^+$ we have

$$|\Delta(manX)|^s = \chi^s(a)|\Delta(X)|^s \quad \text{and} \quad b^+_\tau(manX) = \tau(m)b^+_\tau(X) .$$

Therefore, using the G–invariance of the measure d^*X, we get for a function $F \in \mathcal{C}_c^\infty(\Omega_p^+, \mathcal{H}_\tau)$

$$\begin{aligned}\langle \Theta^{p,\gamma}_{\tau,s}, \ell(man)F\rangle &= \int_{\mathcal{O}^+_{p,\gamma}} \big(b^+_\tau(manX) \,\big|\, F(X)\big)_\tau |\Delta(manX)|^s \, d^*X \\ &= \chi^s(a) \int_{\mathcal{O}^+_{p,\gamma}} \big(b^+_\tau(X) \,\big|\, \tau(m)^{-1}F(X)\big)_\tau |\Delta(X)|^s \, d^*X \\ &= \chi^s(a)\langle \Theta^{p,\gamma}_{\tau,s}, \tau(m)^{-1}F\rangle .\end{aligned}$$

It remains to prove that $\Theta^{p,\gamma}_{\tau,s}$ defines a continuous antilinear form on $\mathcal{C}_c^\infty(\Omega_p^+, \mathcal{H}_\tau)$ which is weakly holomorphic on $\mathbf{U}$. For that we will just show that the antilinear form $\Theta^{p,\gamma}_{\tau,s}$ is closely related to the local Zeta functions defined in section 7.2 and the result will then be a consequence of Proposition 7.2.5.

• Type I and II : In this cases $\mathcal{H}_\tau = \mathbb{C}$ and we have for a function $F \in \mathcal{C}_c^\infty(\Omega_p^+, \mathcal{H}_\tau) = \mathcal{C}_c^\infty(\Omega_p^+)$

$$\langle \Theta_{\tau,s}^{p,\gamma}, F\rangle = \int_{\mathcal{O}_{p,\gamma}^+} \overline{F}(X)|\Delta(X)|^s \, d^*X = Z_{p,\gamma}^+(\overline{F}, s-m, \tau) \ .$$

Thus, for $s \in \mathbf{U}+m$, the map $F \longmapsto \langle \Theta_{\tau,s}^{p,\gamma}, F\rangle$ is a continuous antilinear form on $\mathcal{C}_c^\infty(\Omega_p^+, \mathcal{H}_\tau)$. As the restriction of this form to $\mathcal{C}_c^\infty(\mathcal{O}_{p,\gamma}^+)$ is defined by the function $X \longmapsto |\Delta(X)|^{s-m}$ which is a nonzero continuous function on $\mathcal{O}_{p,\gamma}^+$ (for any $s \in \mathbb{C}^{k+1}$), $\Theta_{\tau,s}^{p,\gamma}$ is nonzero.

• Type III : Taking an orthonormal basis $(R_i)_{i\in I}$ of $\mathcal{H}_\tau$ we write a function $F \in \mathcal{C}_c^\infty(\Omega_p^+), \mathcal{H}_\tau)$ as

$$F = \sum_{i\in I} f_i R_i \quad \text{ where } f_i \in \mathcal{C}_c^\infty(\Omega_p^+) \ .$$

Then we obtain

$$\langle \Theta_{\tau,s}^{p,\gamma}, F\rangle = \sum_{i\in I} \Big(Z_{p,\gamma}^+(\overline{f}_i, s-m, \tau) \,\Big|\, R_i \Big)_\tau \ .$$

Similarly to the Type I and II cases, the result follows fom Proposition 7.2.5 and the fact that $X \longmapsto |\Delta(X)|^{s-m} b_\tau^+(X)$ is continuous on $\mathcal{O}_{p,\gamma}^+$.

□

From Theorem 3.6.1 we have $\chi^s(a) = \prod_{j=0}^k a^{\kappa(s_j\lambda_j + \cdots + s_k\lambda_k)}$. It follows that for $\lambda \in (\mathfrak{a}_{\mathbb{C}}^0)^*$ we have

$$\Big(\forall a \in A, a^{\rho-\overline{\lambda}} = \chi^s(a)\Big) \Longleftrightarrow \rho - \overline{\lambda} = \kappa\big(s_0\lambda_0 + \cdots + (s_0+\cdots+s_k)\lambda_k\big) \ .$$

Definition 7.4.5. *We denote by* $\mathbf{V}$ *the open subset of* $(\mathfrak{a}_{\mathbb{C}}^0)^*$ *defined by*

$$\lambda \in \mathbf{V} \Longleftrightarrow s(\lambda) \in \mathbf{U}+m = \mathbf{U} \ ,$$

where $s(\lambda) = (s_0, \ldots, s_k) \in \mathbb{C}^{k+1}$ *is determined by the relation*

$$\rho - \lambda = \kappa\big(s_0\lambda_0 + \cdots + (s_0+\cdots+s_k)\lambda_k\big) \ ,$$

and $\mathbf{U}$ *is defined in Definition* 7.4.1.

The construction of $\Theta_{\tau,s}^{p,\gamma}$ in the previous Proposition allows us now, for $\overline{\lambda} \in \mathbf{V}$, to construct H_p–invariant distribution vectors for $\pi_{\tau,\lambda}$.

Theorem 7.4.6. *Let* $\pi_{\tau,\lambda}$ *be a representation of the minimal principal spherical series with* $\overline{\lambda} \in \mathbf{V}$. *Then for any* $p \in \{0, \ldots, r\}$ *and any* $\gamma \in \mathcal{W}_p$ *the antilinear form* $a_{\tau,\lambda}^{p,\gamma}$ *defined on* $\mathcal{H}_{\tau,\lambda}^\infty$ *by*

$$\begin{aligned}\langle a_{\tau,\lambda}^{p,\gamma}, L_{\tau,\lambda}\Psi\rangle &= \langle \Theta_{\tau,s(\overline{\lambda})}^{p,\gamma}, F_p\rangle \\ &= \int_{\mathcal{O}_{p,\gamma}^+} \Big(b_\tau^+(X) \,\Big|\, F_p(X)\Big)_\tau |\Delta(X)|^{s(\overline{\lambda})} d^*X \quad \big(\Psi \in \mathcal{C}_c^\infty(G, \mathcal{H}_\tau)\big) \ ,\end{aligned}$$

where F_p *is defined on* Ω_p^+ *by* $F_p(g.I_p^+) = \int_{H_p} \Psi(gh)\, d_p h$, *is a* H_p*–invariant distribution vector for* $\pi_{\tau,\lambda}$.

For fixed τ, λ *and* p *the set* $\{a_{\tau,\lambda}^{p,\gamma}\}_{\gamma\in\mathcal{W}_p}$ *is a set of linearly independant vectors in* $(\mathcal{H}_{\tau,\lambda}^{-\infty})^{H_p}$. *Moreover, for* $\Psi \in \mathcal{C}_c^\infty(G, \mathcal{H}_\tau)$ *the map*

$$\overline{\lambda} \longmapsto \langle a_{\tau,\lambda}^{p,\gamma}, L_{\tau,\lambda}\Psi\rangle$$

is holomorphic on $\mathbf{V}$.

Proof. — As the map $\Psi \longmapsto F_p$ is G–equivariant and continuous from $\mathcal{C}_c^\infty(G, \mathcal{H}_\tau)$ onto $\mathcal{C}_c^\infty(\Omega_p^+, \mathcal{H}_\tau)$ (G acting on the left hand side), it follows from Proposition 7.4.4 that the antilinear form

$$\Psi \longmapsto \langle \Theta_{\tau,s}^{p,\gamma}, F_p \rangle \ ,$$

is continous on $\mathcal{C}_c^\infty(G, \mathcal{H}_\tau)$. As $\overline{\lambda} \in \mathbf{V}$ implies $s(\overline{\lambda}) \in \mathbf{U} + m$ (Definition 7.4.5), this map satisfies relation (7-3-2) for $s = s(\overline{\lambda})$ and $\overline{\lambda} \in \mathbf{V}$. Therefore Proposition 7.3.2 implies that $a_{\tau,\lambda}^{p,\gamma}$ is a well defined element of $\mathcal{H}_{\tau,\lambda}^{-\infty}$.

Let us now show that this element is H_p–invariant. For $g \in G$ we have

$$\begin{aligned} \langle \pi_{\tau,\lambda}(g)^{-1} a_{\tau,\lambda}^{p,\gamma}, L_{\tau,\lambda}\Psi \rangle &= \langle a_{\tau,\lambda}^{p,\gamma}, \pi_{\tau,\lambda}(g) L_{\tau,\lambda}\Psi \rangle \\ &= \langle a_{\tau,\lambda}^{p,\gamma}, L_{\tau,\lambda} r(g)\Psi \rangle \ , \end{aligned}$$

where r is the right regular action. Now if $g = h \in H_p$, the functions F_p associated to Ψ and to $r(h)\Psi$ by right integration on H_p are the same. Therefore for $h \in H_p$ we get

$$\pi_{\tau,\lambda}(h)^{-1} a_{\tau,\lambda}^{p,\gamma} = a_{\tau,\lambda}^{p,\gamma} \ .$$

As the map $\Psi \longmapsto F_p$ is surjective onto $\mathcal{C}_c^\infty(\Omega_p^+, \mathcal{H}_\tau)$, F_p can be taken to be any element of $\mathcal{C}_c^\infty(\mathcal{O}_{p,\gamma}^+, \mathcal{H}_\tau)$. Therefore it follows from Proposition 7.4.4 that $a_{\tau,\lambda}^{p,\gamma}$ is nonzero. As the various $\mathcal{O}_{p,\gamma}^+$ are disjoints, it is also clear that the vectors $(a_{\tau,\lambda}^{p,\gamma})_{\gamma \in \mathcal{W}_p}$ are linearly independant.

Finally it follows also from Proposition 7.4.4 that the map

$$\overline{\lambda} \longmapsto \langle a_{\tau,\lambda}^{p,\gamma}, L_{\tau,\lambda}\Psi \rangle$$

is holomorphic on $\mathbf{V}$.

□

Remark 7.4.7. This Theorem implies that for $\overline{\lambda} \in \mathbf{V}$, the dimension of $\left(\mathcal{H}_{\tau,\lambda}^{-\infty}\right)^{H_p}$ is greater or equal to the cardinal of $\mathcal{W}_p$ *i.e.* to the number of open P–orbits in G/H_p. However it follows from the work of van den Ban ([vdB]–1988 Theorem 5.10) that, in our context, the dimension of $\left(\mathcal{H}_{\tau,\lambda}^{-\infty}\right)^{H_p}$ is generically equal to the number of open P–orbits in G/H_p. This is due the fact that, here, condition (b) reduces to condition (b') and $\dim \mathcal{H}_\tau^{(M \cap H_p)} = 1$ (Corollary 6.1.4). Notice that van den Ban's results are proved under the assumption that every Cartan subgroup of G is abelian, which is true since $G \subset \widetilde{G} \subset GL(\widetilde{\mathfrak{g}})$ is linear.

7.5. Another expression of $a_{\tau,\lambda}^{p,\gamma}$ in the continuous case

As in previous section τ denotes a $M \cap H$–spherical representation of M, p an element of $\{0, \ldots, r\}$ and γ an element of $\mathcal{W}_p$.

Definition 7.5.1. *Let $s \in \mathbb{C}^{k+1}$. Then $P^{p,\gamma}_{\tau,s}$ denotes the $\mathcal{H}_\tau$–valued function defined on Ω^+_p by*

$$P^{p,\gamma}_{\tau,s}(X) = \begin{cases} |\Delta(X)|^s b^+_\tau(X) \text{ if } X \in \mathcal{O}^+_{p,\gamma} \ , \\ 0 \text{ if } X \in \Omega^+_p \backslash \mathcal{O}^+_{p,\gamma} \ . \end{cases}$$

Lemma 7.5.2. *Let $s = (s_0, \ldots, s_k) \in \mathbb{C}^{k+1}$ such that $\operatorname{Re} s_j > 0$ for $j = 1, \ldots, k$. Then function $P^{p,\gamma}_{\tau,s}$ is continuous on Ω^+_p.*

Proof. —

• Type I : In this case $b^+_\tau(X) = 1$ for $X \in \Omega^+_p$. If $\operatorname{Re} s_j > 0$ for $j = 1, \ldots, k$, then the function $X \mapsto |\Delta(X)|^s$ is continuous on Ω^+_p since $\Delta_0(X) \neq 0$ for $X \in \Omega^+_p$. Recall that $\mathcal{O}^+$ denotes the union of the open P–orbits in V^+. On each connected component of the open dense subset $\mathcal{O}^+ \cap \Omega^+_p$, the function $X \longmapsto P^{p,\gamma}_{\tau,s}$ is equal to $|\Delta(X)|^s$ or to zero, hence continuous. But this function is also equal to zero on $\Omega^+_p \backslash \mathcal{O}^+$ which is exactly the set $\{X \in \Omega^+_p \mid \Delta_1(X) \cdots \Delta_k(X) = 0\}$. Therefore it is continuous on Ω^+_p.

• Type II : In this case the function $X \longmapsto P^{p,\gamma}_{\tau,s}(X)$ is equal to $\pm|\Delta(X)|^s$ on each connected component of $\mathcal{O}^+ \cap \Omega^+_p$ and the proof is similar to the proof in Type I case.

• Type III : As in the proof of Lemma 6.2.3 we have for $X = manI^+ \in \mathcal{O}^+$ and for any polynomial $R \in \mathcal{H}_\tau$

$$\big(R \mid b^+_\tau(X)\big)_\tau = \big(R \mid \tau(m)v_\tau\big)_\tau = R(mI^+) \ .$$

Since $M/(M \cap H)$ is compact, the map $m \longmapsto R(mI^+)$ is bounded on M and, using a basis of the finite dimensional space $\mathcal{H}_\tau$, it is easy to see that the $\mathcal{H}_\tau$–valued function $X \longmapsto b^+_\tau(X)$, which is clearly continous on $\mathcal{O}^+$, is also bounded on $\mathcal{O}^+$. Therefore, if $\operatorname{Re} s_j > 0$ for $j = 1, \ldots, k$, the map $X \mapsto P^{p,\gamma}_{\tau,s}(X)$ is continuous on Ω^+.

□

Remark 7.5.3. It may be noticed in Type III case that, using Proposition 6.2.5, we can prove that the map

$$X \longmapsto \Delta(X)^{\frac{\delta}{2}} b^+_\tau(X) \ ,$$

where $\delta_0 = \tau_0$, $\delta_1 = \tau_1 + \tau_0$, ..., $\delta_k = \tau_k + \tau_{k-1}$, is continuous on V^+. But in the previous Lemma we need only the continuity on Ω^+.

Remark 7.5.4. We have for $f \in \mathcal{C}^\infty_c(\Omega^+_p)$

$$Z^+_{p,\gamma}(f, s, \tau) = \int_{\Omega^+_p} f(X) P^{p,\gamma}_{\tau,s}(X) \, dX \ .$$

It follows from the previous Lemma that if $s = (s_0, s_1, \ldots, s_k)$ satisfies $\operatorname{Re} s_j > 0$ for $j = 1, \ldots, k$ then, if $f \in \mathcal{C}^+_c(\Omega^+_p)$, $Z^+_{p,\gamma}(f, s, \tau)$ is defined by a convergent integral which implies that s is an element of $\mathbf{U}$ (or $\mathbf{U} + m$ by Remark 7.4.2).

Definition 7.5.5. *Let $\lambda \in (\mathfrak{a}_{\mathbb{C}}^0)^*$. The Poisson kernel $\mathbb{P}_{\tau,\lambda}^{p,\gamma}$ is defined as the $\mathcal{H}_\tau$–valued function defined on G by*

$$\mathbb{P}_{\tau,\lambda}^{p,\gamma}(g) = P_{\tau,s(\overline{\lambda})}^{p,\gamma}(gI_p^+) \quad \text{for } g \in G \ .$$

It follows from the Definition of $s(\overline{\lambda})$ (Definition 7.4.5) that we have

$$\mathbb{P}_{\tau,\lambda}^{p,\gamma}(g) = \begin{cases} a^{\rho-\overline{\lambda}}\tau(m)v_\tau & \text{if } g \in man\gamma H_p \ , \\ 0 & \text{if } g \notin man\gamma H_p \ , \end{cases}$$

where $v_\tau = 1$ in the Type I and Type II cases. Moreover $\mathbb{P}_{\tau,\lambda}^{p,\gamma}$ is right H_p–invariant.

These properties characterize the Poisson kernel. It is known (see [vdB]-1988, Prop. 5.6) that $\mathbb{P}_{\tau,\lambda}^{p,\gamma}$ is continuous on G for λ in an open domain of $(\mathfrak{a}_{\mathbb{C}}^0)^*$. We deduce from the previous Lemma an explicit domain where it is the case.

Proposition 7.5.6. *Let $\lambda \in (\mathfrak{a}_{\mathbb{C}}^0)^*$ such that*

$$\mathrm{Re}\Big(\lambda(H_{\lambda_j}) - \lambda(H_{\lambda_{j-1}})\Big) < -d \quad \text{for } j = 1, \dots, k \ .$$

Then the Poisson kernel $\mathbb{P}_{\tau,\lambda}^{p,\gamma}$ is a continuous $\mathcal{H}_\tau$–valued function on G. More precisely it belongs to $\mathcal{C}_{\tau,-\overline{\lambda}}(G, \mathcal{H}_\tau)$.

Proof. —Using Lemma 7.5.2, it remains to prove that the announced condition on λ is equivalent to $\mathrm{Re}\, s_j > 0$ for $j = 1, \dots k$. Recall from Remark 7.1.3 that ρ may be considered as an element of $(\mathfrak{a}_{\mathbb{C}}^0)^*$. An elementary computation using the decomposition $\mathfrak{n} = \oplus_{i<j} E_{i,j}(1,-1)$ leads to the equality

$$\rho = \frac{d}{4}\sum_{i<j}(\lambda_i - \lambda_j) = \frac{d}{4}\sum_{j=0}^{k}(k-2j)\lambda_j \ .$$

On the other hand we may write for $\lambda \in (\mathfrak{a}_{\mathbb{C}}^0)^*$

$$\lambda = \sum_{j=0}^{k} \frac{\lambda(H_{\lambda_j})}{2}\lambda_j \ .$$

The relation $-\overline{\lambda} + \rho = \kappa\big(s_0\lambda_0 + (s_0+s_1)\lambda_1 + \cdots + (s_0+s_1+\cdots+s_k)\lambda_k\big)$ is then equivalent to

$$s_0 = -\frac{\overline{\lambda(H_{\lambda_0})}}{2\kappa} + \frac{d}{4\kappa}k \ \text{ and } \ s_j = \frac{\overline{\lambda(H_{\lambda_{j-1}} - H_{\lambda_j})}}{2\kappa} - \frac{d}{2\kappa} \ \ (j = 1, \dots, k),$$

which leads to the result.

□

Theorem 7.5.7. *Suppose that $\lambda \in (\mathfrak{a}_{\mathbb{C}}^0)^*$ satisfies the conditions of Proposition 7.5.6. Then we have for $\Phi \in \mathcal{H}_{\tau,\lambda}^\infty$*

$$\langle a_{\tau,\lambda}^{p,\gamma}, \Phi\rangle = \int_K \big(\mathbb{P}_{\tau,\lambda}^{p,\gamma}(k) \,|\, \Phi(k)\big)_\tau \, dk \ .$$

Proof. — Let $\Phi \in \mathcal{H}^{\infty}_{\tau,\lambda}$ and $\Psi \in \mathcal{C}^{\infty}_{c}(G, \mathcal{H}_{\tau})$ such that $L_{\tau,\lambda}\Psi = \Phi$. As $s(\overline{\lambda})$ is an element of $\mathbf{U} + m$ (see Remark 7.5.4) we apply Theorem 7.4.6 and we get, using the Definition of $P^{p,\gamma}_{\tau,s}$,

$$\langle a^{p,\gamma}_{\tau,\lambda}, \Phi \rangle = \int_{\Omega^{+}_{p}} \left(P^{p,\gamma}_{\tau,s(\overline{\lambda})}(X) \,|\, F_{p}(X) \right)_{\tau} d^{*}X \ ,$$

where F_p is defined on Ω^{+}_{p} by $F_{p}(g.I^{+}_{p}) = \int_{H_p} \Psi(gh)\ d_p h$. Since the map $X \mapsto \left(P^{p,\gamma}_{\tau,s(\overline{\lambda})}(X) \,|\, F_{p}(X) \right)_{\tau}$ is continuous with compact support in Ω^{+}_{p}, we may, introducing $\mathbb{P}^{p,\gamma}_{\tau,\lambda}$, rewrite this convergent integral as an integral on G/H_p. And we get, using the right H_p–invariance of $\mathbb{P}^{p,\gamma}_{\tau,\lambda}$ and the Definition of F_p

$$\begin{aligned} \langle a^{p,\gamma}_{\tau,\lambda}, \Phi \rangle &= \int_{G/H_p} \left(\mathbb{P}^{p,\gamma}_{\tau,\lambda}(\dot{g}) \,|\, F_{p}(gI^{+}_{p}) \right)_{\tau} d\dot{g} \\ &= \int_{G/H_p} \int_{H_p} \left(\mathbb{P}^{p,\gamma}_{\tau,\lambda}(gh) \,|\, \Psi(gh) \right)_{\tau} d_p h \, d\dot{g} \\ &= \int_{G} \left(\mathbb{P}^{p,\gamma}_{\tau,\lambda}(g) \,|\, \Psi(g) \right)_{\tau} dg \ . \end{aligned}$$

Since $\mathbb{P}^{p,\gamma}_{\tau,\lambda}$ belongs to $\mathcal{C}_{\tau,-\overline{\lambda}}(G, \mathcal{H}_{\tau})$ the result follows from Lemma 7.3.3.

□

7.6. The Zeta integral of an H–invariant distribution vector

In this section we consider a continuous representation π of G on a Hilbert space $\mathcal{H}_{\pi}$, the subspace $\mathcal{H}^{\infty}_{\pi}$ of differentiable vectors in $\mathcal{H}_{\pi}$ and the space $\mathcal{H}^{-\infty}_{\pi}$ of distribution vectors.

Definition 7.6.1. *Let π be a continuous representation of G on a Hilbert space $\mathcal{H}_{\pi}$, $a \in \mathcal{H}^{-\infty}_{\pi}$ a distribution vector and $\varphi \in \mathcal{C}^{\infty}_{c}(G)$. It is well known that the map defined on the subspace $\mathcal{H}^{\infty}_{\pi}$ of differentiable vectors by*

$$\mathcal{H}^{\infty}_{\pi} \ni \Phi \mapsto \int_{G} \varphi(g) \langle \pi(g)a, \Phi \rangle dg \ ,$$

defines a continuous anti-linear form on $\mathcal{H}^{\infty}_{\pi}$, in other words an element of $\mathcal{H}^{-\infty}_{\pi}$. This element is denoted by

$$\pi(\varphi)a = \int_{G} \varphi(g)\pi(g)a \, dg$$

If π is a unitary representation it is also well known that $\pi(\varphi)a$ defines in fact an element of $\mathcal{H}^{\infty}_{\pi}$. If π is only a continuous representation we have to consider the conjugate dual representation π^{*} on $\mathcal{H}_{\pi} = \mathcal{H}_{\pi^{*}}$ which is given by

$$(\pi^{*}(g)u \,|\, v)_{\mathcal{H}_{\pi}} = (u \,|\, \pi(g)^{-1}v)_{\mathcal{H}_{\pi}} \quad \text{for } u, v \in \mathcal{H}_{\pi} \ .$$

Each element of $u \in \mathcal{H}_{\pi}$ defines an element a_u in $\mathcal{H}^{-\infty}_{\pi}$ by

$$\langle a_u, \Phi \rangle = (u \,|\, \Phi)_{\mathcal{H}_{\pi}} \quad \text{for } \Phi \in \mathcal{H}^{\infty}_{\pi} \ .$$

And the map $u \longmapsto a_u$ intertwines π^{*} and the representation of G on $\mathcal{H}^{-\infty}_{\pi}$.

Proposition 7.6.2. *Let π be a continuous representation of G on a Hilbert space $\mathcal{H}_\pi$, $a \in \mathcal{H}_\pi^{-\infty}$ a distribution vector and $\varphi \in \mathcal{C}_c^\infty(G)$. Then $\pi(\varphi)a$ belongs to $\mathcal{H}_{\pi^*}^\infty$. More precisely there exists an element b in $\mathcal{H}_\pi$, which is a $\mathcal{C}^\infty$ vector for π^*, such that*

$$\langle \pi(\varphi)a, \Phi \rangle = (b \,|\, \Phi)_{\mathcal{H}_\pi} \quad \text{for any } \Phi \in \mathcal{H}_\pi^\infty \ .$$

Proof. — From the definition of $\pi(\varphi)a$ we have

$$\langle \pi(\varphi)a, \Phi \rangle = \langle a, \pi(\widetilde{\varphi})\Phi \rangle \quad (\Phi \in \mathcal{H}_\pi^\infty) \ ,$$

where $\widetilde{\varphi}$ is the function in $\mathcal{C}_c^\infty(G)$ defined by $\widetilde{\varphi}(g) = \overline{\varphi(g^{-1})}$ for $g \in G$. Since the map $\Phi \longmapsto \pi(\widetilde{\varphi})\Phi$ is continuous from $\mathcal{H}_\pi$ to $\mathcal{H}_\pi^\infty$ ([Br]–1956 p. 115), the antilinear form $\Phi \longmapsto \langle \pi(\varphi)a, \Phi \rangle$ is continuous on $\mathcal{H}_\pi$. The existence of an element $b \in \mathcal{H}_\pi$ such that

$$\langle \pi(\varphi)a, \Phi \rangle = (b \,|\, \Phi)_{\mathcal{H}_\pi} \quad \text{for any } \Phi \in \mathcal{H}_\pi^\infty \ ,$$

follows. It remains to prove that b is a $\mathcal{C}^\infty$ vector for π^*.

An elementary computation shows that for $g \in G$ we have

$$(\pi^*(g)b \,|\, \Phi)_{\mathcal{H}_\pi} = \langle a, \pi\big(\widetilde{\ell(g)\varphi}\big)\Phi \rangle \ ,$$

where ℓ is the left regular representation on $\mathcal{C}_c^\infty(G)$. For $X \in \mathfrak{g}$, $\psi \in \mathcal{C}_c^\infty(G)$ and $\Phi \in \mathcal{H}_\pi^\infty$ we set

$$X * \psi(g) = \frac{d}{dt}\psi(\exp(-tX)g)\big|_{t=0} \text{ and } X\Phi = \frac{d}{dt}\pi(\exp tX)\Phi\big|_{t=0} \ .$$

We want to prove that in $\mathcal{H}_\pi$ we have

$$\lim_{t\to 0} \frac{\pi^*(\exp tX)b - b}{t} = \pi(X * \varphi)a \ .$$

This will imply that b is differentiable along X and finally that b is a $\mathcal{C}^\infty$ vector.

Let us recall ([Pou]–1972) that the topology of $\mathcal{H}_\pi^\infty$ is defined by the family of seminorms

$$||\Phi||_m = \sum_{|\alpha|\leq m} ||X^\alpha \Phi||_{\mathcal{H}_\pi} \ ,$$

where $\alpha = (\alpha_1, \ldots, \alpha_s)$, $|\alpha| = \alpha_1 + \cdots + \alpha_s$, s is the dimension of $\mathfrak{g}$, $(X_1, \ldots, X_s)$ is a basis of $\mathfrak{g}$ and $X^\alpha \Phi = X_1^{\alpha_1} \cdots X_s^{\alpha_s}\Phi$. As a is a continuous linear form on $\mathcal{H}_\pi^\infty$ there exists an integer m and a constant $C > 0$ such that

$$|\langle a, \Phi \rangle| \leq C||\Phi||_m \quad \text{for } \Phi \in \mathcal{H}_\pi^\infty \ .$$

Let $\Phi \in \mathcal{H}_\pi^\infty$, $\psi \in \mathcal{C}_c^\infty(G)$ and K a compact containing the support of ψ. Then for $X \in \mathfrak{g}$ we have $X\pi(\psi)\Phi = \pi(X * \psi)\Phi$ and an easy computation shows that

$$||\pi(\psi)\Phi||_m \leq C(K)C_m(\psi)||\Phi||_{\mathcal{H}_\pi}$$

$$\text{where } \begin{cases} C_m(\psi) = \sum_{|\alpha|\leq m} \int_G |(X^\alpha * \psi)(g)|\, dg \ , \\ C(K) = \sup_{g\in K} ||\pi(g)|| \ . \end{cases}$$

If we set

$$\varphi_t = \frac{\ell(\exp tX)\varphi - \varphi}{t} - X * \varphi \ ,$$

we have, for any $\Phi \in \mathcal{H}_\pi^\infty$,

$$\Big(\frac{\pi^*(\exp tX)b - b}{t} - \pi(X * \varphi)a \mid \Phi\Big)_{\mathcal{H}_\pi} = \langle a, \pi(\widetilde{\varphi}_t)\Phi\rangle \ .$$

As $t \to 0$, the function $\widetilde{\varphi}_t$ tends to 0 in $\mathcal{C}_c^\infty(G)$ and there exists a compact K containing the support of $\widetilde{\varphi}_t$ for t in a neighborood of 0. Therefore we obtain

$$\Big|\Big(\frac{\pi^*(\exp tX)b - b}{t} - \pi(X * \varphi)a \mid \Phi\Big)_{\mathcal{H}_\pi}\Big| \leq CC(K)C_m(\widetilde{\varphi}_t)||\Phi||_{\mathcal{H}_\pi} \ ,$$

which implies

$$\Big|\Big|\frac{\pi^*(\exp tX)b - b}{t} - \pi(X * \varphi)a\Big|\Big|_{\mathcal{H}_\pi} \leq CC(K)C_m(\widetilde{\varphi}_t) \ .$$

As $t \to 0$, the functions $X^\alpha * \varphi_t$ tend also to 0 in $\mathcal{C}_c^\infty(G)$ and finally $C_m(\widetilde{\varphi}_t)$ tends to 0 which leads to the result.

□

Let us consider now an H_p–invariant distribution vector a_p of π (where $p \in \{0, \ldots, r\}$). Then for $\varphi \in \mathcal{C}_c^\infty(G)$ we have

$$\langle \pi(\varphi)a_p, \Phi\rangle = \int_{G/H_p} \Big(\int_{H_p} \varphi(gh)\, d_p h\Big)\langle \pi(\dot{g})a_p, \Phi\rangle\, d\dot{g} \quad (\Phi \in \mathcal{H}_\pi^\infty) \ .$$

The function $g \longmapsto \int_{H_p} \varphi(gh) d_p h$ belongs to $\mathcal{C}_c^\infty(G/H_p)$. The functions f_φ^{p+} and f_φ^{p-} defined respectively on Ω_p^+ and Ω_p^- by

$$f_\varphi^{p+}(gI_p^+) = \int_{H_p} \varphi(gh) d_p h = f_\varphi^{p-}(gI_p^-) \quad \text{for } g \in G$$

are $\mathcal{C}^\infty$ functions with compact support.

On the other hand the function $g \longmapsto \pi(g)a_p$ depends only on the class $\dot{g}$ of g in G/H_p. Therefore this defines a function $X \mapsto \pi(X)a_p$ on $\Omega_p^+ \simeq G/H_p$ and a function $Y \mapsto \pi(Y)a_p$ on $\Omega_p^- \simeq G/H_p$. As the measure $d\dot{g}$ on G/H_p is induced from the measure d^*X on Ω_p^+ and from the measure d^*Y on Ω_p^- (Proposition 4.1.8) we obtain the following equalities

$$\begin{aligned}\pi(\varphi)a_p &= \int_{\Omega_p^+} f_\varphi^{p+}(X)\pi(X)a_p\, d^*X \\ &= \int_{\Omega_p^-} f_\varphi^{p-}(Y)\pi(Y)a_p\, d^*Y \ .\end{aligned}$$

It is therefore quite natural to introduce the following Zeta integrals of an H_p–invariant vector.

Definition 7.6.3. *For $a_p \in (\mathcal{H}_\pi^{-\infty})^{H_p}$ and $f \in \mathcal{C}_c^\infty(\Omega_p^+)$ we define an element $\mathbf{Z}_p^+(f, \pi, a_p) \in \mathcal{H}_\pi^{-\infty}$ by*

$$\mathbf{Z}_p^+(f, \pi, a_p) = \int_{\Omega_p^+} f(X)\pi(X)a_p\, d^*X.$$

Similarly for $h \in \mathcal{C}_c^\infty(\Omega_p^-)$ *we define* $\mathbf{Z}_p^-(h,\pi,a_p) \in \mathcal{H}_\pi^{-\infty}$ *by*

$$\mathbf{Z}_p^-(h,\pi,a_p) = \int_{\Omega_p^-} h(Y)\pi(Y)a_p\, d^*Y \ .$$

Since $\varphi \longmapsto f_\varphi^{p+}$ and $\varphi \longmapsto f_\varphi^{p-}$ are surjective onto respectively $\mathcal{C}_c^\infty(\Omega_p^+)$ and $\mathcal{C}_c^\infty(\Omega_p^-)$ these Zeta integrals are elements of $\mathcal{H}_{\pi^*}^\infty$ (from Proposition 7.6.2) and the relations obtained above may be written as follows.

Proposition 7.6.4. *Let* $p \in \{0,\dots,k\}$ *and* $a_p \in \left(\mathcal{H}_\pi^{-\infty}\right)^{H_p}$. *Then for* $\varphi \in \mathcal{C}_c^\infty(G)$ *we have*

$$\mathbf{Z}_p^+(f_\varphi^{p+},\pi,a_p) = \pi(\varphi)a_p \in \mathcal{H}_{\pi^*}^\infty \quad \text{and} \quad \mathbf{Z}_p^-(f_\varphi^{p-},\pi,a_p) = \pi(\varphi)a_p \in \mathcal{H}_{\pi^*}^\infty \ .$$

7.7. A formal functional equation

Strictly speaking, the result of the present section, will not be used in the sequel. Its essentially heuristic purpose is to explain why a functional equation is expected for $\mathbf{Z}_p^\pm(f,\pi,a_p)$ if the representation π satisfies some additional assumption. Basically, a functional equation for the Zeta function we will consider below, is just a uniqueness result for a homogeneous (vector valued) distribution. This point of view, which was already used in Chapter 5 and 6, goes back to André Weil in a Bourbaki Seminar ([We]–1966).

If f belongs to $\mathcal{C}_c^\infty(\Omega^+)$ the restriction of f to each Ω_p^+ is also compactly supported. It follows that $\mathbf{Z}_p^+(f,\pi,a_p)$ is well defined for an element a_p of $(\mathcal{H}_\pi^{-\infty})^{H_p}$.

Definition 7.7.1. *For* $a = (a_0,\dots,a_r)$ *an element of* $\prod_{p=0}^r (\mathcal{H}_\pi^{-\infty})^{H_p}$ *and* $f \in \mathcal{C}_c^\infty(\Omega^+)$, *we define an element* $\mathbf{Z}^+(f,\pi,a) \in \mathcal{H}_\pi^{-\infty}$ *by*

$$\mathbf{Z}^+(f,\pi,a) = \sum_{p=0}^r \mathbf{Z}_p^+(f,\pi,a_p).$$

Similarly, for $h \in \mathcal{C}_c^\infty(\Omega^-)$ *we define* $\mathbf{Z}^-(h,\pi,a) \in \mathcal{H}_\pi^{-\infty}$ *by*

$$\mathbf{Z}^-(h,\pi,a) = \sum_{p=0}^r \mathbf{Z}_p^-(h,\pi,a_p)$$

We suppose now that the following assumption is satisfied.

Assumption 7.7.2. *For all* $p = 0,\dots,r$ *and every* $a_p \in (\mathcal{H}_\pi^{-\infty})^{H_p}$, *the* $\mathcal{H}_\pi^{-\infty}$*–valued distribution on* $\mathcal{C}_c^\infty(\Omega^-)$ *defined by*

$$h \longmapsto \mathbf{Z}_p^-(h,\pi,a_p)$$

extends to a tempered distribution.

Let us then consider, for all $p,q = 0,\dots,r$ and every $a_q \in (\mathcal{H}_\pi^{-\infty})^{H_q}$ the $\mathcal{H}_\pi^{-\infty}$–valued distribution T_{pq} on Ω_p^+ defined by

$$T_{pq} : \mathcal{C}_c^\infty(\Omega_p^+) \longrightarrow \mathcal{H}_\pi^{-\infty}$$
$$f \longmapsto \int_{\Omega_q^-} \big(\mathcal{F}f\big)(Y)\pi(Y)a_q\, d^*Y = \mathbf{Z}_q^-(\mathcal{F}f,\pi,a_q) \ ,$$

where the Fourier transform $\mathcal{F}f$ is as in Definition 4.1.3. The distribution T_{pq} is well defined by Assumption 7.7.2 above. Let ℓ be the left regular action of G on functions either on V^+ or on V^-.

Lemma 7.7.3. *For $f \in \mathcal{C}_c^\infty(\Omega_p^+)$ and $g \in G$, we have*

$$\langle T_{pq}, \ell(g)f\rangle = \chi_0(g)^m \pi(g)\langle T_{pq}, f\rangle \ .$$

Proof. — From Lemma 4.1.5 we get for $Y \in V^-$

$$\mathcal{F}(\ell(g)f)(Y) = \chi_0(g)^m \Big(\ell(g)\mathcal{F}f\Big)(Y) \ .$$

Therefore

$$\begin{aligned}\langle T_{pq}, \ell(g)f\rangle &= \int_{\Omega_q^-} (\mathcal{F}f)\,(g^{-1}Y)\chi_0(g)^m \pi(Y) a_q \, d^*Y \\ &= \chi_0(g)^m \int_{\Omega_q^-} \mathcal{F}f(Y)\pi(gY)a_q \, d^*Y \\ &= \chi_0(g)^m \pi(g)\langle T_{pq}, f\rangle \ .\end{aligned}$$

□

Let $\mathrm{End}\left(\prod_{p=0}^r (\mathcal{H}_\pi^{-\infty})^{H_p}\right)$ denote the space of endomorphisms of the product $\prod_{p=0}^r (\mathcal{H}_\pi^{-\infty})^{H_p}$ and let $\mathrm{Hom}\left((\mathcal{H}_\pi^{-\infty})^{H_q}, (\mathcal{H}_\pi^{-\infty})^{H_p}\right)$ be the space of homomorphisms from $(\mathcal{H}_\pi^{-\infty})^{H_q}$ into $(\mathcal{H}_\pi^{-\infty})^{H_p}$.
Then an element $A \in \mathrm{End}\left(\prod_{p=0}^r (\mathcal{H}_\pi^{-\infty})^{H_p}\right)$ can be represented by a matrix $A = (A_{pq})$ where $A_{pq} \in \mathrm{Hom}\left((\mathcal{H}_\pi^{-\infty})^{H_q}, (\mathcal{H}_\pi^{-\infty})^{H_p}\right)$.

Theorem 7.7.4. *If the Assumption 7.7.2 is satisfied for the representation π, then there exists an element $A \in \mathrm{End}\left(\prod_{p=0}^r (\mathcal{H}_\pi^{-\infty})^{H_p}\right)$ such that for every $f \in \mathcal{C}_c^\infty(\Omega^+)$ and for every $a = (a_p) \in \prod_{p=0}^r (\mathcal{H}_\pi^{-\infty})^{H_p}$ we have*

$$\mathbf{Z}^-(\mathcal{F}f, \pi, a) = \mathbf{Z}^+(f, \chi_0^{-m} \otimes \pi, Aa) \ .$$

Proof. — Here $\rho = \chi_0^{-m} \otimes \pi$ denotes the representation of G on $\mathcal{H}_\pi$ given by $\rho(g)u = \chi_0(g)^{-m}\pi(g)u$ for $u \in \mathcal{H}_\pi$ and $g \in G$. The differentiable vectors, the distribution vectors and the H_p–invariant distribution vectors are the same for the representations π and $\rho = \chi_0^{-m} \otimes \pi$. Let us notice that, as for π, the representation ρ on $\mathcal{H}_\pi$ induces a representation, here denoted by $\rho_{-\infty}$, on $\mathcal{H}_\pi^{-\infty}$ and given by

$$\langle \rho_{-\infty}(g)a, u\rangle = \langle a, \rho(g)^{-1}u\rangle = \chi_0(g)^m \langle \pi_{-\infty}(g)a, u\rangle \ ,$$

for $a \in \mathcal{H}_\pi^{-\infty}$ and $u \in \mathcal{H}_\pi^\infty$. Thus we have for $f \in \mathcal{C}_c^\infty(\Omega_p^+)$ and $a_p \in (\mathcal{H}_\pi^{-\infty})^{H_p}$

$$\begin{aligned}\mathbf{Z}_p^+(f, \chi_0^{-m} \otimes \pi, a_p) &= \int_{G/H_p} f(gI_p^+)\rho_{-\infty}(g)a_p \, d\dot{g} \\ &= \int_{\Omega_p^+} f(X)|\Delta_0(X)|^m \pi(X)a_p \ d^*X \\ &= \mathbf{Z}_p^+(f|\Delta_0|^m, \pi, a_p) \ .\end{aligned}$$

Consider the before defined $\mathcal{H}_\pi^{-\infty}$–valued distribution T_{pq} on $\mathcal{C}_c^\infty(\Omega_p^+)$. Let us first transform T_{pq} into a $\mathcal{H}_\pi^\infty$–distribution (see [Bru]–1956 p.105),

that is a continuous linear form $\widetilde{T}_{pq}$ on $\mathcal{C}_c^\infty(\Omega_p^+, \mathcal{H}_\pi^\infty) = \mathcal{C}_c^\infty(\Omega_p^+) \otimes \mathcal{H}_\pi^\infty$ by setting

$$\langle \widetilde{T}_{pq}, f \otimes u \rangle = \overline{\left\langle \langle T_{pq}, \overline{f} \rangle, u \right\rangle} \text{ for } u \in \mathcal{H}_\pi^\infty, \ f \in \mathcal{C}_c^\infty(\Omega_p^+) \ .$$

(Recall that an element of $\mathcal{H}_\pi^{-\infty}$ is a anti-linear form). Since $\mathcal{H}_\pi^\infty$ is a Fréchet space, this correspondence between $\mathcal{H}_\pi^{-\infty}$–valued distributions and $\mathcal{H}_\pi^\infty$–distributions is bijective (see [Bru]–1956, Prop. 1.1.b. p.105).

For $F \in \mathcal{C}_c^\infty(\Omega_p^+, \mathcal{H}_\pi^\infty)$ we define $\pi(g)F$ for $g \in G$ and $X \in \Omega_p^+$ by

$$(\pi(g)F)(X) = \pi(g)F(X) \ .$$

As $\chi_0(g)$ is a real positive number, we get from Lemma 7.7.3 above that for $F = f \otimes u$

$$\begin{aligned} \langle \widetilde{T}_{pq}, \ell(g)F \rangle &= \langle \widetilde{T}_{pq}, (\ell(g)f) \otimes u \rangle = \overline{\left\langle \langle T_{pq}, \ell(g)\overline{f} \rangle, u \right\rangle} \\ &= \chi_0(g)^m \overline{\left\langle \pi(g)\langle T_{pq}, \overline{f} \rangle, u \right\rangle} = \chi_0(g)^m \overline{\left\langle \langle T_{pq}, \overline{f} \rangle, \pi(g^{-1})u \right\rangle} \\ &= \chi_0(g)^m \langle \widetilde{T}_{pq}, f \otimes \pi(g^{-1})u \rangle = \langle \widetilde{T}_{pq}, (\chi_0^{-m} \otimes \pi)(g^{-1})F \rangle \ . \end{aligned}$$

The relation $\langle \widetilde{T}_{pq}, \ell(g)F \rangle = \langle \widetilde{T}_{pq}, (\chi_0^{-m} \otimes \pi)(g^{-1})F \rangle$ which we just proved, means that $\widetilde{T}_{pq}$ is a $\mathcal{H}_\pi^\infty$–distribution with multiplier $\rho = \chi_0^{-m} \otimes \pi$ (see [Bru]-1959 Def. 3.3. p.124). By BRUHAT's Theorem 3.1. ([Bru]-1959, p. 124), there exists a unique vector $a_q^p \in \mathcal{H}_\pi^{-\infty}$ such that

$$\forall h \in H_p \quad \rho_{-\infty}(h)a_q^p = a_q^p$$

and

$$\langle \widetilde{T}_{pq}, F \rangle = \int_{\Omega_p^+} \overline{\langle \rho_{-\infty}(X)a_q^p, F(X) \rangle} \, d^*X \ .$$

As χ_0 is trivial on H_p, a_q^p is an H_p–invariant distribution vector in $\mathcal{H}_\pi^{-\infty}$ for the action of π. And it follows that we have for $f \in \mathcal{C}_c^\infty(\Omega_p^+)$

$$\begin{aligned} \langle T_{pq}, f \rangle &= \int_{\Omega_p^+} f(X)\rho_{-\infty}(X)a_q^p \, d^*X \qquad (7\text{-}7\text{-}1) \\ &= \mathbf{Z}_p^+(f, \chi_0^{-m} \otimes \pi, a_q^p) \ . \end{aligned}$$

Let us denote by $A_{pq} \in \mathrm{Hom}\left((\mathcal{H}_\pi^{-\infty})^{H_q}, (\mathcal{H}_\pi^{-\infty})^{H_p}\right)$ the linear map defined by $A_{pq}a_q = a_q^p$. The matrix $A = (A_{pq})$ whose entries are the A_{pq}'s defines an element of $\mathrm{End}\left(\prod_{p=0}^r (\mathcal{H}_\pi^{-\infty})^{H_p}\right)$.

If now $f \in \mathcal{C}_c^\infty(\Omega^+)$, we can write

$$f = \sum_{p=0}^r f_p \qquad (\text{where } f_p \in \mathcal{C}_c^\infty(\Omega_p^+))$$

and

$$\mathcal{F}f = \sum_{p=0}^r \mathcal{F}f_p \ .$$

Then we have

$$\mathbf{Z}^-(\mathcal{F}f, \pi, a) = \sum_{q=0}^{r} \mathbf{Z}_q^-(\mathcal{F}f, \pi, a_q) = \sum_{q=0}^{r}\sum_{p=0}^{r} \mathbf{Z}_q^-(\mathcal{F}f_p, \pi, a_q)$$
$$= \sum_{q=0}^{r}\sum_{p=0}^{r} \langle T_{pq}, f_p \rangle \ .$$

Using (7-7-1) we obtain

$$\mathbf{Z}^-(\mathcal{F}f, \pi, a) = \sum_{q=0}^{r}\sum_{p=0}^{r} \mathbf{Z}_p^+(f_p, \chi_0^{-m} \otimes \pi, A_{pq}a_q)$$
$$= \sum_{p=0}^{r} \mathbf{Z}_p^+(f_p, \chi_0^{-m} \otimes \pi, \sum_{q=0}^{r} A_{pq}a_q)$$
$$= \mathbf{Z}^+(f, \chi_0^{-m} \otimes \pi, Aa) \ .$$

□

7.8. Computation of $\mathbf{Z}_p^{\pm}(f, \pi_{\tau,\lambda}, a_{\tau,\lambda}^{p,\gamma})$

These Zeta integrals are closely related to the local Zeta functions defined in chapter 5 and 6. We will use the notations introduced in section 7.2. It follows from Proposition 7.6.4 that the computation of $\mathbf{Z}_p^{\pm}(f, \pi_{\tau,\lambda}, a_{\tau,\lambda}^{p,\gamma})$ reduces to the computation of $\pi_{\tau,\lambda}(\varphi)a_{\tau,\lambda}^{p,\gamma}$.

Let us notice that for $\pi = \pi_{\tau,\lambda}$ the dual conjugate representation π^* is equivalent to $\pi_{\tau,-\overline{\lambda}}$.

Proposition 7.8.1. *Suppose that $\lambda \in (\mathfrak{a}_{\mathbb{C}}^0)^*$ satisfies the conditions of Proposition 7.5.6. Then, for $\varphi \in \mathcal{C}_c^\infty(G)$, the distribution vector $\pi_{\tau,\lambda}(\varphi)a_{\tau,\lambda}^{p,\gamma}$ is the function in $\mathcal{H}_{\tau,-\overline{\lambda}}^\infty$ given by*

$$\big(\pi(\varphi)a_{\tau,\lambda}^{p,\gamma}\big)(x) = \int_G \varphi(g)\mathbb{P}_{\tau,\lambda}^{p,\gamma}(xg)\,dg \quad \textit{for } x \in G$$

Proof. — As $\pi_{\tau,\lambda}^* \simeq \pi_{\tau,-\overline{\lambda}}$, we already know from Proposition 7.6.2 that $\pi(\varphi)a_{\tau,\lambda}^{p,\gamma}$ is a function in $\mathcal{H}_{\tau,-\overline{\lambda}}^\infty$. From Theorem 7.5.7 we get for $\Phi \in \mathcal{H}_{\tau,\lambda}^\infty$

$$\langle \pi(g)a_{\tau,\lambda}^{p,\gamma}, \Phi \rangle = \int_K \Big(\mathbb{P}_{\tau,\lambda}^{p,\gamma}(kg) \mid \Phi(k)\Big)_\tau \, dk \ .$$

Therefore we have

$$\langle \pi(\varphi)a_{\tau,\lambda}^{p,\gamma}, \Phi \rangle = \int_G \varphi(g) \int_K \Big(\mathbb{P}_{\tau,\lambda}^{p,\gamma}(kg) \mid \Phi(k)\Big)_\tau \, dk\, dg$$
$$= \int_K \Big(\int_G \varphi(g)\mathbb{P}_{\tau,\lambda}^{p,\gamma}(kg)\,dg \mid \Phi(k)\Big)_\tau \, dk$$

The function

$$\Upsilon : x \mapsto \int_G \varphi(g)\mathbb{P}_{\tau,\lambda}^{p,\gamma}(xg)\,dg = \int_G \varphi(x^{-1}g)\mathbb{P}_{\tau,\lambda}^{p,\gamma}(g)\,dg$$

is a $\mathcal{C}^\infty$ function on G. From Definition 7.5.5 Υ satisfies the relation

$$\Upsilon(manx) = a^{\rho-\overline{\lambda}}\tau(m)\Upsilon(x) \quad \text{for } m \in M,\ a \in A,\ n \in N\ .$$

Thus Υ is an element of $\mathcal{C}^\infty_{\tau,-\overline{\lambda}}(G,\mathcal{H}_\tau)$ and we have proved that $\pi_{\tau,\lambda}(\varphi)a^{p,\gamma}_{\tau,\lambda}$ is the image of Υ by the embedding defined in Proposition 7.3.4.

□

Corollary 7.8.2. *Suppose that $\lambda \in (\mathfrak{a}^0_{\mathbb{C}})^*$ satisfies the conditions of Proposition 7.5.6. Then for any $f \in \mathcal{C}^\infty_c(\Omega^+_p)$ and $h \in \mathcal{C}^\infty_c(\Omega^-_p)$ the Zeta integrals $\mathbf{Z}^+_p(f,\pi_{\tau,\lambda},a^{p,\gamma}_{\tau,\lambda})$ and $\mathbf{Z}^-_p(h,\pi_{\tau,\lambda},a^{p,\gamma}_{\tau,\lambda})$ are the functions in $\mathcal{H}^\infty_{\tau,-\overline{\lambda}}$ given for $x \in G$ by*

$$\mathbf{Z}^+_p(f,\pi_{\tau,\lambda},a^{p,\gamma}_{\tau,\lambda})(x) = Z^+_{p,\gamma}(\ell(x)f, s(\overline{\lambda}) - m, \tau)\ ,$$

$$\textit{and} \quad \mathbf{Z}^-_p(h,\pi_{\tau,\lambda},a^{p,\gamma}_{\tau,\lambda})(x) = Z^-_{p,\gamma}(\ell(x)h, t(s(\overline{\lambda})) - m, \tau)\ .$$

Proof. — Let $\varphi \in \mathcal{C}^\infty_c(G)$ such that

$$f(g.I^+_p) = \int_{H_p} \varphi(gh)\, d_p h\ .$$

From the previous Proposition and from Proposition 7.6.4 we get

$$\mathbf{Z}^+_p(f,\pi_{\tau,\lambda},a^{p,\gamma}_{\tau,\lambda})(x) = \big(\pi_{\tau,\lambda}(\varphi)a^{p,\gamma}_{\tau,\lambda}\big)(x) = \int_G \varphi(x^{-1}g)\mathbb{P}^{p,\gamma}_{\tau,\lambda}(g)\, dg\ .$$

Using the relation between $\mathbb{P}^{p,\gamma}_{\tau,\lambda}$ and $P^{p,\gamma}_{\tau,s(\overline{\lambda})}$ (Definition 7.5.5) and the fact that $\mathbb{P}^{p,\gamma}_{\tau,\lambda}$ is invariant under the right action of H_p we obtain

$$\int_G \varphi(x^{-1}g)\mathbb{P}^{p,\gamma}_{\tau,\lambda}(g)\, dg = \int_{G/H_p} f(x^{-1}gI^+_p)P^{p,\gamma}_{\tau,s(\overline{\lambda})}(gI^+_p)\, d\dot{g}\ .$$

And from the definition of $P^{p,\gamma}_{\tau,s}$ we get

$$\begin{aligned}\int_G \varphi(x^{-1}g)\mathbb{P}^{p,\gamma}_{\tau,\lambda}(g)\, dg &= \int_{\mathcal{O}^+_{p,\gamma}} (\ell(x)f)(X)|\Delta(X)|^{s(\overline{\lambda})}b^+_\tau(X)\, d^*X \\ &= Z^+_{p,\gamma}(\ell(x)f, s(\overline{\lambda}) - m, \tau)\ .\end{aligned}$$

The proof on the negative side is similar and relies on the fact that if P^- denotes the $\mathcal{H}_\tau$–valued function defined on Ω^-_p by

$$P^-(gI^-_p) = \mathbb{P}^{p,\gamma}_{\tau,\lambda}(g)\ ,$$

then we have

$$P^-(Y) = \begin{cases} |\nabla(Y)|^{t(s(\overline{\lambda}))}b^-_\tau(Y) \text{ if } Y \in \mathcal{O}^-_{p,\gamma}\ , \\ 0 \text{ if } Y \in \Omega^-_p \backslash \mathcal{O}^-_{p,\gamma}\ . \end{cases}$$

If $s(\overline{\lambda}) = (s_0, s_1, \ldots, s_k)$, then $t(s(\overline{\lambda})) = (-s_0 - s_1 - \cdots, -s_k, s_k, \ldots, s_1)$. Therefore as λ satisfies the conditions of Proposition 7.5.6, we have $s_j > 0$ for $j = 1, \ldots, k$ which implies that P^- is continuous on Ω^-_p (similarly to Lemma 7.5.2) and that $Z^-_{p,\gamma}(\ell(x)h, t(s(\overline{\lambda})) - m, \tau)$ is defined by a convergent integral for $h \in \mathcal{C}^\infty_c(\Omega^-_p)$.

□

And now we extend this result by holomorphic continuation.

Theorem 7.8.3. *Let $\pi_{\tau,\lambda}$ be a representation of the minimal principal spherical series with $\overline{\lambda} \in \mathbf{V}$. Let $a^{p,\gamma}_{\tau,\lambda}$ be the H_p–invariant distribution vector defined in Theorem 7.4.6 where $p \in \{0,\dots,r\}$ and $\gamma \in \mathcal{W}_p$. Then for $f \in \mathcal{C}_c^\infty(\Omega_p^+)$ and $h \in \mathcal{C}_c^\infty(\Omega_p^-)$ the Zeta integrals $\mathbf{Z}_p^+(f,\pi_{\tau,\lambda},a^{p,\gamma}_{\tau,\lambda})$ and $\mathbf{Z}_p^-(h,\pi_{\tau,\lambda},a^{p,\gamma}_{\tau,\lambda})$ are the functions in $\mathcal{H}^\infty_{\tau,-\overline{\lambda}}$ whose values at the point $x \in G$ are given by*

$$\mathbf{Z}_p^+(f,\pi_{\tau,\lambda},a^{p,\gamma}_{\tau,\lambda})(x) = Z^+_{p,\gamma}(\ell(x)f, s(\overline{\lambda}) - m, \tau)\ ,$$
$$\text{and } \mathbf{Z}_p^-(h,\pi_{\tau,\lambda},a^{p,\gamma}_{\tau,\lambda})(x) = Z^-_{p,\gamma}(\ell(x)h, t(s(\overline{\lambda})) - m, \tau)\ .$$

Proof. — For $\overline{\lambda} \in \mathbf{V}$, the map $f \longmapsto Z^+_{p,\gamma}(f, s-m, \tau)$ is a $\mathcal{H}_\tau$–valued distribution and the map $x \longmapsto \ell(x)f$ is continuous from G into $\mathcal{C}_c^\infty(\Omega_p^+)$. Therefore the $\mathcal{H}_\tau$–valued function $x \longmapsto Z^+_{p,\gamma}(\ell(x)f, s(\overline{\lambda}) - m, \tau)$ is continuous on G. Moreover it is easy to see that this function belongs to $\mathcal{C}_{\tau,-\overline{\lambda}}(G,\mathcal{H}_\tau)$. On the other hand we know from Proposition 7.6.4 that $\mathbf{Z}_p^+(f,\pi_{\tau,\lambda},a^{p,\gamma}_{\tau,\lambda})$ is a function in $\mathcal{H}^\infty_{\tau,-\overline{\lambda}}$. Thus it is sufficient to prove that

$$\mathbf{Z}_p^+(f,\pi_{\tau,\lambda},a^{p,\gamma}_{\tau,\lambda})(x) = Z^+_{p,\gamma}(\ell(x)f, s(\overline{\lambda}) - m, \tau) \text{ for } x \in K\ .$$

From the previous Corollary we know that this equality is satisfied for $\overline{\lambda}$ in the open subset of $\mathbf{V}$ where λ satisfies the conditions of Proposition 7.5.6.

Let $\Psi \in \mathcal{C}_c^\infty(G,\mathcal{H}_\tau)$.

The map $(k,\overline{\lambda}) \longmapsto \big(Z^+_{p,\gamma}(\ell(k)f, s(\overline{\lambda}) - m, \tau) \,|\, (L_{\tau,\lambda}\Psi)(k)\big)_\tau$ is continuous in $k \in K$ and holomorphic in $\overline{\lambda} \in \mathbf{V}$. Hence the map

$$\overline{\lambda} \longmapsto \int_K \big(Z^+_{p,\gamma}(\ell(k)f, s(\overline{\lambda}) - m, \tau) \,|\, (L_{\tau,\lambda}\Psi)(k)\big)_\tau\, dk$$

is also holomorphic for $\overline{\lambda}$ in $\mathbf{V}$.

On the other hand the map

$$\overline{\lambda} \longmapsto \big(\mathbf{Z}_p^+(f,\pi_{\tau,\lambda},a^{p,\gamma}_{\tau,\lambda}) \,|\, L_{\tau,\lambda}\Psi\big)_\tau = \langle \pi_{\tau,\lambda}(\varphi)a^{p,\gamma}_{\tau,\lambda}, L_{\tau,\lambda}\Psi\rangle\ ,$$

where φ is a function in $\mathcal{C}_c^\infty(G)$ such that $f(gI_p^+) = \int_{H_p}(\varphi(gh)\, d_p h$, is also holomorphic on $\mathbf{V}$ (see Theorem 7.4.6).

As we have proven that these two maps coincide on an open subset of $\mathbf{V}$, they are equal on $\mathbf{V}$. As this equality is proved for any $\Phi = L_{\tau,\lambda}\Psi \in \mathcal{H}^\infty_{\tau,\lambda}$ the Theorem is proved on the positive side.

The proof on the negative side is similar.

□

7.9. The Zeta function attached to the minimal spherical principal series and its functional equation (Main Theorem)

In order to write the functional equation for the Zeta functions $\mathbf{Z}^\pm$, we will extend the Zeta integrals $\mathbf{Z}_p^\pm$ to functions in the Schwarz spaces $\mathcal{S}(V^+)$ or $\mathcal{S}(V^-)$.

Recall from Proposition 7.2.5 that for $s \in \mathcal{U}^+$ (*resp.* $\mathcal{U}^-$), the map $f \longmapsto Z^+_{p,\gamma}(f,s,\tau)$ (*resp.* $h \longmapsto Z^-_{p,\gamma}(h,s,\tau)$) defines a tempered $\mathcal{H}_\tau$–valued distribution.

Definition 7.9.1. *We denote by $\mathcal{V}^+$ and $\mathcal{V}^-$ the open subsets of $(\mathfrak{a}_{\mathbb{C}}^0)^*$ defined by*

$$\lambda \in \mathcal{V}^+ \Longleftrightarrow s(\lambda) \in \mathcal{U}^+ + m$$
$$\lambda \in \mathcal{V}^- \Longleftrightarrow t(s(\lambda)) \in \mathcal{U}^- + m$$

Let $f \in \mathcal{S}(V^+)$ and $\overline{\lambda} \in \mathcal{V}^+$. As the map $x \longmapsto \ell(x)f$ is continuous from G to $\mathcal{S}(V^+)$, the map $x \longmapsto Z^+_{p,\gamma}(\ell(x)f, s-m, \tau)$ is continuous from G to $\mathcal{H}_\tau$. An easy computation shows that, for $m \in M$, $a \in A$ and $n \in N$, we have

$$Z^+_{p,\gamma}(\ell(manx)f, s(\overline{\lambda}) - m, \tau) = a^{\rho - \overline{\lambda}}\tau(m) Z^+_{p,\gamma}(\ell(x)f, s(\overline{\lambda}) - m, \tau) \ .$$

Hence the map $x \longmapsto Z^+_{p,\gamma}(\ell(x)f, s(\overline{\lambda}) - m, \tau)$ is a function in $\mathcal{C}_{\tau,-\overline{\lambda}}(G, \mathcal{H}_\tau)$. Similarly the map $x \longmapsto Z^-_{p,\gamma}(\ell(x)h, t(s(\overline{\lambda})) - m, \tau)$ is, for $h \in \mathcal{S}(V^-)$ and $\overline{\lambda} \in \mathcal{V}^-$, a function in $\mathcal{C}_{\tau,-\overline{\lambda}}(G, \mathcal{H}_\tau)$.

As $\mathcal{V}^\pm \subset \mathbf{V}$, we will, using the result of Theorem 7.8.3 in the last section, extend $\mathbf{Z}^\pm_p$ to Schwartz functions for $\overline{\lambda} \in \mathcal{V}^\pm$.

Definition 7.9.2.
(1) *Let $\overline{\lambda} \in \mathcal{V}^+$ and $f \in \mathcal{S}(V^+)$. Then $\mathbf{Z}^+_p(f, \pi_{\tau,\lambda}, a^{p,\gamma}_{\tau,\lambda})$ denotes the function in $\mathcal{C}_{\tau,-\overline{\lambda}}(G, \mathcal{H}_\tau)$ given by*

$$\mathbf{Z}^+_p(f, \pi_{\tau,\lambda}, a^{p,\gamma}_{\tau,\lambda})(x) = Z^+_{p,\gamma}(\ell(x)f, s(\overline{\lambda}) - m, \tau) \quad \textit{for } x \in G \ .$$

(2) *Let $\overline{\lambda} \in \mathcal{V}^-$ and $h \in \mathcal{S}(V^-)$. Then $\mathbf{Z}^-_p(h, \pi_{\tau,\lambda}, a^{p,\gamma}_{\tau,\lambda})$ denotes the function in $\mathcal{C}_{\tau,-\overline{\lambda}}(G, \mathcal{H}_\tau)$ given by*

$$\mathbf{Z}^-_p(h, \pi_{\tau,\lambda}, a^{p,\gamma}_{\tau,\lambda})(x) = Z^-_{p,\gamma}(\ell(x)h, t(s(\overline{\lambda})) - m, \tau) \quad \textit{for } x \in G \ .$$

Let us recall that $\mathcal{C}_{\tau,-\overline{\lambda}}(G, \mathcal{H}_\tau)$ is a subset of $\mathcal{H}_{\tau,-\overline{\lambda}}$ which is naturally embedded in the dual of $\mathcal{H}_{\tau,\lambda}$. This dual is the subset of the distribution vectors for $\pi_{\tau,\lambda}$ which are continuous antilinear forms on $\mathcal{H}_{\tau,\lambda}$ (not only on $\mathcal{H}^\infty_{\tau,\lambda}$). Theorem 7.8.3 implies that for $f \in \mathcal{C}^\infty_c(\Omega^+)$ and $\lambda \in \mathcal{V}^+$, $\mathbf{Z}^+_p(f, \pi_{\tau,\lambda}, a^{p,\gamma}_{\tau,\lambda})$ coincides with the distribution vector defined in Definition 7.6.3. Similarly for $h \in \mathcal{C}^\infty_c(\Omega^-)$ and $\lambda \in \mathcal{V}^-$, $\mathbf{Z}^-_p(h, \pi_{\tau,\lambda}, a^{p,\gamma}_{\tau,\lambda})$ coincides with the distribution vector defined in Definition 7.6.3.

Proposition 7.9.3.
(1) *For $\Phi \in \mathcal{H}_{\tau,\lambda}$, the maps*

$$f \longmapsto \langle \mathbf{Z}^+_p(f, \pi_{\tau,\lambda}, a^{p,\gamma}_{\tau,\lambda}), \Phi \rangle \quad \textit{and} \quad h \longmapsto \langle \mathbf{Z}^-_p(h, \pi_{\tau,\lambda}, a^{p,\gamma}_{\tau,\lambda}), \Phi \rangle$$

are tempered distributions for respectively $\overline{\lambda} \in \mathcal{V}^+$ and $\overline{\lambda} \in \mathcal{V}^-$.
(2) *For $\Psi \in \mathcal{C}^\infty_c(G, \mathcal{H}_\tau)$, $f \in \mathcal{S}(V^+)$ and $h \in \mathcal{S}(V^-)$, the maps*

$$\overline{\lambda} \longmapsto \langle \mathbf{Z}^+_p(f, \pi_{\tau,\lambda}, a^{p,\gamma}_{\tau,\lambda}) \, , \, L_{\tau,\lambda}\Psi \rangle \quad \textit{and} \quad \overline{\lambda} \longmapsto \langle \mathbf{Z}^+_p(f, \pi_{\tau,\lambda}, a^{p,\gamma}_{\tau,\lambda}) \, , \, L_{\tau,\lambda}\Psi \rangle$$

are holomorphic respectively on $\mathcal{V}^+$ and $\mathcal{V}^-$.

Proof. — The proof is done on the positive side. From the Definition of $\mathbf{Z}^+_p$ we obtain for $\Phi \in \mathcal{H}_{\tau,\lambda}$

$$\langle \mathbf{Z}^+_p(f, \pi_{\tau,\lambda}, a^{p,\gamma}_{\tau,\lambda}), \Phi \rangle = \int_K \big(Z^+_{p,\gamma}(\ell(k)f, s(\overline{\lambda}) - m, \tau) \,\big|\, \Phi(k) \big)_\tau dk \ .$$

For $s-m \in \mathcal{U}^+$ the distribution $f \longmapsto Z_{p,\gamma}(f, s-m, \tau)$ is tempered. More precisely we get from Lemma 5.2.5 in Types I and II cases or from Proposition 6.4.3 in Type III case the following precisions. For any compact subset $\mathcal{K}$ in $\mathcal{U}^+$ there exist a semi-norm $\nu_{M,N}$ (depending on $\mathcal{K}$) on $\mathcal{S}(V^+)$ and a constant $C(\mathcal{K})$ such that for $s-m \in \mathcal{K}$ and $f \in \mathcal{S}(V^+)$ we have

$$||Z_{p,\gamma}(f, s-m, \tau)||_\tau \leq C(\mathcal{K})\nu_{M,N}(f) \ .$$

As the goup K is compact an elementary computation (using the definition of $\nu_{M,N}$) shows that there exists a constant C' such that for all $k \in K$ and all $f \in \mathcal{S}(V^+)$ we have

$$\nu_{M,N}(\ell(k)f) \leq \ C'\nu_{M,N}(f) \ .$$

Therefore, if $s(\overline{\lambda}) - m \in \mathcal{K}$ we obtain

$$|\langle \mathbf{Z}_p^+(f, \pi_{\tau,\lambda}, a_{\tau,\lambda}^{p,\gamma}), \Phi\rangle| \leq C'C(\mathcal{K})\Big(\int_K ||\Phi(k)||_\tau \ dk\Big) \ \nu_{N,M}(f) \ ,$$

which implies the first part of the Proposition.

On the other hand if $\Phi = L_{\tau,\lambda}\Psi$ with $\Psi \in \mathcal{C}_c^\infty(G, \mathcal{H}_\tau)$ and if λ belongs to an open subset $\mathcal{H}$ of $\mathcal{V}^+$ with $\overline{\mathcal{H}}$ compact there exists a constant C'' such that for all $\overline{\lambda} \in \mathcal{H}$ and $k \in K$ we have

$$||(L_{\tau,\lambda}\Psi)(k)||_\tau \leq C'' \ .$$

It follows that if $\mathcal{K} = \{s(\overline{\lambda}) \mid \lambda \in \overline{\mathcal{H}}\}$ we get

$$\big|\big(Z_{p,\gamma}^+(\ell(k)f, s(\overline{\lambda}) - m, \tau) \ \big| \ (L_{\tau,\lambda}\Psi)(k)\big)_\tau\big| \leq C(\mathcal{K})C'C''\nu_{M,N}(f) \ .$$

As the map $\overline{\lambda} \longmapsto \big(Z_{p,\gamma}^+(\ell(k)f, s(\overline{\lambda}) - m, \tau) \ \big| \ (L_{\tau,\lambda}\Psi)(k)\big)_\tau$ is holomorphic on $\mathcal{H}$, the second part of the Proposition follows.

□

Definition 7.9.4. *Let $\widetilde{\mathbf{V}}$ be the open dense subset of $\mathbf{V}$ where, for every $p \in \{0, \ldots, r\}$, the set $(a_{p,\lambda}^{p,\gamma})_{\gamma \in \mathcal{W}_p}$ is a basis of $(\mathcal{H}_{\tau,\lambda}^{-\infty})^{H_p}$ (see Remark 7.4.7). We set $\mathbb{V} = \widetilde{\mathbf{V}} \cap \mathcal{V}^+ \cap \mathcal{V}^-$.*

If a_p is an element of $(\mathcal{H}_{\tau,\lambda}^{-\infty})^{H_p}$ with $\lambda \in \mathbb{V}$ it follows from Proposition 7.9.3 that the $\mathcal{C}_{\tau,-\overline{\lambda}}^\infty(G, \mathcal{H}_\tau)$–valued distributions $f \longmapsto \mathbf{Z}_p^\pm(f, \pi_{\tau,\lambda}, a_p)$ (defined in Definition 7.6.3) extend to tempered $\mathcal{C}_{\tau,-\overline{\lambda}}(G, \mathcal{H}_\tau)$–valued distributions on $\mathcal{S}(V^\pm)$. Therefore we may extend Definition 7.7.1 to functions in $\mathcal{S}(V^\pm)$ as follows.

Definition 7.9.5. *Let $\overline{\lambda} \in \mathbb{V}$ and let $a = (a_0, a_1, \ldots, a_r)$ be an element of $\prod_{p=0}^r (\mathcal{H}_{\tau,\lambda}^{-\infty})^{H_p}$. For $f \in \mathcal{S}(V^+)$ and $h \in \mathcal{S}(V^-)$, $\mathbf{Z}^+(f, \pi_{\tau,\lambda}, a)$ and $\mathbf{Z}^-(h, \pi_{\tau,\lambda}, a)$ denote the elements of $\mathcal{H}_{\tau,-\overline{\lambda}} \hookrightarrow \mathcal{H}_{\tau,\lambda}^{-\infty}$ given by*

$$\mathbf{Z}^+(f, \pi_{\tau,\lambda}, a) = \sum_{p=0}^r \mathbf{Z}_p^+(f, \pi_{\tau,\lambda}, a_p) \ \text{and} \ \mathbf{Z}^-(h, \pi_{\tau,\lambda}, a) = \sum_{q=0}^r \mathbf{Z}_q^-(h, \pi_{\tau,\lambda}, a_q) \, .$$

These two elements $\mathbf{Z}^+(f, \pi_{\tau,\lambda}, a)$ and $\mathbf{Z}^-(h, \pi_{\tau,\lambda}, a)$ are the Zeta functions associated to $\pi_{\tau,\lambda}$.

Theorem 7.9.6 (Main Theorem). *Let $\pi_{\tau,\lambda}$ be a representation of the minimal principal series with $\overline{\lambda} \in \mathbb{V}$. There exists an element $A^{\tau,\lambda}$ in $End\big(\prod_{p=0}^{r}(\mathcal{H}_{\tau,\lambda}^{-\infty})^{H_p}\big)$ such that the following functional equation holds for $f \in \mathcal{S}(V^+)$ and $a \in \prod_{p=0}^{r}(\mathcal{H}_{\tau,\lambda}^{-\infty})^{H_p}$:*

$$\mathbf{Z}^-(\mathcal{F}f,\, \pi_{\tau,\lambda},\, a) = \mathbf{Z}^+\big(f,\, \chi_0^{-m} \otimes \pi_{\tau,\lambda},\, A^{\tau,\lambda}(a)\big) \ .$$

The endomorphism $A^{\tau,\lambda}$ can be represented as a $(r+1)\times(r+1)$ matrix whose coefficients $A_{p,q}^{\tau,\lambda}$ belong to $\mathrm{Hom}\,\big((\mathcal{H}_{\tau,\lambda}^{-\infty})^{H_q}, (\mathcal{H}_{\tau,\lambda}^{-\infty})^{H_p}\big)$. In the basis $(a_{\tau,\lambda}^{p,\gamma})_{\gamma\in\mathcal{W}_p}$ and $(a_{\tau,\lambda}^{q,\gamma'})_{\gamma'\in\mathcal{W}_q}$ of $(\mathcal{H}_{\tau,\lambda}^{-\infty})^{H_p}$ and $(\mathcal{H}_{\tau,\lambda}^{-\infty})^{H_q}$ respectively, the matrix of $A_{p,q}^{\tau,\lambda}$ is given by

$$\Big(A_{p,q}^{\tau,\lambda}\Big)_{\gamma,\gamma'} = \nu_k\big(t(s(\overline{\lambda})) - m, \tau, q, \gamma', p, \gamma\big) \quad \text{for } \gamma \in \mathcal{W}_p,\ \gamma' \in \mathcal{W}_q \ .$$

The definition of $\nu_k(s,\tau,q,\gamma',p,\gamma)$ is given in Theorem 7.2.6.

Proof. — We have to prove an equality of two functions in $\mathcal{C}_{\tau,-\overline{\lambda}}(G,\mathcal{H}_\tau)$. It is therefore sufficient to prove the equality of the restrictions of these functions to the compact group K. Let $k \in K$. As we have

$$\ell(k)\mathcal{F}f = \mathcal{F}(\ell(k)f) \text{ for } f \in \mathcal{S}(V^+) \ ,$$

we get from Definition 7.9.2 that

$$\mathbf{Z}_q^-(\mathcal{F}f, \pi_{\tau,\lambda}, a_{\tau,\lambda}^{q,\gamma'})(k) = Z_{q,\gamma'}^-(\mathcal{F}(\ell(k)f), t(s(\overline{\lambda})) - m, \tau) \ .$$

Applying the functional equation given in Theorem 7.2.6 and using the relation $t(t(s(\overline{\lambda})) - m) - m = s(\overline{\lambda})$ we obtain

$$\mathbf{Z}_q^-(\mathcal{F}f, \pi_{\tau,\lambda}, a_{\tau,\lambda}^{q,\gamma'})(k) = \sum_{p=0}^{r}\sum_{\gamma\in\mathcal{W}_p} \nu_k\big(t(s(\overline{\lambda})) - m, \tau, q, \gamma', p, \gamma\big)\, Z_{p,\gamma}^+(\ell(k)f, s(\overline{\lambda}), \tau) \ .$$

As $\chi_0(k) = 1$, we may write, using again Definition 7.9.2,

$$\begin{aligned} Z_{p,\gamma}^+(\ell(k)f, s(\overline{\lambda}), \tau) &= Z_{p,\gamma}^+(\ell(k)(f|\Delta_0|^m), s(\overline{\lambda}) - m, \tau) \\ &= \mathbf{Z}_p^+(f|\Delta_0|^m, \pi_{\tau,\lambda}, a_{\tau,\lambda}^{p,\gamma})(k) \end{aligned}$$

Moreover we have (see the proof of Theorem 7.7.4)

$$\mathbf{Z}_p^+(f|\Delta_0|^m, \pi_{\tau,\lambda}, a_{\tau,\lambda}^{p,\gamma}) = \mathbf{Z}_p^+(f, \chi_0^{-m} \otimes \pi_{\tau,\lambda}, a_{\tau,\lambda}^{p,\gamma}) \ .$$

Finally we obtain

$$\mathbf{Z}_q^-(\mathcal{F}f, \pi_{\tau,\lambda}, a_{\tau,\lambda}^{q,\gamma'}) = \sum_{p=0}^{r} \mathbf{Z}_p^+\Big(f, \chi_0^{-m} \otimes \pi_{\tau,\lambda}, \sum_{\gamma\in\mathcal{W}_p} \nu_k\big(t(s(\overline{\lambda})) - m, \tau, q, \gamma', p, \gamma\big) a_{\tau,\lambda}^{p,\gamma}\Big) \ .$$

The existence of the matrix $A^{\tau,\lambda}$ is proved (for $\overline{\lambda} \in \mathbb{V}$) and we get

$$A_{p,q}^{\tau,\lambda}\big(a_{\tau,\lambda}^{q,\gamma'}\big) = \sum_{\gamma\in\mathcal{W}_p} \nu_k\big(t(s(\overline{\lambda})) - m, \tau, q, \gamma', p, \gamma\big) a_{\tau,\lambda}^{p,\gamma} \ ,$$

which gives the result.

□

This Theorem gives the proof in our situation of the formal equation given in section 7.7 and an explicit expression of the matrix A.

7.10. Fourier transform on $L^2(V^+, d^*X)$

In this section we define a slight modification of the Fourier transform $\mathcal{F}$ which gives rise to an unitary and G–equivariant operator $\mathcal{F}^*$ on $L^2(V^+, d^*X)$. This modification was introduced by E.M. STEIN ([St]–1967) for the complex prehomogeneous vector space $\big(GL(n,\mathbb{C})\times GL(n,\mathbb{C}), M_n(\mathbb{C})\big)$ or in other terms for the complex regular graded algebra $(\mathfrak{gl}(2n,\mathbb{C}), H_0)$ where $H_0 = \begin{pmatrix} \mathrm{I}_n & 0 \\ 0 & -\mathrm{I}_n \end{pmatrix}$.

We will use $\mathcal{F}^*$ to give an interpretation of the main Theorem for the harmonic analysis of the representation of G on the Hilbert space $L^2(V, d^*X)$.

Recall that ψ is a G–equivariant map from Ω^+ into Ω^- defined in section 3.9.

Definition 7.10.1. *For $f \in \mathcal{C}_c^\infty(\Omega^+)$ the function $\mathcal{F}^*f$ is defined on Ω^+ by*

$$\big(\mathcal{F}^*f\big)(X) = |\Delta_0(X)|^{-\frac{m}{2}}\Big(\mathcal{F}(|\Delta_0|^{-\frac{m}{2}}f)\Big)\big(\psi(X)\big) \quad \text{for } X \in \Omega^+ \ .$$

Proposition 7.10.2. *The map $\mathcal{F}^*$ extends to a unitary automorphism of the Hilbert space $L^2(V^+, d^*X)$ which commutes with the action of G on V^+.*

Proof. —From Definition 4.1.6, the map $f \longmapsto |\Delta_0|^{-\frac{m}{2}}f$ is a unitary isomorphism from $L^2(V^+, d^*X)$ onto $L^2(V^+, dX)$. Due to the choice of the pairs of measures (dX, dY) (see condition (N_2) in section 4.1), the Fourier transform $\mathcal{F}$ is a unitary isomorphism from $L^2(V^+, dX)$ onto $L^2(V^-, dY)$. From Proposition 4.1.9 and Proposition 3.9.5 we get for $h \in L^2(V^-, dY)$

$$\begin{aligned}\int_{V^+} |\Delta_0(X)|^{-m}|h(\psi(X))|^2 \, d^*X &= \int_{V^-} |\nabla_0(Y)|^m|h(Y)|^2 \, d^*Y \\ &= \int_{V^-} |h(Y)|^2 \, dY \ ,\end{aligned}$$

which implies that the map $h \longmapsto |\Delta_0|^{-\frac{m}{2}}(h\circ\psi)$ is a unitary isomorphism from $L^2(V^-, dY)$ onto $L^2(V^+, d^*X)$. Therefore $\mathcal{F}^*$ is a unitary automorphism of $L^2(V^+, d^*X)$.

Recall that, if f_g $(g \in G)$ denotes the function $X \longmapsto f(g^{-1}X)$, then we have

$$\mathcal{F}(f_g) = \chi_0(g)^m\big(\mathcal{F}f\big)_g \ .$$

Using the G–equivariance of ψ and the fact that χ_0 is the (positive) character of the relative invariant Δ_0 we obtain the G–equivariance of $\mathcal{F}^*$.

□

In order to obtain an expression of $Z^+_{p,\gamma}(\mathcal{F}^*f, s, \tau)$ we need the following properties of ψ.

Lemma 7.10.3.
(1) *For $p \in \{0, \dots, r\}$ and $\gamma \in \mathcal{W}_p$ we have $\psi(\mathcal{O}^+_{p,\gamma}) = \mathcal{O}^-_{p,\gamma}$.*
(2) *If τ is a $(M \cap H)$–spherical representation of M we have*

$$b^-_\tau(\psi(X)) = b^+_\tau(X) \quad (X \in \mathcal{O}^+) \ .$$

(3) *For $h \in \mathcal{C}_c(\Omega^-)$ we have*

$$\int_{V^+} h(\psi(X))\, dX = \int_{V^-} h(Y) |\nabla_0(Y)|^{-2m}\, dY \ .$$

Proof. —
From Proposition 3.9.3 we get $\psi(I^+_{p,\gamma}) = I^-_{p,\gamma}$ which implies (1).

From the definition of $b^\pm_\tau$ (Definition 7.2.2), we have for $m \in M$, $a \in A$ and $n \in N$

$$b^-_\tau\big(\psi(manI^+_{p,\gamma})\big) = \tau(m) b^-_\tau\big(\psi(I^+_{p,\gamma})\big) = \tau(m) b^-_\tau(I^-_{p,\gamma}) \ ,$$
$$b^+_\tau(manI^+_{p,\gamma}) = \tau(m) b^+_\tau(I^+_{p,\gamma}) \ .$$

In Type I and Type II case, $b^-_\tau(I^-_{p,\gamma}) = b^+_\tau(I^+_{p,\gamma}) = 1$, and in Type III case $b^-_\tau(I^-_{p,\gamma}) = b^+_\tau(I^+_{p,\gamma}) = v_\tau$. Hence (2) is proved.

The integral formula (3) follows directly from the integral formula given in Proposition 4.1.9 and from Proposition 3.9.5.

□

Lemma 7.10.4. *Let τ be an irreducible $(M \cap H)$–spherical representation of M, $\gamma \in \mathcal{W}_p$ $(p = 0, \dots, r)$ and $s \in \mathbb{C}^{k+1}$. If f is a function on V^+ such that $X \longmapsto |\Delta_0(X)|^{-\frac{m}{2}} f(X)$ is an element of $\mathcal{S}(V^+)$ then we have*

$$Z^+_{p,\gamma}(\mathcal{F}^* f, s, \tau) = Z^-_{p,\gamma}\Big(\mathcal{F}\big(|\Delta_0|^{-\frac{m}{2}} f\big), t(s) - \frac{3}{2}m, \tau\Big) \ .$$

Proof. —From the hypothesis made on f, the right member of this equality is well defined as a meromorphic function of s.

Let us suppose that this Zeta function is defined by a convergent integral namely that the real part of $t(s) - \frac{3}{2}m$ is large enough. Then setting in this integral $Y = \psi(X)$ and using the previous Lemma we get

$$\begin{aligned} Z^-_{p,\gamma}\Big(\mathcal{F}\big(|\Delta_0|^{-\frac{m}{2}} f\big), t(s) - \frac{3}{2}m, \tau\Big) \\ &= \int_{\mathcal{O}^-_{p,\gamma}} \mathcal{F}\big(|\Delta_0|^{-\frac{m}{2}} f\big)(Y) |\nabla(Y)|^{t(s) - \frac{3}{2}m} b^-_\tau(Y)\, dY \\ &= \int_{\mathcal{O}^+_{p,\gamma}} \big(\mathcal{F}^* f\big)(X) |\Delta_0(X)|^{\frac{m}{2}} |\nabla(\psi(X))|^{t(s) + \frac{m}{2}} b^+_\tau(X)|\, dX \ . \end{aligned}$$

As $|\nabla(\psi(X)|^s = |\Delta(X)|^{t(s)}$ (Proposition 3.9.5) and as $t(t(s) + \frac{m}{2}) = s - \frac{m}{2}$ we obtain

$$Z^-_{p,\gamma}\Big(\mathcal{F}\big(|\Delta_0|^{-\frac{m}{2}} f\big), t(s) - \frac{3m}{2}, \tau\Big) = \int_{\mathcal{O}^+_{p,\gamma}} \big(\mathcal{F}^* f\big)(X) |\Delta(X)|^s b^+_\tau(X)\, dX \ .$$

Thus, for $\mathrm{Re}\, t(s)$ large, $Z^+_{p,\gamma}(\mathcal{F}^* f, s, \tau)$ is well defined as a convergent integral and has a meromorphic continuation on $\mathbb{C}^{k+1}$ given by the meromorphic continuation of the left hand side of the previous equality.

□

We may now write the functional equations obtained in chapters 5 and 6 only on the positive side of the graded algebra. Using the notations introduced in section 7.2 we deduce from Theorem 7.2.6 the following result.

Theorem 7.10.5. *Let τ be an irreducible $(M \cap H)$–spherical representation of M. If f is a function on V^+ such that $X \longmapsto |\Delta_0(X)|^{-\frac{m}{2}} f(X)$ is an element of $\mathcal{S}(V^+)$, then the Zeta function $Z^+_{p,\gamma}(f, s, \tau)$ satisfies the following functional equations*

$$Z^+_{p,\gamma}(f, s, \tau) = \sum_{q=0}^{r} \sum_{\gamma' \in \mathcal{W}_q} \delta_k\big(s + \tfrac{m}{2}, \tau, p, \gamma, q, \gamma'\big) Z^+_{q,\gamma'}(\mathcal{F}^* f, s, \tau) \ ,$$

$$Z^+_{q,\gamma'}(\mathcal{F}^* f, s, \tau) = \sum_{p=0}^{r} \sum_{\gamma \in \mathcal{W}_p} \nu_k\big(t(s) - \tfrac{3m}{2}, \tau, q, \gamma', p, \gamma\big) Z^+_{p,\gamma}(f, s, \tau) \ .$$

Proof. — We first write

$$Z^+_{p,\gamma}(f, s, \tau) = Z^+_{p,\gamma}\big(f|\Delta_0|^{-\frac{m}{2}}, s + \frac{m}{2}, \tau\big) \ .$$

Applying Theorem 7.2.6 we get

$$Z^+_{p,\gamma}(f, s, \tau)$$
$$= \sum_{q=0}^{r} \sum_{\gamma' \in \mathcal{W}_q} \delta_k\big(s + \tfrac{m}{2}, \tau, p, \gamma, q, \gamma'\big) Z^-_{q,\gamma'}\Big(\mathcal{F}\big(f|\Delta_0|^{-\frac{m}{2}}\big), t(s + \tfrac{m}{2}) - m, \tau\Big) \ .$$

Using Lemma 7.10.4 and the relation $t(s + \frac{m}{2}) - m = t(s) - \frac{3}{2}m$, we obtain the first functional equation.

On the other hand, using again Lemma 7.10.4 to express $Z^+_{q,\gamma'}(\mathcal{F}^* f, s, \tau)$, we get from Theorem 7.2.6

$$Z^+_{q,\gamma'}(\mathcal{F}^* f, s, \tau)$$
$$= \sum_{p=0}^{r} \sum_{\gamma \in \mathcal{W}_p} \nu_k\big(t(s) - \tfrac{3}{2}m, \tau, q, \gamma', p, \gamma\big) Z^+_{p,\gamma}\Big(f|\Delta_0|^{-\frac{m}{2}}, t\big(t(s) - \tfrac{3}{2}m\big) - m, \tau\Big) \ .$$

As $t\big(t(s) - \frac{3}{2}m\big) - m = s + \frac{m}{2}$, we obtain the second functional equation.

□

Similarly the main theorem (Theorem 7.9.6) can be rewritten using $\mathcal{F}^*$ instead of $\mathcal{F}$ as follows.

Theorem 7.10.6 (Functional equation with $\mathcal{F}^*$)**.** *Let $\pi_{\tau,\lambda}$ be a representation of the minimal principal series with $\overline{\lambda} \in \mathbb{V}$. There exists an element $B^{\tau,\lambda}$ in $End\big(\prod_{p=0}^{r} (\mathcal{H}^{-\infty}_{\tau,\lambda})^{H_p}\big)$ such that the following functional equation holds :*

$$\mathbf{Z}^+(\mathcal{F}^* f, \pi_{\tau,\lambda}, a) = \mathbf{Z}^+\big(f, \pi_{\tau,\lambda}, B^{\tau,\lambda}(a)\big) \ ,$$

where f is a function on V^+ such that $X \longmapsto |\Delta_0(X)|^{-\frac{m}{2}} f(X)$ is an element of $\mathcal{S}(V^+)$ and a is an element in $\prod_{p=0}^r (\mathcal{H}_{\tau,\lambda}^{-\infty})^{H_p}$. The endomorphism $B^{\tau,\lambda}$ can be represented as a $(r+1)\times(r+1)$ matrix whose coefficients $B_{p,q}^{\tau,\lambda}$ belong to $\mathrm{Hom}\left((\mathcal{H}_{\tau,\lambda}^{-\infty})^{H_q}, (\mathcal{H}_{\tau,\lambda}^{-\infty})^{H_p}\right)$. *In the basis $(a_{\tau,\lambda}^{p,\gamma})_{\gamma\in\mathcal{W}_p}$ and $(a_{\tau,\lambda}^{q,\gamma'})_{\gamma'\in\mathcal{W}_q}$ of $(\mathcal{H}_{\tau,\lambda}^{-\infty})^{H_p}$ and $(\mathcal{H}_{\tau,\lambda}^{-\infty})^{H_q}$ respectively, the matrix of $B_{p,q}^{\tau,\lambda}$ is given by*

$$\left(B_{p,q}^{\tau,\lambda}\right)_{\gamma,\gamma'} = \nu_k\big(t(s(\overline{\lambda})) - \frac{m}{2}, \tau, q, \gamma', p, \gamma\big) \quad \text{for } \gamma \in \mathcal{W}_p,\ \gamma' \in \mathcal{W}_q\ .$$

Proof. — The proof is similar to the proof of Theorem 7.9.6 but now we get from Definition 7.9.2 that

$$\mathbf{Z}_q^+(\mathcal{F}^* f, \pi_{\tau,\lambda}, a_{\tau,\lambda}^{q,\gamma'})(k) = Z_{q,\gamma'}^+(\mathcal{F}^*(\ell(k)f), s(\overline{\lambda}) - m, \tau) \quad (k \in K)\ .$$

Here we used the G–invariance of $\mathcal{F}^*$ proved in Proposition 7.10.2. Then using the relation $t(s(\overline{\lambda}) - m) - \frac{3}{2}m = t(s(\overline{\lambda})) - \frac{m}{2}$ we get from Theorem 7.10.5

$$\begin{aligned}&\mathbf{Z}_q^+(\mathcal{F}^* f, \pi_{\tau,\lambda}, a_{\tau,\lambda}^{q,\gamma'})(k)\\ &\quad = \sum_{p=0}^r \sum_{\gamma\in\mathcal{W}_p} \nu_k\big(t(s(\overline{\lambda})) - \frac{m}{2}, \tau, q, \gamma', p, \gamma\big) Z_{p,\gamma}^+(\ell(k)f, s(\overline{\lambda}) - m, \tau)\ .\end{aligned}$$

As we have from Definition 7.9.2

$$Z_{p,\gamma}^+(\ell(k)f, s(\overline{\lambda}) - m, \tau) = \mathbf{Z}_p^+(f, \pi_{\tau,\lambda}, a_{\tau,\lambda}^{p,\gamma})(k)\ ,$$

the Theorem follows.

□

7.11. Interpretation of the functional equation with $\mathcal{F}^*$

The representation of the reductive group G on the separable Hilbert space $L^2(V^+, d^*X)$ is unitary. Then it is known that such a representation has a direct integral decomposition of the form

$$L^2(V^+, d^*X) \simeq \int_{\widehat{G}}^{\oplus} (V_\pi \otimes \mathcal{H}_\pi)\, d\mu(\pi)\ ,$$

where $\widehat{G}$ is the unitary dual (the set of equivalence classes of unitary irreducible representations) of G, $d\mu$ a Borel measure on $\widehat{G}$, $\mathcal{H}_\pi$ is the space of the representation π and V_π a vector space (possibly of infinite dimension). The representation of G on $V_\pi \otimes \mathcal{H}_\pi$ is $\mathrm{Id} \otimes \pi$.

We know from Proposition 7.10.2 that $\mathcal{F}^*$ is a unitary operator on the space $L^2(V^+, d^*X)$ commuting with the action of G. It follows from a generalized Schur's Lemma (see the proof of Theoreme 14.10.2 in [Wal]–1992) that it is a decomposable operator in the direct integral decomposition. We will see that Theorem 7.10.6 gives the components of $\mathcal{F}^*$ related to the minimal spherical principal series.

In fact the decomposition of $L^2(V^+, d^*X)$ is closely related to the harmonic analysis of the symmetric spaces G/H_p . The space $L^2(V^+, d^*X)$ is a finite sum of G–invariant subspaces given by

$$L^2(V^+, d^*X) = \oplus_{p=0}^r L^2(\Omega_p^+, d^*X) \simeq \oplus_{p=0}^r L^2(G/H_p)\ .$$

Let us notice that these subspaces are not invariant under the action of $\mathcal{F}^*$. The direct integral decomposition for $L^2(G/H_p)$ may be written as

$$L^2(G/H_p) \simeq \int_{\widehat{G}_{H_p}} \mathrm{Hom}\left((\mathcal{H}_\pi^{-\infty})^{H_p}, \mathcal{H}_\pi\right) d\mu_p(\pi) \ ,$$

where $\widehat{G}_{H_p}$ denotes the set of equivalence classes of unitary irreducible H_p–spherical representations *i.e.* representations with a nonzero H_p–invariant distribution vector (see [B-FJ-S]–1994). An explicit description of the measure μ_p and of its support in $\widehat{G}_{H_p}$ is given by the Plancherel formula for the symmetric space G/H_p. This is a difficult problem (see the work of P. Delorme [D]–1998 and the work of E. van den Ban and H. Schlichtkrull [B-S]–2001). Nevertheless the restriction to $\mathcal{C}_c^\infty(G/H_p)$ of this isomorphism is given by the "abstract" Fourier transform which is the map

$$\begin{aligned} \mathcal{C}_c^\infty(G/H_p) &\longrightarrow \int_{\widehat{G}_{H_p}} \mathrm{Hom}\left((\mathcal{H}_\pi^{-\infty})^{H_p}, \mathcal{H}_\pi\right) d\mu_p(\pi) \\ f &\longmapsto \left(\widehat{f}(\pi)\right)_{\pi \in \widehat{G}_{H_p}} \ , \end{aligned}$$

where

$$\widehat{f}(\pi)(a_p) = \int_{G/H_p} f(\dot{g})\pi(\dot{g})a_p \, d_p\dot{g} \quad \left(a_p \in (\mathcal{H}_\pi^{-\infty})^{H_p}\right).$$

This is exactly Definition 7.6.3 namely

$$\widehat{f}(\pi)(a_p) = \mathbf{Z}_p^+(f, \pi, a_p) \ .$$

The Fourier transform related to the harmonic analysis of $L^2(V, d^*X)$ may then be defined as follows. For a function $f \in \mathcal{C}_c^\infty(\Omega^+)$, a H-spherical representation π of G (where H is one of the groups H_p) and an element $a = (a_0, \ldots, a_r) \in \prod_{p=0,\ldots,r}(\mathcal{H}_\pi^{-\infty})^{H_p}$ we set

$$\widehat{f}(\pi)(a) = \sum_{p=0}^{r} \widehat{f_{|\Omega_p^+}}(\pi)(a_p) = \mathbf{Z}^+(f, \pi, a) \ .$$

For a representation $\pi_{\tau,\lambda}$ in the minimal spherical principal series (not necessarily unitary) Theorem 7.10.6 describes the action of $\mathcal{F}^*$ on the corresponding component as

$$\widehat{\mathcal{F}^* f}(\pi_{\tau,\lambda})(a) = \widehat{f}(\pi_{\tau,\lambda}) \circ B(a) \ ,$$

where $B = B^{\tau,\lambda}$ and $\lambda \in \mathbb{V}$.

Appendix : The example of symmetric matrices

As an illustration and a guide to the reader, we will give in this appendix a detailed description, through the example of the symplectic Lie algebra, of all the structure results obtained in the paper. The numbering of the sections will be the same as the numbering in the paper.

A.1.1. The graded algebra

We consider the semisimple algebra $\widetilde{\mathfrak{g}} = \mathfrak{sp}(k+1, \mathbb{R})$ which is given by

$$\widetilde{\mathfrak{g}} = \left\{ \begin{pmatrix} \mathbf{A} & \mathbf{X} \\ \mathbf{Y} & {}^{-t}\mathbf{A} \end{pmatrix} \;\middle|\; \mathbf{X}, \mathbf{Y} \in \mathrm{Sym}(k+1, \mathbb{R}), \mathbf{A} \in \mathrm{M}(k+1, \mathbb{R}) \right\} \ ,$$

where $\mathrm{Sym}(k+1, \mathbb{R})$ denotes the subspace of symmetric matrices in the space $\mathrm{M}(k+1, \mathbb{R})$ of $(k+1) \times (k+1)$ matrices.

Remark. — The integer k has been choosen such that it will correspond to the number k defined in section 1.6.

Let H_0 be given by

$$H_0 = \begin{pmatrix} \mathbf{1}_{k+1} & 0 \\ 0 & -\mathbf{1}_{k+1} \end{pmatrix} \ ,$$

where $\mathbf{1}_{k+1}$ is the identity matrix in $\mathrm{M}(k+1, \mathbb{R})$. The algebra $\widetilde{\mathfrak{g}}$ is graded by $\mathrm{ad}\, H_0$ and we obtain

$$\mathfrak{g} = \{A \in \widetilde{\mathfrak{g}} \mid [H_0, A] = 0\} = \left\{ A = \begin{pmatrix} \mathbf{A} & 0 \\ 0 & {}^{-t}\mathbf{A} \end{pmatrix} \;\middle|\; \mathbf{A} \in \mathrm{M}(k+1, \mathbb{R}) \right\} \ ,$$

$$V^+ = \{X \in \widetilde{\mathfrak{g}} \mid [H_0, X] = 2X\} = \left\{ X = \begin{pmatrix} 0 & \mathbf{X} \\ 0 & 0 \end{pmatrix} \middle| \mathbf{X} \in \mathrm{Sym}(k+1, \mathbb{R}) \right\} \ ,$$

$$V^- = \{Y \in \widetilde{\mathfrak{g}} \mid [H_0, Y] = -2Y\} = \left\{ Y = \begin{pmatrix} 0 & 0 \\ \mathbf{Y} & 0 \end{pmatrix} \middle| \mathbf{Y} \in \mathrm{Sym}(k+1, \mathbb{R}) \right\} \ .$$

The hypothesis $(\mathbf{H_1})$ is therefore verified.

Remark. — As in the previous definition, we will use the same letter to denote an element of $V^{\pm}$ or of $\mathfrak{g}$ and the corresponding matrix in the space $\mathrm{M}(k+1, \mathbb{R})$, which will be distinguished by a bold face font.

The Lie algebra $\widetilde{\mathfrak{g}}$ is simple and the *Killing form* is given by

$$\widetilde{B}(X, X') = (2k+4)\mathrm{Tr}(XX') \text{ for } X, X' \in \mathfrak{sp}(k+1, \mathbb{R}) \ ,$$

(see [H] – 1978 p.190 for a reference).

The *Cartan involution* θ on $\widetilde{\mathfrak{g}}$ is choosen to be :

$$\theta(X) = -{}^tX \quad \text{for } X \in \mathfrak{sp}(k+1, \mathbb{R}) \ .$$

This implies in particular that $\theta(H_0) = -H_0$ and that $\widetilde{B}_\theta$ is positive definite.

The *Cartan subspace* of the pairs $(\widetilde{\mathfrak{g}}, \theta)$ and $(\mathfrak{g}, \theta)$ is choosen to be :

$$\mathfrak{a}^{\mathfrak{p}} = \left\{ H = \begin{pmatrix} \mathbf{H} & 0 \\ 0 & -\mathbf{H} \end{pmatrix} \text{ where } \mathbf{H} = \begin{pmatrix} t_0 & & & \\ & t_1 & & \\ & & \ddots & \\ & & & t_k \end{pmatrix} \in \mathrm{M}(k+1, \mathbb{R}) \right\} \ .$$

A.1.2. Root systems

We denote by η_j $(j = 0, \ldots, k)$ the linear forms on $\mathfrak{a}^{\mathfrak{p}}$ given by

$$\eta_j \begin{pmatrix} \mathbf{H} & 0 \\ 0 & -\mathbf{H} \end{pmatrix} = t_j \quad \text{if } \mathbf{H} = \begin{pmatrix} t_0 & & & \\ & t_1 & & \\ & & \ddots & \\ & & & t_k \end{pmatrix} \ .$$

Then the root system $\widetilde{\Sigma}$ of the pair $(\widetilde{\mathfrak{g}}, \mathfrak{a}^{\mathfrak{p}})$ is of type C_{k+1} and given by

$$\widetilde{\Sigma} = \begin{Bmatrix} \pm\eta_i \pm \eta_j & \text{for } 0 \leq i < j \leq k \\ \pm 2\eta_j & \text{for } j = 0, \ldots, k \end{Bmatrix} \ .$$

The root system Σ of the pair $(\mathfrak{g}, \mathfrak{a}^{\mathfrak{p}})$ is of type A_k and given by

$$\Sigma = \{\eta_i - \eta_j \text{ for } 0 \leq i \neq j \leq k\} \ .$$

The subset $\widetilde{\Pi}$ of $\widetilde{\Sigma}$ given by

$$\widetilde{\Pi} = \Pi \bigcup \{2\eta_k\} \quad \text{where } \Pi = \{\eta_0 - \eta_1, \eta_1 - \eta_2, \ldots, \eta_{k-1} - \eta_k\} \ ,$$

is a set of simple roots for $\widetilde{\Sigma}$ and $\Pi = \{\lambda \in \widetilde{\Pi} \mid \lambda(H_0) = 0)\}$ is a set of simple roots for Σ. The root $\lambda_0 = 2\eta_k$ satisfies $\lambda_0(H_0) = 2$. Moreover the set of positive roots defined by $\widetilde{\Pi}$ is given by

$$\widetilde{\Sigma}^+ = \begin{Bmatrix} \eta_i - \eta_j & \text{for } 0 \leq i < j \leq k \\ \eta_i + \eta_j & \text{for } i, j = 0, \ldots, k \end{Bmatrix} \ .$$

Notation. — As usual, for $i = 0, \ldots, k$ and $j = 0, \ldots, k$, $\mathbf{E}_{ij}$ denotes the $(k+1) \times (k+1)$ matrix whose coefficients are all zero except the coeeficient of index (i, j) which is equal to one.

The root spaces related to the different roots are of dimension one and given by :

$$\widetilde{\mathfrak{g}}^{\eta_i - \eta_j} = \mathbb{R} \begin{pmatrix} \mathbf{E}_{ij} & 0 \\ 0 & -\mathbf{E}_{ji} \end{pmatrix} \text{ for } i \neq j \ ;$$

$$\widetilde{\mathfrak{g}}^{\eta_i + \eta_j} = \mathbb{R} \begin{pmatrix} 0 & \mathbf{E}_{ij} + \mathbf{E}_{ji} \\ 0 & 0 \end{pmatrix} \text{ for } i, j = 0, \ldots, k \ ;$$

$$\widetilde{\mathfrak{g}}^{-\eta_i - \eta_j} = \mathbb{R} \begin{pmatrix} 0 & 0 \\ \mathbf{E}_{ij} + \mathbf{E}_{ji} & 0 \end{pmatrix} \text{ for } i, j = 0, \ldots, k \ .$$

A.1.3. Complexification

It is well known that $\mathfrak{a}^{\mathfrak{p}}$ is also a Cartan subalgebra of $\widetilde{\mathfrak{g}}$. It follows that $\widetilde{\mathfrak{g}}$ is the split real form of it's complexification and therefore the root systems $\widetilde{\Sigma}$ and $\widetilde{\mathcal{R}}$ are the same.

A.1.4. Highest root in $\widetilde{\Sigma}$

The highest root λ^0 in $\widetilde{\Sigma}^+$ is given by

$$\lambda^0 = 2\eta_0 \ .$$

Since λ^0 is the unique root in $\widetilde{\Sigma}$ which satisfies Proposition 1.4.1, the adjoint representation of $\mathfrak{g}$ on $V_{\mathbb{C}}^+$ is irreducible and λ^0 is the highest weight for this representation with respect to $\widetilde{\Sigma}^+$. Thus the hypothesis $(\mathbf{H_2})$ is verified.

We give now the description of the *co-roots* :

- for $\lambda = 2\eta_j$, $\quad H_\lambda = \begin{pmatrix} \mathbf{E}_{jj} & 0 \\ 0 & -\mathbf{E}_{jj} \end{pmatrix}$;
- for $\lambda = \eta_i + \eta_j \ (i \neq j)$, $\quad H_\lambda = \begin{pmatrix} \mathbf{E}_{ii} + \mathbf{E}_{jj} & 0 \\ 0 & -\mathbf{E}_{ii} - \mathbf{E}_{jj} \end{pmatrix}$;
- for $\lambda = \eta_i - \eta_j \ (i \neq j)$, $\quad H_\lambda = \begin{pmatrix} \mathbf{E}_{ii} - \mathbf{E}_{jj} & 0 \\ 0 & -\mathbf{E}_{ii} + \mathbf{E}_{jj} \end{pmatrix}$.

A.1.5. The first step of the descent

The algebra $\widetilde{\mathfrak{l}}_0$ generated by $\widetilde{\mathfrak{g}}^{\lambda_0}$ and $\widetilde{\mathfrak{g}}^{-\lambda_0}$ is isomorphic to $\mathfrak{sl}(2,\mathbb{R})$. More precisely we have

$$\widetilde{\mathfrak{l}}_0 = \left\{ \begin{pmatrix} t\mathbf{E}_{kk} & x\mathbf{E}_{kk} \\ y\mathbf{E}_{kk} & -t\mathbf{E}_{kk} \end{pmatrix} \mid t, x, y \in \mathbb{R} \right\} \ .$$

The centralizer of $\widetilde{\mathfrak{l}}_0$ in $\widetilde{\mathfrak{g}}$ is the reductive Lie algebra $\widetilde{\mathfrak{g}}_1$ isomorphic to $\mathfrak{sp}(k,\mathbb{R})$ and given by

$$\widetilde{\mathfrak{g}}_1 = \left\{ \begin{pmatrix} \mathbf{A}_1 & 0 & \mathbf{X}_1 & 0 \\ 0 & 0 & 0 & 0 \\ \mathbf{Y}_1 & 0 & -{}^t\mathbf{A}_1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \ , \ \mathbf{A}_1 \in \mathrm{M}(k,\mathbb{R}), \mathbf{X}_1, \mathbf{Y}_1 \in \mathrm{Sym}(k,\mathbb{R}) \right\} \ .$$

The Lie algebra $\widetilde{\mathfrak{g}}_1$ is a graded Lie algebra which satisfies $(\mathbf{H_1})$ and $(\mathbf{H_2})$ where the gradation is defined by the element

$$H_1 = H_0 - H_{\lambda_0} = \begin{pmatrix} \mathbf{1}_k & & & \\ & 0 & & \\ & & -\mathbf{1}_k & \\ & & & 0 \end{pmatrix} \ .$$

The root system $\widetilde{\Sigma}_1$ of $(\widetilde{\mathfrak{g}}_1, \mathfrak{a}_1^{\mathfrak{p}})$ where $\mathfrak{a}_1^{\mathfrak{p}} = \mathfrak{a}^{\mathfrak{p}} \cap \widetilde{\mathfrak{g}}_1$ is of type C_k and given by

$$\widetilde{\Sigma}_1 = \left\{ \begin{matrix} \pm\eta_i \pm \eta_j & \text{for } 0 \leq i < j \leq k-1 \\ \pm 2\eta_j & \text{for } j = 0, \ldots, k-1 \end{matrix} \right\} \ .$$

The set of simple roots in $\widetilde{\Sigma}_1 \cap \widetilde{\Sigma}^+$ is given by

$$\begin{aligned}\widetilde{\Pi}_1 &= \{\eta_0 - \eta_1, \eta_1 - \eta_2, \dots, \eta_{k-2} - \eta_{k-1}, 2\eta_{k-1}\} \\ &= \Pi_1 \bigcup \{\lambda_1\}\end{aligned}$$

where $\Pi_1 = \{\lambda \in \Pi \mid \lambda(H_{\lambda_0}) = 0\}$ and $\lambda_1 = 2\eta_{k-1}$.

A.1.6. The descent

The sequence of strongly orthogonal roots given in Theorem 1.6.1 is the set

$$\{\lambda_0, \lambda_1, \dots, \lambda_k\} \quad \text{where } \lambda_j = 2\eta_{k-j} \ .$$

The Lie algebras $\widetilde{\mathfrak{l}}_j$ are isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ and given by

$$\widetilde{\mathfrak{l}}_j = \left\{ \begin{pmatrix} t\mathbf{E}_{k-j,k-j} & x\mathbf{E}_{k-j,k-j} \\ y\mathbf{E}_{k-j,k-j} & -t\mathbf{E}_{k-j,k-j} \end{pmatrix} \mid t,x,y \in \mathbb{R} \right\} \ .$$

The centralizer $\widetilde{\mathfrak{g}}_j$ of $\widetilde{\mathfrak{l}}_0 \oplus \cdots \oplus \widetilde{\mathfrak{l}}_{j-1}$ is given by

$$\widetilde{\mathfrak{g}}_j = \left\{ \begin{pmatrix} \mathbf{A}_j & 0 & \mathbf{X}_j & 0 \\ 0 & 0 & 0 & 0 \\ \mathbf{Y}_j & 0 & -{}^t\mathbf{A}_j & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\} \ ,$$

where $\mathbf{A}_j \in \mathrm{M}(k+1-j,\mathbb{R})$, $\mathbf{X}_j$ and $\mathbf{Y}_j \in \mathrm{Sym}(k+1-j,\mathbb{R})$. The gradation is given by $\operatorname{ad} H_j$ where

$$H_j = H_0 - H_{\lambda_0} - \cdots - H_{\lambda_{j-1}} = \begin{pmatrix} \mathbf{1}_{k+1-j} & & & \\ & 0 & & \\ & & -\mathbf{1}_{k+1-j} & \\ & & & 0 \end{pmatrix} \ .$$

It follows that $(\widetilde{\mathfrak{g}}_j, H_j)$ is a graded algebra from the same series as $(\widetilde{\mathfrak{g}}, H_0)$ with k replaced by $k-j$.

A.I.7. Generic elements in V^+

The group $Sp(k+1,\mathbb{R})$ is simple and connected with center given by $\{\pm\mathrm{Id}\}$. Therefore the adjoint group of $\widetilde{\mathfrak{g}}$ is the group $\widetilde{G} = Sp(k+1,\mathbb{R})/\{\pm\mathrm{Id}\}$. The centralizer of H_0 in $\widetilde{G}$ is given by

$$Z_{\widetilde{G}}(H_0) = \left\{ \begin{pmatrix} \mathbf{g} & 0 \\ 0 & {}^t\mathbf{g}^{-1} \end{pmatrix} \mid \mathbf{g} \in GL(k+1,\mathbb{R}) \right\} /\{\pm\mathrm{Id}\} \ .$$

Therefore $Z_{\widetilde{G}}(H_0)$ is isomorphic to $GL(k+1,\mathbb{R})/\{\pm\mathrm{Id}\}$. If $k+1$ is odd, $GL(k+1,\mathbb{R})/\{\pm\mathrm{Id}\}$ is isomorphic to the connected group $GL^+(k+1,\mathbb{R})$, but if $k+1$ is even, $GL(k+1,\mathbb{R})/\{\pm\mathrm{Id}\}$ has two connected components, the identity component being $GL^+(k+1,\mathbb{R})/\{\pm\mathrm{Id}\}$. It follows that we have

- k even : $G = Z_{\widetilde{G}}(H_0) \simeq GL^+(k+1,\mathbb{R})$;
- k odd : $G \neq Z_{\widetilde{G}}(H_0)$ and $G \simeq GL^+(k+1,\mathbb{R})/\{\pm\mathrm{Id}\}$.

In the sequel of this appendix we will in general not mention the eventual quotient needed to describe $\widetilde{G}$, G or another subgroup of $\widetilde{G}$. For example, if we write that a matrix g belongs to G, the reader should keep in mind that it means that, according to the parity of k, the image of g into the quotient by $\{\pm \mathrm{Id}\}$ belongs to G.

The action of G on V^+ is given by

$$\begin{gathered} G \times V^+ \longrightarrow V^+ \\ \begin{pmatrix} \mathbf{g} & 0 \\ 0 & {}^t\mathbf{g}^{-1} \end{pmatrix}, \begin{pmatrix} 0 & \mathbf{X} \\ 0 & 0 \end{pmatrix} \longmapsto \begin{pmatrix} 0 & \mathbf{g}\mathbf{X}{}^t\mathbf{g} \\ 0 & 0 \end{pmatrix} \ , \end{gathered}$$

where $\mathbf{g} \in GL^+(k+1,\mathbb{R})$ and $\mathbf{X} \in \mathrm{Sym}(k+1,\mathbb{R})$.

Let p and q be positive integers such that $0 \leq p+q \leq k+1$ and let $\mathbf{I}_{qp}$ be the matrix of $\mathrm{Sym}(k+1,\mathbb{R})$ given by

$$\begin{aligned} \mathbf{I}_{qp} &= -\mathbf{E}_{00} - \cdots - \mathbf{E}_{p-1,p-1} + \mathbf{E}_{p,p} + \cdots + \mathbf{E}_{p+q-1,p+q-1} \\ &= \begin{pmatrix} -1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & -1 & & & & & & \\ & & & 1 & & & & & \\ & & & & \ddots & & & & \\ & & & & & 1 & & & \\ & & & & & & 0 & & \\ & & & & & & & \ddots & \\ & & & & & & & & 0 \end{pmatrix} . \end{aligned}$$

If $\mathbf{X}$ belongs to $\mathrm{Sym}(k+1,\mathbb{R})$, it is well known by Sylvester's law that there exist integers p, q and $\mathbf{g} \in GL^+(k+1,\mathbb{R})$, such that

$$\mathbf{X} = \mathbf{g}\mathbf{I}_{qp}\,{}^t\mathbf{g} \ .$$

Let us write :

$$I^+_{qp} = \begin{pmatrix} 0 & \mathbf{I}_{qp} \\ 0 & 0 \end{pmatrix} \ \text{ and } X = \begin{pmatrix} 0 & \mathbf{X} \\ 0 & 0 \end{pmatrix} \ .$$

Then $\mathrm{ad}\, X$ maps $\mathfrak{g}$ onto V^+ if and only if $\mathrm{ad}\, I^+_{qp}$ maps $\mathfrak{g}$ onto V^+. An easy calculation proves that this is the case if and only if $p+q = k+1$. Therefore we have

$$X \in V^+ \text{ is generic } \iff \det(\mathbf{X}) \neq 0 \ .$$

Moreover, if $p+q = k+1$, then $\{\theta(I^+_{qp}), H_0, I^+_{qp}\}$ is an $\mathfrak{sl}_2$–triple. It follows that the graded algebra $(\widetilde{\mathfrak{g}}, H_0)$ is regular *i.e.* satisfies the hypothesis $(\mathbf{H_3})$.

The regular irreducible prehomogeneous space that we are considering here is the real form $(GL^+(k+1),\mathbb{R}), \mathrm{Sym}(k+1,\mathbb{R}))$ of the prehomogeneous space $(GL(k+1,\mathbb{C}), \mathrm{Sym}(k+1,\mathbb{C}))$.

A.1.8. The structure of $(\widetilde{\mathfrak{g}}, H_0)$.

The abelian subspace $\mathfrak{a}^0$ is equal to $\mathfrak{a}^\mathfrak{p}$ and we have

$$\mathfrak{z}_\mathfrak{g}(\mathfrak{a}^0) = \mathfrak{z}_\mathfrak{g}(\mathfrak{a}^\mathfrak{p}) = \mathfrak{a}^0 = \mathfrak{a}^\mathfrak{p} \ .$$

The decomposition of Theorem 1.8.1 is the root-spaces decomposition related to the system $\widetilde{\Sigma}$. More precisely we have for $i \neq j$

$$\begin{aligned} E_{ij}(1,1) &= \widetilde{\mathfrak{g}}^{\eta_{k-i}+\eta_{k-j}} \ , \\ E_{ij}(1,-1) &= \widetilde{\mathfrak{g}}^{\eta_{k-i}-\eta_{k-j}} \ , \\ E_{ij}(-1,-1) &= \widetilde{\mathfrak{g}}^{-\eta_{k-i}-\eta_{k-j}} \ . \end{aligned}$$

For $A = \{j, \dots, k\}$, the graded algebra $\widetilde{\mathfrak{g}_A}$ defined in Proposition 1.8.5 has the following form :

$$\widetilde{\mathfrak{g}}_A = \left\{ \begin{pmatrix} \mathbf{A}_j & 0 & \mathbf{X}_j & 0 \\ 0 & a_j & 0 & 0 \\ \mathbf{Y}_j & 0 & -{}^t\mathbf{A}_j & 0 \\ 0 & 0 & 0 & -{}^t a_j \end{pmatrix} \right\} \ ,$$

where $\mathbf{A}_j \in \mathrm{M}(k+1-j, \mathbb{R})$, $a_j \in \mathrm{M}(j, \mathbb{R})$, $\mathbf{X}_j$ and $\mathbf{Y}_j \in \mathrm{Sym}(k+1-j, \mathbb{R})$. This algebra contains strictly $\widetilde{\mathfrak{g}_j}$ but $V_A^+ = V^+ \cap \widetilde{\mathfrak{g}_A}$ is equal to $V^+ \cap \widetilde{\mathfrak{g}_j}$ and the action on V_A^+ of the orthogonal subspace of $\mathfrak{g}_j$ in $\mathfrak{g}_A = \mathfrak{g} \cap \widetilde{\mathfrak{g}_A}$ is trivial.

A.1.9. Properties of the spaces $E_{ij}(\pm 1, \pm 1)$

Here the dimension of these spaces is clearly equal to 1 and the result of Proposition 1.9.3 is therefore trivial. We will however explicit the Weyl group elements which interchange the roots λ_i and λ_j. These elements are essential ingredients in the proof of Proposition 1.9.3 in the general situation.

The Weyl group $\widetilde{W}$ of $\widetilde{\Sigma}$ is isomorphic to the semi–direct product of $\mathfrak{S}_{k+1}$ by $(\mathbb{Z}/2\mathbb{Z})^{k+1}$. The action of a permutation τ of $\mathfrak{S}_{k+1}$ on the roots is given by $\eta_j \mapsto \eta_{\tau(j)}$ and the action of $(\varepsilon_0, \dots, \varepsilon_k) \in \{\pm 1\}^{k+1}$ is given by $\eta_j \mapsto \varepsilon_j \eta_j$. The Weyl group W of Σ is the subgroup of $\widetilde{W}$ isomorphic to $\mathfrak{S}_{k+1}$. Therefore the action of the element $s_0 \in W \subset \widetilde{W}$ which interchanges Σ^+ and Σ^- is given by the permutation τ_0 of $\{0, \dots, k\}$ such that

$$\tau_0(\eta_j) = \eta_{k-j} \ ,$$

which implies the result of Proposition 1.9.2 namely that $s_0.\lambda_j = \lambda_{k-j}$.

Let n be the integer such that $k+1 = 2n$ if k odd and $k+1 = 2n+1$ if k even. Let J_n be the matrix in $\mathrm{M}(n, \mathbb{R})$ given by

$$J_n = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix} \ .$$

A representative element of s_0 is given by

$$s_0 = \begin{pmatrix} \mathbf{L} & 0 \\ 0 & \mathbf{L} \end{pmatrix} \text{ where } \mathbf{L} = \begin{cases} \begin{pmatrix} 0 & J_n \\ -J_n & 0 \end{pmatrix} \text{ if } k+1 = 2n \ ; \\ \\ \begin{pmatrix} 0 & 0 & J_n \\ 0 & 1 & 0 \\ -J_n & 0 & 0 \end{pmatrix} \text{ if } k+1 = 2n+1 \ . \end{cases}$$

Since $\mathbf{L}$ belongs to $GL^+(k+1, \mathbb{R})$, this representative belongs to G.

The constants defined in this section are given here by

$$\begin{aligned} \ell &= \dim \widetilde{\mathfrak{g}}^{\lambda_j} = 1 \ ; \\ d &= \dim E_{ij}(\pm 1, \pm 1) = 1 \ ; \\ \dim V^+ &= \frac{1}{2}(k+1)(k+2) \ . \end{aligned}$$

A.1.10. Normalization of the Killing form

As $\dim V^+ = \frac{1}{2}(k+1)(k+2)$, the normalized Killing form is given here by

$$\begin{aligned} b(X,Y) &= -\frac{k+1}{2(k+1)(k+2)} \widetilde{B}(X,Y) \\ &= \mathrm{Tr}(XY) \ , \end{aligned}$$

fo rall $X, Y \in \widetilde{\mathfrak{g}} = \mathfrak{sp}(k+1, \mathbb{R})$.

A.1.11. The relative invariant Δ_0

The determinant is an irreducible polynomial on the space of symmetric matrices which satisfies

$$\det(\mathbf{g} X^t \mathbf{g}) = \det(\mathbf{g})^2 \det(\mathbf{X}) \ ,$$

for $\mathbf{X} \in \mathrm{Sym}(k+1, \mathbb{R})$ and $\mathbf{g} \in GL(k+1, \mathbb{R})$. Therefore the polynomial Δ_0 is, up to a multiplicative constant, given by

$$\Delta_0(X) = \det(\mathbf{X}) \text{ for } X = \begin{pmatrix} 0 & \mathbf{X} \\ 0 & 0 \end{pmatrix} \in V^+ \ .$$

In the proof of Theorem 1.11.2 we obtained a relative invariant polynomial P on V^+ given by the formula

$$P(X) = \det_{(V^-, V^+)}(\mathrm{ad}\, X)^2 \text{ for} X \in V^+ \ .$$

Here it is the determinant of the endomorphism of $\mathrm{Sym}(k+1, \mathbb{R})$ given by

$$\mathbf{Y} \longmapsto -2\mathbf{XYX} \ .$$

It follows that

$$P = (-2)^{\frac{(k+1)(k+2)}{2}} \Delta_0^{k+2}.$$

A.1.12. The case $k = 0$

In this case $\widetilde{\mathfrak{g}}$ is the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$. The action of $G \simeq \mathbb{R}^{+*}$ on $V^+ \simeq \mathbb{R}$ is given by

$$(g, x) \mapsto g^2 x \ .$$

The G–orbits are $\{0\}$, $]0, +\infty]$ and $]-\infty, 0[$ and the $G_{\mathbb{C}}$–orbits in $V_{\mathbb{C}}^+$ are $\{0\}$ and $\mathbb{C}^*$. The fundamental invariant is of degree 1 and given by the map $x \mapsto x$ on $\mathbb{R}$. Therefore we obtain

$$\kappa = 1 \ .$$

A.1.13. Properties of Δ_0

The properties described in section 1.12 are here trivial consequences of the properties of the determinant. Since an element X_j of $\widetilde{\mathfrak{g}}^{\lambda_j}$ is of the form

$$X_j = \begin{pmatrix} 0 & \xi_j \mathbf{E}_{k-j,k-j} \\ 0 & 0 \end{pmatrix} \text{ where } \xi_j \in \mathbb{R} \ ,$$

the formula (2) of Theorem 1.13.1 means that

$$\det \begin{pmatrix} x_k \xi_k & & \\ & \ddots & \\ & & x_0 \xi_0 \end{pmatrix} = x_0 \cdots x_k \det \begin{pmatrix} \xi_k & & \\ & \ddots & \\ & & \xi_0 \end{pmatrix} \ .$$

The character χ_0 of the group G is given by

$$\chi_0(g) = \det(\mathbf{g})^2 \text{ where } \mathbf{g} \in GL^+(k+1, \mathbb{R}) \ .$$

A.1.14. The polynomials Δ_j

Since the graded algebra $\widetilde{\mathfrak{g}}_j$ is obtained in the same way as $\widetilde{\mathfrak{g}}$ with k replaced by $k+1-j$, the fundamental relative invariants are also determinants. It follows that the polynomials Δ_j are principal minors. More precisely for the element $X \in V^+$ determined by a symmetric matrix $\mathbf{X} = (x_{i,j})_{i,j=0,\ldots,k}$ we have

$$\Delta_j(X) = \det \begin{pmatrix} x_{0,0} & \cdots & x_{0,k-j} \\ \vdots & & \vdots \\ x_{k-j,0} & \cdots & x_{k-j,k-j} \end{pmatrix} \ ,$$

and the properties of Δ_j given in Theorem 1.14.2 follow easily.

A.2 The orbits of G in V^+

In the case we are considering here, it is easy to obtain the G–orbits in V^+. It is equivalent to obtain the orbits of $GL^+(k+1, \mathbb{R})$ in $\mathrm{Sym}(k+1, \mathbb{R})$ for the action $(\mathbf{g}, \mathbf{X}) \mapsto \mathbf{g}\mathbf{X}\,{}^t\mathbf{g}$. These orbits are, of course, given by Sylvester's law of inertia for real quadratic forms.

Nevertheless, we will follow the sections of this chapter to exhibit the different objects used in the proof of the general case.

A.2.2. First reduction

It is well known that each matrix $\mathbf{X} \in \mathrm{Sym}(k+1, \mathbb{R})$ is conjugate under an orthogonal matrix $\mathbf{g}$ to a diagonal matrix $\mathbf{D}$ *i.e.*

$$\mathbf{X} = \mathbf{g}\mathbf{D}\mathbf{g}^{-1} = \mathbf{g}\mathbf{D}\,{}^t\mathbf{g} \ .$$

Moreover it is possible to choose $\mathbf{g}$ with positive determinant by replacing, if necessary, the matrix $\mathbf{g}$ with the matrix

$$\mathbf{g}.\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \ .$$

It follows that each element of V^+ belongs to the G–orbit of an element of the "diagonal" $\widetilde{\mathfrak{g}}^{\lambda_0} \oplus \cdots \oplus \widetilde{\mathfrak{g}}^{\lambda_k}$, which is the result of Theorem 2.2.2.

A.2.3. An involution which permutes the roots in $E_{i,j}(\pm 1, \pm 1)$

The maximal compact group K of G is here isomorphic to $SO(k+1, \mathbb{R})$ eventually quotiented by $\{\pm \mathrm{Id}\}$ according to the parity of k. The maximal compact subgroup $\widetilde{K}$ of $\widetilde{G}$ is the quotient by $\{\pm \mathrm{Id}\}$ of the set of matrices

$$\begin{pmatrix} \mathbf{a} & \mathbf{b} \\ -\mathbf{b} & \mathbf{a} \end{pmatrix} \text{ where } \mathbf{a} + i\mathbf{b} \in U(k+1) \ .$$

In particular the elements of the form

$$\begin{pmatrix} \mathbf{a} & 0 \\ 0 & {}^t\mathbf{a}^{-1} \end{pmatrix} \text{ where } \mathbf{a} \in O(k+1) \ ,$$

belong to the group $\widetilde{K}$ and not to the group K if $\det(\mathbf{a}) = -1$.

An element X_i of $\widetilde{\mathfrak{g}}^{\lambda_i}$ such that $\{\theta(X_i), H_{\lambda_i}, X_i\}$ is a $\mathfrak{sl}_2$–triple is of the form

$$X_i = \eta_i \begin{pmatrix} 0 & \mathbf{F_i} \\ 0 & 0 \end{pmatrix} \text{ where } \mathbf{F}_i = \mathbf{E}_{k-i,k-i} \text{ and } \eta_i = \pm 1 \ .$$

It follows that the Weyl group element related to this $\mathfrak{sl}_2$–triple is given by

$$w_i = \begin{pmatrix} \mathbf{1}_{k+1} - \mathbf{F}_i & \eta_i \mathbf{F}_i \\ -\eta_i \mathbf{F}_i & \mathbf{1}_{k+1} - \mathbf{F}_i \end{pmatrix} \ ,$$

which belongs to $\widetilde{K}$ since $w_i^{-1} = {}^t w_i$. For $i \neq j$, we have $\mathbf{F}_i \mathbf{F}_j = 0$ and we obtain

$$w_{i,j} = w_i w_j = \begin{pmatrix} \mathbf{1}_{k+1} - (\mathbf{F}_i + \mathbf{F}_j) & \eta_i \mathbf{F}_i + \eta_j \mathbf{F}_j \\ -(\eta_i \mathbf{F}_i + \eta_j \mathbf{F}_j) & \mathbf{1}_{k+1} - (\mathbf{F}_i + \mathbf{F}_j) \end{pmatrix} = w_j w_i \ .$$

If X is an element of $E_{ij}(1,1)$ or of $E_{ij}(1,-1)$ we have

$$w_{i,j}.X = w_{i,j} X {}^t w_{i,j} = \eta_i \eta_j \theta(X) \ .$$

It follows that Proposition 2.3.2 is trivially satisfied in this case and that the involution of $E_{i,j}(p,q), (p = \pm 1, q = \pm 1)$ given by $\tau_{ij} = w_{ij} \circ \theta$ is, up to the sign, the identity. The sign depends on the choice of X_i and of X_j.

A.2.4. Construction of elements interchanging λ_i and λ_j

Here we have

$$H_{\lambda_i} - H_{\lambda_j} = H_{\frac{\lambda_i - \lambda_j}{2}} \ .$$

Therefore an element $X \in E_{i,j}(1,-1)$ such that $\{\theta(X), H_{\lambda_i} - H_{\lambda_j}, X\}$ is an $\mathfrak{sl}_2$–triple is of the form

$$X = \begin{pmatrix} \mathbf{A} & 0 \\ 0 & -{}^t\mathbf{A} \end{pmatrix} \text{ where } \mathbf{A} = \pm \mathbf{E}_{k-i,k-j} \ .$$

According to the choice of X_i and of X_j, this element will satisfy the equality $\tau_{ij}(X) = X$ or the equality $\tau_{ij}(X) = -X$. The Weyl group element $\gamma_{i,j}$ associated to this $\mathfrak{sl}_2$–triple is given by

$$\gamma_{i,j} = e^{\mathrm{ad}X} e^{\mathrm{ad}\theta(X)} e^{\mathrm{ad}X} = \begin{pmatrix} \mathbf{g}_{ij} & 0 \\ 0 & {}^t\mathbf{g}_{ij}^{-1} \end{pmatrix} \ ,$$

where $\mathbf{g}_{ij} = \mathbf{1}_{k+1} - \mathbf{E}_{k-i,k-i} - \mathbf{E}_{k-j,k-j} \pm (\mathbf{E}_{k-i,k-j} - \mathbf{E}_{k-j,k-i})$. For simplicity, we will write

$$\mathbf{g}_{ij} \longleftrightarrow \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ ,$$

where the 2×2 matrix on the right hand side is the square submatrix of $\mathbf{g}_{ij}$ corresponding to the indices in $\{k-i, k-j\} \times \{k-i, k-j\}$. The other coeefficients of $\mathbf{g}_{ij}$ are equal to one on the diagnonal and zero elsewhere. Proposition 2.4.3 is easily verified, namely that $\gamma_{i,j}$ interchanges the roots λ_i and λ_j and stabilizes the other roots λ_s.

The action of γ_{ij}^2 is not trivial on $\widetilde{\mathfrak{g}}$ since we have (using the notation of the previous section)

$$\mathbf{g}_{ij}^2 = \mathbf{1}_{k+1} - 2\mathbf{E}_{k-i,k-i} - 2\mathbf{E}_{k-j,k-j} = \mathbf{1}_{k+1} - 2\mathbf{F}_i - 2\mathbf{F}_j \ ,$$

which can be written

$$\mathbf{g}_{ij}^2 \longleftrightarrow \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \ .$$

To obtain an involution of $\widetilde{\mathfrak{g}}$ we use w_i^2 which does not depend on the choice of X_i since

$$w_i^2 = \begin{pmatrix} \mathbf{1}_{k+1} - 2\mathbf{F}_i & 0 \\ 0 & \mathbf{1}_{k+1} - 2\mathbf{F}_i \end{pmatrix} \ .$$

We set

$$\widetilde{\gamma_{i,j}} = \gamma_{i,j} w_i^2 = -w_i^2 \gamma_{i,j} = -\gamma_{,ij} w_j^2 = \begin{pmatrix} \widetilde{\mathbf{g}}_{ij} & 0 \\ 0 & {}^t\widetilde{\mathbf{g}}_{ij}^{-1} \end{pmatrix} \ ,$$

where $\widetilde{\mathbf{g}}_{ij} = \pm \left(\mathbf{1}_{k+1} - \mathbf{E}_{k-i,k-i} - \mathbf{E}_{k-j,k-j} + \mathbf{E}_{k-i,k-j} + \mathbf{E}_{k-j,k-i}\right)$ *i.e.*

$$\widetilde{\mathbf{g}}_{ij} \longleftrightarrow \pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ .$$

This element belongs to $\widetilde{K}$ but not to K since $\det(\widetilde{\mathbf{g}}_{ij}) = -1$ and we verify the result of Proposition 2.4.4 namely that

$$\widetilde{\gamma}_{ij}^2 = \mathrm{Id} \ .$$

A.2.5. Quadratic forms

Let $X \in V^+$ (*resp.* $Y \in V^-$) and let $\mathbf{X}$ (*resp.* $\mathbf{Y}$) be the associated symmetric matrix. Then the quadratic form $Q_X(Y)$ on V^- is given by

$$Q_X(Y) = b(e^{\mathrm{ad}X}Y, Y) = \mathrm{tr}(\mathbf{XYXY}) \ .$$

If X belongs to $\oplus_{j=0}^k \widetilde{\mathfrak{g}}^{\lambda_j}$ then we have $\mathbf{X} = \sum_{i=0}^k \xi_i \mathbf{E}_{ii}$ with $\xi_i \in \mathbb{R}$. It follows that, if $\mathbf{Y} = (y_{ij})$, we have

$$Q_X(Y) = \sum_{i=0}^k \xi_i^2 y_{ii}^2 + 2 \sum_{0 \le i < j \le k} \xi_i \xi_j y_{ij}^2 \ .$$

And we verify that the rank of this quadratic form is given by

$$\mathrm{rank}\, Q_X = m + \frac{m(m-1)}{2} \ ,$$

where m is the number of indices i such that $\xi_i \neq 0$. This is the result of Theorem 2.5.1.

Since the dimension of each root space is 1 and since $\frac{\lambda_i + \lambda_j}{2}$ *is the unique root in* $E_{ij}(\pm 1, \pm 1)$*, the case we are considering here is of* Type I.

A.2.8. The signature of the quadratic forms q_{X_i,X_j}

Let $(X_j)_{j=0,\dots,k}$ be a sequence satisfying condition (C). It means that X_j belongs to $\widetilde{\mathfrak{g}}^{\lambda_j}$ and that $\{\theta(X_j), H_{\lambda_j}, X_j\}$ is a $\mathfrak{sl}_2$–triple. Then X_j is given by the symmetric matrix

$$\mathbf{X}_j = \epsilon_j \mathbf{E}_{k-j,k-j} \quad \text{where} \quad \epsilon_j = \pm 1 \ .$$

The quadratic form q_{X_i,X_j} is defined on $E_{ij}(-1,-1)$ by

$$q_{X_i,X_j}(Y) = -\frac{1}{2} b(\mathrm{ad}X_i Y, \mathrm{ad}X_j Y) \ ,$$

and given for $\mathbf{Y} = y(\mathbf{E}_{k-i,k-j} + \mathbf{E}_{k-j,k-i})$ by

$$q_{X_i,X_j}(Y)) = \begin{cases} \epsilon_i \epsilon_j y^2 & \text{if } i \neq j \ , \\ 4y^2 & \text{if } i = j \ . \end{cases}$$

It follows that, for $i = j$, this quadratic form is positive definite on $\widetilde{\mathfrak{g}}^{-\lambda_j}$ (Proposition 2.8.2) and that, for $i \neq j$, it is positive definite (*resp.* negative definite) on $E_{ij}(-1,-1)$ if $\epsilon_i \epsilon_j > 0$ (*resp.* $\epsilon_i \epsilon_j < 0$) (Corollary 2.8.7). Therefore the sequence $(X_j)_{j=0,\dots,k}$ satisfies the condition (D) if and only if $\epsilon_i = \epsilon_j$ for $i, j \in \{0, \cdots, k\}$, which implies that there exist only two sequences of this type obtained for $\epsilon_j = 1$ for each $j \in \{0, \dots, k\}$ or $\epsilon_j = -1$ for each $j \in \{0, \dots, k\}$ (Remark 2.8.13). This can be written here

$$(X_j)_{j=0,\dots,k} \text{ satisfies (D)} \iff \mathbf{X}_0 + \mathbf{X}_1 + \cdots + \mathbf{X}_k = \pm \mathbf{1}_{k+1} \ .$$

Using the explicit value of γ_{ij} obtained in section A.2.4, we have

$$X_j = \gamma_{ij}(X_i) \iff \epsilon_i = \epsilon_j \ .$$

It follows that the quadratic form $q_{X_i,\gamma_{ij}(X_i)}$ is positive definite (Lemma 2.8.8).

For $\varepsilon = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_k)$ we will denote by I^+_ε the element of V^+ determined by the symmetric matrix

$$\mathbf{I}_\varepsilon = \sum_{j=0}^{k} \varepsilon_j \mathbf{E}_{k-j,k-j} \ .$$

In particular, for $\varepsilon = (1, 1, \ldots, 1)$, we will write I^+ instead of I^+_ε. The associated symmetric matrix is then

$$\mathbf{I}^+ = \sum_{j=0}^{k} \mathbf{E}_{j,j} = \mathbf{1}_{k+1} \ .$$

A.2.9. $Z_G(I^+)$–orbits

The quadratic forms Q_{I^+} and Q_{-I^+} are positive definite on V^- and the centralizer of I^+ (and of $-I^+$) in G is exactly the maximal compact subgroup of G *i.e.* K. Theorem 2.9.3 is here the consequence of the fact that each symmetric matrix is conjugate under the group $K = SO(k+1, \mathbb{R})$ to a diagonal matrix.

A.2.10. The G–orbits in V^+

The result of Theorem 2.10.1 can here be written as follows.

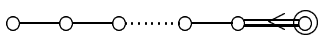
The $GL^+(k+1, \mathbb{R})$–orbits in $\mathrm{Sym}(k+1, \mathbb{R})$ *are in one to one correspondance with the orbits of the elements I^+_ε where ε is of the form :*

$$\varepsilon = (\underbrace{0, \ldots, 0}_{k+1-p-q}, \underbrace{1, \ldots, 1}_{q}, \underbrace{-1, \ldots, -1}_{p}) \text{ with } p \geq 0 \, ; q \geq 0 \, ; p+q \leq k+1 \ .$$

It is Sylvester's law of inertia and (q, p) is the signature of the quadratic form associated to the symmetric matrices in the orbit under consideration.

A.2.11. The classification

In the table of the end of chapter 2 we find the example studied here in the following form :

- $\widetilde{\mathfrak{g}} = \mathfrak{sp}(k+1, \mathbb{R})$; $\mathfrak{g}' = \mathfrak{sl}(k+1, \mathbb{R})$; $V^+ = \mathrm{Sym}(k+1, \mathbb{R})$;
- $\widetilde{\mathcal{R}} = \widetilde{\Sigma}$ of type C_{k+1} ;
- $d = \ell = e = 1$;
- Type I.
- 

A.3. The symmetric spaces G/H

The open G–orbits in V^+ are of the form $G.I^+_\epsilon$ where $\epsilon = (\epsilon_0, \ldots, \epsilon_k)$ with $\epsilon_j = \pm 1$ and

$$I^+_\epsilon = \begin{pmatrix} 0 & \mathbf{I}_\epsilon \\ 0 & 0 \end{pmatrix} \quad \text{with} \quad \mathbf{I}_\epsilon = \begin{pmatrix} \epsilon_k & & \\ & \ddots & \\ & & \epsilon_0 \end{pmatrix} \ .$$

Moreover the orbits $G.I_\epsilon$ and $G.I_{\epsilon'}$ are the same if and only if

$$\#\{j \mid \epsilon_j = 1\} = \#\{j \mid \epsilon'_j = 1\} \ .$$

A.3.1. The involutions

The Weyl group element w_ϵ of the $\mathfrak{sl}_2$–triple $\{\theta(I_\epsilon^+), H_0, I_\epsilon^+\}$ is of the form

$$w_\epsilon = \begin{pmatrix} 0 & \mathbf{I}_\epsilon \\ -\mathbf{I}_\epsilon & 0 \end{pmatrix} \ .$$

The adjoint action of w_ϵ on $\widetilde{\mathfrak{g}}$ is given by the involution σ_ϵ where

$$\sigma_\epsilon \begin{pmatrix} \mathbf{A} & \mathbf{X} \\ \mathbf{Y} & -{}^t\mathbf{A} \end{pmatrix} = \begin{pmatrix} -\mathbf{I}_\epsilon\, {}^t\mathbf{A}\mathbf{I}_\epsilon & -\mathbf{I}_\epsilon\mathbf{Y}\mathbf{I}_\epsilon \\ -\mathbf{I}_\epsilon\mathbf{X}\mathbf{I}_\epsilon & \mathbf{I}_\epsilon\mathbf{A}\mathbf{I}_\epsilon \end{pmatrix} \ .$$

It is easy to verify here the results of Theorem 3.1.1. In particular it can be noticed that :

• The involution σ_ϵ stabilizes $\mathfrak{g}$ and the subalgebra of fixed points under this involution is given by

$$\mathfrak{h}_\epsilon = \left\{ \begin{pmatrix} \mathbf{A} & 0 \\ 0 & -{}^t\mathbf{A} \end{pmatrix} \text{ where } \mathbf{I}_\epsilon\mathbf{A}\mathbf{I}_\epsilon = -{}^t\mathbf{A} \right\} \ .$$

Therefore $\mathfrak{h}_\varepsilon$ is isomorphic to the Lie algebra of the orthogonal group of the quadratic form defined on $\mathbb{R}^{k+1}$ by $(x_0, x_1, \ldots, x_k) \longmapsto \sum_{j=0}^k \epsilon_j x_j$.

• The application $\mathrm{ad}I_\epsilon^+ : \mathfrak{g} \longrightarrow V^+$ is given by

$$\begin{pmatrix} \mathbf{A} & 0 \\ 0 & -{}^t\mathbf{A} \end{pmatrix} \longmapsto \begin{pmatrix} 0 & -\mathbf{I}_\epsilon\, {}^t\mathbf{A} - \mathbf{A}\mathbf{I}_\epsilon \\ 0 & 0 \end{pmatrix} \ .$$

This application has a kernel given by $\mathfrak{h}_\epsilon$ and is onto. More precisely let $\mathfrak{q}_\epsilon$ be the set of elements $X \in \mathfrak{g}$ such that $\sigma_\epsilon(X) = -X$. Then the application $-\frac{1}{2}\mathrm{ad}\theta(I_\epsilon^+) : V^+ \longrightarrow \mathfrak{q}_\epsilon$ is given by

$$\begin{pmatrix} 0 & \mathbf{X} \\ 0 & 0 \end{pmatrix} \longmapsto \begin{pmatrix} -\frac{1}{2}\mathbf{X}\mathbf{I}_\epsilon & 0 \\ 0 & \frac{1}{2}\mathbf{I}_\epsilon\mathbf{X} \end{pmatrix} \ .$$

Then we have for $X \in V^+$

$$X = \mathrm{ad}I_\epsilon^+ \circ \left(-\frac{1}{2}\mathrm{ad}\theta(I_\epsilon^+) \right)(X) \ .$$

Therefore the image of the application $\mathrm{ad}I_\epsilon^+$ is V^+.

• The involutions σ_ε, $\sigma_{\varepsilon'}$ and θ commute (results of Theorem 3.1.1 (5) and Proposition 3.1.3).

Here we have $\mathfrak{a}^0 = \mathfrak{a}^\mathfrak{p}$ and these subspaces are maximal abelian in

$$\mathfrak{p} = \{X \in \mathfrak{g} \mid \theta(X) = -X\} = \left\{ X = \begin{pmatrix} \mathbf{A} & 0 \\ 0 & -{}^t\mathbf{A} \end{pmatrix} \text{ where } {}^t\mathbf{A} = \mathbf{A} \right\}$$

and in

$$\mathfrak{q}_\varepsilon = \{X \in \mathfrak{g} \mid \sigma_\varepsilon(X) = -X\} = \left\{ X = \begin{pmatrix} \mathbf{A} & 0 \\ 0 & -{}^t\mathbf{A} \end{pmatrix} \text{ where } {}^t\mathbf{A} = \mathbf{I}_\varepsilon\mathbf{A}\mathbf{I}_\varepsilon \right\} \ .$$

This implies the result of Proposition 3.1.4.

A.3.2. An involution σ for Type I

In section A.2.8. we have proved that the sequence $(X_j)_{j=0,\dots,k}$ satisfies the condition (D) if and only if $\mathbf{X}_0 + \mathbf{X}_1 + \cdots + \mathbf{X}_k = \pm \mathbf{1}_{k+1}$. In this case $X_0 + X_1 + \cdots + X_k = I_\varepsilon^+$ with $\varepsilon = (1, 1, \dots, 1)$ or $\varepsilon = (-1, -1, \dots, -1)$ and it follows immediately that we have

$$\sigma_\varepsilon = \theta \ .$$

A.3.3. Root systems

The root system $\widetilde{\Sigma}^0 = \Sigma(\widetilde{\mathfrak{g}}, \mathfrak{a}^0)$ is equal to the root system $\widetilde{\Sigma} = \Sigma(\widetilde{\mathfrak{g}}, \mathfrak{a}^{\mathfrak{p}})$ and therefore is of type C_{k+1} (see A.1.2).

The algebra $\mathfrak{g}$ is isomorphic to $\mathfrak{gl}(k+1, \mathbb{R})$ and $\mathfrak{a}^0$ is a Cartan subalgebra of $\mathfrak{g}$. It follows that the root system $\Sigma^0 = \Sigma(\mathfrak{g}, \mathfrak{a}^0)$ is of type A_k, which is the result of Theorem 3.3.3 (1).

Let now $\underline{\sigma}_\varepsilon$ be the involution of $\widetilde{\mathfrak{g}}$ given by

$$\underline{\sigma}_\varepsilon(X) = \begin{cases} \sigma_\varepsilon(X) & \text{for } X \in \mathfrak{g} \ ; \\ -\sigma_\varepsilon(X) & \text{for } X \in V^+ \oplus V^- \ . \end{cases}$$

Then we have

$$\theta\underline{\sigma}_\epsilon \begin{pmatrix} \mathbf{A} & \mathbf{X} \\ \mathbf{Y} & -{}^t\mathbf{A} \end{pmatrix} = \begin{pmatrix} \mathbf{I}_\epsilon \, \mathbf{A} \mathbf{I}_\epsilon & -\mathbf{I}_\epsilon \mathbf{X} \mathbf{I}_\epsilon \\ -\mathbf{I}_\epsilon \mathbf{Y} \mathbf{I}_\epsilon & -\mathbf{I}_\epsilon \, {}^t\mathbf{A} \mathbf{I}_\epsilon \end{pmatrix} \ .$$

Let $\widetilde{\mathfrak{g}}_{\theta\underline{\sigma}_\varepsilon}$ be the set of fixed points of $\widetilde{\mathfrak{g}}$ under the involution $\theta\underline{\sigma}_\varepsilon$. Theorem 3.3.2 (2) asserts that the root system $\widetilde{\Sigma}^{0\underline{\sigma}_\varepsilon} = \Sigma(\widetilde{\mathfrak{g}}_{\theta\underline{\sigma}_\varepsilon}, \mathfrak{a}^0))$ is here of type A_k.

Up to conjugation by an element of K we may choose ε of the form

$$\varepsilon = (\underbrace{1, \dots, 1}_{q}, \underbrace{-1, \dots, -1}_{p}) \quad \text{with } p + q = k + 1 \ .$$

Then we have

$$\mathbf{I}_\varepsilon = \begin{pmatrix} -\mathbf{1}_p & 0 \\ 0 & \mathbf{1}_q \end{pmatrix}.$$

It follows that $\widetilde{\mathfrak{g}}_{\theta\underline{\sigma}_\varepsilon}$ is the set of elements of the form

$$X = \begin{pmatrix} a_1 & 0 & 0 & x \\ 0 & a_2 & {}^t x & 0 \\ 0 & y & -{}^t a_1 & 0 \\ {}^t y & 0 & 0 & -{}^t a_2 \end{pmatrix} \ ,$$

where $a_1 \in \mathrm{M}(p, \mathbb{R})$, $a_2 \in \mathrm{M}(q, \mathbb{R})$, $x, y \in \mathrm{M}(p, q, \mathbb{R})$. Therefore the application

$$X \longmapsto \begin{pmatrix} a_1 & x \\ {}^t y & -{}^t a_2 \end{pmatrix} \ ,$$

is an isomorphism from the Lie algebra $\widetilde{\mathfrak{g}}_{\theta\underline{\sigma}_\varepsilon}$ onto $\mathfrak{gl}(k+1, \mathbb{R})$. Since $\mathfrak{a}^0$ is a Cartan subalgebra of $\widetilde{\mathfrak{g}}_{\theta\underline{\sigma}_\varepsilon}$ we verify here the result of Theorem 3.3.2 (2).

Moreover $\mathfrak{g}_{\theta\sigma_\varepsilon}$ which is equal to $\widetilde{\mathfrak{g}}_{\theta\sigma_\varepsilon} \cap \mathfrak{g}$ is isomorphic to $\mathfrak{gl}(p,\mathbb{R}) \times \mathfrak{gl}(q,\mathbb{R})$. Therefore the root system $\Sigma^{0\sigma_\varepsilon} = \Sigma(\mathfrak{g}_{\theta\sigma_\varepsilon}, \mathfrak{a}^0)$ is of type $A_{p-1} \times A_{q-1}$ which is the result of Theorem 3.3.3 (3).

A.3.4. The groups K, H_ε and G^{σ_ε}

We recall (see A.1.7) that the group G is isomorphic to $GL^+(k+1,\mathbb{R})$ for k even and to $GL^+(k+1,\mathbb{R})/\{\pm \mathrm{Id}\}$ if k odd. More precisely

$$G = \left\{ g = \begin{pmatrix} \mathbf{g} & 0 \\ 0 & {}^t\mathbf{g}^{-1} \end{pmatrix} \right\} ,$$

where $\mathbf{g} \in GL^+(k+1,\mathbb{R})$ eventually quotiented by $\pm\mathrm{Id}$.

The involution θ is well defined on G by

$$\theta \begin{pmatrix} \mathbf{g} & 0 \\ 0 & {}^t\mathbf{g}^{-1} \end{pmatrix} = \begin{pmatrix} {}^t\mathbf{g}^{-1} & 0 \\ 0 & \mathbf{g} \end{pmatrix} .$$

By definition, an element g belongs to G^θ if $\theta(g) = g$. This means that ${}^t\mathbf{g}^{-1} = \mathbf{g}$ and eventually ${}^t\mathbf{g}^{-1} = -\mathbf{g}$ if k odd. But this last equality, equivalent to $\mathbf{g}\,{}^t\mathbf{g} = -\mathbf{1}_{k+1}$, is impossible. It follows that the group $K = G^\theta$ is isomorphic to

$$\begin{aligned} &SO(k+1) && \text{if } k \text{ even} , \\ &SO(k+1)/\{\pm\mathrm{Id}\} && \text{if } k \text{ odd} , \end{aligned}$$

which is a connected group whatever the parity of k is.

Moreover K is the centralizer in G of $I^+ = \begin{pmatrix} 0 & \mathbf{1}_{k+1} \\ 0 & 0 \end{pmatrix}$ and of $I^- = \theta(I^+)$.

We suppose now that, as in section A.3.3, ε is of the form

$$\varepsilon = (\underbrace{1,\dots,1}_{q},\underbrace{-1,\dots,-1}_{p}) \quad \text{with } p+q = k+1 .$$

The involution σ_ε is well defined on G by conjugation by the element w_ε defined in A.3.1 and we get

$$\sigma_\varepsilon \begin{pmatrix} \mathbf{g} & 0 \\ 0 & {}^t\mathbf{g}^{-1} \end{pmatrix} = \begin{pmatrix} \mathbf{I}_\varepsilon\, {}^t\mathbf{g}^{-1} \mathbf{I}_\varepsilon & 0 \\ 0 & \mathbf{I}_\varepsilon \mathbf{g} \mathbf{I}_\varepsilon \end{pmatrix} .$$

By definition an element g belongs to G^{σ_ε} if $\sigma_\varepsilon(g) = g$. This means that $\mathbf{g}\mathbf{I}_\varepsilon\, {}^t\mathbf{g} = \mathbf{I}_\varepsilon$ and eventually $\mathbf{g}\mathbf{I}_\varepsilon\, {}^t\mathbf{g} = -\mathbf{I}_\varepsilon$ if k odd. The first equality means that g centralizes I_ε^+ and the second that I_ε^+ and $-I_\varepsilon^+$ are in the same G-orbits in V^+. By the classification of the G–orbits this last case occurs if and only if $p = q$. More precisely we have for $p = q$

$$\mathbf{g}\mathbf{I}_\varepsilon\, {}^t\mathbf{g} = -\mathbf{I}_\varepsilon \Longleftrightarrow \mathbf{g} = \mathbf{h} \times \begin{pmatrix} 0 & -\mathbf{1}_p \\ \mathbf{1}_p & 0 \end{pmatrix} ,$$

where $\mathbf{h}\mathbf{I}_\varepsilon\, {}^t\mathbf{h} = \mathbf{I}_\varepsilon$ and $\mathbf{1}_p$ is the identity matrix in $\mathrm{M}(p,\mathbb{R})$.
We obtain therefore that

- $G^{\sigma_\varepsilon} = Z_G(I_\varepsilon)$ if $p \neq q$,
- $G^{\sigma_\varepsilon} \supset Z_G(I_\varepsilon)$ and $G^{\sigma_\varepsilon} \neq Z_G(I_\varepsilon)$ if $p = q$.

This is the result of Theorem 3.4.2 (3).

The centralizer of I_ε^+ in G is given by

$$Z_G(I_\varepsilon^+) = \begin{cases} SO(q,p) & \text{if } k = p+q-1 \text{ even} \\ SO(q,p)/\{\pm \mathrm{Id}\} & \text{if } k = p+q-1 \text{ odd} \end{cases} .$$

If $p > 0$ and $q > 0$ the group $SO(q,p)$ has two connected components and therefore $Z_G(I_\varepsilon^+)$ is not necessarily connected. More precisely the number of connected components of $Z_G(I_\varepsilon^+)$ is equal to

- 1 if $(q = 0)$ or $(p = 0)$ or (p odd and q odd) ;
- 2 if ($p+q$ odd with $q > 0, p > 0$) or (p even > 0 and q even > 0) .

The subgroup $Z_G(I_\varepsilon^+)$ is open in G^{σ_ε} and will be denoted by H_ε. Therefore we obtain a collection of reductive symmetric spaces G/H_ε which have a realization as open subsets of V^+, namely $G.I_\varepsilon^+$. The union of these open subset for $\varepsilon = (1,\dots,1,\underbrace{-1,\dots,-1}_{p})$ and $p \in \{0,\dots,k+1\}$ is the open dense subset of generic elements of V^+ which is associated to the set of invertible symmetric matrices.

A.3.5. A parabolic subgroup of G

The parabolic subalgebra of $\mathfrak{g}$ defined by

$$\mathfrak{P} = \mathfrak{z}_\mathfrak{g}(\mathfrak{a}^0) \oplus (\oplus_{i<j} E_{ij}(1,-1))$$

is here a minimal parabolic subalgebra of $\mathfrak{g}$ which is given by

$$\mathfrak{P} = \left\{ \begin{pmatrix} \mathbf{A} & 0 \\ 0 & -{}^t\mathbf{A} \end{pmatrix} \text{ where } \mathbf{A} \text{ is an under triangular matrix} \right\} .$$

From the definition of θ in A.1.1 and of σ_ε in A.3.1 we get

$$\sigma_\varepsilon\theta \begin{pmatrix} \mathbf{A} & 0 \\ 0 & -{}^t\mathbf{A} \end{pmatrix} = \begin{pmatrix} \mathbf{I}_\varepsilon \mathbf{A} \mathbf{I}_\varepsilon & 0 \\ 0 & -\mathbf{I}_\varepsilon^t \mathbf{A} \mathbf{I}_\varepsilon \end{pmatrix} .$$

The subalgebra $\mathfrak{P}$ is $\sigma_\varepsilon\theta$–stable for every involution σ_ε associated to a generic element I_ε^+ of the diagonal $\oplus_{j=0}^k \widetilde{\mathfrak{g}}^{\lambda_j}$ and is therefore a minimal $\sigma_\varepsilon\theta$–stable parabolic subalgebra of $\mathfrak{g}$.

The LANGLANDS decomposition $\mathfrak{P} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ of $\mathfrak{P}$ is here of the form

$$\mathfrak{P} = \mathfrak{a}^0 \oplus \mathfrak{n} ,$$

since $\mathfrak{z}_\mathfrak{g}(\mathfrak{a}^0) = \mathfrak{a}^0$. Here $\mathfrak{n} = \oplus_{i<j} E_{ij}(1,-1)$ is the set of matrices

$$\begin{pmatrix} \mathbf{A} & 0 \\ 0 & -{}^t\mathbf{A} \end{pmatrix} \text{ such that } \mathbf{A} = \begin{pmatrix} 0 & \dots & 0 \\ * & \ddots & \vdots \\ * & * & 0 \end{pmatrix} .$$

We will now describe the LANGLANDS decomposition $P = MAN$ of the parabolic subgroup P of G with Lie algebra $\mathfrak{P}$. Each of the groups M, A and N will be in a one to one correspondence with some subgroups $\mathbf{M}$, $\mathbf{A}$ and $\mathbf{N}$ of $GL^+(k+1,\mathbb{R})$ where the correspondence goes as follows :

$$g = \begin{pmatrix} \mathbf{g} & 0 \\ 0 & -{}^t\mathbf{g}^{-1} \end{pmatrix} \in M, A, N \Leftrightarrow \mathbf{g} \in \mathbf{M}, \mathbf{A}, \mathbf{N} \text{ (respectively)} .$$

With this correspondence the LANGLANDS decomposition is given by

$$\mathbf{N} = \left\{ \mathbf{n} \in GL^+(k+1,\mathbf{R}) \;\middle|\; \mathbf{n} = \begin{pmatrix} 1 & 0 & 0 \\ * & \ddots & 0 \\ * & * & 1 \end{pmatrix} \right\} ,$$

$$\mathbf{A} = \left\{ \mathbf{a} \in GL^+(k+1,\mathbf{R}) \;\middle|\; \mathbf{a} = \begin{pmatrix} a_k & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & a_0 \end{pmatrix} (a_j > 0) \right\} ,$$

$$\mathbf{M} = \left\{ \mathbf{m} \in GL^+(k+1,\mathbf{R}) \;\middle|\; \mathbf{m} = \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \pm 1 \end{pmatrix} \right\} .$$

The action of M on each I_ε^+ is trivial which implies that M is a subgroup of H_ε. This is the result of Theorem 3.5.4.

A.3.6. The prehomogeneous vector space (P, V^+)

We will study the polynomials Δ_j $(j = 0, \ldots, k)$. We recall that if $X \in V^+$ is determined by a symmetric matrix $\mathbf{X}$ then $\Delta_j(X)$ is a principal minor of $\mathbf{X}$ which will be denoted by δ_j. More precisely for $\mathbf{X} = (x_{ij})_{i,j=0,\ldots,k}$ we have

$$\Delta_j(X) = \delta_j(\mathbf{X}) = \begin{vmatrix} x_{0,0} & \cdots & x_{0,k-j} \\ \vdots & & \vdots \\ x_{k-j,0} & \cdots & x_{k-j,k-j} \end{vmatrix} .$$

For $\mathbf{n} \in \mathbf{N}$ we have

$$\delta_j(\mathbf{n}\mathbf{X}{}^t\mathbf{n}) = \delta_j(\mathbf{X}) ,$$

and for $\mathbf{d}$ a diagonal matrix in $\mathbf{MA}$ of the form $\begin{pmatrix} d_k & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & d_0 \end{pmatrix}$ we have

$$\delta_j(\mathbf{d}\mathbf{X}\mathbf{d}) = (d_k \ldots d_j)^2 \delta_j(\mathbf{X}) .$$

It follows that for $p = man \in P$ with $m \in M$, $n \in N$ and $a \in A$ associated to a diagonal matrix $\mathbf{a} = \begin{pmatrix} a_k & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & a_0 \end{pmatrix}$ with $a_j > 0$ we obtain

$$\Delta_j(p.X) = \chi_j(p)\Delta_j(X) \text{ where } \chi_j(p) = (a_k \ldots a_j)^2 .$$

This proves that Δ_j is relatively invariant under the action of P and gives the corresponding character χ_j. This is the result of Theorem 3.6.1.

Let $\mathbf{X}$ be a symmetric matrix such that $\delta_j(\mathbf{X}) \neq 0$ for $j = 0, \ldots, k$. By GAUSS Lemma there exists a matrix $\mathbf{n} \in \mathbf{N}$ and real numbers $x_0, \ldots, x_k$ such that

$$\mathbf{X} = \mathbf{n} \begin{pmatrix} x_k & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & x_0 \end{pmatrix} {}^t\mathbf{n} .$$

Moreover since $\delta_j(\mathbf{X}) = x_k \dots x_j$, the elements x_j are nonzero. Therefore we can write $x_j = \varepsilon_j a_j^2$ with $a_j > 0$ and $\varepsilon_j = \pm 1$ for $j = 0, \dots, k$. Then there exists $\mathbf{a} \in \mathbf{A}$ and ε such that

$$\mathbf{X} = \mathbf{na}\mathbf{I}_\varepsilon \, {}^t\mathbf{a}\,{}^t\mathbf{n} \ .$$

In other words, if X is an element of V^+ such that $\Delta_j(X) \neq 0$ for $j \in \{0, \dots, k\}$ there exists ε such that

$$X \in NA.I_\varepsilon^+ \ .$$

Since M centralizes I_ε^+ we have

$$NA.I_\varepsilon^+ = P.I_\varepsilon^+ \ .$$

It follows that

$$P.I_\varepsilon^+ = \{X \in V^+ \mid \Delta_j(X)\varepsilon_j \dots \varepsilon_k > 0 \text{ for } j = 0, \dots, k\} \ ,$$

which is the result of Theorem 3.6.3.

If $\mathcal{O}^+$ denotes the set of $X \in V^+$ such that $\Delta_j(X) \neq 0$ for $j = 0, \dots, k$, then $\mathcal{O}^+$ is the union of the open orbits of P in V^+. Therefore (P, V^+) is a prehomogeneous vector space.

The open P–orbits in G/H_ε are in a one to correspondance with the open orbits $P.I_\eta^+$ contained in $G.I_\varepsilon^+$. From the description of the open G–orbits in V^+ we have $P.I_\eta^+ \subset G.I^+$ if and only if

$$\#\{j \mid \eta_j = 1\} = \#\{j \mid \varepsilon_j = 1\} \ .$$

It follows that for $\varepsilon = (1, \dots, 1, \underbrace{-1, \dots, -1}_{p})$ the number of these open orbits is given by $\binom{k+1}{p}$. This is the result of Theorem 3.5.3.

A.3.7. The involution γ of $\widetilde{\mathfrak{g}}$

We are looking for an element $\gamma \in \widetilde{G}$ such that $\gamma^2 = \mathrm{Id}$ and
(i) γ stabilizes $\mathfrak{g}$ and interchanges V^+ and V^-;
(ii) γ normalizes P;
(iii) γ normalizes $\mathfrak{a}^0$.
The following element satisfies these properties :

$$\gamma = \begin{pmatrix} 0 & \Gamma \\ -\Gamma & 0 \end{pmatrix} \text{ where } \Gamma = \begin{pmatrix} 0 & \cdots & & 1 \\ & & \iddots & \\ \vdots & & 1 & \vdots \\ & \iddots & & \\ 1 & \cdots & & 0 \end{pmatrix} \ .$$

It may be noted that the square of γ is equal to -Id in $Sp(k+1, \mathbb{R})$ and then to +Id in $\widetilde{G} = Sp(k+1, \mathbb{R})/\{\pm \mathrm{Id}\}$. Therefore γ induces an involution on $\widetilde{\mathfrak{g}}$. More precisely a direct calculation shows the result of Theorem 3.7.1, namely that for $j = 0, \dots, k$ we have

$$\gamma.H_{\lambda_j} = -H_{\lambda_{k-j}} \ ,$$

and for each of the two sequences $(X_j)_{j=0,\dots,k}$ satisfying the condition (D) we have

$$\gamma.X_j = \theta(X_{k-j}) \ .$$

Therefore we have $\gamma.I^+ = \theta(I^+)$.

We want now show the tools used for the general construction of γ and prove that such an element is not unique.
We start with elements $\widetilde{\gamma_{i,j}}$ obtained in section A.2.4 :

$$\widetilde{\gamma_{i,j}} = \begin{pmatrix} \widetilde{\mathbf{g}}_{ij} & 0 \\ 0 & \widetilde{\mathbf{g}}_{ij} \end{pmatrix} \text{ where } \widetilde{\mathbf{g}}_{ij} \longleftrightarrow \pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ .$$

The sign of $\widetilde{\mathbf{g}}_{ij}$ depends on the choice of an element $X \in E_{i,j}(1,-1)$ such that $\{\theta(X), H_{\lambda_i} - H_{\lambda_j}, X)$ is a $\mathfrak{sl}_2$–triple.

Let n be the integer defined by $k = 2n+2$ if k even and $k = 2n+1$ if k odd. Then we define

$$\widetilde{\gamma} = \widetilde{\gamma_{0,k}} \circ \widetilde{\gamma_{1,k-1}} \circ \cdots \circ \widetilde{\gamma_{n,k-n}} \ .$$

This element depends on the choice of $n+1$ signs. Therefore there exists $\eta = (\eta_0, \dots, \eta_n)$ with $\eta_i = \pm 1$ such that

$$\widetilde{\gamma} = \widetilde{\gamma_\eta} = \begin{pmatrix} \Gamma_\eta & 0 \\ 0 & \Gamma_\eta \end{pmatrix} \ ,$$

where, according to the parity of k, we have

$$\Gamma_\eta = \begin{pmatrix} & & & & \eta_0 \\ & & & \iddots & \\ & & \eta_n & & \\ & \iddots & & & \\ \eta_0 & & & & \end{pmatrix} \text{ or } \begin{pmatrix} & & & & \eta_0 \\ & & & \iddots & \\ & & \eta_n & & \\ & 1 & & & \\ & & \eta_n & & \\ & \iddots & & & \\ \eta_0 & & & & \end{pmatrix} \ .$$

If $(X_j)_{j=0,\dots,k}$ satisfies the condition (D) then $\sum_{j=0}^k X_j$ is equal to $\pm I^+$ and is fixed by $\widetilde{\gamma}_\eta$. The Weyl group element w associated to I^+ or $-I^+$ is given by

$$w = \begin{pmatrix} 0 & \mathbf{1}_{k+1} \\ -\mathbf{1}_{k+1} & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & -\mathbf{1}_{k+1} \\ \mathbf{1}_{k+1} & 0 \end{pmatrix} \ .$$

If we set $\gamma_\eta = w\widetilde{\gamma}_\eta = \widetilde{\gamma}_\eta w$ we obtain

$$\gamma_\eta = \begin{pmatrix} 0 & \Gamma_\eta \\ -\Gamma_\eta & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & -\Gamma_\eta \\ \Gamma_\eta & 0 \end{pmatrix} \ .$$

These elements satisfy also the properties of Theorem 3.7.1.

A.3.8. The orbits of P in V^- and the polynomials ∇_j

The action of P on V^- is given by

$$\begin{pmatrix} \mathbf{p} & 0 \\ 0 & {}^t\mathbf{p}^{-1} \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ \mathbf{Y} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ {}^t\mathbf{p}^{-1}\mathbf{Y}\mathbf{p}^{-1} & 0 \end{pmatrix} \ .$$

The polynomials ∇_j defined on V^- for $j = 0, \dots, k$ by

$$\nabla_j(Y) = \Delta_j(\gamma.Y) \ ,$$

are relative invariants under the action of P since γ normalizes P. Again the polynomials ∇_j are given by some minors.

Let $\mathbf{Y} = (y_{ij})_{i,j=0,\dots,k}$ be the symmetric matrix associated to an element Y of V^-. Then we obtain

$$\nabla_j(Y) = \delta_j(-\Gamma\mathbf{Y}\Gamma) = \begin{vmatrix} -y_{k,k} & \cdots & -y_{k,j} \\ \vdots & & \vdots \\ -y_{j,k} & \cdots & -y_{j,j} \end{vmatrix} = (-1)^{k+1-j} \begin{vmatrix} y_{j,j} & \cdots & y_{j,k} \\ \vdots & & \vdots \\ y_{k,j} & \cdots & y_{k,k} \end{vmatrix} .$$

It may be noted that the polynomials ∇_j do not depend on the choice of the elements γ obtained in A.3.7 since (using the notations of the previous section) we have

$$\delta_j(-\Gamma_\eta\mathbf{Y}\Gamma_\eta) = \delta_j(-\Gamma\mathbf{Y}\Gamma) .$$

A more general argument proving this result is given in Remark 3.8.3.

The character of ∇_j for the action of P is given by

$$\chi_j^-(p) = \chi_j(\gamma p\gamma^{-1}) ,$$

where χ_j is the character of Δ_j. If $\mathbf{p}$ is the under triangular matrix associated to an element $p \in P$, then the matrix associated to $\gamma p\gamma^{-1}$ is the under triangular matrix given by $\Gamma^t\mathbf{p}^{-1}\Gamma$. Therefore

$$\nabla_j(p.Y) = \delta_j\big(-{}^t(\Gamma\mathbf{p}^{-1}\Gamma)(\Gamma\mathbf{Y}\Gamma)(\Gamma\mathbf{p}^{-1}\Gamma)\big) .$$

If $\mathbf{p} = \mathbf{nam}$ with $\mathbf{n} \in \mathbf{N}$, $\mathbf{a} \in \mathbf{A}$ and $\mathbf{m} \in \mathbf{M}$, we have $\Gamma\mathbf{n}^{-1}\Gamma \in \mathbf{N}$, $\Gamma\mathbf{m}^{-1}\Gamma \in \mathbf{M}$ and

$$\mathbf{a} = \begin{pmatrix} a_k & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & a_0 \end{pmatrix} \Longrightarrow \Gamma\,\mathbf{a}^{-1}\Gamma = \begin{pmatrix} a_0^{-1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & a_k^{-1} \end{pmatrix} .$$

It follows that

$$\nabla_j(p.Y) = (a_0 \cdots a_{k-j})^{-2}\delta_j(-\Gamma\mathbf{Y}\Gamma) = (a_0 \cdots a_{k-j})^{-2}\nabla_j(Y) ,$$

and consequently that

$$\chi_j^-(p) = (a_0 \cdots a_{k-j})^{-2} .$$

This is the result of Theorem 3.8.2. The open orbits of P in V^- are characterized by the signs of the polynomials ∇_j as explained in Theorem 3.8.6. Representatives of these orbits are given by the elements

$$I_\varepsilon^- = \theta(I_\varepsilon^+) = \begin{pmatrix} 0 & 0 \\ -\mathbf{I}_\varepsilon & 0 \end{pmatrix} .$$

A.3.9. A G–equivariant map from Ω^+ onto Ω^-

From the definitions the sets Ω^+ and Ω^- of generic elements in respectively V^+ and V^- are defined as follows:

$$\Omega^+ = \{X = \begin{pmatrix} 0 & \mathbf{X} \\ 0 & 0 \end{pmatrix} \mid \mathbf{X} \in \mathrm{Sym}(k+1,\mathbb{R}),\ \Delta_0(X) = \det(\mathbf{X}) \neq 0\},$$

$$\Omega^- = \{Y = \begin{pmatrix} 0 & 0 \\ \mathbf{Y} & 0 \end{pmatrix} \mid \mathbf{Y} \in \mathrm{Sym}(k+1,\mathbb{R}),\ \nabla_0(Y) = \det(\mathbf{Y}) \neq 0\}.$$

The G–equivariant map $\Psi : \Omega^+ \longrightarrow \Omega^-$ is defined by the fact that $(\Psi(X), H_0, X)$ is an $\mathfrak{sl}_2$–triple. An easy computation shows that therefore we get:

$$\Psi\Big(\begin{pmatrix} 0 & \mathbf{X} \\ 0 & 0 \end{pmatrix}\Big) = \begin{pmatrix} 0 & 0 \\ -\mathbf{X^{-1}} & 0 \end{pmatrix}.$$

A.3.10. Intersection of N–orbits with the diagonal

We know from Theorem 3.10.4 that an element in $\mathcal{O}^+$ is N–conjugate to a unique element $d(X) = f_0(X)X_0 + f_1(X)X_1 + \cdots + f_k(X)X_k$. As the polynomials Δ_j are N–invariant, we have for all $j = 0, \ldots, k$

$$\Delta_j(d(X)) = \Delta_j(X).$$

Hence we obtain:

$$f_j(X)f_{j+1}(X)\ldots f_k(X) = \Delta_j(X).$$

Therefore $f_j(X) = \dfrac{\Delta_j(X)}{\Delta_{j+1}(X)}$ for all $j = 0, \ldots, k-1$ and $f_k(X) = x_{0,0}$ if $\mathbf{X} = (x_{i,j})_{i,j=0,\ldots,k}$. This proves that the polynomials $P_j(X)$ in Theorem 3.10.4 are here given by

$$P_j(X) = \Delta_j(X).$$

This is in fact true (with the same proof) for Type I and Type II.

A.4. Integral formulas

A.4.1. Integrals on V^+ and V^-

Let us choose the Lebesgue measures dX on V^+ and dY on V^- defined by

$$dX = (\sqrt{2})^{\frac{k(k+1)}{2}} \prod_{i \le j} dx_{i,j} \quad \text{if } \mathbf{X} = (x_{i,j}) \in \mathrm{Sym}(k+1, \mathbb{R}) \ ,$$

$$dY = (\sqrt{2})^{\frac{k(k+1)}{2}} \prod_{i \le j} dy_{i,j} \quad \text{if } \mathbf{Y} = (y_{i,j}) \in \mathrm{Sym}(k+1, \mathbb{R}).$$

Of course $d\mathbf{X}$ will denote the corresponding measure on $\mathrm{Sym}(k+1, \mathbb{R})$.

Then, obviously we have

$$\begin{aligned} \int_{V^+} f(X)\, dX &= \int_{V^+} f(\begin{pmatrix} 0 & \mathbf{X} \\ 0 & 0 \end{pmatrix})\, d\mathbf{X} \\ &= \int_{V^-} f(-{}^t\begin{pmatrix} 0 & 0 \\ \mathbf{Y} & 0 \end{pmatrix})\, d\mathbf{Y} \\ &= \int_{V^-} f(\theta(Y))\, dY \ , \end{aligned}$$

Therefore the pair (dX, dY) satisfies the condition (N_1).

The Fourier transform of a function $f \in \mathcal{S}(V^+)$ is then defined by

$$(\mathcal{F}f)(Y) = \int_{V^+} f(X) e^{2i\pi b(X,Y)} dX = \int_{V^+} f(X) e^{2i\pi \mathrm{Tr}(\mathbf{XY})} dX.$$

An easy calculation shows that if $\overline{\mathcal{F}}$ is defined on $h \in \mathcal{S}(V^-)$ by

$$\overline{\mathcal{F}}h(X) = \int_{V^-} h(Y) e^{-2i\pi b(X,Y)} dY = \int_{V^-} h(Y) e^{-2i\pi \mathrm{Tr}(\mathbf{XY})} dY,$$

then $\overline{\mathcal{F}} \circ \mathcal{F}$ is the identity on $\mathcal{S}(V^+)$.
Hence the pair (dX, dY) satisfies condition (N_1) and (N_2).

For $\mathbf{g} \in GL^+(k+1, \mathbb{R})$, the endomorphism of $\mathrm{Sym}(k+1, \mathbb{R})$ given by $\mathbf{X} \longmapsto \mathbf{gX^t g}$ has a determinant equal to

$$\det(\mathbf{g})^{2\frac{\dim V^+}{k+1}} = \chi_0(g)^{\frac{\dim V^+}{k+1}} .$$

Therefore if we set $m = \dfrac{\dim V^+}{k+1} = \frac{1}{2}(k+2)$ we get

$$\int_{V^+} f(g.X)\, dX = \int_{\mathrm{Sym}(k+1,\mathbb{R})} f\left(\begin{pmatrix} 0 & \mathbf{gX^t g} \\ 0 & 0 \end{pmatrix}\right) d\mathbf{X} = \chi_0(g)^{-m} \int_{V^+} f(X)\, dX$$

which is Lemma 4.1.5.

A.4.2. Two diffeomorphisms

To describe the decomposition of $\widetilde{\mathfrak{g}}$ given by the eigenspaces of $\mathrm{ad}\, H_{\lambda_k}$ let us first notice that here we have

$$H_{\lambda_k} = \begin{pmatrix} \mathbf{H_k} & \\ 0 & -\mathbf{H_k} \end{pmatrix} \text{ where } \mathbf{H_k} = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \mathbf{0}_k \\ & \end{array}\right) .$$

For an element $X \in V^+$ we write

$$X = \begin{pmatrix} 0 & \mathbf{X} \\ 0 & 0 \end{pmatrix} \text{ where } \mathbf{X} = \left(\begin{array}{c|c} \mathbf{v}_1 & \mathbf{w}_1 \\ \hline {}^t\mathbf{w}_1 & \mathbf{u}_1 \\ & \end{array}\right)$$

with $\mathbf{v}_1 \in \mathbb{R}$, $\mathbf{w}_1 \in \mathrm{M}(1, k, \mathbb{R})$ and $\mathbf{u}_1 \in \mathrm{Sym}(k, \mathbb{R})$. Then we obtain

$$X \in V^+(0) \Longleftrightarrow \mathbf{v}_1 = 0 \text{ and } \mathbf{w}_1 = 0 ,$$
$$X \in V^+(1) \Longleftrightarrow \mathbf{v}_1 = 0 \text{ and } \mathbf{u}_1 = 0 ,$$
$$X \in V^+(2) \Longleftrightarrow \mathbf{u}_1 = 0 \text{ and } \mathbf{w}_1 = 0 .$$

Moreover, if $X \in V^+(2)$, then $X \in V^+(2)' \Longleftrightarrow \mathbf{v}_1 \neq 0$.

Similarly for an element $Y \in V^-$ we write

$$Y = \begin{pmatrix} 0 & 0 \\ \mathbf{Y} & 0 \end{pmatrix} \text{ where } \mathbf{Y} = \left(\begin{array}{c|c} \mathbf{v}'_1 & \mathbf{w}'_1 \\ \hline {}^t\mathbf{w}'_1 & \mathbf{u}'_1 \\ & \end{array}\right)$$

with $\mathbf{v}_1' \in \mathbb{R}$, $\mathbf{w}_1' \in \mathrm{M}(1,k,\mathbb{R})$ and $\mathbf{u}_1' \in \mathrm{Sym}(k,\mathbb{R})$. Then we obtain

$$Y \in V^-(0) \Longleftrightarrow \mathbf{v}_1' = 0 \text{ and } \mathbf{w}_1' = 0 \ ,$$
$$X \in V^-(-1) \Longleftrightarrow \mathbf{v}_1' = 0 \text{ and } \mathbf{u}_1' = 0 \ ,$$
$$X \in V^-(-2) \Longleftrightarrow \mathbf{u}_1' = 0 \text{ and } \mathbf{w}_1' = 0 \ .$$

Moreover, if $Y \in V^-(0)$, then $Y \in V^-(0)' \Longleftrightarrow \det(\mathbf{u}_1') \neq 0$.

And for an element $A \in \mathfrak{g}$ we write

$$A = \begin{pmatrix} \mathbf{A} & 0 \\ 0 & -{}^t\mathbf{A} \end{pmatrix} \text{ where } \mathbf{A} = \left(\begin{array}{c|c} \mathbf{c} & \mathbf{B}_1 \\ \hline \mathbf{A}_1 & \mathbf{C} \end{array}\right)$$

with $\mathbf{c} \in \mathbb{R}$, $\mathbf{B}_1 \in \mathrm{M}(1,k,\mathbb{R})$, $\mathbf{A}_1 \in \mathrm{M}(k,1,\mathbb{R})$ and $\mathbf{C} \in \mathrm{M}(k,\mathbb{R})$. Then we obtain

$$A \in \mathfrak{g}(0) \Longleftrightarrow \mathbf{A}_1 = 0 \text{ and } \mathbf{B}_1 = 0 \ ,$$
$$X \in \mathfrak{g}(1) \Longleftrightarrow \mathbf{A}_1 = 0, \mathbf{c} = 0 \text{ and } \mathbf{C} = 0 \ ,$$
$$X \in \mathfrak{g}(-1) \Longleftrightarrow \mathbf{B}_1 = 0, \mathbf{c} = 0 \text{ and } \mathbf{C} = 0 \ .$$

The map Φ from $V^+(0) \times \mathfrak{g}(-1) \times V^+(2)'$ into V^+ defined by

$$\Phi(u, A, v) = e^{\mathrm{ad}\, A}(u + v)$$

corresponds (with the previous notations) to the map

$$\mathbf{\Phi} : \mathrm{Sym}(k,\mathbb{R}) \times \mathrm{M}(k,1,\mathbb{R}) \times \mathbb{R}^* \longrightarrow \mathrm{Sym}(k+1,\mathbb{R})$$
$$(\mathbf{u}_1 \ , \ \mathbf{A}_1 \ , \ \mathbf{v}_1) \longmapsto \mathbf{X}$$

where

$$\mathbf{X} = \left(\begin{array}{c|c} 1 & 0 \\ \hline \mathbf{A}_1 & \mathbf{1}_k \end{array}\right) \times \left(\begin{array}{c|c} \mathbf{v}_1 & 0 \\ \hline 0 & \mathbf{u}_1 \end{array}\right) \times \left(\begin{array}{c|c} 1 & {}^t\mathbf{A}_1 \\ \hline 0 & \mathbf{1}_k \end{array}\right)$$
$$= \left(\begin{array}{c|c} \mathbf{v}_1 & \mathbf{v}_1{}^t\mathbf{A}_1 \\ \hline \mathbf{A}_1\mathbf{v}_1 & \mathbf{A}_1\mathbf{v}_1{}^t\mathbf{A}_1 + \mathbf{u}_1 \end{array}\right) \ .$$

This map is clearly a diffeomorphism onto the set of $\mathbf{X} \in \mathrm{Sym}(k+1,\mathbb{R})$ such that $\mathbf{v}_1 \neq 0$ which corresponds to the set of $X \in V^+$ such that $\Delta_k(X) \neq 0$. This is the first result of Proposition 4.2.3.

Similarly the map Ψ from $V^-(0)' \times \mathfrak{g}(-1) \times V^-(-2)$ into V^- defined by

$$\Psi(u', A, v') = e^{\mathrm{ad}\, A}(u' + v')$$

corresponds (with the previous notations) to the map

$$\mathbf{\Psi} : \{\mathbf{u}_1' \in \mathrm{Sym}(k,\mathbb{R}) \mid \det(\mathbf{u}_1') \neq 0\} \times \mathrm{M}(k,1,\mathbb{R}) \times \mathbb{R}^* \longrightarrow \mathrm{Sym}(k+1,\mathbb{R})$$
$$(\mathbf{u}_1' \ , \ \mathbf{A}_1 \ , \ \mathbf{v}_1') \longmapsto \mathbf{Y}$$

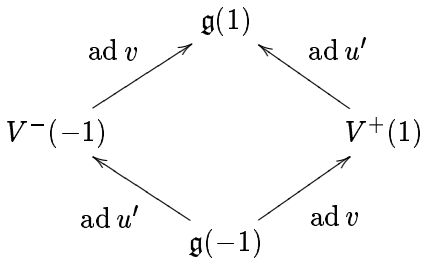
where

$$\mathbf{Y} = \left(\begin{array}{c|c} \mathbf{v}'_1 + {}^t\mathbf{A}_1\mathbf{u}'_1\mathbf{A}_1 & -{}^t\mathbf{A}_1\mathbf{u}'_1 \\ \hline -\mathbf{u}'_1\mathbf{A}_1 & \mathbf{u}'_1 \end{array}\right) .$$

This map is clearly a diffeomorphism onto the set of $\mathbf{Y} \in \mathrm{Sym}(k+1, \mathbb{R})$ such that $\det(\mathbf{u}'_1) \neq 0$ which corresponds to the set of $Y \in V^-$such that $\nabla_1(Y) \neq 0$. This is the second result of Proposition 4.2.3.

A.4.3. Isomorphisms between $\mathfrak{g}(1), \mathfrak{g}(-1), V^+(1)$ and $V^-(-1)$

We consider the following diagram where v is a generic element in $V^+(2)$ and u' is a generic element in $V^-(0)$.



Let us make explicit these arrows in order to prove directly that they are isomorphisms and that the diagram commutes. For this purpose we will use the same notations as in section **A.4.2**

Let $u' \in V^-(0)'$. Under the just mentioned notational conventions the map

$$\mathrm{ad}\, u' : V^+(1) \longrightarrow \mathfrak{g}(1)$$

corresponds to the map:

$$\begin{aligned} M(1,k,\mathbb{R}) &\longrightarrow M(1,k,\mathbb{R}) \\ \mathbf{w}_1 &\longmapsto -\mathbf{w}_1\mathbf{u}'_1 \end{aligned}$$

which is obviously an isomorphism since $\det(\mathbf{u}'_1) \neq 0$.
Similarly the isomorphism:

$$\mathrm{ad}\, u' : \mathfrak{g}(-1) \longrightarrow V^-(-1)$$

corresponds to the isomorphism:

$$\begin{aligned} M(k,1,\mathbb{R}) &\longrightarrow M(1,k,\mathbb{R}) \\ \mathbf{A}_1 &\longmapsto {}^t\mathbf{A}_1\mathbf{u}'_1. \end{aligned}$$

Let now $v \in V^+(2)'$. Then the map

$$\mathrm{ad}\, v : V^-(-1) \longrightarrow \mathfrak{g}(1)$$

corresponds to the map

$$\begin{aligned} M(1,k,\mathbb{R}) &\longrightarrow M(1,k,\mathbb{R}) \\ \mathbf{w}'_1 &\longmapsto \mathbf{v}_1\mathbf{w}'_1 \end{aligned}$$

since $\mathbf{v}_1 \neq 0$, as v is generic in $V^+(2)$.

Similarly the map:

$$\operatorname{ad} v : \mathfrak{g}(-1) \longrightarrow V^+(1)$$

corresponds to the map

$$M(k,1,\mathbb{R}) \longrightarrow M(1,k,\mathbb{R})$$
$$\mathbf{A}_1 \longmapsto -\mathbf{v}_1{}^t\mathbf{A}_1$$

which again is obviously an isomorphism. And it follows easely that the diagram commutes.

We define a Lebesgue measure on the spaces $V^+(1)$, $V^-(-1)$, $\mathfrak{g}(1)$ *and* $\mathfrak{g}(-1)$, *which are all isomorphic to* $\mathbb{R}^k$, *by setting*

$$dw \text{ (resp. } dw',\ dA,\ dB) = (\sqrt{2})^k d\gamma \ ,$$

where $d\gamma = d\gamma_1 \dots d\gamma_k$ *if*

$$\gamma = (\gamma_1, \dots, \gamma_k) \in M(1,k,\mathbb{R}) \quad \text{or } \gamma = \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_k \end{pmatrix} \in M(k,1,\mathbb{R}) \ .$$

Since the relation $w = [A, v]$ is equivalent to $\mathbf{w}_1 = -\mathbf{v}_1{}^t\mathbf{A}_1$, it follows that, for example

$$\int_{V^+(1)} f(w)\, dw = |\mathbf{v}_1|^k \int_{\mathfrak{g}(-1)} f([A,v])\, dA \ .$$

Similarly as the relation $w' = [A, u']$ is equivalent to $\mathbf{w}'_1 = {}^t\mathbf{A}_1\mathbf{u}'_1$, it follows that

$$\int_{V^-(-1)} f(w')\, dw = |\det \mathbf{u}'_1| \int_{\mathfrak{g}(-1)} f([A,u'])\, dA \ .$$

Since $\mathbf{v}_1 = \Delta_k(v)$ and $\det \mathbf{u}'_1 = \nabla_1(u')$, we get the result of Proposition 4.3.1 with the constants α, β, γ and γ equal to 1.

A.4.4. A first normalization and its consequence

For the normalization made in **A.4.3**, the relations (4-4-1) are clearly satisfied. Moreover the relations (4-4-2) are also satisfied since, here, the involution σ associated to $I^+_{(1,\dots,1)}$ is equal to θ (see **A.3.2**). As we have seen in **A.4.3** that the constants appearing in the integral formulas of Proposition 4.3.1 are all equal to 1, Proposition 4.4.2 is proved in our case.

A.4.5. A second normalization and its consequence

The form b (**A.1.10**) induces a duality between the spaces $V^+(1)$ and $V^-(-1)$, and between the spaces $\mathfrak{g}(-1)$ and $\mathfrak{g}(1)$. More precisely we have for $w \in V^+(1)$, $w' \in V^-(-1)$, $\mathbf{w}_1 = (w^1, \dots, w^k)$ and $\mathbf{w}'_1 = (w'^1, \dots, w'^k)$

$$b(w,w') = \operatorname{Tr} ww' = 2\sum_{j=1}^{k} w^j w'^j \ .$$

Similarly for $A \in \mathfrak{g}(-1)$ and $B \in \mathfrak{g}(1)$ we get

$$b(A,B) = \operatorname{Tr} AB = 2\sum_{j=1}^{k} a^j b^j \quad \text{if } \mathbf{A}_1 = \begin{pmatrix} a^1 \\ \vdots \\ a^k \end{pmatrix},\ \mathbf{B}_1 = (b^1, \dots, b^k)\ .$$

It is now clear that with the normalization made in **A.4.3**, the pairs of measures (dw, dw') and (dA, dB) are dual for the Fourier transform and its inverse defined as in **A.4.1.** *i.e.* the relation (4-5-1) is satisfied.

A.4.6. Integral formulas on V^+ and V^-

We define a pair of measures $(du,\ du')$ on $V^+(0) \times V^-(0)$ and a pair of measures $(dv,\ dv')$ on $V^+(2) \times V^-(-2)$ by conditions similar to (N_1) and (N_2), as we did in **A.4.1**. If the pair of measures (dw, dw') on $V^+(1) \times V^-(-1)$ is normalized as in **A.4.3**, then an easy computation on the powers of $\sqrt{2}$ lead to

$$dX = dudwdv \quad \text{and} \quad dY = du'dw'dv'\ .$$

The integral formulas obtained in Theorem 4.6.2 are here an easy consequence of the explicit expression of the diffeomorphisms $\mathbf{\Phi}$ and $\mathbf{\Psi}$ obtained in **A.4.2**. As the jacobian of $\mathbf{\Phi}$ at the point $(\mathbf{u}_1, \mathbf{A}_1, \mathbf{v}_1)$ is equal to $|\mathbf{v}_1|^k$ we get for $\mathbf{f} \in L^1(\mathrm{Sym}(k+1,\mathbb{R}))$ associated to $f \in L^1(V^+)$

$$\begin{aligned}
&\int_{Sym(k+1,\mathbb{R})} \mathbf{f}(\mathbf{X})\ d\mathbf{X} \\
&= \int_{Sym(k,\mathbb{R})} \int_{\mathbb{R}^k} \int_{\mathbb{R}} \mathbf{f}\Big(\left(\begin{array}{c|c} \mathbf{v}_1 & {\mathbf{v}_1}^t \mathbf{A}_1 \\ \hline \mathbf{A}_1\mathbf{v}_1 & \mathbf{A}_1{\mathbf{v}_1}^t\mathbf{A}_1 + \mathbf{u}_1 \end{array}\right) \Big) |\mathbf{v}_1|^k d\mathbf{u}_1 d\mathbf{A}_1 d\mathbf{v}_1 \\
&= \int_{Sym(k,\mathbb{R})} \int_{\mathbb{R}} T_f^+(\mathbf{u}_1, \mathbf{v}_1) |\mathbf{v}_1|^{\frac{k}{1}}\ d\mathbf{u}_1 d\mathbf{v}_1\ ,
\end{aligned}$$

where

$$T_f^+(\mathbf{u}_1, \mathbf{v}_1) = |\mathbf{v}_1|^{\frac{k}{2}} \int_{\mathbb{R}^k} \mathbf{f}\Big(\left(\begin{array}{c|c} \mathbf{v}_1 & {\mathbf{v}_1}^t \mathbf{A}_1 \\ \hline \mathbf{A}_1\mathbf{v}_1 & \mathbf{A}_1{\mathbf{v}_1}^t\mathbf{A}_1 + \mathbf{u}_1 \end{array}\right) \Big) d\mathbf{A}_1\ .$$

Similarly, as the jacobian of $\mathbf{\Psi}$ at the point $(\mathbf{u'}_1, \mathbf{A}_1, \mathbf{v}'_1)$ is equal to $|\det \mathbf{u}'_1|$ we get for $\mathbf{g} \in L^1(\mathrm{Sym}(k+1,\mathbb{R}))$ associated to $g \in L^1(V^-)$

$$\int_{Sym(k+1,\mathbb{R})} \mathbf{g}(\mathbf{Y})\ d\mathbf{Y} = \int_{Sym(k,\mathbb{R})} \int_{\mathbb{R}} T_g^-(\mathbf{u}'_1, \mathbf{v}'_1) |\det \mathbf{u}'_1|^{\frac{1}{2}} d\mathbf{u'}_1 d\mathbf{v}'_1\ ,$$

where

$$T_g^-(\mathbf{u}'_1, \mathbf{v}'_1) = |\det(\mathbf{u'}_1)|^{\frac{1}{2}} \int_{\mathbb{R}^k} \mathbf{g}\Big(\left(\begin{array}{c|c} \mathbf{v}'_1 + {}^t\mathbf{A}_1\mathbf{u}'_1\mathbf{A}_1 & -{}^t\mathbf{A}_1\mathbf{u}'_1 \\ \hline -\mathbf{u}'_1\mathbf{A}_1 & \mathbf{u}'_1 \end{array}\right) \Big) d\mathbf{A}_1\ .$$

The formulas in Theorem 4.6.2 and Proposition 4.6.4 follow since $\mathbf{v}_1 = \Delta_k(v)$ and $\det(\mathbf{u}'_1) = \nabla_1(u')$.

A.4.7. Fourier transform of a quadratic character

Recall from Definition 4.7.3 that, if $(u',\, v) \in V^-(0)' \times V^-(-2)$, and if (p', q') is the signature of the quadratic form $Q_{u',v}$ on $\mathfrak{g}(-1)$ defined by

$$Q_{u',v}(A) = \frac{1}{2} b((\mathrm{ad}\ A)^2 u', v)\,,$$

then we set

$$\gamma(u', v) = e^{i\frac{\pi}{4}(p'-q')}\ .$$

Using the parametrizations introduced in **A.4.2**, the explicit expression of $\gamma(u', v)$ can be written as follows in the case of symmetric matrices. If $A \in \mathfrak{g}(-1)$ corresponds to $\mathbf{A_1} \in M(k, 1, \mathbb{R})$, if $u' \in V^-(0)'$ corresponds to $\mathbf{u_1'} \in M(k, \mathbb{R})$ $(\det(\mathbf{u_1'}) \neq 0)$ and if $v \in V^+(2)$ corresponds to $\mathbf{v_1} \in \mathbb{R}$, we get from **A.4.3**

$$\frac{1}{2} b((\mathrm{ad}\ A)^2 u', v) = -\frac{1}{2} b(\mathrm{ad}\, u' A, \mathrm{ad}\, v A) = \mathbf{v_1}{}^t \mathbf{A_1} \mathbf{u_1'} \mathbf{A_1}\ .$$

Therefore we obtain

$$\gamma(u',\ v) = e^{i\frac{\pi}{4}\mathrm{sg}(\mathbf{v}_1)(p-q)}\ ,$$

where (p, q) is the signature of the quadratic form associated to the symmetric matrix $\mathbf{u_1'}$ $(p + q = k)$.

The Weil character formula (Corollary 4.7.4) may be written here is follows. Let $(\mathbf{u}_1', \mathbf{v}_1) \in \mathbb{R}^k \times \mathbb{R}^*$ with $\det \mathbf{u}_1' \neq 0$. If (p, q) is the signature of the quadratic form associated to matrix $\mathbf{u_1'}$, then for $\phi \in \mathcal{S}(\mathbb{R}^k)$ we have

$$\int_{\mathbb{R}^k} (\mathcal{F}\phi)(\mathbf{v}_1{}^t \mathbf{A}_1 \mathbf{u}_1') e^{2i\pi \mathbf{v}_1{}^t \mathbf{A}_1 \mathbf{u}_1' \mathbf{A}_1}\, d\mathbf{A}_1$$
$$= e^{i\frac{\pi}{4}\mathrm{sg}(\mathbf{v}_1)(p-q)} |\mathbf{v}_1|^{-\frac{k}{2}} |\det \mathbf{u}_1'|^{-\frac{1}{2}} \int_{\mathbb{R}^k} \phi(\mathbf{A}_1) e^{-2i\pi \mathbf{v}_1{}^t \mathbf{A}_1 \mathbf{u}_1' \mathbf{A}_1}\, d\mathbf{A}_1\ ,$$

where $\mathcal{F}$ is the Fourier transform defined on $\mathbb{R}^k$ by

$$(\mathcal{F}\phi)(\mathbf{A}_1) = \int_{\mathbb{R}^k} \phi(\mathbf{A}_2) e^{2i\pi\, {}^t \mathbf{A}_1 \mathbf{A}_2}\, d\mathbf{A}_2\ ,$$

and the measure on $\mathbb{R}^k \simeq M(1, k, \mathbb{R})$ is normalized as in **A.4.3**.

A.4.8. A relation between $\mathbf{T}^-_{\mathcal{F}f}$ and T^+_f

The partial Fourier transform $\mathcal{F}_u$ with respect to the u variable is given here for a function $f \in \mathcal{S}(V^+)$ associated to $\mathbf{f} \in \mathcal{S}(\mathrm{Sym(k+1, \mathbb{R})}$ by

$$\mathcal{F}_u \mathbf{f}\Big(\left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \mathbf{u}_1' \end{array}\right) \Big) = \int_{\mathrm{Sym(k,\mathbb{R})}} \mathbf{f}\Big(\left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \mathbf{u}_1 \end{array}\right) \Big) e^{2i\pi\, \mathrm{Tr}(\mathbf{u}_1 \mathbf{u}'_1)}\, d\mathbf{u}_1\ ,$$

where $d\mathbf{u}_1 = (\sqrt{2})^{\frac{(k-1)k}{2}} \prod_{1\leq i,j\leq k} dx_{ij}$ for $\mathbf{u}_1 = (x_{ij})_{1\leq i,j\leq k}$. Similarly the (partial) Fourier transform $\mathcal{F}_v$ with respect to the v variable is given by

$$\mathcal{F}_v\mathbf{f}\Big(\left(\begin{array}{c|c} \mathbf{v}_1' & 0 \\ \hline 0 & 0 \end{array}\right)\Big) = \int_{\mathbb{R}} \mathbf{f}\Big(\left(\begin{array}{c|c} \mathbf{v}_1 & 0 \\ \hline 0 & 0 \end{array}\right)\Big)\, e^{2i\pi \mathbf{v}_1\mathbf{v}_1'}\, d\mathbf{v}_1 \ ,$$

where $d\mathbf{v}_1$ is the Lebesgue measure on $\mathbb{R}$.

The proof of the formula

$$\big(T^-_{\mathcal{F}f}\big)(u',\, v') = \mathcal{F}_v\Big(\gamma(u',\, v)\mathcal{F}_u T^+_f(u',\, v)\Big)(v')$$

is not easier to obtain in the case of the symmetric matrices than in the general case (see Theorem 4.8.1).

A.5. Functional equation of the Zeta function

There is no improvement or simplification of the proof or the result of the functional equation in the specific case of symmetric matrices. Let us just mention, that as $d = e = 1$, we have

$$\delta_0(z, \varepsilon, \eta) = (2\pi)^{-(z+1)} e^{-i\frac{\pi}{2}\varepsilon\eta(z+1)}\Gamma(z+1)\,.$$

As we have seen in sections **5.3** and **5.4**, this formula allows then the explicit computation of the coefficients $\delta_k(s, \varepsilon, \eta)$ and $\nu_k(s, \eta, \varepsilon)$.

A.7. Zeta function attached to a representation in the minimal spherical series

For $p \in \{0, \ldots, k+1\}$, we set $I_p^+ = I_\varepsilon^+$ and $\sigma_p = \sigma_\varepsilon$ with

$$\varepsilon = (\underbrace{1, \ldots, 1}_{k+1-p}, \underbrace{-1, \ldots, -1}_{p})\ .$$

Then Ω_p^+ and Ω_p^- denote the G–orbits of respectively I_p^+ and $I_p^- = \theta(I_p^+)$ where $G \simeq GL(k+1, \mathbb{R})^+$ (eventually quotiented by $\pm\mathrm{Id}$). The group $H_p = Z_G(I_p^+)$ is open in G^{σ_p} and isomorphic to $SO(k+1-p, p)$ eventually quotiented by $\pm\mathrm{Id}$ (see **A.3.4**). The G–orbit $\mathbf{\Omega}_p$ of $\mathbf{I}_p$ in $\mathrm{Sym}(k+1, \mathbb{R})$ is the set of symmetric matrices of signature $(k+1-p, p)$ and we have here

$$\Omega_p^+ \simeq \Omega_p^- \simeq GL(k+1, \mathbb{R})^+ / SO(k+1-p, p)$$

A.7.1 The minimal spherical principal series

Recall that we have here

$$\mathfrak{a} = \mathfrak{a}^{\mathfrak{p}} = \mathfrak{a}^0 = \sum_{j=0}^{k} \mathbb{R} H_{\lambda_j}\ .$$

As $\mathfrak{a} = \mathfrak{a}^0 \subset \{X \,|\, \sigma(X) = -X\}$ (for any involution $\sigma = \sigma_p$) we have

$$\mathfrak{h}_p \cap \mathfrak{a} = \{0\}\ .$$

Hence for any $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ the first condition of van den Ban (namely $\lambda_{|\mathfrak{h}_p\cap\mathfrak{a}} = 0$) is always fullfilled.

On the other hand, as $M/M\cap H=\{e\}$, the second condition of van den Ban (condition b') in section 7.1. is not needed here: any $(M\cap H)$–spherical irreducible representation of M is trivial.

Therefore the representations of the minimal spherical principal are parametrized here by $\lambda\in\mathfrak{a}_{\mathbb{C}}^*$ and denoted $(\pi_\lambda,\mathcal{H}_\lambda)$. They are obtained as follows.

Recall that

$$\mathfrak{a}=\left\{H=\begin{pmatrix}\mathbf{H}&0\\0&-\mathbf{H}\end{pmatrix}\text{ with }\mathbf{H}=\begin{pmatrix}t_k&&&\\&t_{k-1}&&\\&&\ddots&\\&&&t_0\end{pmatrix}\in\mathfrak{gl}(k+1,\mathbb{R})\right\}\ .$$

Then we have for an element $H\in\mathfrak{a}$

$$\begin{aligned}\rho(H)=\frac{1}{2}\operatorname{tr}(\operatorname{ad}H_{|\mathfrak{n}})&=\frac{1}{2}\sum_{0\le i<j\le k}(t_i-t_j)\\&=\frac{1}{2}(kt_0+(k-2)t_1+\cdots-kt_k)\ .\end{aligned}$$

Therefore if $\mathbf{a}=\begin{pmatrix}a_k&0&0\\0&\ddots&0\\0&0&a_0\end{pmatrix}$ with $a_j>0$ is an element in $\mathbf{A}$ we get

$$a^\rho=\prod_{j=0}^k a_j^{\frac{k-2j}{2}}\ .$$

Similarly, if $\lambda\in\mathfrak{a}_{\mathbb{C}}^*$ is defined by

$$\lambda\Big(\begin{pmatrix}t_k&&&\\&t_{k-1}&&\\&&\ddots&\\&&&t_0\end{pmatrix}\Big)=\sum_{j=0}^k\lambda_jt_j$$

then we get

$$a^\lambda=\prod_{j=0}^k a_j^{\lambda_j}.$$

For $\lambda\in\mathfrak{a}_{\mathbb{C}}^*$, the representation π_λ is defined by the right regular action on $\mathcal{C}_\lambda(G)$ where $\mathcal{C}_\lambda(G)$ is the space of continuous functions $\Phi:G\longrightarrow\mathbb{C}$ such that

$$\Phi(mang)=a^{\lambda+\rho}\Phi(g)\quad\text{ for }g\in G,\ m\in M,\ a\in A,\ n\in N\ ,$$

and is extented to the Hilbert space $\mathcal{H}_\lambda$ obtained by completion with respect to the scalar product

$$(\Phi_1,\Phi_2)=\int_K\Phi_1(k)\overline{\Phi_2(k)}\,dk\ ,$$

where $K\simeq SO(k+1,\mathbb{R})$ is the maximal compact subgroup of G.

Therefore the set of representations $(\pi_\lambda,\mathcal{H}_\lambda)$ $(\lambda\in\mathfrak{a}_{\mathbb{C}}^*)$ is in fact the set of K–spherical principal series of $GL(k+1,\mathbb{R})^+$ (see Remark 7.1.7).

Remark. — We dot not continue to investigate the specific case of symmetric matrices, because the sequel would not be more simpler than the general Type I case treated in sections 7.3-7.9.

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