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Gracinda M. S. Gomes†, Jean-Eric Pin‡ and Helena Sezinando†

ggomes@cii.fc.ul.pt, Jean-Eric.Pin@liafa.jussieu.fr,
mhelena@lmc.fc.ul.pt

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Abstract

In this paper, we first consider $n \times n$ upper-triangular matrices with entries in a given semiring k. Matrices of this form with invertible diagonal entries form a monoid $B_n(k)$. We show that $B_n(k)$ splits as a semidirect product of the monoid of unitriangular matrices $U_n(k)$ by the group of diagonal matrices. When the semiring is a field, $B_n(k)$ is actually a group and we recover a well-known result from the theory of groups and Lie algebras. Pursuing the analogy with the group case, we show that $U_n(k)$ is the ordered set product of $n(n-1)/2$ commutative monoids (the root subgroups in the group case). Finally, we give two different presentations of the Schützenberger product of n groups G_1, $\ldots$, G_n, given a monoid presentation $\langle A_i \mid R_i \rangle$ of each group G_i. We also obtain as a special case presentations for the monoid of all $n \times n$ unitriangular Boolean matrices.

There is a huge literature on presentations of groups, see for example [6, 15]. In particular, presentations are known for virtually every "classical" group and presentations arise naturally in such areas as knot theory, topology and geometry. For monoids, presentations play a crucial role in decidability problems, but although presentations of a number of "classical" monoids are known, the catalog is far from being complete. Computing presentations of a given monoid might also be crucial for the study of its structural complexity.

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†Centro de Álgebra da Universidade de Lisboa, 1649-003 Lisboa, Portugal and Departamento de Matemática, Faculdade de Ciências da Universidade de Lisboa, 1749-016, Lisboa, Portugal.

‡LIAFA, Université Paris VII and CNRS, Case 7014, 2 Place Jussieu, 75251 Paris Cedex 05, France.

An example of this situation occurred for instance in the study of the monoid of all injective order-preserving partial transformations on a chain [7, 9]. Another motivation for studying presentations is the advent of softwares for symbolic computations like GAP [10]. Providing algorithms to compute presentations of given monoids is a great help for the developers of these softwares.

In this paper, we study the Schützenberger product of n groups G_1, $\ldots$, G_n, each of which is given by a monoid presentation $\langle A_i \mid R_i \rangle$. The Schützenberger product is an operation on monoids that was originally introduced for solving questions in automata theory.

The Schützenberger product is a submonoid of the monoid of $n \times n$ upper-triangular matrices with entries in the semiring of finite subsets of the group $G_1 \times \cdots G_n$. This lead us to first consider monoids of triangular matrices over an arbitrary semiring. These monoids would actually deserve a systematic investigation and our paper is just a first step into this direction. Let us briefly mention the motivation for such a study.

First, representation by triangular matrices over fields is a classical topic in Lie algebras [4, 5] and group theory [11, 24], and it is also important to semigroup theory. For instance Okniński has given in [20, Section 4.4] a triangularizability criterion for semigroups of matrices. This result was recently used by Almeida, Margolis and Volkov [1] to show that finite semigroups triangularizable over a fixed finite field form a variety of finite semigroups and to provide a finite basis of identities for it.

Next, the theory of automata gives strong motivation to study triangular representations over semirings as well. For instance, it is known that monoids of upper-triangular matrices over the Boolean semiring are intimately related with the Straubing-Thérien's hierarchy of recognizable languages. The first level of this hierarchy is the variety of piecewise testable languages, and a celebrated result of Simon [27] states that the corresponding variety of finite monoids is the variety of $\mathcal{J}$-trivial monoids. As a consequence, Straubing [28], (see also [21, p. 85]) proved that a finite monoid is $\mathcal{J}$-trivial if and only if it divides a monoid of unitriangular Boolean matrices. The second level of this hierarchy is also a variety of languages and Pin and Straubing [22] have shown that the corresponding variety of monoids is generated by the monoids of upper-triangular Boolean matrices. Finally, a result of Reutenauer [23] states that the syntactic algebra of a rational formal power series on a field k is triangularizable if and only if the series belongs to the subalgebra of rational series generated by the letters and the series of rank 1.

Let us come back to our results. In this paper, we consider $n \times n$ upper-triangular matrices with entries in a given semiring k. Matrices of this form with invertible diagonal entries form a monoid $B_n(k)$. We show that $B_n(k)$ splits as a semidirect product of the monoid of unitriangular matrices $U_n(k)$ by the group of diagonal matrices. When the semiring is a field, $B_n(k)$

is actually a group and we recover a well-known result from the theory of groups and Lie algebras. Pursuing the analogy with the group case, we show that $U_n(k)$ is the ordered set product of $n(n-1)/2$ commutative monoids, which are the root subgroups in the group case.

Next we give two different presentations of the Schützenberger product of n groups $G_1, \ldots, G_n$, each of which is given by a monoid presentation $\langle A_i \mid R_i \rangle$.

As a corollary, corresponding to the case where all groups are trivial, we obtain a simple presentation for the monoid U_n of all unitriangular Boolean matrices of order n. Our presentation for U_n has a strong combinatorial flavour, somewhat reminiscent of the presentation of the plactic monoid [14, 13]. It would be interesting to know whether our relations have a combinatorial interpretation, in the same way as Young tableaux are the combinatorial counterpart of the plactic monoid.

1 The Schützenberger product

One of the most useful tools for studying the concatenation product is the *Schützenberger product* of n monoids, which was originally defined by Schützenberger for two monoids [26], and extended by Straubing [29] for any number of monoids.

Given a monoid M, the set of *finite* subsets of M, denoted $\mathcal{P}_f(M)$, is a semiring under union as addition and the product of subsets as multiplication, defined by $XY = \{xy \mid x \in X \text{ and } y \in Y\}$, for all $X, Y \subseteq M$. Unless otherwise specified, we shall use in the sequel the word "subset" for "finite subset" without any further warning. We also identify the singleton $\{x\}$ and the element x.

Let $M_1, \ldots, M_n$ be monoids and let $M = M_1 \times \cdots \times M_n$. For $1 \leq i, j \leq n$, let

$$M_{i,j} = \begin{cases} 1 \times \cdots \times 1 \times M_i \times \cdots \times M_j \times 1 \times \cdots \times 1 & \text{if } i \leq j \\ \{(1, 1, \ldots, 1)\} & \text{if } j < i \end{cases}$$

Let k be the semiring $\mathcal{P}_f(M)$ and $\mathcal{M}_n(k)$ be the semiring of square matrices of size n with entries in k. The Schützenberger product of $M_1, \ldots, M_n$, denoted by $\diamondsuit_n(M_1, \ldots, M_n)$, is the submonoid of the multiplicative monoid $\mathcal{M}_n(k)$ composed of all the matrices P satisfying the three following conditions:

(1) If $i > j$, $P_{i,j} = 0$;

(2) If $1 \leq i \leq n$, $P_{i,i} = \{(1, \ldots, 1, s_i, 1, \ldots, 1)\}$ for some $s_i \in M_i$;

(3) If $1 \leq i < j \leq n$, $P_{i,j} \subseteq M_{i,j}$.

Condition (1) indicates that the matrices of the Schützenberger product are upper triangular, condition (2) enables one to identify the diagonal coefficient $P_{i,i}$ with an element s_i of M_i and condition (3) shows that if $i < j$,

$P_{i,j}$ can be identified with a subset of $M_{i,j}$. With this convention, a matrix of $\diamondsuit_3(M_1, M_2, M_3)$ will have the form

$$\begin{pmatrix} s_1 & P_{1,2} & P_{1,3} \\ 0 & s_2 & P_{2,3} \\ 0 & 0 & s_3 \end{pmatrix}$$

with $s_i \in M_i$ for $1 \leq i \leq 3$, $P_{1,2} \subseteq M_{1,2}$, $P_{1,3} \subseteq M_{1,3}$ and $P_{2,3} \subseteq M_{2,3}$.

Notice that the Schützenberger product is not associative, in the sense that in general the monoids $\diamondsuit_2(M_1, \diamondsuit_2(M_2, M_3))$, $\diamondsuit_2(\diamondsuit_2(M_1, M_2), M_3)$ and $\diamondsuit_3(M_1, M_2, M_3)$ are pairwise distinct.

2 Presentations

For completion, we start by recalling some basic definitions and well-known results about monoid presentations. In the sequel, we denote by A^* (respectively $FG(A)$) the free monoid (respectively free group) over A.

A *monoid presentation* is a pair $\langle A \mid R \rangle$, where A is an alphabet and R is a subset of $A^* \times A^*$. The elements of A are called *generators* and the ones of R *relations*. The monoid *presented by* $\langle A \mid R \rangle$ is the quotient of the free monoid A^* by the congruence $\sim_R$ generated by R. In other words, it is the monoid generated by the set A submitted to the relations R. This intuitive meaning is suggested by the notation. Indeed $\langle X \rangle$ traditionally denotes the monoid generated by a set X and the vertical bar used as a separator can be interpreted as "such that", as in a definition like $\{n \in \mathbb{N} \mid \text{n is prime}\}$.

By extension, a monoid is said to be *defined by a presentation* $\langle A \mid R \rangle$ if it is isomorphic to the monoid presented by $\langle A \mid R \rangle$. Usually, we write $u = v$ instead of $(u, v) \in R$.

There is a corresponding notion of *group presentation*. This is a pair $\langle A \mid R \rangle$, where A is an alphabet and R is a subset of $FG(A)$. The group *presented by* $\langle A \mid R \rangle$ is the quotient of the group $FG(A)$ by the congruence generated by the relations $w = 1$, for $w \in R$. In fact, it is easy to pass from a group presentation to a monoid presentation defining the same group. Indeed, if $\langle A \mid R \rangle$ is a group presentation, let $\bar{A} = \{\bar{a} \mid a \in A\}$ be a disjoint copy of A, and let $\tilde{A}$ be the disjoint union of A and $\bar{A}$. The letters of $\bar{A}$ will represent formal inverses of the letters of A. Now, since every word of $FG(A)$ is the product of elements of A and inverses of elements of A, it can be identified with a word of $\tilde{A}^*$. Therefore, the set R can be identified with a subset of $\tilde{A}^* \times \tilde{A}^*$, and the monoid presentation $\langle \tilde{A} \mid (R \times \{1\}) \cup \{(a\bar{a}, 1), (\bar{a}a, 1) \mid a \in A\} \rangle$ defines the same group as the group presentation $\langle A \mid R \rangle$. In the sequel, we shall always assume that groups are given by monoid presentations.

Let M be a monoid and let $\varphi : A^* \mapsto M$ be a surjective morphism and $R \subseteq A^* \times A^*$ a set of relations. In general it is undecidable to check whether

$\langle A \mid R\rangle$ is a presentation of M, but at least two standard techniques are commonly used, both first introduced in group theory. The first one is the *Guess and Prove method*, described in the next proposition (several examples of this technique can be found for instance in [25]).

Proposition 2.1 *Suppose there exists a subset W of A^* satisfying the following conditions:*

(1) *for each relation $(u, v) \in R$, $\varphi(u) = \varphi(v)$,*
(2) *for each word $w \in A^*$, there exists a word $w' \in W$ such that the relation $w = w'$ is a consequence of R,*
(3) *the restriction of φ to W is one-to-one.*

Then, M is defined by the presentation $\langle A \mid R\rangle$.

An important consequence of the Guess and Prove method is the following corollary.

Corollary 2.2 *Let M be a monoid defined by a presentation $\langle A \mid R\rangle$ and let R' be a set of relations on A^*. Suppose that M satisfies all the relations of R' and that each relation in R is a consequence of R'. Then $\langle A \mid R'\rangle$ is a presentation of M.*

The second method consists in using *Tietze transformations*. Indeed, a standard result in group theory states that, given a finite group presentation for a group, any other finite group presentation of that group can be obtained from the given presentation by applying the so-called Tietze transformations [15]. This result, however, does not fully extend to the monoid case, but the following weak version still holds.

Proposition 2.3 *Given a presentation $\langle A \mid R\rangle$ for a monoid M, a new presentation for M can be obtained by a repeated application of the following transformations:*

(1) *If the words $u, v \in A^*$ are $\sim_R$-equivalent, add the pair (u, v) to R;*
(2) *If the words $u, v \in A^*$ are $\sim_{R'}$-equivalent, where $R' = R \setminus \{(u, v)\}$, replace R by R';*
(3) *If u is any word of A^*, add a new letter c to A and add the pair (u, c) to R;*
(4) *If R contains a relation of the form (a, x), where $a \in A$ and x is a word containing no occurrence of a, delete a from A and replace every pair (u, v) of R by (u', v'), where u' (resp. v') is obtained from u (resp. v) by substituting x for each occurrence of a.*

We now consider the following situation: $G_1, \ldots, G_n$ are groups with monoid presentations $\langle A_1 \mid R_1\rangle$, $\langle A_2 \mid R_2\rangle$, $\ldots \langle A_n \mid R_n\rangle$, respectively. We shall assume that the A_i's are pairwise disjoint.

The main aim of this paper is to provide a presentation for the Schützenberger product of $G_1, \ldots, G_n$, that will be denoted by S in the rest of the paper. We shall actually propose two solutions for this problem.

3 Decompositions of upper-triangular matrices

In this section, we establish a general decomposition result on monoids of upper-triangular matrices over a semiring, which extends a classical result of group theory on upper-triangular matrices over a field.

Let k be a commutative semiring with identity. Recall that a matrix is *diagonal* if all its nondiagonal entries are 0 and *unitriangular* if it is upper triangular and its diagonal entries are all equal to 1, the identity of the semiring k.

Recall that we denote by $B_n(k)$ the monoid of $n \times n$ upper-triangular matrices with invertible elements on the diagonal. For instance, if G is a group and k is the semiring $\mathcal{P}_f(G)$, the monoid $B_n(k)$ consists of the upper-triangular matrices whose diagonal entries are singletons. Note that a matrix in $B_n(k)$ is not in general invertible: for instance, if k is the Boolean semiring, the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is not invertible.

Let $D_n(k)$ be the group of all diagonal matrices of $B_n(k)$ and let

$$\pi : B_n(k) \to D_n(k)$$

be the monoid morphism which maps a matrix m onto the diagonal matrix having the same diagonal entries as m. For instance, if $k = \mathcal{P}_f(G)$, the group $D_n(k)$ is isomorphic to G^n, since the only invertible elements of $\mathcal{P}_f(G)$ are the singletons.

The monoid $\pi^{-1}(1)$ is the monoid $U_n(k)$ of all *unitriangular* matrices of $B_n(k)$. If k is a field, all monoids considered so far are groups and $B_n(k)$ splits as a semidirect product of $U_n(k)$ and $D_n(k)$.

If k is a semiring, a similar result holds, but we shall state it in a more general setting in order to cover the case of the Schützenberger products. We say that a submonoid M of $B_n(k)$ is *splittable* if $\pi(M)$ is a group equal to the set of all diagonal matrices in M. In other words, we require M to project onto $M \cap D_n(k)$ by π and the inverse of a matrix in $\pi(M)$ to be also in $\pi(M)$. Observe, in particular, that the Schützenberger product of n groups $G_1, \ldots, G_n$ is splittable. Indeed, $\diamondsuit_n(G_1, \ldots, G_n)$ is a submonoid of $B_n(k)$, where $k = \mathcal{P}_f(G)$ and $G = G_1 \times \cdots \times G_n$. Furthermore, in this case, $\pi(M)$ is isomorphic to G.

We are now ready to state our decomposition results. Let M be a splittable submonoid of $B_n(k)$, let G be the group $\pi(M)$ and let U be the monoid $M \cap U_n(k)$.

Proposition 3.1 *Every element of M admits a unique decomposition as the product of a diagonal matrix in G by a unitriangular matrix in U.*

Proof. Let m be an element of M. Then a decomposition of the required form is $m = ud$ where $d = \pi(m)$ is the diagonal matrix having the same diagonal entries as m and $u = md^{-1}$. The condition that M is splittable ensures that both d and d^{-1} belong to M. Therefore u belongs to U.

Suppose now that $u_1d_1 = u_2d_2$, where d_1, d_2 are diagonal and u_1, u_2 are unitriangular. Then $\pi(u_1d_1) = \pi(u_2d_2)$, whence $\pi(d_1) = \pi(d_2)$ since $\pi(u_1) = \pi(u_2) = 1$ and thus $d_1 = d_2$ since the restriction of π to diagonal matrices is the identity. As diagonal matrices are invertible, it follows also that $u_1 = u_2$. Therefore the decomposition is unique. □

Theorem 3.2 *The monoid M is isomorphic to the semidirect product $U*G$, where G acts on U by conjugacy.*

Proof. Let $m, m' \in M$ and let $m = ud$ and $m' = u'd'$ be their unique decompositions given by Proposition 3.1, with $d, d' \in G$ and $u, u' \in U$. In the semidirect product $U*G$, the product $(u, d)(u', d')$ is equal to $(udu'd^{-1}, dd')$, which is indeed the unique decomposition of mm' in $U \times G$. Thus M is clearly isomorphic to $U * G$. □

4 Decompositions of unitriangular matrices

In this section, we give further decomposition results for the monoid $U_n(k)$, which again extend some well-known results on unitriangular matrices over a field.

Let us define an order $\leq$ on the set

$$I = \{(i, j) \mid 1 \leq i < j \leq n\}$$

by setting $(i, j) \leq (k, \ell)$ if and only if $j > \ell$ or $j = \ell$ and $i \leq k$. Thus we have

$$(1, n) < (2, n) < \cdots < (n-1, n) < (1, n-1) < \cdots < (n-2, n-1) < \cdots < (1, 2)$$

If k is a field, the group $U_n(k)$ is known to be generated by the unitriangular matrices with at most one nonzero off-diagonal entry. In fact, for each $(i, j) \in I$, the set of all unitriangular matrices with at most one nonzero entry in position (i, j) is a commutative group, called a *root group*, and the set product of these groups (in increasing order) is equal to $U_n(k)$.

This is the field analogue to our Lemma 4.1 and Proposition 4.2. However, in the semiring case, the analogue of the root groups are just commutative monoids.

Given an element x of k, define, for each $(i,j) \in I$, the *elementary matrices* $e_{i,j,x}$ and $c_{i,j,x} = 1 + e_{i,j,x}$

$$e_{i,j,x} = \begin{pmatrix} 0 & & & & 0 \\ & \ddots & & & \\ & & 0 & x & \\ & & & \ddots & \\ 0 & & & & 0 \end{pmatrix} \qquad c_{i,j,x} = \begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & 1 & x & \\ & & & \ddots & \\ 0 & & & & 1 \end{pmatrix}$$

where the x entry is of course in the position (i,j).

Denote by $R_{i,j}$ the set of matrices of the form $c_{i,j,x}$ for some $x \in k$. Now, observing that, for any $x, y \in k$, we have $c_{i,j,x} = 1 + e_{i,j,x}$ and $e_{i,j,x}e_{i,j,y} = 0$, the following lemma is easily proved.

Lemma 4.1 *For each $(i,j) \in I$ and for any $x, y \in k$,*

$$c_{i,j,x} + c_{i,j,y} = c_{i,j,x+y} = c_{i,j,x}c_{i,j,y} = c_{i,j,y}c_{i,j,x}$$

and the set $R_{i,j}$ forms a commutative submonoid of $U_n(k)$.

By analogy with the field case, we shall call the monoids $R_{i,j}$ the *root monoids*. We now show that the monoid $U_n(k)$ is equal to the set product $\prod_{(i,j)\in(I,\leq)} R_{i,j}$.

Proposition 4.2 *Every unitriangular matrix m can be decomposed as an ordered product*

$$\prod_{(i,j)\in(I,\leq)} c_{i,j,m_{i,j}}$$

Proof. A trivial but useful observation is that if $(i,j),(k,\ell) \in I$ and $j \neq k$ (in particular if $(i,j) < (k,\ell)$), then $e_{i,j,x}e_{k,\ell,y} = 0$ for every $x, y \in k$. Now, let m be a unitriangular matrix defined by $m = 1 + \sum_{(i,j)\in I} e_{i,j,m_{i,j}}$. Then we have

$$m = \prod_{(i,j)\in(I,\leq)} (1 + e_{i,j,m_{i,j}}) = \prod_{(i,j)\in(I,\leq)} c_{i,j,m_{i,j}} \qquad \square$$

For instance,

$$\begin{pmatrix} 1 & m_{1,2} & m_{1,3} & m_{1,4} \\ 0 & 1 & m_{2,3} & m_{2,4} \\ 0 & 0 & 1 & m_{3,4} \\ 0 & 0 & 0 & 1 \end{pmatrix} = c_{1,4,m_{1,4}}c_{2,4,m_{2,4}}c_{3,4,m_{3,4}}c_{1,3,m_{1,3}}c_{2,3,m_{2,3}}c_{1,2,m_{1,2}}$$

5 Decompositions of matrices

In this section, we return to the Schützenberger product $S = \diamondsuit_n(G_1, \ldots, G_n)$ of n groups $G_1, \ldots, G_n$. We complete the results of Sections 3 and 4 by giving a few elementary results on matrices that will help us with the computations of the next sections.

Since the diagonal matrices play a special role, it is convenient to identify, for $1 \leq i \leq n$, each element $g \in G_i$ with the diagonal matrix

$$d_{i,g} = \begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & g & & \\ & & & \ddots & \\ 0 & & & & 1 \end{pmatrix}$$

where the g entry is in position (i, i). As in Section 4, we define for $(i, j) \in I$ and $P \subseteq G_{i,j}$, the *elementary matrices* $e_{i,j,P}$ and $c_{i,j,P} = 1 + e_{i,j,P}$

$$e_{i,j,P} = \begin{pmatrix} 0 & & & & 0 \\ & \ddots & & & \\ & & 0 & P & \\ & & & \ddots & \\ 0 & & & & 0 \end{pmatrix} \qquad c_{i,j,P} = \begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & 1 & P & \\ & & & \ddots & \\ 0 & & & & 1 \end{pmatrix}$$

where the P entry is in the position (i, j). When P is a singleton $\{p\}$, we simplify the notations $e_{i,j,\{p\}}$ and $c_{i,j,\{p\}}$ to $e_{i,j,p}$ and $c_{i,j,p}$ respectively. We also define, for each pair $(i, j) \in I$, a (possibly infinite) alphabet

$$C_{i,j} = \{c_{i,j,p} \mid p \in G_{i,j}\}$$

Finally, we set

$$C = \bigcup_{(i,j) \in I} C_{i,j} \quad \text{and} \quad D = \{d_{i,g} \mid 1 \leq i \leq n, g \in A_i\}$$

Note that the one-to-one map $g \mapsto d_{i,g}$ from A_i into D allows one to identify A_i with a subset of D. We shall implicitly use this identification in the sequel without any further warning. In particular, $D = A_1 \cup \ldots \cup A_n$.
It is well known that given a diagonal matrix $d_{i,g}$, the result of multiplying a matrix m by $d_{i,g}$ on the left is the matrix deduced from m by multiplying the i-th row by g on the left.

$$\begin{pmatrix} 1 & & 0 & & 0 \\ & \ddots & & & \\ 0 & & g & & 0 \\ & & & \ddots & \\ 0 & & 0 & & 1 \end{pmatrix} \begin{pmatrix} m_{1,1} & \ldots & m_{1,n} \\ \vdots & \ddots & \vdots \\ m_{n,1} & \ldots & m_{n,n} \end{pmatrix} = \begin{pmatrix} m_{1,1} & \ldots & m_{1,n} \\ gm_{i,1} & \ldots & gm_{i,n} \\ m_{n,1} & \ldots & m_{n,n} \end{pmatrix}$$

Similarly, the result of multiplying a matrix m by $d_{i,g}$ on the right is the matrix deduced from m by multiplying the i-th column by g on the right.

$$\begin{pmatrix} m_{1,1} & \dots & m_{1,n} \\ \vdots & \ddots & \vdots \\ m_{n,1} & \dots & m_{n,n} \end{pmatrix} \begin{pmatrix} 1 & & 0 & & 0 \\ & \ddots & & & \\ 0 & & g & & 0 \\ & & & \ddots & \\ 0 & & 0 & & 1 \end{pmatrix} = \begin{pmatrix} m_{1,1} & m_{1,i}g & m_{1,n} \\ \vdots & \vdots & \vdots \\ m_{n,1} & m_{n,i}g & m_{n,n} \end{pmatrix}$$

For an elementary matrix $e_{i,j,P}$, the result of multiplying a matrix m by $e_{i,j,P}$ on the left is the matrix where the i-th row is the j-th row of m multiplied by P on the left and all other entries are 0.

$$\begin{pmatrix} 0 & & & & 0 \\ & \ddots & & & \\ & & 0 & P & \\ & & & \ddots & \\ 0 & & & & 0 \end{pmatrix} \begin{pmatrix} m_{1,1} & \dots & m_{1,n} \\ \vdots & \ddots & \vdots \\ m_{n,1} & \dots & m_{n,n} \end{pmatrix} = \begin{pmatrix} 0 & \dots & 0 \\ Pm_{j,1} & \dots & Pm_{j,n} \\ 0 & \dots & 0 \end{pmatrix}$$

Similarly, the result of multiplying a matrix m by $e_{i,j,P}$ on the right is the matrix where the j-th column is the i-th column of m multiplied by P on the right and all other entries are 0.

$$\begin{pmatrix} m_{1,1} & \dots & m_{1,n} \\ \vdots & \ddots & \vdots \\ m_{n,1} & \dots & m_{n,n} \end{pmatrix} \begin{pmatrix} 0 & & & & 0 \\ & \ddots & & & \\ & & 0 & P & \\ & & & \ddots & \\ 0 & & & & 0 \end{pmatrix} = \begin{pmatrix} 0 & m_{1,i}P & 0 \\ \vdots & \vdots & \vdots \\ 0 & m_{n,i}P & 0 \end{pmatrix}$$

Denote by U the submonoid of all unitriangular matrices of S. The submonoid of all diagonal matrices can be identified with the group $G = G_1 \times \cdots \times G_n$. Since S is a splittable monoid, Theorem 3.2 can be directly applied to it.

Corollary 5.1 *The monoid S is isomorphic to the semidirect product $U*G$, where G acts on U by conjugacy.*

The next results are the counterpart of Lemma 4.1 and Proposition 4.2 for the Schützenberger product of n groups. Although they cannot be formally derived from these analoguous statements, because of the restrictions on the entries in the definition of a Schützenberger product, their proofs are exactly the same, and are therefore omitted.

Lemma 5.2 *For any subsets P, Q of $G_{i,j}$,*

$$c_{i,j,P} + c_{i,j,Q} = c_{i,j,P+Q} = c_{i,j,P}c_{i,j,Q} = c_{i,j,Q}c_{i,j,P}$$

In particular, for a given subset P of $G_{i,j}$, the elements of the form $c_{i,j,p}$, with $p \in P$, are idempotents and commute. Furthermore,

$$c_{i,j,P} = \sum_{p \in P} c_{i,j,p} = \prod_{p \in P} c_{i,j,p}$$

Proposition 5.3 *Every unitriangular matrix can be written as*

$$\prod_{(i,j) \in (I,\leq)} c_{i,j,P_{i,j}}$$

where $P_{i,j} \subseteq G_{i,j}$, for each $(i,j) \in I$.

For the Schützenberger product of n groups, the root monoids are idempotent and are thus semilattices. Therefore, the following result holds:

Proposition 5.4

(1) *The monoid U is idempotent generated,*

(2) *U is equal to the ordered product of the semilattices $\mathcal{P}_f(G_{i,j})$, with $(i,j) \in I$.*

Proof. (1) In view of Proposition 5.3 and Lemma 5.2, it only remains to show that all the idempotents of S are in U. Let m be an idempotent. Then $\pi(m)$ is an idempotent in G and thus equal to 1, which means that m is unitriangular.

(3) The result follows from Proposition 5.3 since, by Lemma 5.2, the lattices $\{c_{i,j,P} \mid P \in \mathcal{P}_f(G_{i,j})\}$ and $\mathcal{P}_f(G_{i,j})$ are isomorphic. □

6 A first presentation

In this section, we give a first presentation for the Schützenberger product. The first thing to do is to find a reasonable set of generators.

Proposition 6.1 *The set $C \cup D$ is a set of generators for S.*

Proof. Since S is splittable, Proposition 3.1 shows that every element of S is the product of a diagonal matrix by a unitriangular matrix. Now, since for each i, the set A_i generates G_i, each diagonal matrix is a product of matrices of D. Furthermore, by Proposition 5.3, every unitriangular matrix can be written in the form

$$\prod_{(i,j) \in (I,\leq)} c_{i,j,P_{i,j}}$$

and, by Lemma 5.2, each $c_{i,j,P_{i,j}}$ is a product of elements of C. □

Proposition 6.1 provides a set of generators for S. In order to obtain a presentation, we look for the relations among these generators. We start by presenting some slightly more general relations concerning $c_{i,j,P}$'s and then others involving $d_{i,g}$'s.

Lemma 6.2 *The following relations hold in S:*

(1) $c_{i,j,P}c_{i,j,Q} = c_{i,j,P+Q}$, *for* $(i,j) \in I$ *and* $P, Q \subseteq G_{i,j}$;
(2) $c_{i,j,P}^2 = c_{i,j,P}$, *for* $(i,j) \in I$ *and* $P \subseteq G_{i,j}$;
(3) $c_{i,j,P}c_{k,\ell,Q} = c_{k,\ell,Q}c_{i,j,P}$, *for* $(i,j) \in I$, $P \subseteq G_{i,j}$, $(k,\ell) \in I$, $Q \subseteq G_{k,\ell}$, $k \neq j$ *and* $\ell \neq i$;
(4) $c_{i,j,P}c_{j,k,Q} = c_{i,k,PQ}c_{j,k,Q}c_{i,j,P}$, *for* $1 \leq i < j < k \leq n$, $P \subseteq G_{i,j}$ *and* $Q \subseteq G_{j,k}$.

Proof. First observe that, if $k \neq j$ and $\ell \neq i$, the products $e_{i,j,P}e_{k,\ell,Q}$ and $e_{k,\ell,Q}e_{i,j,P}$ are the zero matrix. In particular, $e_{i,j,P}e_{i,j,Q}$ is always the zero matrix and thus

$$c_{i,j,P}c_{i,j,Q} = (1 + e_{i,j,P})(1 + e_{i,j,Q}) = 1 + e_{i,j,P} + e_{i,j,Q} = 1 + e_{i,j,P+Q}$$

giving the first relation. The second relation is a special case of the first one. For the third relation, observe that

$$c_{i,j,P}c_{k,\ell,Q} = (1 + e_{i,j,P})(1 + e_{k,\ell,Q}) = 1 + e_{i,j,P} + e_{k,\ell,Q} + e_{i,j,P}e_{k,\ell,Q}$$

and thus $c_{i,j,P}c_{k,\ell,Q} = 1 + e_{i,j,P} + e_{k,\ell,Q} = c_{k,\ell,Q}c_{i,j,P}$.

A similar argument gives the fourth relation. □

Lemma 6.3 *The following relations hold in S:*

(1) $d_{i,g}d_{j,h} = d_{j,h}d_{i,g}$, *for* $(i,j) \in I$, $g \in G_i$ *and* $h \in G_j$;
(2) $d_{i,g}c_{j,k,P} = c_{j,k,P}d_{i,g}$, *for* $(j,k) \in I$, $g \in G_i$, $P \subseteq G_{j,k}$, $i \notin \{j,k\}$;
(3) $d_{i,g}c_{i,j,P} = c_{i,j,gP}d_{i,g}$, *for* $(i,j) \in I$, $g \in G_i$, $P \subseteq G_{i,j}$;
(4) $c_{i,j,P}d_{j,g} = d_{j,g}c_{i,j,Pg}$, *for* $(i,j) \in I$, $g \in G_j$, $P \subseteq G_{i,j}$.

Proof. This lemma follows from the multiplication rules given in Section 5. □

For $1 \leq i \leq n$, let $\varphi_i : A_i^* \mapsto G_i$ be the morphism defining G_i and let U_i be a section of φ_i, that is, a subset of A_i^* such that φ_i induces a bijection from U_i onto G_i. We now give our first presentation for S, given a monoid presentation $\langle A_i \mid R_i \rangle$ for each group G_i.

Theorem 6.4 *A presentation for S is $\langle C \cup D \mid R \rangle$, where R is the union of the following sets of relations:*

(P1) $d_{i,g}d_{j,h} = d_{j,h}d_{i,g}$ *for* $(i,j) \in I$, $g \in A_i$ *and* $h \in A_j$;

(P2) $d_{i,g}c_{j,k,p} = c_{j,k,p}d_{i,g}$ *for* $(j,k) \in I$, $i \notin \{j,k\}$, $g \in A_i$, $p \in G_{j,k}$;

(P3) $d_{i,g}c_{i,j,p} = c_{i,j,gp}d_{i,g}$ *for* $(i,j) \in I$, $g \in A_i$, $p \in G_{i,j}$;

(P4) $c_{i,j,p}d_{j,g} = d_{j,g}c_{i,j,pg}$ *for* $(i,j) \in I$, $g \in A_j$, $p \in G_{i,j}$;

(P5) $c_{i,j,p}^2 = c_{i,j,p}$ *for* $(i,j) \in I$ *and* $p \in G_{i,j}$;

(P6) $c_{i,j,p}c_{k,\ell,q} = c_{k,\ell,q}c_{i,j,p}$ *for* $(i,j) \in I$, $(k,\ell) \in I$, $p \in G_{i,j}$, $q \in G_{k,\ell}$, $\ell \neq i$ *and* $k \neq j$;

(P7) $c_{i,j,p}c_{j,k,q} = c_{i,k,pq}c_{j,k,q}c_{i,j,p}$ *for* $1 \leq i < j < k \leq n$, $p \in G_{i,j}$ *and* $q \in G_{j,k}$;

(P8) $R_1 \cup \cdots \cup R_n$.

Proof. We shall use the "guess and prove" method described in Proposition 2.1. Let φ be the morphism from $(C \cup D)^*$ onto S which maps the letter $d_{i,a}$ onto the diagonal matrix $d_{i,\varphi_i(a)}$ and the letter $c_{i,j,p}$ onto the matrix $c_{i,j,p}$. We shall denote by $\sim$ the congruence modulo R on $(C \cup D)^*$.

Proposition 6.1 shows that $C \cup D$ is a generating set for S. Next, Lemmas 6.3 and 6.2 tell that the relations (P1)-(P7) are satisfied. The relations $R_1 \cup \cdots \cup R_n$ are also satisfied by definition of φ. This gives the first condition of Proposition 2.1.

Note that (P6) applied with $i = k$ and $j = \ell$ shows that any two letters of $C_{i,j}$ commute modulo R. Furthermore, (P5) shows that if c is a letter of $C_{i,j}$, then $c \sim c^2$. It follows at once that any two words of $C_{i,j}^*$ with the same content[1] are $\sim$-equivalent. For each subset P of $C_{i,j}$, we fix a word $c_{i,j,P}$ of content P. Note that, if $P = \emptyset$, then $c_{i,j,P}$ is necessarily the empty word.

Before proving the second condition of Proposition 2.1, we need to recover the relations between the $c_{i,j,P}$'s given by Lemmas 6.2 and 6.3.

Lemma 6.5 *The following relations are consequences of the relations* (P1)-(P8)*:*

(C1) $c_{i,j,P}c_{i,j,Q} \sim c_{i,j,P+Q}$, *for* $(i,j) \in I$ *and* $P, Q \subseteq G_{i,j}$;

(C2) $c_{i,j,P}^2 \sim c_{i,j,P}$, *for* $(i,j) \in I$ *and* $P \subseteq G_{i,j}$;

(C3) $c_{i,j,P}c_{k,\ell,Q} \sim c_{k,\ell,Q}c_{i,j,P}$ *for* $(i,j) \in I$, $(k,\ell) \in I$, $\ell \neq i$ *and* $k \neq j$, $P \subseteq G_{i,j}$ *and* $Q \subseteq G_{k,\ell}$;

(C4) $c_{i,j,P}c_{j,k,Q} \sim c_{i,k,PQ}c_{j,k,Q}c_{i,j,P}$ *for* $1 \leq i < j < k \leq n$, $P \subseteq G_{i,j}$ *and* $Q \subseteq G_{j,k}$;

(C5) $d_{i,g}c_{j,k,P} \sim c_{j,k,P}d_{i,g}$, *for* $(j,k) \in I$, $g \in G_i$, *and* $P \subseteq G_{j,k}$, $i \notin \{j,k\}$;

(C6) $d_{i,g}c_{i,j,P} \sim c_{i,j,gP}d_{i,g}$, *for* $(i,j) \in I$, $g \in G_i$, *and* $P \subseteq G_{i,j}$;

(C7) $c_{i,j,P}d_{j,g} \sim d_{j,g}c_{i,j,Pg}$, *for* $(i,j) \in I$, $g \in G_j$, *and* $P \subseteq G_{i,j}$.

[1]Recall that the *content* of a word is the set of letters occurring in it.

Proof. Observe that (C3), applied with $i = k$ and $j = \ell$, states that if $P, Q \subseteq G_{i,j}$, then $c_{i,j,P}$ and $c_{i,j,Q}$ commute modulo $\sim$.

We just prove (C4), the other formulas being analogous and easier. We prove the result by induction on $|P| + |Q|$. First, if P or Q is empty, the formula is trivial. If $P = \{p\}$ and Q is the disjoint union of Q' and $\{q\}$, then $c_{j,k,Q} \sim c_{j,k,Q'}c_{j,k,q}$ since the letters of $C_{j,k}$ commute. Therefore

$$\begin{array}{lcll} c_{i,j,p}c_{j,k,Q} & \sim & c_{i,j,p}c_{j,k,Q'}c_{j,k,q} & \\ & \sim & c_{i,k,pQ'}c_{j,k,Q'}c_{i,j,p}c_{j,k,q} & \text{by the induction hypothesis} \\ & \sim & c_{i,k,pQ'}c_{j,k,Q'}c_{i,k,pq}c_{j,k,q}c_{i,j,p} & \text{by condition (7)} \\ & \sim & c_{i,k,pQ'}c_{i,k,pq}c_{j,k,Q'}c_{j,k,q}c_{i,j,p} & c_{i,k,pq} \text{ and } c_{j,k,Q'} \text{ commute} \\ & \sim & c_{i,k,pQ}c_{j,k,Q}c_{i,j,p} & \end{array}$$

Finally, if P is the disjoint union of P' and $\{p\}$, then

$$\begin{array}{lcll} c_{i,j,P}c_{j,k,Q} & \sim & c_{i,j,P'}c_{i,j,p}c_{j,k,Q} & \\ & \sim & c_{i,j,P'}c_{i,k,pQ}c_{j,k,Q}c_{i,j,p} & \\ & \sim & c_{i,k,pQ}c_{i,j,P'}c_{j,k,Q}c_{i,j,p} & c_{i,j,P'} \text{ and } c_{i,k,pQ} \text{ commute} \\ & \sim & c_{i,k,pQ}c_{i,k,P'Q}c_{j,k,Q}c_{i,j,P'}c_{i,j,p} & \text{by the induction hypothesis} \\ & \sim & c_{i,k,PQ}c_{j,k,Q}c_{i,j,P} & \end{array}$$

Next we prove the second condition of Proposition 2.1. Consider the subset of D^* defined by

$$U = \{\ u_1 \cdots u_n \mid u_1 \in U_1, \ldots, u_n \in U_n\}$$

and the set W of words of $(C \cup D)^*$ defined by

$$W = \Big\{ \prod_{(i,j)\in(I,\leq)} c_{i,j,P_{i,j}} \mid P_{i,j} \text{ is a finite subset of } G_{i,j} \Big\}$$

We claim that every word of $(C \cup D)^*$ is equivalent, modulo R, to a word of UW. The claim holds for the empty word, since, if $u_1 \in U_1, \ldots, u_n \in U_n$ are words representing the identity of $G_1, \ldots, G_n$, respectively, then the empty word is equivalent to $u_1 \cdots u_n$, a word of U. Assume by induction that the claim holds for every word of length $\leq n$. If v is a word of length n and b is a letter of $C \cup D$, then, by the induction hypothesis, $v \sim w$ for some word $w \in UW$. Put $w = u_1 \cdots u_n w'$, with $u_1 \in U_1, \ldots, u_n \in U_n$ and

$$w' = \prod_{(i,j)\in(I,\leq)} c_{i,j,P_{i,j}}$$

where each $P_{i,j}$ is a subset of $G_{i,j}$.

Suppose first that $b \in A_i$. Relations (C5), (C6) and (C7) can be used to "push" b to the left hand side of the $c_{i,j,P_{i,j}}$'s, obtaining

$$wb \sim u_1 \ \cdots \ u_n b \prod_{(i,j)\in(I,\leq)} c_{i,j,Q_{i,j}}$$

where $Q_{i,j}$ is either $P_{i,j}$, $g^{-1}P_{i,j}$ or $P_{i,j}g$ depending on applying (C5), (C6) or (C7). Now, relations (P1) can be used to permute b with the letters of A_j, for $j > i$. Thus $wb \sim u_1 \cdots u_i b u_{i+1} u_n w'$. But $u_i b \in A_i^*$ and thus $u_i b$ is equivalent modulo R_i to some word $u_i' \in U_i$. It follows that $vb \sim wb \sim u_1 \cdots u_{i-1} u_i' u_{i+1} u_n w'$, a word of UW.

Suppose now that $b \in C_{r,s}$, and more precisely, let $b = c_{r,s,g}$ for some $g \in G_{r,s}$. It suffices to establish that $w'b$ is equivalent to a word of W. This result will follow from a slightly more general lemma. For each ordered subset $(J, \leq)$ of $(I, \leq)$, set

$$W_J = \Big\{ \prod_{(i,j)\in(J,\leq)} c_{i,j,P_{i,j}} \mid P_{i,j} \text{ is a subset of } G_{i,j} \Big\}$$

In particular, $W_\emptyset = \{1\}$ and $W_I = W$. It will suffice to take $J = I$ and $P_{r,s} = \{g\}$ in the next lemma to conclude that vb is $\sim$-equivalent to a word of UW. Hence condition (2) of Proposition 2.1 holds.

Lemma 6.6 *For each initial segment J of $(I, \leq)$ and each subset $P_{r,s}$ of $G_{r,s}$, each word of $W_J c_{r,s,P_{r,s}}$ is $\sim$-equivalent to a word of $W_{J+\{(r,s)\}}$.*

Proof. We prove the lemma by induction on the cardinality of J. The result is trivial if J is empty. If J is non empty, let $(k, \ell) = \max J$ and $J' = J \setminus \{(k, \ell)\}$. Then each $w \in W_J$ can be written as $w' c_{k,\ell,P_{k,\ell}}$ for some $w' \in W_{J'}$. If $(k, \ell) < (r, s)$, the result is trivial. It is also easy if $(k, \ell) = (r, s)$, since, by (C2)

$$w c_{r,s,P_{r,s}} = w' c_{k,\ell,P_{k,\ell}} c_{k,\ell,P_{k,\ell}} \sim w' c_{k,\ell,P_{k,\ell}}$$

and $w' c_{k,\ell,P_{k,\ell}} \in W_J$.

If $(r, s) < (k, \ell)$, then $r < s$, $k < \ell$ and either $\ell < s$ or $\ell = s$ and $r < k$. If $\ell \neq r$ (this covers in particular the case $\ell = s$ and $r < k$), then by (C3)

$$w c_{r,s,P_{r,s}} = w' c_{k,\ell,P_{k,\ell}} c_{r,s,P_{r,s}} \sim w' c_{r,s,P_{r,s}} c_{k,\ell,P_{k,\ell}}$$

Now, by the induction hypothesis and since $(r, s) \in J'$, we have that $w' c_{r,s,P_{r,s}}$ is $\sim$-equivalent to a word w'' of $W_{J'}$ and $w'' c_{k,\ell,P_{k,\ell}}$ is a word of W_J.

When $\ell < s$, there is one remaining case to consider: when $k < r = \ell < s$. In this case, by (C4)

$$w c_{r,s,P_{r,s}} = w' c_{k,\ell,P_{k,\ell}} c_{\ell,s,P_{\ell,s}} \sim w' c_{k,s,P_{k,\ell}P_{\ell,s}} c_{\ell,s,P_{\ell,s}} c_{k,\ell,P_{k,\ell}}$$

Now, since $(k, s) < (\ell, s) < (k, \ell)$, both (k, s) and (ℓ, s) are in J' and the induction hypothesis can be used twice to show that $w' c_{k,s,P_{k,\ell}P_{\ell,s}} c_{\ell,s,P_{\ell,s}}$ is $\sim$-equivalent to a word w'' of $W_{J'}$. The result follows, since $w'' c_{k,\ell,P_{k,\ell}}$ is a word of W_J. □

The last step (condition (3) of Proposition 2.1) consists in proving that φ induces a bijection from UW onto $\diamondsuit_n(G_1, \ldots, G_n)$. But this is clear, since

$$\varphi\Big(u_1 \cdots u_n \prod_{(i,j)\in(I,\leq)} c_{i,j,P_{i,j}}\Big) = \begin{pmatrix} \varphi_1(u_1) & P_{1,2} & \ldots & P_{1,n-1} & P_{1,n} \\ 0 & \varphi_2(u_2) & & P_{2,n-1} & P_{2,n} \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & \varphi_{n-1}(u_{n-1}) & P_{n-1,n} \\ 0 & 0 & \ldots & 0 & \varphi_n(u_n) \end{pmatrix}$$

Hence the proof of Theorem 6.4 is concluded. □

Corollary 6.7 *A presentation for $\diamondsuit_2(G_1, G_2)$ is $\langle C \cup D \mid R' \rangle$, where R' is the union of the following relations:*

(1) $d_{1,g}d_{2,h} = d_{2,h}d_{1,g}$ *for* $g \in A_1$ *and* $h \in A_2$;
(2) $d_{1,g}c_{1,2,p} = c_{1,2,gp}d_{1,g}$ *for* $g \in A_1$, $p \in G_{1,2}$;
(3) $c_{1,2,p}d_{2,g} = d_{2,g}c_{1,2,pg}$ *for* $g \in A_2$, $p \in G_{1,2}$;
(4) $c_{1,2,p}^2 = c_{1,2,p}$ *for* $p \in G_{1,2}$;
(5) $c_{1,2,p}c_{1,2,q} = c_{1,2,q}c_{1,2,p}$ *for* $p, q \in G_{1,2}$, $q \in G_{1,2}$;
(6) $R_1 \cup R_2$.

7 A second presentation

We first show that it is possible to obtain a slightly different set of generators for $S = \diamondsuit_n(G_1, \ldots, G_n)$ and then obtain a second presentation. This, in general, has less generators than the first one, which can be of interest for computational purposes.

The presentations given by Theorems 6.4 and 7.3 are usually incomparable. Indeed, the set of generators for the second presentation is a subset of the set of generators of the first presentation, but when the two presentations are finite, the second one may have a larger number of relations than the first one. However, for $n = 2$, the second presentation is more transparent than the first one (see Corollary 7.5 below).

Let us set, for each $(i, j) \in I$,

$$C'_{i,j} = \{c_{i,j,p} \mid p \in G_{i+1,j-1}\}$$

Recall that for $i > j$, $G_{i+1,j-1}$ reduces to the trivial subgroup of $G_1 \times \ldots \times G_n$ and thus for $j = i + 1$, the set $C'_{i,j}$ is a singleton. Let

$$C' = \bigcup_{(i,j)\in I} C'_{i,j}$$

We also introduce the map $p \to p'$ from $G_{i,j}$ onto $G_{i+1,j-1}$ defined by setting, for $p = (1, \ldots, 1, g_i, \ldots, g_j, 1, \ldots, 1)$,

$$p' = g_i^{-1} p g_j^{-1} = (1, \ldots, 1, g_{i+1}, \ldots, g_{j-1}, 1, \ldots, 1)$$

A direct computation shows that the following relations hold in S

$$c_{i,j,p} = d_{j,g_j^{-1}} d_{i,g_i} c_{i,j,p'} d_{i,g_i^{-1}} d_{j,g_j} \qquad (E_{i,j,p})$$

Proposition 7.1 *The set $C' \cup D$ is a set of generators for S.*

Proof. In view of Proposition 6.1, it remains to prove that each unitriangular matrix of the form $c_{i,j,p}$, where $p \in G_{i,j}$, can be written as a product of elements of $C' \cup D$. But this follows immediately from Formula $(E_{i,j,p})$. □

We now propose a second presentation for S, given a monoid presentation $\langle A_i \mid R_i \rangle$ of each group G_i. For all $i \in \{1, \ldots, n\}$, and for each word $u = g_1 \cdots g_k \in A_i^*$, set

$$d_{i,u} = d_{i,g_1} \cdots d_{i,g_k}$$

We also fix for each $i \in \{1, \ldots, n\}$ and for each $g \in A_i$, a word $\bar{g} \in A_i^*$ such that $d_{i,\bar{g}} = d_{i,g}^{-1}$ modulo R_i. By extension, we set, for each word $u = g_1 \cdots g_k \in A_i^*$,

$$d_{i,\bar{u}} = d_{i,\bar{g}_k} \cdots d_{i,\bar{g}_1}$$

The next lemma is now an easy extension of Lemma 6.3.

Lemma 7.2 *The following relations hold in S:*

(1) $d_{i,u} d_{j,v} = d_{j,v} d_{i,u}$ *for* $(i,j) \in I$, $u \in A_i^*$ *and* $v \in A_j^*$;
(2) $d_{i,u} c_{j,k,P} = c_{j,k,P} d_{i,u}$ *for* $(j,k) \in I$, $u \in A_i^*$, $P \subseteq G_{j,k}$, $i \notin \{j,k\}$;
(3) $d_{i,u} c_{i,j,P} d_{i,\bar{u}} = c_{i,j,uP}$, *for* $(i,j) \in I$, $u \in A_i^*$, $P \subseteq G_{i,j}$;
(4) $d_{j,\bar{v}} c_{i,j,P} d_{j,v} = c_{i,j,Pv}$, *for* $(i,j) \in I$, $v \in A_j^*$, $P \subseteq G_{i,j}$.

We are now ready to give our second presentation.

Theorem 7.3 *A presentation for S is $\langle C' \cup D \mid R' \rangle$, where R' is the union of the following relations:*

(Q1) $d_{i,g} d_{j,h} = d_{j,h} d_{i,g}$ *for* $(i,j) \in I$, $g \in A_i$ *and* $h \in A_j$;
(Q2) $d_{i,g} c_{j,k,p} = c_{j,k,p} d_{i,g}$ *for* $(j,k) \in I$, $i \notin \{j,k\}$, $g \in A_i$ *and* $p \in G_{j+1,k-1}$;
(Q3) $c_{i,j,p}^2 = c_{i,j,p}$ *for* $(i,j) \in I$ *and* $p \in G_{i+1,j-1}$;
(Q4) $c_{i,j,p}(d_{i,u} c_{i,k,q} d_{i,\bar{u}}) = (d_{i,u} c_{i,k,q} d_{i,\bar{u}}) c_{i,j,p}$ *for* $(i,j),(i,k) \in I$, $j \neq k$, $v \in A_j^*$, $p \in G_{i+1,j-1}$, $q \in G_{i+1,k-1}$, $u \in A_i^*$;
(Q5) $c_{i,j,p}(d_{j,\bar{v}} c_{k,j,q} d_{j,v}) = (d_{j,\bar{v}} c_{k,j,q} d_{j,v}) c_{i,j,p}$ *for* $(i,j),(k,j) \in I$, $i \neq k$, $p \in G_{i+1,j-1}$, $q \in G_{k+1,j-1}$, $v \in A_j^*$;

(Q6) $(d_{i,u}c_{i,j,p}d_{i,\bar{u}})(d_{j,v}c_{i,j,q}d_{j,\bar{v}}) = (d_{j,v}c_{i,j,q}d_{j,\bar{v}})(d_{i,u}c_{i,j,p}d_{i,\bar{u}})$ *for* $(i,j) \in I$, $u \in A_i^*$, $v \in A_j^*$, $p, q \in G_{i+1,j-1}$;

(Q7) $c_{i,j,p}c_{k,\ell,q} = c_{k,\ell,q}c_{i,j,p}$ *for* $(i,j),(k,\ell) \in I$, $i \neq k$, $i \neq \ell$, $j \neq k$, $j \neq \ell$, $p \in G_{i+1,j-1}$, $q \in G_{k+1,\ell-1}$;

(Q8) $(d_{j,\bar{u}}c_{i,j,p}d_{j,u})(d_{j,v}c_{j,k,q}d_{j,\bar{v}}) = c_{i,k,puvq}(d_{j,v}c_{j,k,q}d_{j,\bar{v}})(d_{j,\bar{u}}c_{i,j,p}d_{j,u})$ *for* $1 \leq i < j < k \leq n$, $p \in G_{i+1,j-1}$, $q \in G_{j+1,k-1}$, $u, v \in A^*$;

(Q9) $R_1 \cup \cdots \cup R_n$.

Proof. By Proposition 7.1, we know that $C' \cup D$ is a set of generators for S. We now proceed in two steps. First we verify that each relation of R' is satisfied by S. Next we introduce an auxiliary presentation $\langle C' \cup D \mid R'' \rangle$ for S, obtained from the presentation $\langle C \cup D \mid R \rangle$ by Tietze transformations, and we show that every relation of R'' is a consequence of R'. Then, using Corollary 2.2, we conclude that $\langle C' \cup D \mid R' \rangle$ is a presentation for S.

For the first step, we observe that the relations (Q1), (Q2), (Q3) and (Q7) are special cases of (P1), (P2), (P5) and (P6) respectively. Relations (Q4) (resp. (Q5), (Q6), (Q8)) follow easily from Lemma 7.2 and 6.2. For instance, for (Q4)

$$c_{i,j,p}(d_{i,u}c_{i,k,q}d_{i,\bar{u}}) = c_{i,j,p}c_{i,k,uq} = c_{i,k,uq}c_{i,j,p} = (d_{i,u}c_{i,k,q}d_{i,\bar{u}})c_{i,j,p}$$

and for (Q8),

$$\begin{aligned}(d_{j,\bar{u}}c_{i,j,p}d_{j,u})(d_{j,v}c_{j,k,q}d_{j,\bar{v}}) &= c_{i,j,pu}c_{j,k,vq} = c_{i,k,puvq}c_{j,k,vq}c_{i,j,pu} \\ &= c_{i,k,puvq}(d_{j,v}c_{j,k,q}d_{j,\bar{v}})(d_{j,\bar{u}}c_{i,j,p}d_{j,u})\end{aligned}$$

We now proceed to the second step. Formula $(E_{i,j,p})$ shows that for each $(i,j) \in I$ and $p = (1,\ldots,1,g_i,\ldots,g_j,1,\ldots,1) \in G_{i,j}$, the relation

$$c_{i,j,p} = d_{j,\bar{g}_j}d_{i,g_i}c_{i,j,p'}d_{i,\bar{g}_i}d_{j,g_j} \qquad (R_{i,j,p})$$

holds in S. Therefore, by Proposition 2.3 (4), a new presentation $\langle C' \cup D \mid R'' \rangle$ is obtained by considering the new set of generators $C' \cup D$ and by substituting the right hand side of $(R_{i,j,p})$, for each letter $c_{i,j,p}$ of $C \setminus C'$, in each relation of R.

We claim that every relation of R'' is a consequence of R'. To prove this claim, it is useful to have in mind a few elementary consequences of R'.

Lemma 7.4 *For all* $(i,j) \in I$, $u \in A_i^*$ *and* $v \in A_j^*$, *the relations*

$$d_{i,u}d_{j,\bar{v}} = d_{j,\bar{v}}d_{i,u} \qquad (T)$$

are consequences of the relations (Q1). Also, if $u, v \in A_i^*$ *are* $\sim_{R_i}$*-equivalent, then*

$$d_{i,u} = d_{i,v} \quad \text{and} \quad d_{i,u}d_{i,\bar{u}} = 1 = d_{i,\bar{u}}d_{i,u} \qquad (U)$$

are consequences of R_i.

Let us show that each relation obtained by performing the substitution $c_{i,j,p} \to d_{j,\bar{g}_j} d_{i,g_i} c_{i,j,p'} d_{i,\bar{g}_i} d_{j,g_j}$ on each relation of R is a relation of R''. Relations (P1) are not changed by the substitution and are identical to the relations (Q1). Relations (P2) give either rise to a relation of (Q2), if $p \in G_{j+1,k-1}$, or to a relation of the form

$$d_{i,g} d_{k,\bar{g}_k} d_{j,g_j} c_{j,k,p'} d_{j,\bar{g}_j} d_{k,g_k} = d_{k,\bar{g}_k} d_{j,g_j} c_{j,k,p'} d_{j,\bar{g}_j} d_{k,g_k} d_{i,g}$$

which follows from (Q2) and (T). Relations (P3) lead to

$$d_{i,g} d_{j,\bar{g}_j} d_{i,g_i} c_{i,j,p'} d_{i,\bar{g}_i} d_{j,g_j} = d_{j,\bar{g}_j} d_{i,gg_i} c_{i,j,(gp)'} d_{i,\bar{g}_i \bar{g}} d_{j,g_j} d_{i,g}$$

As $g \in A_i$, we have $(gp)' = p'$ and these relations follow from Lemma 7.4. Relations (P4) are dual of (P3) and can be treated in a similar way.

Relations (P5) give rise to a relation of the form

$$d_{j,\bar{g}_j} d_{i,g_i} c_{i,j,p'} d_{i,\bar{g}_i} d_{j,g_j} = d_{j,\bar{g}_j} d_{i,g_i} c_{i,j,p'} d_{i,\bar{g}_i} d_{j,g_j} d_{j,\bar{g}_j} d_{i,g_i} c_{i,j,p'} d_{i,\bar{g}_i} d_{j,g_j}$$

which is a consequence of (Q3) and (U).

Relations (P6) lead to various possibilities. We remind the reader that these relations are of the form $c_{i,j,p} c_{k,\ell,q} = c_{k,\ell,q} c_{i,j,p}$, with $\ell \neq i$ and $k \neq j$. If $i \neq k$, $j \neq \ell$, we obtain a relation of the form

$$\begin{aligned} d_{j,\bar{g}_j} d_{i,g_i} c_{i,j,p'} d_{i,\bar{g}_i} d_{j,g_j} d_{\ell,\bar{g}_\ell} d_{k,g_k} c_{k,\ell,q'} d_{k,\bar{g}_k} d_{\ell,g_\ell} = \\ d_{\ell,\bar{g}_\ell} d_{k,g_k} c_{k,\ell,q'} d_{k,\bar{g}_k} d_{\ell,g_\ell} d_{j,\bar{g}_j} d_{i,g_i} c_{i,j,p'} d_{i,\bar{g}_i} d_{j,g_j} \end{aligned}$$

which follows from (Q2), (Q7) and (T). If $i = k$, but $j \neq \ell$, we obtain a relation of the form

$$\begin{aligned} (d_{j,\bar{g}_j} d_{i,g_i} c_{i,j,p'} d_{i,\bar{g}_i} d_{j,g_j})(d_{\ell,\bar{g}_\ell} d_{i,h_i} c_{i,\ell,q'} d_{i,\bar{h}_i} d_{\ell,g_\ell}) = \\ (d_{\ell,\bar{g}_\ell} d_{i,h_i} c_{i,\ell,q'} d_{i,\bar{h}_i} d_{\ell,g_\ell})(d_{j,\bar{g}_j} d_{i,g_i} c_{i,j,p'} d_{i,\bar{g}_i} d_{j,g_j}) \end{aligned}$$

Proving that this relation follows from (R') requires a short argument. The starting point is the following consequence of (Q4):

$$c_{i,j,p'} (d_{i,\bar{g}_i} d_{i,h_i} c_{i,\ell,q'} d_{i,\bar{h}_i} d_{i,g_i}) = (d_{i,\bar{g}_i} d_{i,h_i} c_{i,\ell,q'} d_{i,\bar{h}_i} d_{i,g_i}) c_{i,j,p'}$$

which gives, by multiplying both sides on the left by d_{i,g_i} and on the right by $d_{i,\bar{g}_i}$ and applying (U)

$$d_{i,g_i} c_{i,j,p'} d_{i,\bar{g}_i} d_{i,h_i} c_{i,\ell,q'} d_{i,\bar{h}_i} = d_{i,h_i} c_{i,\ell,q'} d_{i,\bar{h}_i} d_{i,g_i} c_{i,j,p'} d_{i,\bar{g}_i} \tag{1}$$

Multiply on the left by $d_{j,\bar{g}_j} d_{\ell,\bar{g}_\ell}$ and on the right by $d_{\ell,g_\ell} d_{j,g_j}$ to obtain

$$\begin{aligned} d_{j,\bar{g}_j} d_{\ell,\bar{g}_\ell} d_{i,g_i} c_{i,j,p'} d_{i,\bar{g}_i} d_{i,h_i} c_{i,\ell,q'} d_{i,\bar{h}_i} d_{\ell,g_\ell} d_{j,g_j} = \\ d_{j,\bar{g}_j} d_{\ell,\bar{g}_\ell} d_{i,h_i} c_{i,\ell,q'} d_{i,\bar{h}_i} d_{i,g_i} c_{i,j,p'} d_{i,\bar{g}_i} d_{\ell,g_\ell} d_{j,g_j} \end{aligned}$$

Next, using (Q2) (since $j \notin \{i, \ell\}$) and (T), we can shift $d_{\ell,\bar{g}_\ell}$ to the right and d_{j,g_j} to the left on the left hand side and d_{ℓ,g_ℓ} to the left and $d_{j,\bar{g}_j}$ to the right on the right hand side, as shown in Figure 1 to obtain the desired relation.

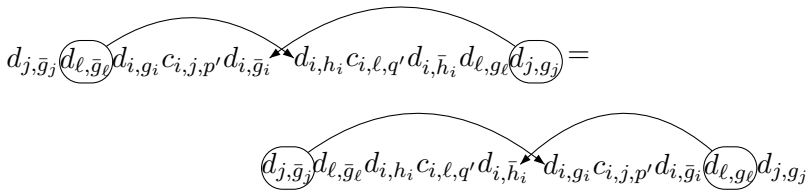

Figure 1: Shifting around.

The case $j = \ell$ but $i \neq k$ can be treated in a similar way, using (Q5) instead of (Q4). Finally, if $i = k$ and $j = \ell$, we obtain a relation of the form

$$(d_{j,\bar{g}_j} d_{i,g_i} c_{i,j,p'} d_{i,\bar{g}_i} d_{j,g_j})(d_{j,\bar{h}_j} d_{i,h_i} c_{i,j,q'} d_{i,\bar{h}_i} d_{j,h_j}) = (d_{j,\bar{h}_j} d_{i,h_i} c_{i,j,q'} d_{i,\bar{h}_i} d_{j,h_j})(d_{j,\bar{g}_j} d_{i,g_i} c_{i,j,p'} d_{i,\bar{g}_i} d_{j,g_j})$$

which follows from (Q6) and Lemma 7.4.

Finally, relations (P7) produce relations of the form

$$(d_{j,\bar{g}_j} d_{i,g_i} c_{i,j,p'} d_{i,\bar{g}_i} d_{j,g_j})(d_{k,\bar{h}_k} d_{j,h_j} c_{j,k,q'} d_{j,\bar{h}_j} d_{k,h_k}) = (d_{k,\bar{h}_k} d_{i,g_i} c_{i,k,(pq)'} d_{i,\bar{g}_i} d_{k,h_k})(d_{k,\bar{h}_k} d_{j,h_j} c_{j,k,q'} d_{j,\bar{h}_j} d_{k,h_k})(d_{j,\bar{g}_j} d_{i,g_i} c_{i,j,p'} d_{i,\bar{g}_i} d_{j,g_j})$$

where $i < j < k$. To derive these relations from (R'), we start with the following instance of (Q8)

$$(d_{j,\bar{g}_j} c_{i,j,p'} d_{j,g_j})(d_{j,h_j} c_{j,k,q'} d_{j,\bar{h}_j}) = c_{i,k,p' g_j h_j q'} (d_{j,h_j} c_{j,k,q'} d_{j,\bar{h}_j})(d_{j,\bar{g}_j} c_{i,j,p'} d_{j,g_j})$$

Observing that $(pq)' = p' g_j h_j q'$ and inserting $d_{i,\bar{g}_i} d_{i,g_i} d_{k,h_k} d_{k,\bar{h}_k}$, which, by Lemma 7.2, is equivalent to 1, we obtain

$$(d_{j,\bar{g}_j} c_{i,j,p'} d_{j,g_j})(d_{j,h_j} c_{j,k,q'} d_{j,\bar{h}_j}) = c_{i,k,(pq)'} d_{i,\bar{g}_i} d_{i,g_i} d_{k,h_k} d_{k,\bar{h}_k} (d_{j,h_j} c_{j,k,q'} d_{j,\bar{h}_j})(d_{j,\bar{g}_j} c_{i,j,p'} d_{j,g_j})$$

Now, multiplying on the left by $d_{k,\bar{h}_k} d_{i,g_i}$ and on the right by $d_{i,\bar{g}_i} d_{k,h_k}$ both sides of the previous relation, we get

$$d_{k,\bar{h}_k} d_{i,g_i} d_{j,\bar{g}_j} c_{i,j,p'} d_{j,g_j} d_{j,h_j} c_{j,k,q'} d_{j,\bar{h}_j} d_{i,\bar{g}_i} d_{k,h_k} = d_{k,\bar{h}_k} d_{i,g_i} c_{i,k,(pq)'} d_{i,\bar{g}_i} d_{i,g_i} d_{k,h_k} d_{k,\bar{h}_k} d_{j,h_j} c_{j,k,q'} d_{j,\bar{h}_j} d_{j,\bar{g}_j} c_{i,j,p'} d_{j,g_j} d_{i,\bar{g}_i} d_{k,h_k}$$

To conclude, we use (Q2) and (T) to shift, on the left hand side, $d_{k,\bar{h}_k}$ and d_{i,g_i} to the right and $d_{i,\bar{g}_i}$ to the left and, on the right hand side, d_{i,g_i} to the right and d_{k,h_k} and $d_{i,\bar{g}_i}$ to the left, as shown in Figure 2. □

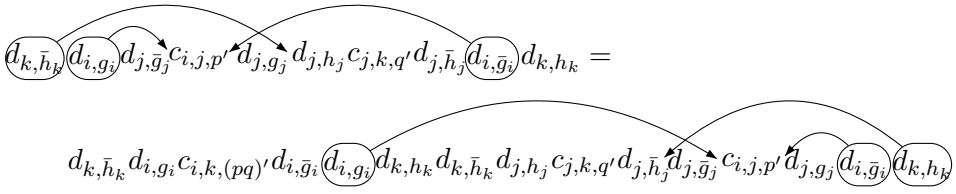

Figure 2: Shifting the elements.

If $n = 2$, the alphabet C' contains a single letter, simply denoted by c in the sequel. Thus, applying Theorem 7.3, we are lead to the following corollary:

Corollary 7.5 *A presentation for $\diamondsuit_2(G_1, G_2)$ is $\langle \{c\} \cup D \mid R' \rangle$, where R' is the union of the following relations:*

(1) $d_{1,g}d_{2,h} = d_{2,h}d_{1,g}$ *for* $g \in A_1$ *and* $h \in A_2$*;*
(2) $c^2 = c$*;*
(3) $(d_{1,u}cd_{1,\bar{u}})(d_{j,v}cd_{j,\bar{v}}) = (d_{j,v}cd_{j,\bar{v}})(d_{1,u}cd_{1,\bar{u}})$ *for* $u \in A_1^*$, $v \in A_2^*$*;*
(4) $R_1 \cup R_2$*.*

8 An application to the monoid of unitriangular Boolean matrices

If all groups $G_1, \ldots, G_n$ are trivial, their Schützenberger product is exactly the monoid U_n of all unitriangular Boolean square matrices of order n [21]. As it was mentionned in the introduction, this monoid is $\mathcal{J}$-trivial and thus has a unique minimal set of generators [8]. This is actually another analogy between unitriangular matrices over the Boolean semiring and over the field F_p. Indeed, $U_n(F_p)$ is a p-group and as such, all of its minimal sets of generators have the same size.

Observe that, in this case, there is, for each $(i, j) \in I$, a unique element $c_{i,j,p}$ with $p \in G_{i,j}$, namely the elementary matrix with only one nondiagonal 1-entry (in position (i, j)):

$$\begin{pmatrix} 1 & & & & & & & 0 \\ & \ddots & & & & & & \\ & & 1 & \cdots & 1 & & & 0 \\ & & & \ddots & \vdots & & & \\ & & & & 1 & & & 0 \\ & & & & & \ddots & & \\ 0 & & & & & & & 1 \end{pmatrix}$$

Identifying this matrix with the pair (i, j), one can give an improved version of Proposition 5.3 as follows.

Proposition 8.1 *Let m be a matrix of U_n and let $J = \{(i,j) \in I \mid m_{i,j} = 1\}$. Then $m = \prod_{(i,j)\in(J,\leq)}(i,j)$.*

We also obtain, as a consequence of Theorem 6.4, the following presentation for U_n:

Theorem 8.2 *A presentation for the monoid U_n is $\langle I \mid T\rangle$, where T is the union of the following sets of relations:*

(T1) $(i,j)^2 = (i,j)$ *for* $(i,j) \in I$;

(T2) $(i,j)(k,\ell) = (k,\ell)(i,j)$ *for* $(i,j),(k,\ell) \in I$, $(i,j) \neq (k,\ell)$, $i \neq \ell$ *and* $j \neq k$;

(T3) $(i,j)(j,k) = (i,k)(j,k)(i,j)$ *for* $1 \leq i < j < k \leq n$.

Actually, our proof gives a simple rewriting system for U_n (see [3] for references on string rewriting systems).

Theorem 8.3 *The monoid U_n is presented by the following rewriting system $\langle I \mid R\rangle$, where R is the union of the following rules:*

(R1) $(i,j)^2 \to (i,j)$ *for* $(i,j) \in I$;

(R2) $(i,j)(k,\ell) \to (k,\ell)(i,j)$ *for* $(i,j) > (k,\ell)$, $j \neq k$;

(R3) $(i,j)(j,k) \to (i,k)(j,k)(i,j)$ *for* $(i,j) > (j,k)$.

Note that this rewriting system is clearly converging since the right hand side of each rule is strictly smaller that its left hand side for the lexicographic order induced by the order on I.

An example might help the reader to understand the mechanism of the rewriting rules. Take $n = 6$, and consider the following element, written in the normal form

$$(2,6)(3,6)(5,6)(2,5)(1,4)(2,4)(3,4)(1,3)(1,2)$$

Let us multiply this element on the right by $(4,6)$. After rewriting using the rules of (R), the following normal form is obtained

$$(1,6)(2,6)(3,6)(4,6)(5,6)(2,5)(1,4)(2,4)(3,4)(1,3)(1,2)$$

Notice that, in this particular case, Theorem 7.3 leads to the same presentation for U_n.

9 Conclusion and open problems

We conclude this paper by rising a number of questions involving Schützenberger products.

We have given two presentations of the Schützenberger product of n groups given by a presentation. One of our original goals was to recover the

description of the Schützenberger product of three free groups obtained in [19]. However, the computations involved are more intricate than expected, and this objective has been postponed to a future research.

Another interesting situation occurs when the groups $G_1, \ldots, G_n$ have monoid presentations on the same alphabet A, say $\langle A \mid R_1\rangle$, $\langle A \mid R_2\rangle$, $\ldots$,$\langle A \mid R_n\rangle$, respectively. Denote by $\mu_1 : A^* \to G_1, \ldots, \mu_n : A^* \to G_n$, respectively, the morphisms defined by these presentations.

In this case, we are not interested in the full Schützenberger product $\diamondsuit_n(G_1, \ldots, G_n)$, but in a submonoid, obtained by "cutting down to generators". This monoid, denoted by ${}_A\diamondsuit_n(G_1, \ldots, G_n)$, is the image of A^* under the morphism $\mu : A^* \to \diamondsuit_n(G_1, \ldots, G_n)$ where, for each $u \in A^*$, the matrix $\mu(u)$ is defined as follows:

(1) for $1 \leq i \leq n$, $\mu_{i,i}(u) = \mu_i(u)$;

(2) for $1 \leq i \leq j \leq n$,

$$\mu_{i,j}(u) = \{(\mu_i(u_i), \mu_{i+1}(u_{i+1}), \ldots, \mu_j(u_j)) \mid u_i \cdots u_j = u\}$$

In particular, ${}_A\diamondsuit_n(G_1, \ldots, G_n)$ is generated by the matrices $\mu(a)$, for $a \in A$, where

(1) for $1 \leq i \leq n$, $\mu_{i,i}(a) = \mu_i(a)$;

(2) for $1 \leq i \leq j \leq n$,

$$\mu_{i,j}(a) = \{(\mu_i(a), 1, \ldots, 1), (1, \mu_{i+1}(a), \ldots, 1), \ldots, (1, \ldots, 1, \mu_j(a))\}$$

It would be interesting to find a presentation of this monoid, given the presentations of the groups $G_1, \ldots, G_n$. Note that A is now the natural set of generators, a major difference with the case considered in the previous sections.

The solution to this problem does not seem to be trivial. To simplify a little bit, we may only consider the case where the alphabet is of the form $\tilde{A}$ and the monoid presentation arises from a group presentation, as explained in Section 2. For instance, in the special case where $n = 2$ and $G_1 = G_2 = G$, it is well known [2, 18, 17] that the monoid ${}_{\tilde{A}}\diamondsuit_2(G, G)$ is isomorphic to the prefix expansion $\tilde{G}^{\mathcal{R}}$ developed by Birget and Rhodes [2, 12]. Now, in the even more special case where G is the free group on A, the monoid ${}_{\tilde{A}}\diamondsuit_2(G, G)$ is the free inverse monoid on A, for which $A \cup \bar{A}$ together with the set of the following relations forms a well-known presentation:

(1) $u\bar{u}u = u$, for $u \in (A \cup \bar{A})^*$;

(2) $(u\bar{u})(v\bar{v}) = (v\bar{v})(u\bar{u})$, for $u, v \in (A \cup \bar{A})^*$.

It might be tempting to guess that, when G is not necessarily free, a presentation for ${}_{\tilde{A}}\diamondsuit_2(G, G)$ is obtained by adding the relations:

(3) $u^2 = u$ for each $u \in (A \cup \bar{A})^*$ such that $u = 1$ is a relation in G.

This situation is considered in detail in [18, 16] but the answer is negative, except when G is the free group. Thus the problem of finding a complete set of relations in this case is still open.

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