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INTRODUCTION TO FATIGUE ANALYSIS IN MECHANICAL DESIGN BY THE MULTISCALE APPROACH

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ABSTRACT

Crack initiation under high cycle fatigue is a localized phenomenon that occurs in some highly stressed grains of the metallic material. Therefore, the study of high cycle fatigue problems in a rational way is best performed by the introduction of the mesoscopic scale of material description (i.e. the scale of the metal grains of a metallic aggregate) in addition to the usual macroscopic scale of continuum mechanics. The principles of such a multiscale approach in high-cycle fatigue of metallic structures are presented in this work. The multiscale approach is settled on the assumption that under high-cycle fatigue loading a structure will not be fractured by fatigue if an elastic shakedown state is reached at the macroscopic as well as at the mesoscopic scale. The concept of the fatigue limit criterion coincides thus with the possibility of a cyclically loaded structure to tend to an elastic shakedown state at all scales. Extensions of Melan's elastic shakedown theorem to realistic material behaviour are discussed in this article. These theorems allow an easy estimation of the mechanical parameters at the elastic shakedown state at both the macroscopic and mesoscopic scales. Some examples of application of the extended Melan's theorem are provided. The relationships between mesoscopic and macroscopic quantities are studied within the framework of undamaged as well as damaged materials. In the case of undamaged materials, some additional assumptions allow to link the mesoscopic quantities to the usual (macroscopic) stresses and strains through closed form relationships.

1. HISTORICAL PERSPECTIVE IN THE CONTEXT OF MECHANICAL DESIGN

The first accident due to fatigue happened at the beginning of the use of railways. The Paris Versailles accident of 1842 was the first one to cause numerous victims. This accident was followed by many discussions and conjectures in the scientific world about the real origin of the failure. Among different hypotheses, one can mention the influence of heat, magnetism, or crystallization of iron. Some persons thought that these phenomena take place instantaneously. The first man to suppose that the development of these unexpected fractures was gradual was William Rankine. The term of fatigue was first used by Poncelet in France and by Frederick Braithwaite in England. Since then, many studies have been done during the nineteenth century, in which the work of Wöhler, a German railways engineer, was outstanding. He established the S-N curve approach and introduced the concept of fatigue limit, that is the concept of a limiting stress range below which failure will never occur. This concept was particularly important for applications and had a great influence on the German school of fatigue. The role of the mean stress (i.e. the influence of the loading path characteristics) on the fatigue limit was discovered only later, because Wöhler experiments were of rotating bending types so that the mean value of the stress was zero. During nearly one hundred years, most of the fatigue researches were devoted to the high cycle fatigue regime and in establishing S-N curves for metallic materials. The S-N curve approach is still often used as a design tool to predict the fatigue life of mechanical structures. One can notice that before the sixties, very few studies were done on topics that are now well diffused, like low cycle fatigue or application of fracture mechanics to fatigue crack propagation behaviour. The development of aeronautical industries followed by the nuclear program were then powerful stimulants to promote fatigue research.

Instead of developing stress approaches, Coffin at General Electric and Manson at NASA proposed fatigue models based on strain amplitude, and particularly plastic strain amplitude for low cycle fatigue. These approaches work well in the cases of aeronautical and nuclear energy structures because these structures have to sustain a few number of severe cycles. For instance, a cycle for an airplane consists of a take-off-landing cycle, without taking account of the other small cycles. Nearly at the same time, Paris overcame the doubts of reviewers on the application of linear fracture mechanics approach to study fatigue crack propagation. This new way for studying this old science found a great audience. With the help of new experimental devices, the development of research on fatigue crack propagation became exponential. Nowadays, many authors consider that fatigue is only a crack propagation phenomenon. Therefore, the fatigue limit corresponds to a fatigue crack propagation threshold. This trend is favored by increasingly precise observation devices. These new tools for fatigue research had so much success that the more classical domain of high cycle fatigue was more or less neglected. Indeed, few new ideas appeared in this field, despite the fact that numerous fatigue criteria have been proposed. Surprisingly, most of the time the theoretical tools, used by the mechanical industry, are still those developed during the 19th century and the beginning of this

century. These tools are mainly the S-N curve and the Goodman diagram, even if they are only applicable in uniaxial situations. The design of structures, which have to resist fatigue and particularly high cycle fatigue, is still a problem: engineers have to perform difficult and time consuming experiments to find the fatigue limit of a mechanical component. This is done generally on the structure itself, which necessitates iterative experiments on full-scale test models. The need of more efficient design tools and the generalization of the finite elements method favored the development of new approaches in high cycle fatigue. To be useful, these methods must be applicable under multiaxial loading conditions that are frequently encountered on modern mechanical structures.

The main purpose of this paper is to present a non classical way for high cycle fatigue modeling, based on the use of a multiscale approach. Foundations of this method are settled on the following remarks. The first fatigue phenomena, which shall control the future behaviour of the mechanical structure, occur at the level of grain size (microscopic or mesoscopic scale). Engineering computations suppose implicitly a representative elementary volume of definite size surrounding the point where the computations are done. This macroscopic volume contains a vast amount of grains, so that the local mechanical parameters differ from the macroscopic ones. Estimating the links between those quantities is a difficult question to which we try to answer, thanks to some reasonable assumptions. The main hypothesis is that before fatigue crack nucleation, the material shakes down elastically at all scales. To strengthen our argumentation, we shall recall the main difficulties we have met in studying different fatigue phenomena by the classical existing approaches, within a structural computational framework.

2. REMARKS ON SCALES OF MATERIAL DESCRIPTION IN FATIGUE

Fatigue failure is the final step of a complex physical process difficult to model. It begins with the appearance of slip bands in some grains that broaden progressively during the following cycles. These phenomena occur simultaneously with the development of localized damage. The processes mentioned above result in the formation of intragranular microcracks at a stage determined by the imposed load level. After a certain number of cycles, a main crack initiates, then propagates and shields the other defects and subsequently continues to the final rupture of the mechanical component. These different stages of fatigue damage are represented on Fig. 1(a). However, this description of the fatigue damage phenomenon is schematic, since it does not make reference to the many types of fatigue which are depending on the imposed stress and strain level.

Low cycle fatigue and high cycle fatigue are traditionally distinguished. Low cycle fatigue involves significant plastic deformation (of the order of few percent or a fraction of percent) and corresponds to short lifetime (thousand to some ten thousands of cycles). In this loading regime, the metal grains suffer deformation in a homogeneous manner, Fig.1(b). Moreover, the initiation of the first microcracks in the persistent slip bands happens quite early in the life of the structure. Most of the life is then spent in propagation.

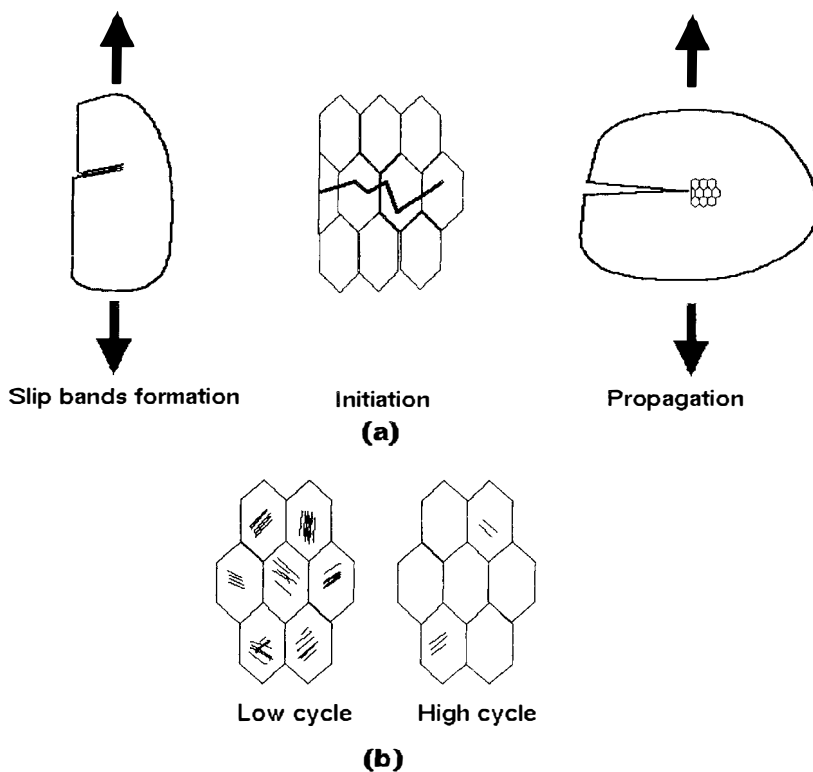


Fig. 1, (a) Different stages of fatigue life,
(b) Plastic strain aspects in low and high cycle fatigue

In the high-cycle fatigue regime, usually no irreversible deformation at the macroscopic level can be detected. This type of fatigue is characterized by a large heterogeneity of plastic deformation from grain to grain. Only certain misoriented crystals undergo plastic slip as shown in Fig. 1(b). Indeed, a very heterogeneous distribution of microcracks can be observed even if the macroscopic loading is homogeneous. In high-cycle fatigue, the initiation of the first visible cracks takes a large part of the fatigue life of the structure. This is the reason of the great importance of studying the conditions governing the crack initiation process. Many researches have thus been undertaken on this topic for many years. They allow a better understanding of some aspects of the high cycle fatigue phenomena. However, many unsolved questions remain and even controversies arise with the development of sophisticated experimental techniques and observation tools. Thus, the defects can be detected more early, so that the widespread current tendency consists of considering that fatigue is essentially a propagation phenomenon. The fatigue

limit of a material can then be defined as a non-propagation limit. Nevertheless, this way to consider the problem is questionable for several reasons.

In fact, numerous studies on short cracks (that is the ones of very small size, generally inferior to 50μ) have shown that these short cracks can propagate, even if they are submitted to $\Delta K < \Delta K_{th}$. For this propagation to occur it is sufficient that the amplitude of the applied stress is high enough (i.e. superior to the fatigue limit of the material). Kitagawa, [1], proposed to use this property to establish the limit between the initiation phase and the propagation phase. On Kitagawa's diagram, which essentially represents the locus of non-propagating cracks, one can distinguish two regions corresponding to two regimes. A Kitagawa's diagram for low carbon steel is reproduced in Fig. 2. Regime I, limited by a horizontal dashed line in Fig. 2, corresponds to the fatigue limit, whereas regime II corresponds to the long crack threshold of linear fracture mechanics. The intersection of these two lines defines the separation between the initiation phase and the propagation phase.

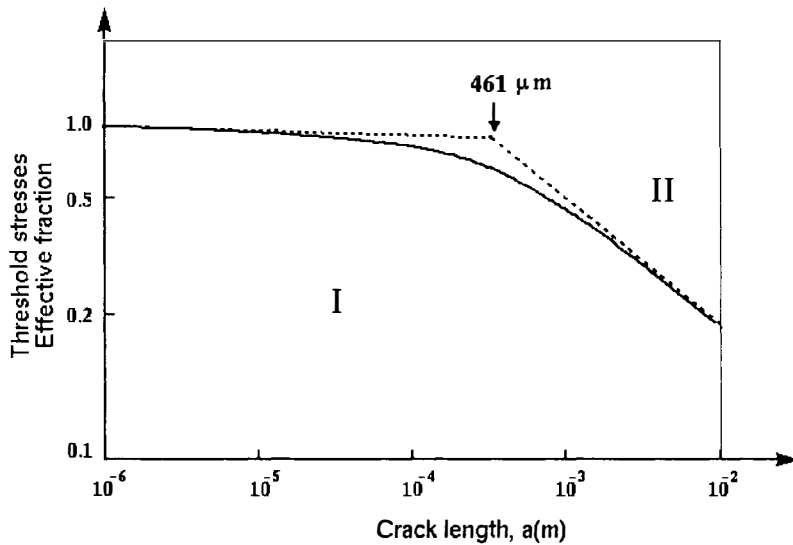


Fig. 2, Kitagawa diagram

Some authors try to describe this evolution with the help of macroscopic parameters. Thus, one proposes « corrections » to the stress intensity factor to correlate the results. See for instance, El Haddad's correction, which is nothing but a technique to fit experimental curves in a particular uniaxial loading. Furthermore, numerous empirical formulas have been proposed. In these formulas intervene macroscopic parameters such as ΔJ and CTOD. Even the shear stress acting on the plane of the microcracks and/or the

plastic sliding along these planes have been taken into consideration (for a review see D.L. McDowell [2]). Nevertheless, the way to evaluate these quantities is questionable because, as will be explained later, these local parameters differ from the macroscopic quantities. The use of macroscopic parameters at the scale of these defects does not seem pertinent. Indeed, the material at that scale of description cannot be considered any more as homogeneous. Consequently, the mechanical quantities at the microscopic scale (the scale of dislocations or a little beyond) or at the mesoscopic scale (the scale of grains), differ from those used in these theories. The difference is mainly due to the local stress redistribution induced by the inhomogeneous incompatible strains developed in the grains. This basic feature of the metals behaviour is not taken into account in the aforementioned models. Finally, considering the fatigue phenomenon only as a propagation event does not simplify the problem. It rather introduces complications in practical applications because the mechanical structures are frequently submitted to multiaxial loading. Nowadays, fracture mechanics cannot treat these problems accurately.

This is the reason why we choose a different way to proceed, while trying to bring back the problem of the evaluation of the fatigue resistance to the field of classical continuum mechanics, but using a pertinent scale of description of the fatigue phenomenon, which is the mesoscopic scale. To establish our proposal, we first recall the general relations between macroscopic quantities, which are usually used by the engineers, and local quantities. Because these quantities differ, we will propose next an approach to evaluate the local mechanical parameters for loading conditions near the fatigue limit. For that purpose, it is assumed that the material is in an elastic shakedown state before the initiation of a fatigue crack. This is natural hypothesis. Indeed, the fatigue limit considered from this point of view consists simply to assume that it is not possible to find a state where the material remains elastic at the end of a certain number of cycles.

3. RELATIONS BETWEEN MACROSCOPIC AND MESOSCOPIC PARAMETERS

Two different viewpoints can be adopted for studying the properties of materials:

- The engineer approach, which is purely macroscopic and phenomenological.
- The physicist approach, according to which the crystal grain (mesoscopic scale), or even the dislocations (microscopic scale) are considered.

These two research directions are developed in parallel and their quantitative links for applications are often difficult to establish. Without considering such a fine scale as the physicist scale, it is clear that the global mechanical properties of the materials depend on their constituents and their defects (pores, microcavities, cracks, oxidized zones etc.). However, the macroscopic properties are the only properties, which can be evaluated by the engineers with ordinary direct experimental measurements. The constitutive relations are then formulated by means of macroscopic mechanical variables such as the stress Σ and the total strain \mathbf{E} or plastic strain \mathbf{E}^p , as well as their rates. These relations are

sufficient for a large number of engineering applications, such as common design of structures. Nevertheless, when one is interested in finer scale properties, as for example when trying to characterize the fatigue damage and its evolution, then such approaches are not appropriate. This is because damage is controlled by phenomena at a smaller scale; their direct effects cannot be detected by macroscopic measurements. What the engineer perceives is already some how filtered by the representative volume of the macro-element, and, a fortiori, by the test specimen. At the local scale that is of interests to us, it is necessary to take these micro-effects into account in a way similar to structural effects which are well known by the engineers. One of the principal difficulties in the study of fatigue is the difficulty to discriminate between work-hardening, usually beneficial to the material, and damage, which could be defined as a general degradation of the mechanical properties. This is the reason why a direct macroscopic approach based on experimental observations and phenomenological modeling is insufficient. It is essential to identify clearly the parameters characterizing material damage, independently of work-hardening effects. These damage parameters are tensorial by nature and need to be averaged to obtain macroscopic effects that are measurable by the engineers. These parameters must be, as much as possible, consistent with the principles of mechanics and thermodynamics.

3.1 Macroscopic scale and representative elementary volume

The engineer evaluates mechanical parameters (for example the stress field Σ or the strain field \mathbf{E}) not at a point of a structure, but over a finite surface S or volume V that defines the scale which is used. More precisely, this volume can be of the order of one cubic millimeter in relation with the dimension of usual strain gauges, or it can also be related to the local mesh size in a finite elements calculation. In this volume, the mechanical quantities Σ and \mathbf{E} are assumed constant. Nevertheless, this volume is heterogeneous and even anisotropic since it is composed by a large number of crystal grains. For instance, if the average size of the grains is about 10μ , V contains roughly one million grains, which have different properties, because of their orientation, their initial strain hardening state etc. Furthermore, there are other origins of heterogeneity, such as grain boundaries, inclusions, precipitates, so that within this volume, the local parameters differ considerably from the macroscopic values. Actually, it is precisely the local parameters, which are at the origin of fatigue crack initiation. It is therefore interesting to know the relations between the macroscopic quantities and the local quantities that intervene in the fatigue initiation process. Furthermore, this knowledge allows to critically review various approaches proposed so far, in a rational way. For that purpose, it is necessary to define the representative elementary volume concept. The representative elementary volume is denoted as RVE. An intuitive definition of RVE could be as follows: *the representative elementary volume must be sufficiently small so that one can distinguish the microscopic heterogeneities and sufficiently large to be representative of the overall macroscopic behaviour*. In the theories of polycrystalline aggregates, it is generally assumed that the geometry of each constituent phase is known. This allows the formulation

of statistical hypotheses on homogeneity and ergodicity, which are used to provide a more precise definition of RVE (Kröner, 1980; Willis, 1981; Hashin, 1983; Stolz, 1996). Here, we will restrict ourselves to the intuitive definition of RVE given before.

In the following, we are interested on the characterization of damaged media, which contain defects such as cavities, cracks, oxidized zones, etc. Our task is to detect the onset of crack initiation. Then it is necessary to distinguish defects for which the material cannot be considered as completely separated, (although one can observe discontinuities of material properties as for example persistent slip bands), from the cracks which are really initiated. This is the reason why, we propose to characterize the latter by the fact that, these defects (i.e. cracks) cannot transmit traction under the effect of external loading. Generally, one can characterize the damaged medium only by this property, which is different from work hardening that precedes the onset of damage phenomena in a metal free of defects. In this manner, a line of physical properties discontinuity, onto which interatomic cohesion forces corresponding to the Bareblatt's theory (Barenblatt, 1967) are still acting, is not considered as a crack, since $\sigma \cdot \mathbf{n} \neq 0$ on this line.

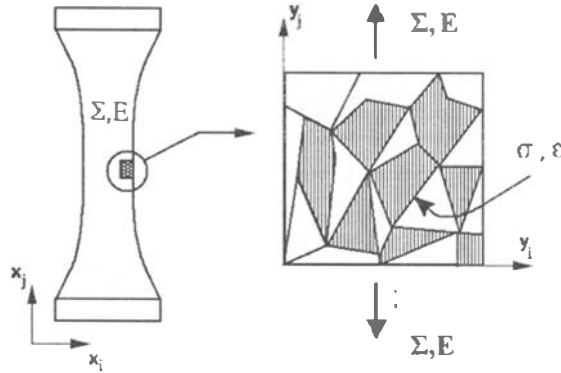


Fig. 3, Representative volume element, macroscale and mesoscale

Let us then consider a representative volume element V with boundary ∂V , Fig. 3. This "macroscopic" volume element is for an external observer apparently homogenous and is submitted to homogenous loading. In the following, capital letters are reserved for quantities corresponding to V that are qualified as macroscopic, such as Σ stress tensor, E strain tensor, U displacement vector.

By definition, the quantities Σ and E are constant in V . Nevertheless, at the local scale V is inhomogeneous and contains defects. The corresponding local fields will be denoted by $\sigma(\mathbf{x})$, $\varepsilon(\mathbf{x})$, and $\mathbf{u}(\mathbf{x})$.

According to what was said before, it is natural to assume that a damaged zone or a cracked zone is characterized by the fact that the stress vector is zero on its boundary. If the volume V contains at the local scale damaged zones of volume $Z(t)$, then $D(t) = V - Z(t)$ represents the sound material volume at time t and ∂Z the boundary between D and Z . In the following, one denotes by $\langle f \rangle$ the average of $f(x)$ in the sound material volume, i.e.:

$$\langle f \rangle = \frac{1}{V} \int_{D(t)} f dV$$

A set of relations linking the macroscopic quantities to the local ones will be demonstrated (see for instance [3], [4]). Our reasoning is based on the theorem of virtual work.

3.2 Relation between stress tensors

Let us impose on the boundary ∂V of the RVE the traction forces $\mathbf{T}(x) = \mathbf{\Sigma} \cdot \mathbf{n}$, where $\mathbf{\Sigma}$ is a constant tensor that corresponds to the macroscopic stress field. The resulting local field is statically admissible with \mathbf{T} , i.e. it verifies $\mathbf{\Sigma} \cdot \mathbf{n} = \mathbf{T}$ on ∂V and $\boldsymbol{\sigma} \cdot \mathbf{n} = 0$ on $\partial Z(t)$. One can show easily that:

$$\mathbf{\Sigma} = \langle \boldsymbol{\sigma} \rangle$$

Indeed, whatever the constant strain field \mathbf{E} deriving from the displacement field $\mathbf{U} = \mathbf{E} \cdot \mathbf{x}$, one has:

$$\int_V (\mathbf{\Sigma} - \boldsymbol{\sigma}) : \mathbf{E} dV = \int_{\partial V} (\mathbf{\Sigma} - \boldsymbol{\sigma}) \cdot \mathbf{n} \cdot \mathbf{U} dS - \int_{\partial Z} \boldsymbol{\sigma} \cdot \mathbf{n} \cdot \mathbf{U} dS$$

The preceding (classical) result means that *the average of local stress is equal to the macroscopic stress*. It is reminded that due to local heterogeneities these two fields may differ. However, their difference is in average (on the RVE) equal to zero. *Nevertheless, in fatigue studies, one has to characterize this difference since, the cracks initiate in the most critical places relative to the level of loading or to the resistance characteristics*. It is precisely what it is proposed to be done in the fatigue nucleation model developed later. It will be shown hereafter that *the equality in average of the local and macroscopic stress fields cannot be extended to the other mechanical quantities*.

3.3 Relation between total strain tensors

In particular, concerning the strains, one has the following relation between the total macroscopic strain \mathbf{E} and the local strains $\boldsymbol{\varepsilon}$.

$$\mathbf{E} = \langle \boldsymbol{\varepsilon} \rangle - \frac{1}{2V} \int_{\partial Z} (\mathbf{u} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{u}) dS \quad (1)$$

Strain rates verify an analogous relation. The above relation seems intuitively evident. Indeed, it indicates that the macroscopic deformation is the sum, of the average of the local deformation of the material and of the deformation induced by the holes and cracks contained in V (imagine the deformation of a sponge for instance). *It is only when these defects do not exist in the RVE that one has $\mathbf{E} = \langle \boldsymbol{\varepsilon} \rangle$.* The demonstration of Eq. (1) can be performed as follows. Consider the specific stress tensor $\Sigma_{ij} = (\delta_i^p \delta_j^q + \delta_i^q \delta_j^p) / 2$. Obviously, this tensor verifies the equilibrium equation. Moreover, let $\mathbf{U}(\mathbf{x})$ be the displacement field from which the strain tensor $\mathbf{E}(\mathbf{x})$ is derived. These stress and strain fields being respectively statically and kinematically admissible, one has by the virtual work equation,

$$\int_V \boldsymbol{\Sigma} \cdot (\nabla \mathbf{U}) dV = \int_{\partial V} \boldsymbol{\Sigma} \cdot \mathbf{n} \cdot \mathbf{U} dS$$

from which one deduces easily:

$$\mathbf{E} = \frac{1}{2} [\nabla \mathbf{U} + (\nabla \mathbf{U})^T] = \frac{1}{2V} \int_{\partial V} (\mathbf{U} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{U}) dS \quad (2)$$

On the other hand, let $\boldsymbol{\varepsilon}$ be the local strain field associated to the displacement field \mathbf{u} . The virtual work equation gives:

$$\int_V (\boldsymbol{\Sigma} : \boldsymbol{\varepsilon}) dV = \int_{\partial V} (\boldsymbol{\Sigma} \cdot \mathbf{n} \cdot \mathbf{u}) dS + \int_{\partial Z} (\boldsymbol{\Sigma} \cdot \mathbf{n} \cdot \mathbf{u}) dS$$

The stress $\boldsymbol{\Sigma}$ is constant within V . Then the above relationship leads to:

$$\frac{1}{V} \int_V \boldsymbol{\varepsilon} dV = \frac{1}{2V} \int_{\partial V} (\mathbf{u} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{u}) dS + \frac{1}{2V} \int_{\partial Z} (\mathbf{u} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{u}) dS$$

In addition, by subtracting the above relation from Eq.(2), one obtains Eq.(1) if it is admitted that the following equality holds:

$$\int_{\partial V} (\mathbf{U} - \mathbf{u}) \otimes \mathbf{n} dS = 0$$

The above relationship means that the real displacement field \mathbf{u} , fluctuating on the boundary of the RVE, is in average equal to the macroscopic field \mathbf{U} , which is calculated using the average macroscopic material properties. This relation results from the macrohomogeneity hypothesis of Hill-Mandel, which is used, in the theory of polycrystalline aggregates (Stolz, 1996 [5]).

3.4 Conservation of the total energy and the total energy rate

Concerning the energy, as $\boldsymbol{\sigma} \cdot \mathbf{n} = 0$ on the boundary of internal damaged zones, one has:

$$\langle \boldsymbol{\sigma} : \boldsymbol{\varepsilon} \rangle = \int_V \boldsymbol{\sigma} : \boldsymbol{\varepsilon} \, dV = \int_{\partial V} \boldsymbol{\sigma} \cdot \mathbf{n} \cdot \mathbf{u} \, dS = \int_{\partial V} \mathbf{T} \cdot \mathbf{u} \, dS = \int_{\partial V} \mathbf{T} \cdot \mathbf{U} \, dS = \boldsymbol{\Sigma} : \mathbf{E}$$

This corresponds to the energy conservation. The same reasoning applies to the work rate:

$$\langle \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} \rangle = \boldsymbol{\Sigma} : \dot{\mathbf{E}}$$

It is noticed that these identities apply equally well to non-associated fields, provided that the stresses are statically admissible and the strains are kinematically admissible.

3.5 Relations between the elastic and plastic macroscopic and mesoscopic strain tensors

Let us consider the elastic and plastic strain tensors, in the case *where the RVE does not contain damaged zones*. Let us recall that in this case the relation between total strains is simply $\mathbf{E} = \langle \boldsymbol{\varepsilon} \rangle$. Let us admit that the local strain as well as the macroscopic strain can be split into elastic and plastic parts according to the formulas:

$$\boldsymbol{\varepsilon}(\mathbf{x}) = \boldsymbol{\varepsilon}^e(\mathbf{x}) + \boldsymbol{\varepsilon}^p(\mathbf{x}) \quad \boldsymbol{\varepsilon}^e(\mathbf{x}) = \mathbf{M}(\mathbf{x}) : \boldsymbol{\sigma}(\mathbf{x})$$

$$\mathbf{E}(\mathbf{x}) = \mathbf{E}^e(\mathbf{x}) + \mathbf{E}^p(\mathbf{x}) \quad \mathbf{E}^e = \mathbf{M}^O : \boldsymbol{\Sigma}$$

$\mathbf{M}(\mathbf{x})$ is the local elastic compliance tensor (which depends on \mathbf{x}), and \mathbf{M}^O the macroscopic elastic compliance tensor (which is constant). Then, one can deduce from the previous formulas the following relationship:

$$\mathbf{E}^p = \langle \boldsymbol{\varepsilon}(\mathbf{x}) \rangle - \mathbf{E}^e = \langle \boldsymbol{\varepsilon}^p(\mathbf{x}) \rangle + \langle \mathbf{M}(\mathbf{x}) : \boldsymbol{\sigma}(\mathbf{x}) \rangle - \mathbf{M}^O : \boldsymbol{\Sigma}$$

Let us notice that:

$$\langle \mathbf{M}(\mathbf{x}) : \boldsymbol{\sigma}(\mathbf{x}) \rangle \neq \mathbf{M}^O : \boldsymbol{\Sigma}$$

The above relationship has the (usual) meaning that the mean value of a product is different from the product of the mean values. Here, this is due to the heterogeneity of the material at the mesoscopic scale. Indeed, *there is no simple direct relation between the macroscopic plastic strain and the average value of the mesoscopic plastic strain*. One has in corollary the same property for the elastic parts.

3.6 Relation between macroscopic and mesoscopic plastic dissipation

In the same way, *there is no link between the macroscopic plastic dissipation and the local plastic dissipation*:

$$\Sigma : \mathbf{E}^p \neq \langle \boldsymbol{\sigma} : \boldsymbol{\varepsilon}^p \rangle$$

Indeed, $\boldsymbol{\varepsilon}^p$ is not kinematically admissible and \mathbf{E}^p is different from $\boldsymbol{\varepsilon}^p$. The second member of this relation is equal to the macroscopic plastic dissipation only if the solid is a perfectly plastic material. If the material behaviour is not perfectly plastic, then there is no a link with the plastic macroscopic dissipation represented by $\Sigma : \mathbf{E}^p$. It is reminded that some fatigue models are based precisely on the plastic dissipation. Naturally, the plastic dissipation involved in the fatigue crack initiation phenomenon, is the mesoscopic one. In view of the above result, this quantity cannot be accessed by global measures on test specimens. However, the difference between macroscopic and mesoscopic plastic dissipation decreases with increasing plastic strain because the material behaviour tends to perfect plasticity. Thus the use of this parameter to model fatigue becomes more adequate for higher values of plastic strain range. The case where the RVE presents damaged zones is more complex. The reader can find the main results in Bui et al., [4].

3.7 Conclusion

Most of the existing fatigue models are based on engineering macroscopic parameters, even if they treat of mesoscopic or microscopic phenomena. In some cases, this approach is justified (e.g. when the problem of *fatigue propagation of long cracks* is considered). In that case, the *stress field, which is governed by the stress intensity factors, is used as the principal part of an asymptotic solution*. Let us notice that the local field is evaluated under the assumption of small strain and linear elastic behaviour. In the close vicinity of the crack tip, this basic hypothesis does not hold any more. However, if one suppresses the outer part of some RVE surrounding the crack tip, and replace the external loading by a tension $\mathbf{T} = \Sigma \cdot \mathbf{n}$, then the global response is not changed. This RVE is sufficiently large to represent the mean properties of the material. If the active stress intensity factor range ΔK is of the order of $10 \text{ MPa}\sqrt{m}$, for a material with a yield strength of about 500 MPa, the radius r_0 of the RVE can be estimated by:

$$\sigma_y \sqrt{r_0} \approx \Delta K \Rightarrow 500 \sqrt{r_0} \approx 10 \Rightarrow r_0 \approx 400 \mu$$

This value is already a large compared to the grain size. If ΔK is about $5 \text{ MPa}\sqrt{m}$, r_0 is about 100μ . It is then expected that the corresponding RVE contains at least 1000 grains. This is sufficiently large for a statistical estimation. It is then reasonable to use the macroscopic stress for calculating ΔK . However, in many other cases the use of macroscopic mechanical quantities does not seem pertinent. In particular in the models dealing with short cracks propagation, this use is not appropriate because at that scale the local stress or strain can be very different from the corresponding macroscopic quantities.

4. MAIN PROBLEMS IN FATIGUE OF STRUCTURES

Before studying the high cycle fatigue domain, it is worthwhile to comprehend some important problems encountered by engineers in mechanical design. Furthermore, it is interesting to present the main trends in the research conducted by them to obtain answers to their questions. During the last decades fatigue problems were studied most of the time within the framework of the nuclear and the aeronautical engineering, because these are the only two high technology industries, which have enough finances to support extensive experimental researches. It is one of the reasons why fatigue research has mainly focused on low cycle fatigue and fatigue crack growth. This trend is further favored by the development of sophisticated test machines, which allow in a relatively easy way to carry out tests under constant strain amplitude. Fatigue tests are performed in tension-compression under controlled constant plastic strain amplitude or constant total strain amplitude. The parameters which are characteristic of this particular loading path are then related to the number of cycles N_f at failure of the test specimen. For instance, the relationship between plastic strain amplitude and number of cycles to failure, known as Manson-Coffin law, describes the fatigue behaviour of the specimen if plasticity is predominant. The Basquin's law relating the elastic part of the strain amplitude to the number of cycles expresses better the fatigue behaviour if global elastic regime with very limited plasticity is the concern. These two types of law apply respectively to low cycle fatigue (some hundred to some ten thousand of cycles) and to ordinary fatigue. High cycle fatigue (also called polycyclic fatigue) which will be studied later in this paper is concerned with longer life duration from hundred thousands to millions of cycles.

If the temperature is an essential parameter, the problems are even more complex both from the experimental point of view and from the theoretical point of view. The non-linear behaviour of the material varies with temperature. Generally, viscosity effects (which imply loading rate influence) are important at higher temperatures whereas plasticity effects are dominant at lower temperatures (no loading rate influence). At intermediate temperatures, one can have combination of these two effects. Thus, the evaluation of the stresses and strains at every point of a structure subjected to thermomechanical cycling is a very difficult problem. Indeed, it seems that this problem has not yet been solved in a satisfactory way. Extensions of the Manson-Coffin law to take into account the simultaneous action of temperature and mechanical cycling have been proposed by the American school. These methodologies are the strain range partitioning method and the modified frequency method. However, these proposals are mainly curve fitting techniques for uniaxial mechanical loading at different constant temperatures. It is our opinion that the applicability in a predictive way of these methodologies to real industrial structures has not been demonstrated today.

Application of the Manson-Coffin law or of the Basquin law to the fatigue analysis of structures requires the transformation of these uniaxial relations into multiaxial relations. For this purpose, generally the equivalent stress and/or strain amplitude of von Mises is used. It is noticed that in the case where the loading is characterized by fixed principal strain/stress directions relatively to the structure, the Tresca's equivalent stress and/or strain can also be used without ambiguity. In both cases, it is necessary to calculate the cyclic response at any point of the structure. This requires a good description of the elastoplastic behaviour of the material. The first idea is to conduct an elastoplastic calculation of the structure for one loading cycle using the cyclic stress-strain curve. The cyclic stress-strain curve is constructed from experiments performed at different plastic strain ranges by reporting the value of the peak stress in the stabilized cycle against half the strain range. This method, though very popular among engineers, has no theoretical foundation. Indeed, a cyclic curve is obtained under a particular loading path which perhaps has nothing to do with the loading path observed at each point of a cyclically loaded structure. For instance, for a given strain amplitude, the peak stress reached in a stabilized cycle can depend on the mean strain, or on the maximum strain reached in a previous cycle. The interesting reader can consult reference [6], by Chaboche, Cordier and Dang Van, on this subject. Later, more elaborated constitutive equations were proposed based on researches performed in various countries: Mroz in Poland, Armstrong and Friedrichs in England, Chaboche and Lemaitre in France, Eisenberg and Phillips, Dafalias and Popov, McDowell in the USA and many others. For the description of the materials behaviour, the main ideas appearing in all the existing models are:

- The introduction of a set of kinematic hardening parameters, denoted by \mathbf{A} . These parameters describe the translation of the center of the current elastic domain.
- The description of the evolution of the diameter of the current elastic domain through an isotropic hardening parameter $K(E_{eq}^p)$, generally depending on the accumulated equivalent plastic strain defined by:

$$E_{eq}^p = \int_0^t \sqrt{\frac{3}{2} \mathbf{E}_{ij}^p \mathbf{E}_{ij}^p} dt \quad (3)$$

In the uniaxial case (tension-compression cycles for instance), the evolution of the elastic domain shown in figure 4 can be represented by the following equations:

$$f(\Sigma, \mathbf{A}) = |\Sigma - \mathbf{A}| - K \leq 0$$

$$(d/dt) \mathbf{A} = \dots$$

$$(d/dt) K = \dots$$

The second member of the two last equations above depends on the particular model proposed by each author.

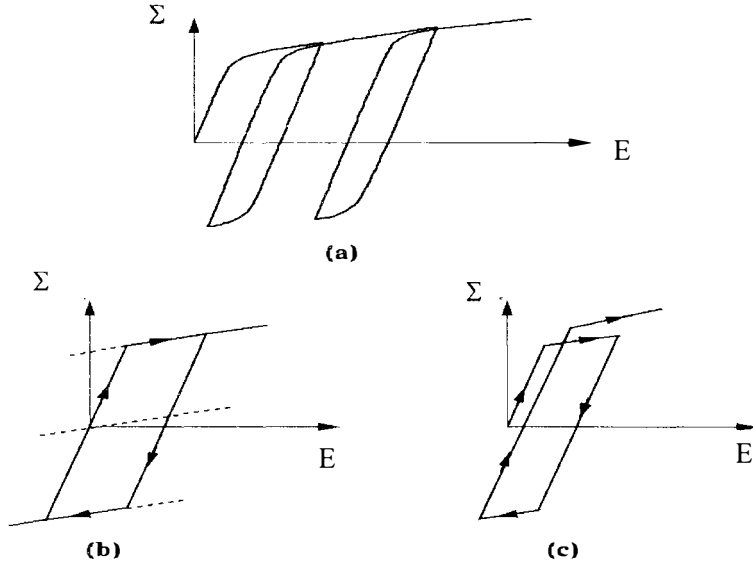


Fig. 4, (a) Typical stress-strain curve, (b) Linear kinematic hardening behaviour, (c) Linear isotropic hardening behaviour

Generalizations in three dimensions are usually obtained by choosing the von Mises equivalent stress as the function f . Moreover, the scalar cinematic hardening parameter A is replaced by tensorial quantities \mathbf{A} . Let us notice that under two dimensional loading like tension-shear (Σ_{xx}, Σ_{xy}), the above choices imply that the initial elastic domain represented by an ellipse E , will transform to an ellipse E' having its axes in the same directions as the initial one. Indeed, no rotation or changes of the shape of the elastic domain are allowed by these models. The only transformations permitted are the translation of the center of the yield locus by \mathbf{A} , along with its expansion defined by the evolution of the parameter K . However, the behaviour of materials is very often much more complex and cannot be accurately described by such assumptions. This has been demonstrated by the pioneering experiments performed by Bui in the late sixties, [3]. In these experiments thin-walled cylindrical specimens have been submitted to combined tension and torsion loading. Two materials were studied, Armco iron and aluminum. After performing a test following a particular loading path, the current elastic domain has been accurately determined through many tests along radial loading paths. The current yield locus is depicted in Fig.5(a) for Armco iron and in Fig.5(b) for aluminum. It can be seen that the current elastic domain in addition to its translation and expansion has also suffered heavy distortion. Indeed, the initially elliptic domain remains no longer an ellipse but deforms in the direction of loading and the description of its evolution is much more complex. In his thesis, Bui has proposed

a model to explain such kind of behaviour generalizing an idea of J. Mandel on vortex formation. This idea was also employed later by other researchers but it did not lead to any practical development of constitutive equations suitable for structural analysis. The real behaviour of materials is too complex, so that even now no completely satisfactory theory exists. Modeling of the material behaviour is a compromise between required precision of the description, type of the loading path imposed to the structure, and ease of identification of the parameters that are introduced in the model.

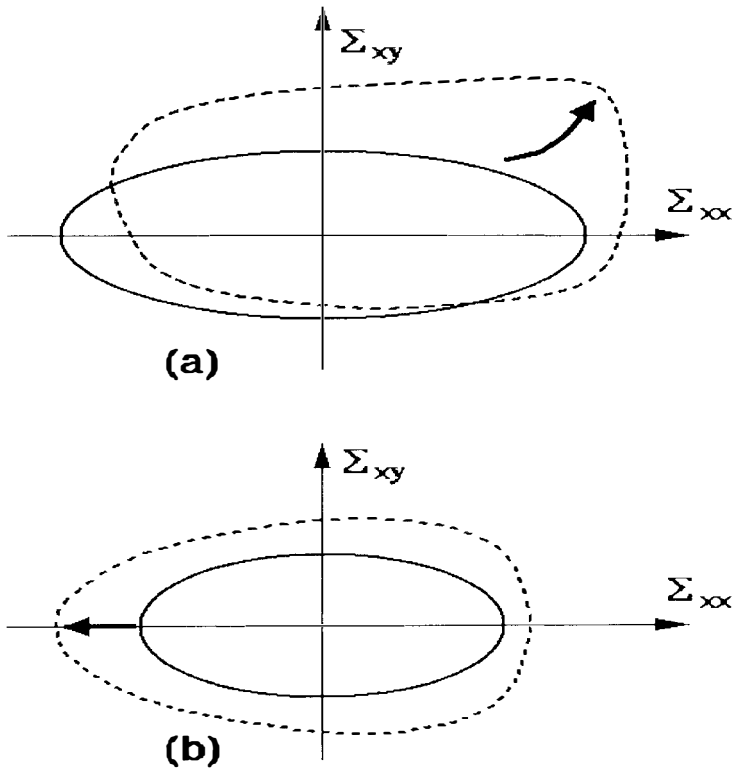


Fig. 5, Influence of loading path on current yield locus.
(a) Armco iron, (b) Aluminum

This is why some engineers prefer to use simpler constitutive equations, Fig. 4(b), like:

- Isotropic hardening :

$$f(\Sigma) = J_2(\Sigma) - K^2(E_{eq}^p)$$

- Linear cinematic hardening (Prager- Ziegler model) :

$$f(\Sigma, A) = J_2(\Sigma - CE^p) - K^2$$

- A combination of these two models:

$$f(\Sigma, A) = J_2(\Sigma - CE^p) - K^2(E_{eq}^p)$$

Under cyclic loading, the limit state is of elastic shakedown with the first and the third models. With the second model, depending on the level of external loading, the predicted limit state can either be elastic or plastic shakedown. Ratcheting behaviour, i.e. no stabilized strain state, cannot be obtained with this type of model (Halphen, [7]). As it is shown by many authors (see for instance Bower and Johnson, [8], [9]), ratchet can be obtained by models of non-linear kinematic hardening. In conclusion the structural asymptotic response obtained by calculation is very much dependent on the chosen constitutive equations and, of course, on the level of the external loading. Despite a great number of investigations, general constitutive equations which are applicable to all situations do not exist. The choice of a constitutive equation for the computation of a structure submitted to cyclic loading is a compromise between simplicity, feasibility and precision of the description. This is why the evaluation of the structural life under plastic fatigue (and a fortiori under transient thermo-plastic fatigue) is still now a difficult problem despite the relative simplicity of the low cycle fatigue laws.

In the case of high cycle fatigue, the difficulties are of a different nature because most of the time the whole structure remains in an apparent elastic state. Only contained and localized plasticity can occur. This means that after a certain number of cycles an apparent elastic stabilized state is reached, i.e. the structure shakes down elastically. In our approach, we postulate that before fatigue crack initiation the structure is in an elastic shakedown state at all scales. Under this assumption, the precise knowledge of the local constitutive equations is not necessary. Indeed, thanks to Melan's theorem and its generalization by different authors (Mandel, Zarka, and Halphen, [8]) we can derive a method for estimating the apparent stabilized cycle of the stress and strain tensors at the macroscopic and the mesoscopic level. This way, the mesoscopic quantities, which intervene in the proposed fatigue criteria, can be estimated, provided that the RVE of the material is considered as a structure made of different grains.

5. Elastic shakedown of an elastoplastic structure

In this section, Melan's elastic shakedown theorem and its extensions are presented.

5.1 Residual stress and residual strain

Let us consider an elastoplastic structure submitted to cyclic loads $F(t) = F(t+\Theta)$, Θ being the period and eventually to cyclic displacements of same period imposed on part of its boundary. Let $\Sigma(x,t)$ denotes the current stress response at a point x . Moreover, $\Sigma^{el}(x,t)$ represents the stress response at the same point, under the assumption of a pure elastic behaviour. The residual stress field at the same time t is defined by the difference between these two stress fields:

$$\mathbf{R}(x,t) = \Sigma(x,t) - \Sigma^{el}(x,t) \quad (4)$$

$\mathbf{R}(x,t)$ is thus the remaining stress field in the structure after removal of the loads F and after release of the imposed displacements at the time t , provided that this unloading is purely elastic. Thus, $\mathbf{R}(x,t)$ is a field of self equilibrated stress, because both of Σ and Σ^{el} are statically admissible. Let $\mathbf{E}(x,t)$ be the strain field in the elastoplastic structure at time t . One has :

$$\mathbf{E}(x,t) = \mathbf{E}^e(x,t) + \mathbf{E}^p(x,t) = \mathbf{M}(x) : \Sigma(x,t) + \mathbf{E}^p(x,t)$$

or

$$\mathbf{E}(x,t) = \mathbf{M}(x) : (\Sigma^{el}(x,t) + \mathbf{R}(x,t)) + \mathbf{E}^p(x,t) = \mathbf{E}^{el}(x,t) + \mathbf{M}(x) : \mathbf{R}(x,t) + \mathbf{E}^p(x,t) \quad (5)$$

where $\mathbf{E}^{el}(x,t) = \mathbf{M}(x) : \Sigma^{el}(x,t)$ is the strain response to the loading $F(t)$ under the assumption of a pure elastic behaviour. The equation (5) can be written :

$$\mathbf{E}(x,t) = \mathbf{E}^{el}(x,t) + \mathbf{E}^r(x,t) \quad \text{with} \quad \mathbf{E}^r(x,t) = \mathbf{M}(x) : \mathbf{R}(x,t) + \mathbf{E}^p(x,t) \quad (6)$$

where $\mathbf{E}^r(x,t)$ is the residual strain field at the time t . It is the remaining strain field in the structure after total unloading provided that this unloading is purely elastic. $\mathbf{E}^r(x,t)$ is compatible, i.e. derives from a residual displacement field \mathbf{U}^r because $\mathbf{E}(x,t)$ as well as $\mathbf{E}^{el}(x,t)$ are kinematically admissible. Thanks to the last equation, one can see that the residual stress field $\mathbf{R}(x,t)$ and displacement field $\mathbf{U}^r(x,t)$ are solutions of an elastic problem for the structure submitted to initial strain $\mathbf{E}^p(x,t)$ and to zero external loading. The field $\mathbf{R}(x,t)$ vanishes if $\mathbf{E}^p(x,t)$ derives from a displacement field compatible with zero imposed displacements. Thus, the origin of residual stress is : either the fact

that $E^p(x, t)$ does not derive from a displacement field or the fact that this displacement field does not vanish where the displacements are imposed in the loading process at time t .

5.2 Elastic shakedown

The structure is in elastic shakedown state if after a certain number of cycles corresponding to a time θ , the field of plastic strain tensor becomes constant i.e. $E^p(x)$. This number of cycles can be more or less important depending on the structure and the loading. Hence, the residual stress field becomes also constant, $R(x)$, so that thanks to equation (6) for $t \geq \theta$, the stress cycle $\Sigma(x, t)$ is purely elastic. If the elastic shakedown is not possible because the imposed load cycles are too important, one can have plastic shakedown or ratchet. These three limit states are schematically depicted in Fig. 6.

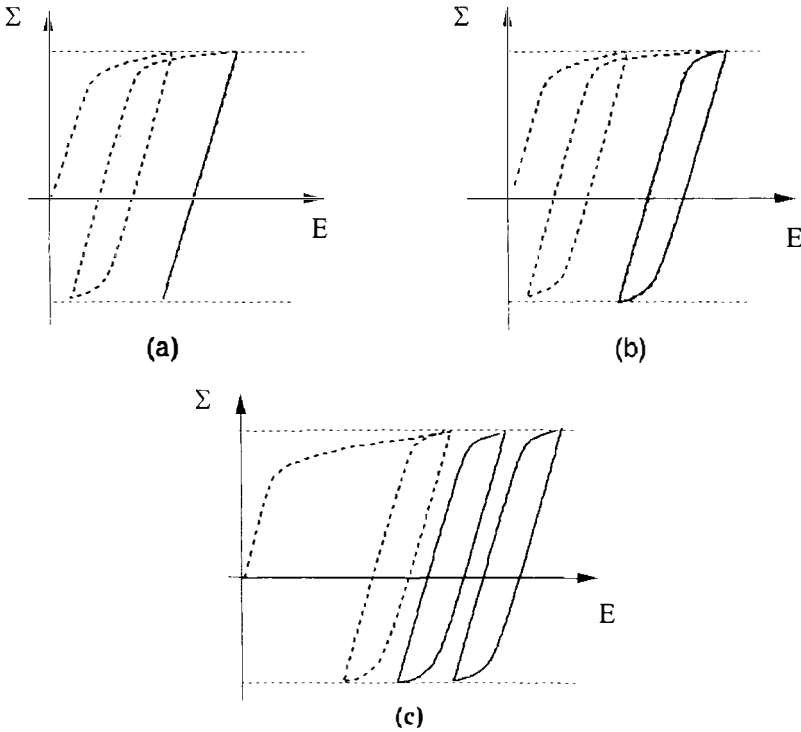


Fig. 6, (a) Elastic shakedown, (b) Plastic shakedown, (c) Ratchet

Plastic shakedown corresponds to stabilization of the plastic strain on a fixed strain cycle $E^*(x, t)$. Residual stress cycles are also stabilized on a fixed stress cycle $R^*(x, t)$.

This case corresponds most of the time to low cycle fatigue regime. If there are no possible shakedown (neither elastic nor plastic), then there is ratchet which corresponds to no stabilization. In this case, the equivalent plastic strain increases monotonously and failure happens rapidly. We are interested by high cycle fatigue that corresponds to elastic shakedown, which means that after a certain number of periods, there is no further dissipation.

5.3 Static elastic shakedown theorems

The original static theorem of Melan gives a sufficient condition for elastic shakedown for a structure made of an elastic, perfectly plastic material.

Theorem 1 : *If there exists a time θ and a fixed (i.e. independent of t) self equilibrated stress field $\mathbf{R}(\mathbf{x})$ such that $\forall \mathbf{x}$ and $t > \theta$, $g(\boldsymbol{\Sigma}^{\text{el}}(\mathbf{x}, t) + \mathbf{R}(\mathbf{x})) < K^2$, the structure will shakedown elastically.*

This theorem was then extended by different authors to account for more realistic material behaviour. We retain the proposal of Mandel, Zarka and Halphen [10] which is expressed in a quite simple way and lead to interesting applications. The elastoplastic material is a combination of linear kinematic and isotropic hardening. The yield surface is defined by the potential :

$$f = g(\boldsymbol{\Sigma} - C\mathbf{E}^p) - K^2(B)$$

C being a positive scalar and B a scalar strain hardening parameter which is an increasing function of time. In general B is the equivalent plastic strain E_{eq}^p defined by equation (3). The current yield limit K is supposed to be a non decreasing function of B and also that it is a convex function homogeneous of even degree. Moreover, the deformations are supposed to be small (which is always the case in high cycle fatigue). Then Mandel et al. proved the following theorem, generalizing Melan's theorem to a new class of materials.

Theorem 2 : *If it exist a time θ , a constant in time (i.e. independent of t) self equilibrated stress field $\mathbf{R}(\mathbf{x})$ and a fixed field of strain hardening parameters $\{\mathbf{E}^p(\mathbf{x}), B(\mathbf{x})\}$ such that $\forall \mathbf{x}$ and $t > \theta$, $g(\boldsymbol{\Sigma}^{\text{el}}(\mathbf{x}, t) + \mathbf{R}(\mathbf{x}) - C\mathbf{E}^p(\mathbf{x})) - K^2(B) < 0$, the structure will shakedown elastically.*

These theorems are theoretically interesting, however, difficult to apply because the fixed stress field $\mathbf{R}(\mathbf{x})$ must be self equilibrated, a condition which is difficult to fulfill. It is why Mandel et al. proposed an other theorem, which is a necessary condition of elastic shakedown, provided the deformations are small (which is the case in high cycle fatigue) and the stress space is of finite dimension. This last condition is fulfilled by solutions obtained by the finite elements method.

Theorem 3: *In the case where the problem is of finite dimension, if K is a strictly increasing function of B and g is a uniform continuous function, a necessary elastic shakedown condition is that it exists a time θ and at each point x of the structure a fixed stress tensor $\Sigma_1(x)$ such that:*

$$\forall t > \theta, \quad g(\Sigma^{\text{el}}(x, t) - \Sigma_1(x)) - K^2(B_s) \leq 0 \quad (7)$$

where B_s is the greatest value of B compatible with the condition of small strain.

This theorem is very interesting because of its *local character*, to the contrary of the two first theorems. If one knows at each point x , $\Sigma^{\text{el}}(x, t)$ and if the real stress field $\Sigma(x, t)$ is supposed to be at the shakedown limit, thanks to this theorem, it is possible to construct locally $\Sigma_1(x)$ and then $\Sigma(x, t)$. This is illustrated by the following interpretation, which is provided by the same authors. For a given x , the quantity $\sqrt{g(\Sigma^{\text{el}}(x, t) - \Sigma_1(x))}$ represents a measure of the deviation between the elastic stress $\Sigma^{\text{el}}(x, t)$ and the fixed point $\Sigma_1(x)$ in the stress space. Let us maximize this deviation for $t > \theta$. Then let us consider the variation of $\Sigma_1(x)$ such as to minimize that maximum. The obtained point $\Sigma^*(x)$ corresponds to the smallest maximum deviation of the stress amplitude relative to the chosen stress norm defined by the function g . This deviation $F(x, t)$ is thus given by:

$$\sup_{t > \theta} \sqrt{g(\Sigma^{\text{el}}(x, t) - \Sigma_1(x))} = \inf_{\Sigma^*} \sup_{t > \theta} \sqrt{g(\Sigma^{\text{el}}(x, t) - \Sigma^*(x))} = F(x, \theta)$$

Theorem 3-bis: *The inequality of theorem 3 is equivalent to the inequality $F(x, \theta) < K_s$, where K_s is the value of K for $B = B_s$.*

The demonstration by Mandel et al. is reproduced here because of its importance for fatigue applications. If the last inequality is satisfied, the point Σ_1 is such that:

$$\sqrt{g(\Sigma^{\text{el}} - \Sigma_1)} \leq \sup \sqrt{g(\Sigma^{\text{el}} - \Sigma_1)} = F(x, \theta) < K_s$$

so that theorem 3 is satisfied. Conversely, if this inequality is satisfied, then there exists a Σ^* such that $\sqrt{g(\Sigma^{\text{el}} - \Sigma^*)} < K_s$ and consequently also its supremum for $t > \theta$, i.e.:

$$F(x, \theta) \leq \sup \sqrt{g(\Sigma^{\text{el}} - \Sigma^*)}$$

Therefore, relation (7) is satisfied. Theorem 3 is used to estimate the elastic shakedown state of the whole structure at the macroscopic level. It is also used to estimate the mesoscopic stress and strain within a RVE, provided that the RVE is considered as a structure made from many grains which tend to an elastic shakedown state under the action of the macroscopic stress $\Sigma(x, t)$ acting on it. This use of Theorem 3 is illustrated by the examples presented in the next section.

5.4 Simple illustration examples

These mathematical results may seem abstract to engineers. It is thus interesting to illustrate these by more intuitive examples in direct relation with material properties.

Example 1

Let us consider a structure made of a polycrystalline elastoplastic material. The behaviour of a typical grain of this material is represented in Fig.7(a). The structure is submitted to cyclic loading, which induces in the neighborhood of a point x of the structure the stress $\Sigma(x, t)$. Therefore, the RVE surrounding point x is submitted to the macroscopic stress $\Sigma(x, t)$ evaluated under the assumption of pure elastic behaviour. Let us assume that the macroscopic stress is a pure shear described by two non-zero components Σ_{zx} and Σ_{zy} , where xyz is a fixed coordinate system with respect to the material. Our task is to estimate the local stresses in the plastically flowing grains within RVE using Theorem 3. The load cycle of period Θ is divided into n parts t_i , $1 \leq i \leq n$. The stress vector $\mathbf{T}(t)$ acting on the plane normal to z , is pure shear. The stress $\mathbf{T}(t)$, associated with various time instants t_i defines a curve Γ , which is divided into $(n-1)$ parts delimited by the end of vectors $\mathbf{T}_i = \mathbf{T}(t_i)$, Fig.7(b). Critically oriented grains in the volume V will slip. If there are not residual stresses, the initial elastic domain C_0 is featured by the circle of radius k_0 , centered on O (coinciding with x). As \mathbf{T} follows Γ , let \mathbf{T}_j be the first point outside C_0 , at which plastic strains occur. This results in an evolution of the elastic domain, which is now represented by C_j . The translation of the center from O to O_j corresponds to kinematic hardening. The growth of the radius from k_0 to k_j is due to isotropic hardening. The corresponding uni-dimensional response is represented in Fig. 7(a).

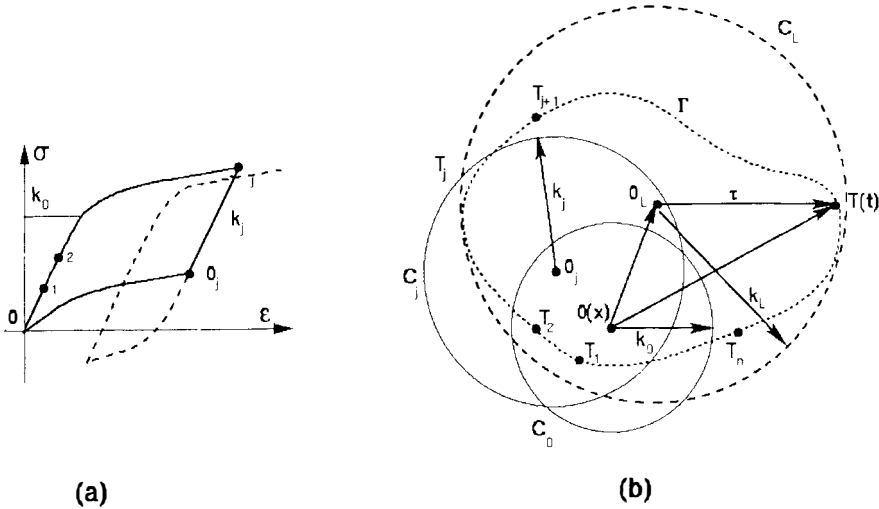


Fig. 7, Illustration of the scheme to compute the stabilized local residual stresses

As the load keeps varying, active parts of the loading path will maintain the elastic domain changing in the way described above. After some cycles a limit circle C_L with center O_L , and radius k_L is found. The circle C_L circumscribes the whole curve Γ defined from the stress vector $\mathbf{T}(t)$. An elastic shakedown state is thus obtained.

It is noticed that the vector $\vec{OO_L}$ corresponds to the local stabilized residual stress $\boldsymbol{\rho}^*$. More explicitly, the following interpretation can be given:

$$\mathbf{T} = \boldsymbol{\tau} + \boldsymbol{\rho} - c\boldsymbol{\varepsilon}^P \quad \text{corresponds to} \quad \mathbf{T}(t) = \mathbf{T} - \vec{OO_L}$$

The stress $\boldsymbol{\tau}$ (represented by $\vec{O_L T}$) is the local shear stress vector at time t in the shakedown state. For progressive isotropic hardening, the stabilized state occurs for the smallest diameter circle enclosing Γ . Note that $c\boldsymbol{\varepsilon}^P$ is in general small compared to $\boldsymbol{\rho}$, because the kinematic hardening c is small compared to elastic moduli. Then it is reasonable to assume that $\boldsymbol{\rho} \approx \vec{OO_L}$. This result corresponds to the statement of *Theorem 3-bis*, which means that among all possible circles surrounding Γ , we choose as the most suitable the one with the smallest radius.

Note that in this example, we do not consider the redistribution of the stresses induced by plastic yielding in the structure as $\boldsymbol{\Sigma}$ is evaluated under the assumption of a pure elastic behaviour. However, in our model we shall suppose that shakedown happens at all scales before fatigue. Then this redistribution will lead to a stabilized elastic macroscopic stress state in the RVE. This construction gives an estimation of the fluctuation of the local mesoscopic stress in a critically oriented grain of V .

Generalizing the previous example for more complex loading paths, we choose the simple von Mises' equivalent stress, which represents a norm in the deviatoric stress space. The circles C must be replaced by hyper-spheres in the deviatoric stress space defined by this equivalent stress.

Example 2

Let us consider an inclusion submitted to uniform plastic strain $\boldsymbol{\varepsilon}^P$ and embedded in an elastic matrix. By \mathbf{L} and \mathbf{I} are denoted the elastic compliance of the matrix and of the inclusion, respectively. Suppose that under external loading the total strain of the matrix and of the inclusion is the same (i.e., Lin-Taylor model):

$$\mathbf{E} = \boldsymbol{\varepsilon}^c + \boldsymbol{\varepsilon}^P$$

Multiplying both sides of the above relationship by \mathbf{I} and taking into account the elastic stress-strain relations:

$$\boldsymbol{\sigma} = \mathbf{l} : \boldsymbol{\varepsilon}^c, \quad \mathbf{E} = \mathbf{L}^{-1} : \boldsymbol{\Sigma}$$

one obtains:

$$\boldsymbol{\sigma} = \mathbf{l} : \mathbf{L}^{-1} : \boldsymbol{\Sigma} - \mathbf{l} : \boldsymbol{\varepsilon}^p$$

The tensor $\mathbf{l} : \mathbf{L}^{-1}$ is a localization tensor that concentrates the stress and $-\mathbf{l} : \boldsymbol{\varepsilon}^p$ corresponds to the local residual stress $\boldsymbol{\rho}$. The construction presented in the previous example can be used to derive the stabilized plastic strain tensor $\boldsymbol{\varepsilon}^p$ and the local residual stress $\boldsymbol{\rho}$ when the inclusion, which has a mixed kinematic and isotropic hardening, is submitted to controlled strain \mathbf{E} . If the inclusion and the matrix have same elastic compliance, then the localization tensor is the fourth order unit tensor, i.e. $\mathbf{l} : \mathbf{L}^{-1} = \text{Identity}$. For isotropic elastic material characterized by the same Lamé coefficients λ, μ , one has,

$$\sigma_{ij} = \Sigma_{ij} - 2\mu \varepsilon_{ij}^p$$

since $\varepsilon_{ii}^p = 0$. This example serves to better understand our approach for studying fatigue, which will be presented in the next chapter. Indeed, the above example is based on the following concept: crack initiation near the fatigue limit occurs in some critically oriented grains which have undergone plastic deformation. The hypothesis that an elastic shakedown state is reached means that $\boldsymbol{\varepsilon}^p$ becomes independent of time. In this example, the residual stress $\boldsymbol{\rho}$ measuring the local fluctuation of the stress tensor is directly related to the plastic strain $\boldsymbol{\varepsilon}^p$, so that $\text{trace}(\boldsymbol{\rho}) = 0$. Thus, the macroscopic P , and the mesoscopic p , hydrostatic stresses are equal, i.e. $P = \text{trace}(\boldsymbol{\Sigma})/3 = \text{trace}(\boldsymbol{\sigma})/3 = p$.

6. HIGH CYCLE FATIGUE

High cycle fatigue represents the most important field of industrial applications in mechanical design. However, despite of investigations for over 150 years, there is still a large gap between academic researches and the needs of the design engineers. Usually, researchers try to describe phenomena by doing microscopic observations and proposing laws which are in fact best fit data from ideal experiments with simple specimens subjected to simple loads (bending, tension or torsion). Engineers have to apply these proposals in a more complex context which is due to complicated geometry of components, multiaxiality of stress state, etc. Most of the time, this turns to be a quite impossible task. In fact, very often the laws that are derived from specific loading paths are not applicable under different stress conditions. This is why there is a lack of analytical and/or numerical procedures for fatigue analysis, in spite of the great numbers of proposed laws and models for fatigue behaviour (see for instance, Garud [11]).

Engineering structures undergo generally complex states of multiaxial stresses. The origin of multiaxiality comes from different factors such as:

- External loading
- Geometry of the structure: it is well known that the stress state can be multiaxial even if the external applied load is pure tension
- Residual stresses, which are multiaxial by nature. The superposition of a fixed residual stress field with a varying stress field due to external loading gives a multiaxial varying stress state

Many industrial structures combine all these factors so that fatigue life assessment requires a multiaxial fatigue limit computation methodology. In the following, different fatigue limit criteria based on the multiscale scale approach are presented. It is reminded that these criteria are based on the assumption of elastic shakedown at all scales (macroscale as well as mesoscale) before fatigue. Some of these criteria are already extensively used in industries for fatigue design.

6.1 Definition of fatigue limit

For constant stress or strain range, the fatigue limit corresponds to the stress state just enough to produce the initiation of a macroscopic crack. This crack will then propagate and will be at the origin of final fracture. For complex multiaxial periodic loads, this concept is not so clear and a generalized definition of the fatigue limit is needed. For this purpose, let us consider a loading agency λQ_i that generates at a critical point of a structure the following multiaxial stresses depending on the parameter λ :

$$\Sigma_{ij}(t) = \lambda (A_{ij} \sin(\omega_{ij} t + \varphi_{ij})) + B_{ij}$$

It is clear that if λ is small, there is no fatigue at all. It is also evident that when λ grows, it will reach a critical value λ_c for which fatigue will happen. Therefore, λ_c corresponds to the fatigue limit for this loading path. Any periodic loading can be recast in a similar form to the relationship given above. Therefore, it is possible to define a fatigue limit for any multiaxial loading, following the approach depicted here.

6.2 Orowan's model

Fatigue is generally due to stress concentrations and heterogeneities. The first fatigue phenomena are microscopic and local. They usually occur in some grains, which have undergone plastic deformation localized in characteristic intra-crystalline (shear) bands. In high cycle fatigue, the rest of the matrix behaves elastically because the macroscopic plastic strain is negligible. Fatigue crack initiation will happen very often in the shear bands of plastically deformed grains.

Using this image, Orowan [12] proposed the fatigue nucleation model depicted in Fig.8. A weak plastic element is embedded between two elastic springs, which impose their deformation on the whole system. The weak element undergoes plastic strain and hardens. If the plastic behaviour of the weak element is governed by pure isotropic hardening, as shown in Fig. 8, then its response tends to an elastic shakedown state precluding fatigue. The limit loading path in that case oscillates between A (the corresponding shear stress is τ) and B (the corresponding shear stress is $-\tau$). The local shear loading is symmetric even if the prescribed external loading is not symmetric. Then, if the limiting range AB is less than some definite value, there is no fatigue. Indeed, fatigue occurs only if this condition is not verified.

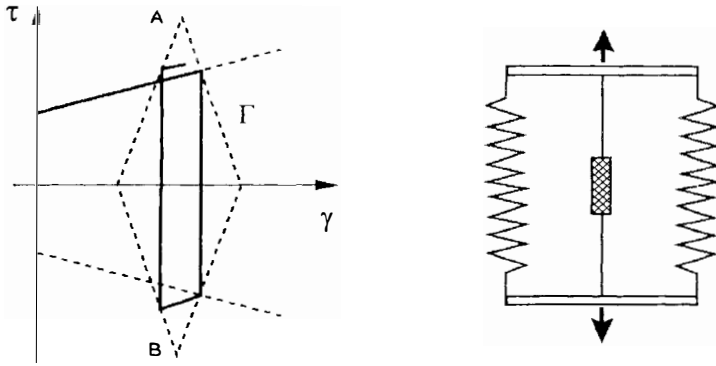


Fig. 8, Orowan's model

7. FATIGUE MODELS BASED ON THE MULTISCALE APPROACH

The macro-meso approach, which is used here to derive fatigue limit criteria, is originally based on the ideas depicted in the previous section. Following further the macro-meso approach, we introduce the two scales of material description represented in Fig. 3. These are:

- A macroscopic scale characterized by an arbitrary representative volume element $V(x)$, surrounding the point x , where fatigue analysis is made. This scale corresponds for instance to the size of an element of a finite element mesh or to the dimension of a strain gauge. The macroscopic scale is the usual scale considered by engineers. Mechanical macroscopic variables $\Sigma(x, t)$, $E(x, t)$, are assumed to be homogeneous at any time t in the RVE surrounding x .

- A mesoscopic scale of the order of grain size corresponding to a subdivision of $V(x)$. The mesoscopic quantities σ and ε at a point m are not homogeneous in $V(x)$ and differ from Σ and E ; even if the mean value of σ equals Σ as it has been shown, the local stress can fluctuate. These two quantities are related by the following relation:

$$\sigma_{ij} = A_{ijkl}(x, m) \Sigma_{kl}(x, t) + \rho_{ij}(m, t)$$

The above equation is indeed Eq.(4), where $\Sigma^{el}(x, t)$ is replaced by $A_{ijkl}(x, m) \Sigma_{kl}(x, t)$ and Σ is replaced by σ . Thus, the RVE, $V(x)$, is considered as a structure submitted to a homogeneous loading defined by $\Sigma_{kl}(x, t)$. Therefore, the elastic localization tensor $A_{ijkl}(x, m)$ can be correlated to the microstructure by solving six elastic problems (one for each (kl) stress component). However, this last point is not addressed here. In the following, it is assumed that $A_{ijkl}(x, m) \Sigma_{kl}(x, t)$ can be approximated by the local macroscopic stress tensor $\Sigma(x, t)$ at the stabilized macroscopic elastic shakedown state.

Let us discuss these assumptions. The localization tensor is chosen for the sake of simplicity equal to the identity tensor. Consequently, we shall consider that the material properties present the same degree of isotropy at the meso and the macro levels. Because we are considering load states close to the fatigue limit, the applied stresses are rather low and it seems reasonable to evaluate the macroscopic stress field using elastic computation. This is valid for plain smooth structures. When a structure has geometric discontinuities like notches, grooves etc., which concentrate the stresses, macroscopic plastic deformation can occur during the initial loading phases. Nevertheless, this plastic strain is contained by the surrounding elastic material and reaches a stabilized state after a certain number of cycles. High cycle resistance means that after that period, the plastic dissipation at the macroscopic scale decays, that is, macroscopic elastic shakedown occurs. Consequently, the macroscopic stress tensor Σ is given by:

$$\Sigma(x, t) = \Sigma^{el}(x, t) + R(x)$$

where $R(x)$ is the macroscopic residual stress field. It is reminded that the macroscopic residual stresses $R(x)$ and thus the current macroscopic stresses $\Sigma(x, t)$ can be estimated by using Theorem 3 established previously.

Additionally, it is also assumed that near the fatigue limit, elastic shakedown happens at the mesoscopic level. Then the mesoscopic stress tensor $\sigma(m, t)$ is given by:

$$\sigma(m, t) = \Sigma(x, t) + \rho(m)$$

where $\rho(m)$ is the local residual stress, which can be estimated by the way presented in *Examples 1-2* studied previously. It represents the local stress fluctuation $\Sigma(x, t) - \sigma(m, t)$. If the loading is below the corresponding fatigue limit, this fluctuation after a certain number of cycles, does not vary any more.

7.1 Fatigue limit criterion based on the current stress state

It has been assumed that in the high cycle fatigue regime, elastic shakedown at all scales takes place, at least for loading paths near the corresponding fatigue limit. Under this assumption, it is possible to estimate the local stress cycle in grains that may suffer damage. As the local stress is approximately known at any time t , it is natural to try (as in plasticity) to take account of the loading path. Thus, a reasonable fatigue criterion, which generalizes the Orowan's proposal, could be stated as follows, [13-15]:

Crack initiation will occur in a critically oriented grain within $V(x)$ that has undergone plastic deformation, if, for at least one time instant t of the stabilized cycle, one has :

$$f(\sigma(m, t)) \geq 0 \quad \text{for } m \in V(x)$$

In such a criterion, the current stress state is considered. Damage arises over a precise portion of the loading path (or equivalently, over a precise time interval of the loading period). As cracks usually occur in transgranular slip bands, the local shear acting on these planes is an important parameter. Moreover, the normal stress acting on these planes accelerates damage formation. However, this quantity is rather difficult to compute in a general case because it depends of the considered plane. For this reason, hydrostatic stress is preferred because it is much easier to use being an invariant scalar. Furthermore, it can be interpreted as the mean value of the normal stresses acting on all the planes that pass through the considered point of the structure. Based on these remarks, we choose for $f(\sigma)$ a relation between the local shear stress τ and the local hydrostatic stress p . The simplest criterion that can be conceived is a linear relationship between these quantities,

$$f(\sigma) = \tau + a p - b \quad (10)$$

where a and b are material parameters. The safety domain (no fatigue crack initiation) is delimited by two straight lines, which can be determined by simple experiments such as uniaxial tests of tension-compression and torsion on classical fatigue test machines.

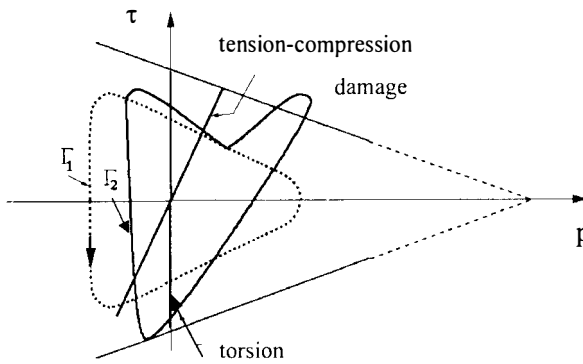


Fig. 9, Fatigue limit domain and typical loading paths

The fatigue limit domain is represented in Fig. 9. In this figure, the ordinate is the algebraic shear stress τ acting on an oriented direction and the abscissa is the hydrostatic stress p . On the same figure, we represent the loading paths for the fatigue limit in torsion, and in tension-compression. These loading paths can be used to identify the constants a and b of relation (10). Two more loading paths are shown in Fig.9. Path Γ_1 is non-damaging because it entirely lies within the straight lines that delimit the safe domain, whereas a small portion of path Γ_2 induces damage.

To check automatically the fatigue resistance of a structure is a rather difficult task, because at each point one has to consider the plane on which the loading path $(\tau(t), p(t))$ is "maximum" relative to the criterion. This computation can be simplified as following, [14]. The maximum shear stress according to Tresca's measure is calculated over the cycle period:

$$\tau(t) = \text{Tresca}(\boldsymbol{\sigma}(t))$$

For this, it is useful to notice that:

$$\text{Tresca}(\boldsymbol{\sigma}(t)) = \text{Tresca}(\mathbf{s}(t)) = \underset{I,J}{\text{Max}} \left| \sigma_I(t) - \sigma_J(t) \right| / 2$$

The stresses $\sigma_i(t)$ and $\sigma_j(t)$ are principal local mesoscopic stresses at time t . The quantity d that quantifies the danger of fatigue failure defined by,

$$d = \underset{t}{\text{Max}} \frac{\tau(t)}{b - a p(t)}$$

is calculated over the loading period. The maximum is to be taken over the cycle ($0 < t < \Theta$). If $d > 1$, fatigue failure will occur. Working this way, all couples (τ, p) are situated in the positive part of τ (see Fig. 9). All facets, which could be involved by the crack initiation, are automatically reviewed.

Couples (τ, p) verifying the condition $d > 1$ are associated with specific facets. Therefore, the criterion provides also the direction of crack initiation.

An other possibility is to use the octahedral shear $J_2(\boldsymbol{\sigma}(t))$ instead of $\tau(t)$. However, this method does not give the critical facets.

7.2 Fatigue limit criterion based on the limit of the local elastic shakedown state

A different way to interpret Orowan's proposal is to consider that the fatigue limit for a given periodic load agency $\lambda Q_i(t)$, corresponds to the limit of the intensity λ such that elastic shakedown is possible. Beyond this limit, plastic shakedown or ratchet phenomena will induce damage and fracture because of subsequent softening. The limit size of the hyper-sphere surrounding the loading path, (see sections 5.3 and 5.4), is one possible and natural way to characterize this state. In the proposed criterion intervenes the radius of this

hyper-sphere k^* and the maximum hydrostatic stress in the cycle. If k^* is greater than some limit value k_{lim} , which depends of the local maximum hydrostatic stress in the cycle, fatigue will occur. The limit value k_{lim} is chosen as:

$$k_{lim} = \beta - \alpha p_{max}$$

The fatigue criterion is (Papadopoulos 1987, [16]):

$$k^* + \alpha p_{max} - \beta > 0$$

The parameters α and β can be identified like previously by two different tests. By this method, it is no more necessary to describe the whole loading path, once k^* is determined. In many cases, the predictions are very similar to the current stress state methodology described before. Different extensions are possible for a better description of strongly non-proportional stress conditions (e.g. very high out-of-phase loads). They are discussed in some recent papers by Papadopoulos [17-20] who remarks that these types of loads activate many slip systems so that there are more possibilities of crack nucleation. The problem is to differentiate between multiaxial loading paths proportional or nearly proportional on one hand, and strongly non-proportional on the other hand, by a better characterization of the stress trajectory. Very good agreement between highly out-of-phase test results and predictions by the new criterion of Papadopoulos is obtained [21]. However, this new methodology is more difficult to apply in the fatigue design of structures, than the simple (τ, p) criterion presented here.

8. CONCLUSION

The fundamental problem in mechanical engineering is whether or not a metallic mechanical component of given geometry, submitted to a periodic load system will break after a finite number of cycles. Despite the great number of researches already done since the middle of the 19th century, the practical answer to this question is still very difficult to obtain by the classical approaches. A new theoretical framework for studying fatigue is proposed in the present work. It is based on a multiscale description of the fatigue phenomenon. The elaborated fatigue assessment methodology has already proved its efficiency in industrial applications. This method after being checked by comparing its predictions against experimental results on real components is now chosen as fatigue computation method by some automotive industries. Different high-cycle fatigue CAD systems based on this approach exist which facilitate the task of designing against long term fatigue (see for instance [21]). It is also noticed that recent developments obtained within the proposed framework allow considering the problem of structural design in a global way, by taking into account at the computation stage the effects of various surface treatments. Examples are given in other contributions of this volume.

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