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# Developments of fracture mechanics in France in the last decades

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**ABSTRACT:** The purpose of the present paper is to give a survey on some developments of fracture mechanics in France in the last decades. Until about 1970, the subject was practically unknown at the universities. The topics were growing up rapidly, because of French industrial projects or programs in the fields of aeronautics, aerospace industries, nuclear reactor technology. In France, fracture mechanics became an important subject for engineers and researchers starting with the first Summer School (1970) devoted to fracture mechanics held at La Celle Saint Loup (France) (François & Joly 1972). At this meeting, engineers from the French Atomic Centre (CEA), Electricité de France, the Nuclear Vessel Builders (Framatome, Creusot-Loire) and from other industries gave lectures on linear elastic fracture mechanics, Griffith's energy balance, Irwin's energy release rate  $G$  and Kies' stress intensity factors. Obviously, fracture mechanics is strongly motivated by industrial applications. Ten years after the first meeting at La Celle Saint Loup, scientists and engineers in France organised the ICF-5 conference in Cannes, in April 1981.

Contributions to the experimental determination of toughness of different materials, the study of the influence of hydrostatic pressure on the toughness of aluminium alloys (Auger & François 1974), microscopic observations of cracks in metals, fatigue crack propagation (Lemaitre & Chaboche 1978) date back to the 1970's. But most of the papers in this period are devoted to computational aspects in linear and non-linear fracture mechanics with the emphasis on stress intensity factors calculations, 3D calculation of SIF by the weight function method (Labbens et al. 1976), path-independent integrals (Bui 1993), fatigue crack propagation and fatigue criterion (Dang Van 1973), crack tip plastic zones (Nguyen & Bui 1974).

In the 1980's, studies in non-linear fracture mechanics were concerned with ductile fracture, with applications to nuclear energy technology (Rousselier 1981), crack initiation using damage mechanics (Bui & Ehrlacher 1980; Lemaitre & Chaboche 1985), thermodynamical aspects of fracture (Nguyen 1980; Ehrlacher 1985), homogenisation of periodic structures containing cavities or cracks (Marigo 1981, 1985; Sanchez-Palencia 1980; Suquet 1990). The contributions of Beremin, a pseudonym for a group of researchers from the Ecole des Mines de Paris, working in ductile fracture and the physics of metals (Mudry 1982; Pineau & Joly 1991; Pineau 1992) shall be mentioned in particular.

Then, contributions on new subjects emerged since the mid 1980's: shear band lo-

calisation (Fressengeas & Molinari 1985, 1987; Benallal et al. 1989, 1991, Doghri & Billardon 1995), stability analyses of systems of cracks (Nguyen et al. 1990), scale effects in micro-macro analyses (Leroy & Molinari 1993; Lalai & Sab 1996), delamination of composites (Allix & Ladeveze 1992, 1996; Allix et al. 1995), asymptotic analyses (Geymonat et al. 1986; Leguillon & Sanchez-Palencia 1987), inverse problems and identification of cracks (Bui 1993; Andrieux & Ben Abda 1992; Andrieux et al. 1997), weak singularities of stresses at interfaces such as the 'Epine' singularities (French word for the 'thorn' like shaped function characterised by its boundedness, discontinuity and unbounded gradient) (Bui & Taheri 1989; Deperrois & Dang Van 1990; Leguillon 1992; Santacreu 1994), experimental approaches in dynamic fracture (Bui & Maigre 1988; Bui et al. 1992; Maigre & Rittel 1993) etc. In the sequel, some of these contributions will be discussed in more detail.

**KEYWORDS:** Linear elastic fracture mechanics, stress intensity factors, path-independent integral, integral equation method, plasticity, damage mechanics, scale effects, inverse problems, fluid flow.

## 1 CONSERVATION LAWS

The  $J$ -integral, developed independently by J.R. Rice (1968) and G.P. Cherepanov (1968), has motivated and stimulated work in static as well as in dynamic fracture mechanics, in small strain as well as in finite deformation. The first paper on this subject is about the dual conservation law in classical elasticity (dual to the Eshelby law), using the dual law of elasticity  $\varepsilon = \partial U(\sigma)/\partial \sigma$  instead of the primal one  $\sigma = \partial W$ , (Bui 1973). The dual  $I$ -integral is then introduced as a path-independent integral equivalent to the  $J$ -integral for exact solutions to the boundary value problems of cracked body. Considerations of dual approaches in computational mechanics provide bounds for the exact potential energy and, assuming some monotonic conditions for the meshes used in finite element approximations, also provide approximate bounds for the derivative of the potential energy with respect to the crack length.

The problem of separation and resolution of mixed modes by using the path-independent integrals  $J_I$  and  $J_{II}$  is dealt with in (Bui 1983). The use of adjoint elastic fields, symmetrical and antisymmetric fields, leads to the same results (Bui 1993). The same idea can be found in the paper by H.D. Bui and H. Maigre (1988) who consider the adjoint dynamic fields for introducing the dynamic  $H$ -integral, which is a true path-independent integral in elastodynamics, while classical extensions of the  $J$ -integral to take account of dynamic effects, as known in the literature, include area integral terms over the crack tip region.

The analysis of fracture in thermo-elasticity using adjoint elastic fields and adjoint thermal fields, leads to a new class of conservation laws which do not contain source terms. Thus, the true path-independent  $T$ -integrals in thermo-elasticity were developed (Bui & Proix 1984).

Area integrals over ring-shaped domains, the boundaries of which consist of two non intersecting closed curves  $G_{int}$  and  $G_{ext}$  not containing the crack tip, were proposed for the static  $J$ -integral and the thermo-elastic  $T$ -integral. Independently, Ph. Destuynder and M. Djaoua (1981) and K. Ohtsuka (1981) introduced the La-

grangian derivative of energy and defined the  $G_q$ -integral (the so-called  $q$ -method), where  $q(x)$  is the velocity field of the cracked solid, parallel to the crack tip propagation velocity inside the curve  $G_{int}$ , and equal to zero outside the curve  $G_{ext}$ . The ring-shaped area integral corresponding to the Lagrangian version of the  $T$ -integral is called the  $A$ -integral (Bui 1993).

Stability analyses of crack propagation can be done either by the Eulerian second derivative of energy (Nguyen et al. 1990) or by the Lagrangian second derivative of energy (Suo & Combescure 1989). In addition to the first field  $q(x)$ , there is a second velocity field  $p(x)$  corresponding to a second crack, which justifies the naming of the so-called  $qp$ -method.

Stability analysis of crack propagation has been generalised to include propagation of damage in brittle elastic solids (Fedelich & Ehrlacher 1989; Francfort & Marigo 1993) and delamination of thin membranes (Bérest 1987).

Conservation laws in finite strain analysis have been derived for elasticity, thermoelasticity and electroelasticity (Maugin & Epstein 1991; Maugin & Trimarco 1992; Dascalu & Maugin 1994). Also, the Lagrangian approach has been extended to take into account geometrical and physical non-linearities.

## 2 THEORY AND EXPERIMENT

New theoretical methods of analyses and new experimental techniques have to be developed in parallel for studying crack deviation in mixed modes, temperature rise during crack propagation, fatigue crack propagation under microscopic observations and dynamic fracture. Developments of new mathematical and numerical methods are necessary.

For example, the deviation of a crack in linear fracture mechanics can be studied by using an uncoupled theory, where the angle of deviation quite well predicted when the criterion  $s_{qq}$  max is used. However, the uncoupled theory cannot predict the critical applied force, which is required for the determination of the true stress-intensity factor at the tip of a kinked crack (in Mode I). Therefore a coupled theory such as developed in (Amestoy et al. 1979, 1985; Leblond 1989; Amestoy & Leblond 1992; Leguillon 1993), using asymptotic matching of solutions, is necessary. A particular example is given by the exact solution for the infinitesimal kinked crack problem. The crack deviation angle, predicted by the criterion  $k_2^* = 0$  of a pure local Mode I, does not differ significantly from the angle obtained from applying the criterion  $s_{qq}$  max, with the stress field  $s_{qq}$  of the uncoupled theory. If the Irwin criterion holds true in any circumstance, then for the consistency reason, the critical value of applied force can be derived from the same formula  $k_1^* = K_{Ic}$ . No such criterion can be found in the uncoupled theory.

Another example is given by the experimental determination of the dynamic stress-intensity factors associated with crack initiation. Based on theoretical analyses (Bui & Maigre 1988), a convolution equation is derived for the dynamic stress-intensity factors  $K_I(t)$  or  $K_{II}(t)$ , using only data available on the boundary of the cracked solid, the traction vector  $T(x, t)$  and the associated velocity vector  $v(x, t)$ . For Mode I, the time-convolution equation is of the following form  $K_I(t) * k_1'(t) = H(T, v; T, v')$  where the prime stands for a known dynamic adjoint field in Mode I (velocity

$v'(x)$  and traction field  $T'(x)$  on the boundary) and the  $H$ -integral is explicitly known from the boundary data  $T$  and  $v$ .

Experiments on dynamic fracture of metals, polymers, and glass using the convolution method and the Hopkinson bar technique, which provide both the force and the velocity of the impact point, have been performed by Maigre and Rittel (1993).

In the 1980's, attention has been paid to the problem of a dynamic crack acting as a moving heat source. Some experimental evidence has been given by previous researchers, Weichert and Schönert (1974). Based on thermodynamical considerations, Nguyen (1980), Bui et al. (1980, 1981), Ehrlacher (1985), have proved that, in 2D, the moving crack tip in an elastic material, due to the dissipation of energy at the crack tip, induces a logarithmic singularity of the temperature field. Experiments were performed in thin sheets of steel, using a thermographical set-up. Indeed, we observed a hot point before the occurrence of diffuse plastic dissipation and before the development of damage, which is the result of shear band localisation throughout the thickness, as shown in Figure 1. The final damage process induces the greatest dissipation rate.

Today, a large body of experimental work is devoted to the study of micromechanisms of damage, fracture and fatigue in metals, as well as fracture and delamination in composites. Pinna and Doquet (1996) have concentrated on Mode II fatigue micro-crack propagation in metals, and based their work on observations made during testing in a microscope.

### 3 INTEGRAL EQUATION METHODS FOR CRACK PROBLEMS

Computational analyses of cracked bodies in 2D as well as in 3D have motivated work on integral equations methods. The first papers on this topic by Bui (1975, 1977) treated a plane crack of arbitrary shape in a 3D infinite elastic medium. A set of three integral equations was established: one uncoupled equation for the crack opening displacement of Mode I, and two coupled equations for Modes II and III.

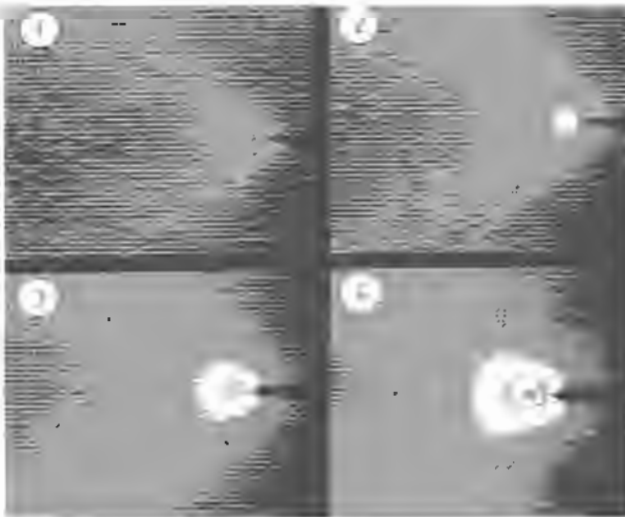


Figure 1. Thermographic observation of crack propagation in a thin sheet of steel (Bui et al. 1981).

These equations establish the relationship between the stress intensity factor  $K_I(s)$  and the intensity of the crack discontinuity near the crack front  $[u_3(s, r)] \equiv C(s)r^{1/2}$ , where  $s, r$  are the local co-ordinates near the crack front. It is shown that the relationship between  $C(s)$  and the stress intensity  $K_I(s)$  is the same as that obtained in 2D 'plane strain' problems. This relationship holds true even for an edge crack, when the co-ordinate  $s$  approaches perpendicularly the free surface, Bui and Putot (1979).

It is worth noting that the usual statement of a 2D state of plane stress at a free surface is incorrect. The proof can be derived directly from the equation of elasticity in plane stress for which the singular field  $\epsilon_{33} \approx O(r^{-1/2})$  at the crack tip, in plane stress, violates the compatibility equations which imply that  $\epsilon_{33}(x_1, x_2)$  is an affine function of the co-ordinates,  $\epsilon_{33} = ax_1 + bx_2 + c$ .

Integral equations for arbitrarily curved cracks in 3D infinite space were established by Sladek and Sladek (1983), Levan and Royer (1986), Bui and Bonnet (1987), Levan (1988), Bonnet and Bui (1993).

Many difficulties arise when dealing with numerical aspects of integral equations for crack problems. Firstly, the kernel is hypersingular (in the finite part sense of the integral), for which regularisation techniques are needed (Xiao 1994; Bonnet 1995). The method of regularisation consists of subtracting and adding some singularity to the kernel. The additional term can be integrated exactly, while the remaining terms, when combined, are weakly singular and can be computed with a better accuracy. Mathematically, this procedure comes from the very definition of finite part integral in the distributional sense. Secondly, the structure of equations is asymmetric, for which variational approaches are used (Nedelec 1982). Here, the symmetry is restored by a double boundary integration, which requires large computer memories. Variational approaches preserve both the symmetry and the regularity of integrals.

The integral equations method, in its variational form, has been used for solving 2D problems too (Yin & Ehrlacher 1993) where the components of the crack displacement discontinuity were the unknown quantities.

Dynamic cracks problems, particularly diffraction or scattering of elastic waves in infinite space, are efficiently studied by means of integral equation methods (Cortey-Dumont 1985; Ha Duong 1987; Becache 1991, 1993; Bonnet & Bui 1993).

#### 4 DUCTILE FRACTURE

This subject is motivated by potential applications in nuclear reactor technology. Considerable efforts have been paid in both mechanics of materials and metallurgy, particularly for studying stainless steels and duplex stainless steels. Contrary to some initial hope, ductile fracture mechanics is not a simple generalisation of linear elastic fracture mechanics to non linear problems, nor can we state that the  $J$ -integral is the right parameter in ductile fracture in every case. Experiments in ductile fracture show evidence of a specimen size effect and theoretical considerations indicate that not only the crack length, but also the geometry of the solid affect the value of  $J$  (the so-called resistance curve  $J-\Delta a$  depends on the geometry of the solid). This structural dependence is due to physical non-linearities such as plasticity, viscoplasticity or damage and hence restricts the applicability of the resistance curve concept to very special cases (small scale yielding hypotheses). The interpretation of  $J da/dt$  as the

rate of energy dissipated inside the contour of the  $J$ -integral (taken in the elastic region) leads to the decomposition  $Jda/dt = D_R + D_p$ , with the dissipation rate by fracture  $D_R$  at the crack tip and the plastic dissipation rate  $D_p$  inside the same contour. In the classical model of a crack in plasticity, the term  $D_R$  vanishes (Rice's Paradox). It is proved that  $D_R$  vanishes for a large class of elasto-plastic materials, including hardening materials (Nguyen 1980). For a power law of plasticity, the paradox  $D_R = 0$  still holds true (Bui & Dang Van 1987).

In the classical approach, the path-independent  $J$ -integral is defined by means of the work density  $W(e) = \int \sigma d\varepsilon$ . It has no simple thermodynamical interpretation, unlike the case where the elastic strain energy density  $W_{\text{elas}} = 1/2 (\varepsilon - \varepsilon^p) \cdot E \cdot (\varepsilon - \varepsilon^p)$  is used for defining the path-dependent  $J$ -integral. Precisely, to obtain a non zero fracture dissipation rate,  $D_R \neq 0$ , the contour of the  $J$ -integral must be taken as the boundary of the damage process zone.

Analytical solutions in brittle damage of elastic solids (elastic law  $\sigma = E\varepsilon$  when  $|\varepsilon| < \varepsilon_R$  and  $\sigma = 0$  otherwise) provided by (Bui et al. 1981), lead to  $D_R \neq 0$  and show that the damage process zone has some thickness. In (Bui et al. 1981), a particular solution in damage plasticity is given for the stationary case of damage in Mode 3. The interface between damage and plastic zones is the cycloid  $N$  ( $N_1 = h(\cos 2\phi + 1)/\pi$ ,  $N_2 = h(\sin 2\phi + 2\phi)/\pi$ ). The elastic-plastic interface, depending on the remote stress intensity factor  $K_3$ , is the cycloid  $M$  ( $M_1 = K_3^2 (\cos 2\phi + 1)/2\pi\tau_0^2$ ,  $M_2 = K_3^2 \sin 2\phi/2\pi\tau_0^2 + 2h\phi/\pi$ ). For classical Mode 3 in plasticity, the cycloid  $M$  becomes the classical Rice circle and the damage zone reduces to the crack  $h = 0$ , Figure 2.

For a more general loading in damage plasticity, unfortunately there is no exact solution. Numerical works are only available, based on the introduction of damage evolution law on a microscopic scale. The Rousselier (1981) criterion appears to be the best one for predicting the growth rate of cavities in ductile materials like stainless steels of the French nuclear vessels. It is an extension of the Gurson law. Finite elements solutions in plasticity, including ductile fracture damage, can be found in many works (D'Escatha & Devaux 1979; Mudry 1982; Beremin 1983; Wadier 1989; Debryne 1995). The macroscopic behaviour of solids containing cavities or cracks periodically distributed at microscopic scale can be analysed by the homogenisation method (Andrieux 1981; Suquet 1982; Marigo 1985). A continuous approach to ductile fracture, using damage mechanics, is provided by (Dragon & Mroz 1979; Lemaitre & Chaboche 1985).

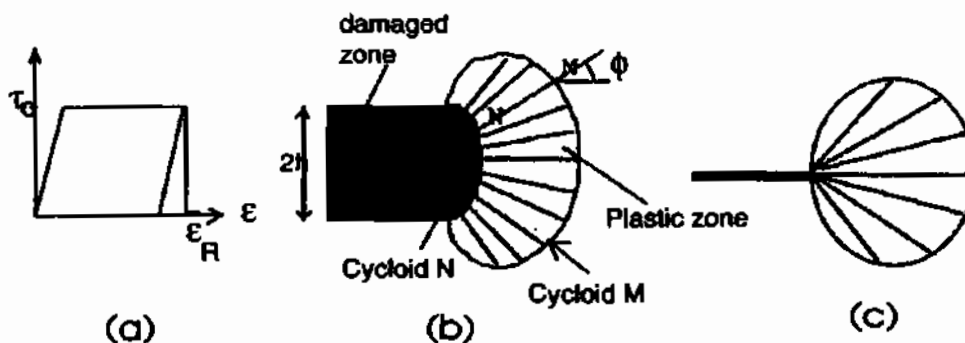


Figure 2. (a) Damage plasticity; (b) Damage and plastic zones; (c) The limiting case of a crack as  $\varepsilon_R \rightarrow \infty$  (Rice's circular plastic zone) (Bui & Ehrlacher 1981).

The strong dependence of local damage criterion on the materials, and perhaps the dependence on the loading conditions too, prevents its applicability to different cases. However, a methodology exists for analysing ductile fracture of components. It consists of global and local approaches to ductile fracture which are based on the transferability of laboratory tests results, on specific materials, to specific components, with the use of numerical computations in non-linear mechanics (Pineau 1992). This methodology justifies the development of French Computer Codes for industrial applications, such as the CASTEM 2000 of the French Atomic Centre, the Aster Code of Electricité de France (Mialon 1996) and the Systus Code of Framatome, and requires also sophisticated equipment when laboratory testing metals.

## 5 DAMAGE MECHANICS

Damage in materials occurs due to initiation of micro-cracks and microcavities in highly deformed zones, e.g. at the crack tip, at grain boundaries, at interfaces between inclusions and matrix of composites. On the meso-scale of several grains, the homogenised mechanical properties are characterised by a decrease of the elastic stiffness  $E$ , softening or a decrease of the yield strength, for increasing strain. A continuum theory of damage can be developed within the framework of the theory of plasticity, using the concept of generalised standard materials introduced by B. Halphen and Q.S. Nguyen (1975). Internal parameters,  $\omega_i$ , are introduced into the constitutive laws such that their evolution rates  $d\omega_i/dt = \lambda g(\sigma, \omega_i, \epsilon^p)$  and the rate of plastic deformation  $d\epsilon^p/dt = \lambda f(\sigma, \omega_i, \epsilon^p)$ , with the same plastic multiplier  $\lambda > 0$  and the familiar yield function  $f(\sigma, \omega_i, \epsilon^p)$ , are concomitant.

Some assumptions are necessary for specifying the forms of functions  $g(\sigma, \omega_i, \epsilon^p)$  and  $\Lambda(\omega_i)$  (Dragon & Mroz 1979). The simplest model (Lemaitre & Chaboche 1978) allows the determination of damage by stiffness measurement. The damage parameter is a scalar  $0 < \omega < 1$  and the dependence of the elastic stiffness on  $\omega$  is assumed to be of the simple form  $\Lambda(\omega) = (1 - \omega)\Lambda_0$ .

In some contributions, internal damage parameters are considered as microscopic variables which correspond to new stress fields satisfying additional equilibrium equations (Frémond & Nedjar 1996). Such models using different size scales are aimed at the stabilisation of localisation phenomena.

A simpler model of elastic-brittle damage is provided by the following damage criterion:  $\omega = 0$  if  $f = |\sigma| - \sigma_0 < 0$  and  $\omega = 1$  if  $f = |\sigma| - \sigma_0 = 0$ . This allows for analytic solutions to damage propagation problems. There exist two distinct zones in the solid, the undamaged material zone ( $\omega = 0$ ), which is elastic, and the zone associated with complete damage, ( $\omega = 1$ ). Since the boundary between these zones is unknown, we are left to free boundary value problems, in quasi-statics as well as in dynamics. For example, the steady state dynamic propagation of damage in an infinite medium, in Mode III fracture, has been solved analytically (Bui & Ehrlacher 1980).

Several papers are devoted to the propagation of discontinuities in solids due to a sudden change of properties (Stolz 1994) or due to phase transformations (Buisson et al. 1991; Chateau 1994). Thermodynamical analysis shows evidence of internal entropy production, concentrated on the moving damage front which behaves like a heat source (Stolz & Pradeilles-Duval 1996). The familiar energy release rate in



fracture mechanics can be generalised to damage propagation or to propagation of discontinuities in terms of the jumps of free energy and velocity gradients across the damage front.

## 6 CRACK INITIATION

Prior to rupture, localisation of deformation and damage by instabilities may occur and these phenomena can be used to define the onset of fracture. In a general context, the deformation and damage may be localised in the bulk of a structure, at the boundary or at interfaces. For a given loading process, the localisation conditions, related and based on earlier works of Hadamard (Mandel 1963), define the instant of occurrence of these phenomena but also the orientation of an eventual crack associated with a maximum tensile stress plane. Depending on the constitutive behaviour, the type of boundary conditions applied to the structure and the type of interfacial behaviour, they also allow to predict rupture to start inside, at the free surface or at an interface (Benallal et al. 1989, 1991).

Cracks frequently initiate at the interface of two materials (Desmorat 1996), at the junction of dissimilar elastic materials and the free surface (Zwiers et al. 1982). Most developments have been devoted to such interface singularities. However, there are new kinds of singularities, related to the interface between dissimilar thermal or plastic properties and also at discontinuity points of the loading distribution. At the intersection line between the free surface and the interface where the thermal strain  $\varepsilon_{22}^T$  or the plastic strain  $\varepsilon_{22}^P$  are discontinuous, there are weak singularities. (Bui & Taheri 1989) coined the term 'Epine' singularity, called after the French word for the 'thorn' like shaped function. The 'Epine' singularities of the stress components  $\sigma_{11}$  are characterised by their boundedness, discontinuity and unbounded gradient. They occur when the interface normally intersects the free surface (Deperrois & Dang Van 1990; Leguillon 1992; Santacreu 1994). The same kind of singularities arise at points of discontinuous normal stress applied to the boundary of elastic solids (Barton 1941). They are exactly the local effects solutions in plates or shells considered as 3D bodies, defined as the difference between a 3D solution and the Kirchhoff plate or shell solution. Thermal striping, crack initiation at the interface of a weld can be explained by the 'Epine' normal stress singularity. For example at a plastic interface, the jump relations at the free surface are of the form  $[\sigma_{11}] = 2\mu[\varepsilon_{22}^P]$  in plane strain (Bui & Taheri 1989),  $[\sigma_{11}] = 2\mu([\varepsilon_{22}^P] + \nu[\varepsilon_{33}^P])/(1-\nu)$  in 3D (Santacreu 1994). If the thermal and plastic interfaces are inclined (angle  $\alpha$  between the interface and the normal to the free surface), the 'Epine' singularity exists with the factor  $\cos 2\alpha$ , but it is masked by a stronger logarithmic singularity with the factor  $\sin 2\alpha$ . The 'Epine' singularity appears also in the asymptotic analysis of an adhesive joint between two elastic solids, intersecting the free surface, as the joint thickness tends to zero (Geymonat 1996).

Studies of delamination in composites have important applications in material science. Damage modelling of an elementary ply in laminates composites (Allix & Ladeveze 1992, 1996, Allix et al. 1995) consists of the characterisation of the layer by homogeneous internal parameters, describing both, transverse cracking of the

matrix and fiber rupture, and of the identification of interface properties by classical experiments (DCB specimens, traction on plates with holes).

Crack initiation originates at points near the boundary of solids or inside subsurface regions of high residual stresses, when the solids are subjected to plastic strain. Classical finite elements methods in non-linear computational mechanics provide efficient means for analysing monotonic loadings. However, these methods cannot be applied to shake-down analyses, plastic accommodation and ratchetting studies. Significant progress has recently been made (Dang Van 1993; Dang Van et al. 1996) for the computations of residual stresses associated with moving contact loads, repetitive contact loads, cyclic surface loads etc., (Fouvry 1996; Stalin-Muller 1996). These new computational methods have very important applications in fatigue and wear of materials and structures.

Computational work is required for determining the macroscopic cyclic stress and strain, as well as the cumulative plastic strain. For a given load path, it is particularly important to determine the endurance limit. This is provided by the well-known Dang Van criterion in multiaxial loadings (1973) which is now widely used in French industrial applications. The Dang Van criterion expressed in the form  $f(\tau, \sigma_m) = 0$ , where  $\tau$  is the maximum shear stress and  $\sigma_m$  the hydrostatic stress, is based on a micro-macro analysis of polycrystals with various localisation formulae (Lin-Taylor assumption, self-consistent model etc.). Microscopic models developed for metal plasticity (Zarka 1969; Zaoui 1972) are used only to justify the theoretical form of the fatigue law  $f(\tau, \sigma_m) = 0$ . The constants in the fatigue laws may be determined directly by experiments. Recent work in fatigue damage of metals using 'double scale models' renew the ideas of Dang Van's model developed in the 1970's.

## 7 SCALE EFFECTS IN FRACTURE MECHANICS

Fracture of quasi-brittle materials such as concrete, rock and ceramics presents a very complex feature because of the heterogeneity at microscopic scale and of different properties of the constituents (matrix, inclusions, cavities, etc.). Experiments on a macroscopic scale show evidence of softening behaviour and dependence of the fracture stress,  $\sigma'$ , on the specimen size,  $L$ .

There are two kinds of scale effects. The first one corresponds to the *volume* scale effect of a body with size  $L$ , having cracks at the microscopic scale. The fracture stress is of the form  $\sigma' = AL^{-\alpha}$  (where  $A$  and  $\alpha > 0$  are characteristics of the material). This scale effect can be explained by the heterogeneity at the microscopic scale.

The second one, called *structural* scale effect, is due to the non-linearity of macroscopic behaviour as already mentioned in ductile fracture of a body with a macroscopic crack.

The simplest models of non-linear fracture mechanics are associated with small scale yielding hypotheses. They are based on classical elasto-plasticity on the macroscopic scale (resistance curve model, Hillerborg's fictitious crack model). More complex models take into account the evolution of damage in the vicinity of the process zone which exhibits softening behaviour. To circumvent difficulties arising in numerical instabilities, such as the mesh size dependence or the localisation in one finite element etc., one introduces a certain characteristic width. It results in localisa-

tion limiters as can be provided by higher-order gradients theories to account for micro-structural effects (Leroy & Molinari 1993) and a non-local damage theory (Pijaudier-Cabot & Bazant 1987; Mazars et al. 1991). These models can be analysed either numerically by means of finite element methods or, in some special cases (hyperelastic models), analytically (Leroy & Molinari 1993).

Stochastic models were developed in order to take into account the heterogeneity of materials and the random character of mechanical constants on a micro-scale. H.J. Hermann et al. (1973) describe micro-cracking and the process of fracturing in heterogeneous solids by a lattice model of beams. Stochastic finite methods have been developed (Rossi & Piau 1983) for modelling the volume scale effect observed in experiments, and to study the influence of heterogeneities. In the latter model, micro-cracking is the result of decohesion between interface elements at some critical stress. Both, volume and structural scale effects have been accounted for in the stochastic non-local model of quasi-brittle fracture (Lalai & Sab 1996), as shown in Figure 3. Three size scales have been introduced in the so-called m3 model (micro-meso-macro). On the microscale of the grain size  $d$ , the non-local elastic-brittle law is assumed. On the meso-scale of the representative volume of size  $L' \gg d$ , the assumption of heterogeneity of the grains and randomness of critical stresses are assumed. The sizes  $d$  and  $L'$  are small in comparison with the size  $L$  of the structure (macro-scale)  $L \gg L' \gg d$ . The model by Lalai and Sab is based on Weibull's statistical law (with constant ' $m$ ') and thus depends on three constants ( $d, L', m$ ).

Beyond some critical size  $L'$ , the volume size effect can no longer be noticed. This corresponds to the minimum volume size at the meso-scale for which a macro-homogeneity throughout the body is observed.

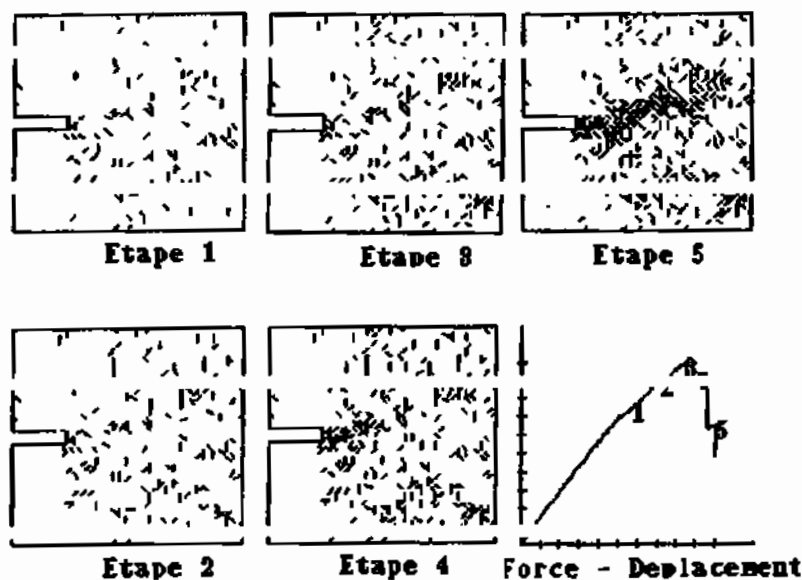


Figure 3. Numerical simulation of quasi-brittle fracture with 3600 5-Nodes square elements, with random critical stress in the sides and diagonals of elements. Steps 1, 2, 3: diffuse damage with increasing load. Steps 4 and 5: localisation in a crack-like zone, with decreasing load (Lalai & Sab 1996).

## 8 INVERSE PROBLEMS IN FRACTURE MECHANICS

Inverse problems in fracture mechanics are essentially concerned with non-destructive-testing of materials and structures for the detection of cracks, cavities or inclusions in solids (Cielo et al. 1986; Zorgat et al. 1991). Classical approaches to solve such an inverse problem include ultrasonics, Eddy current methods, X-ray or  $\Gamma$ -ray tomography, and thermography. Recently, some theoretical methods for crack detection have been worked out for inverse boundary value problems.

Inverse boundary value problems address the identification of cracks in three dimensions, where, similar to situations in potential theory as well as in the theory of elasticity, a surplus of data along the boundary is available. Such problems have been widely studied in 2D. We shall report here some new results in 3D elasticity.

In elasticity theory, the data consist of the pairs  $\{u, T(u)\}$ , which denote the displacement and the traction vector fields on the exterior boundary  $\Sigma$ , respectively. These pairs correspond to some plane crack to be determined. In the classical solution method, a family of solutions,  $v$ , of the elasticity equation for the uncracked body is considered, and the functional  $RG(v) := \int_{\Sigma} \{T(v) \cdot u - T(u) \cdot v\} dS$  is defined, with the integral taken over the exterior boundary of the cracked solid,  $\Sigma$ . If there is no crack, the Betti reciprocal theorem states that  $RG(v) = 0$ . Therefore the functional  $RG(v)$  provides a mechanical measure of the 'signature' of the crack and is called 'Reciprocity gap' functional. The  $RG(v)$  functional being known from the pair of data, can be expressed in terms of the displacement discontinuity  $[u]$  as follows  $\int_{\Gamma} \{T(v) \cdot [u]\} dS = RG(v)$ , where the integral is taken over the unknown crack surface  $F$  and the right hand side  $RG(v)$  is the known linear functional. This equation, for appropriate choices of basis functions  $v$ , determines completely the crack plane  $F$  (normal vector  $N$  and position in 3D space) (Andrieux & Ben Abda 1992). Recently, the complete identification of the crack has been obtained by determining the support of the displacement discontinuity, with the fields,  $v$ , chosen so as to obtain the Fourier transform of  $[u]$  (Andrieux et al. 1997). Mathematically, the crack identification problem is solved explicitly by determining the projections of the singularity of  $u$ , or its discontinuity  $[u]$ , on appropriate basis vector fields  $v$  of an Hilbert space.

## 9 FLUID-FILLED CRACKS

Fluid-filled cracks have been the subject of many papers. Previous studies of a fluid-filled crack in an elastic medium have been based on approximate global equations of continuity and motion, and on an assumption of one-dimensional flow in the crack. Such investigations have led to the existence of singularities in the fluid pressure and hence the conclusion that a crack cannot be completely filled with fluid. The last assumption can be later introduced in another natural way by considering the surface force (no complete filling is possible when the radius of curvature of the crack tip is smaller than some water molecules). However, at a macroscopic scale, without considering the surface force, the assumption of partial filling of fluid is not necessary if we introduce the two-dimensional viscous flow near the crack tip. Such an extension from one-dimensional to two-dimensional fluid flow has been given

(Bui & Parnes 1992). It is shown that the exact equations for 2D fluid-filled cracks do not give rise to a singular pressure.

The case of crack propagation with a fluid-filled crack is most interesting, from both, a physical and mechanical point of view. As the fluid is assumed to be viscous, the physical adherence condition at the crack surface is not compatible with the assumption of a fluid completely filling a Griffith crack, because of the discontinuity and singularity of the vertical components of velocity,  $u_2(0, \pm 0) = \pm \infty$ , of the solid points lying at the moving crack tip  $D$  ( $x_1 = 0$ ). The elastic solid point at the crack tip  $D$  breaks up into two parts, moving in opposite directions. Therefore, in order to avoid flow singularities the viscous fluid is assumed to partially occupy the crack zone, with a convex (or concave) meniscus in equilibrium with a surface tension force. A model of steady state flow of a viscous fluid-filled crack has been provided (Bui et al. 1991). Using the N3S Codes (Navier-Stokes computer codes of Electricité de France), the coupled problem of viscous fluid flow in the moving frame at the crack tip has been solved numerically as shown in Figure 4.

The case of a stationary crack has been studied by H.D. Bui (1996) who gives the analytical solution of the problem of a partially fluid-filled crack in an infinite medium, with the uniform traction at infinity. It is shown that for a convex meniscus, a stress intensity factor exists even when the traction at infinity vanishes. Consequently, the existence of a 'capillary' stress intensity factor lowers the applied effective stress intensity factor at rupture. This model provides a simple explanation of the well known Rehbinder effect (Bui 1996). The toughness of wet material is generally smaller than that of a dry material, as shown experimentally (Rossi & Boulay 1990). Most interpretations invoke the chemical change of material properties by water or moisture. Here, the interpretation of the Rehbinder effect is based on a purely mechanical argument and on the assumption of capillary force.

Viscous fluid flow through a longitudinal crack in a pressurised cylindrical shell has been studied with applications to the LBB problem (Leak-before-break) (Bui 1985). It is shown that there are two kinds of fluid flow corresponding to the values of Folias' parameter  $\lambda = a[12(1 - \nu^2)]^{1/4} (hR)^{-1/2}$ , where '2a' represents the crack length, R is the mean radius of the cylindrical shell, h the thickness of the shell,  $\nu$  is Poisson ratio. For  $\lambda > 4.78$ , the semi-axis of the elliptical crack opening displacement decreases from the inner radius to the outer radius of the tube (convergent flow) and for  $\lambda < 4.78$ , the semi-axis of the elliptical crack opening increases (divergent flow). Theoretical predictions of the fluid flow rate agrees quite well with experiments on

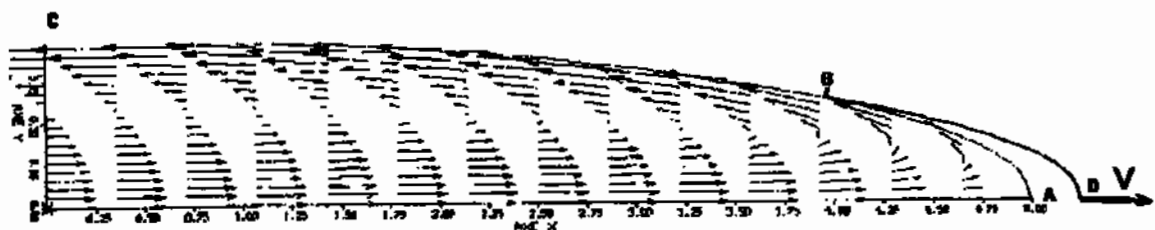


Figure 4. Steady state viscous fluid flow in the moving frame at the crack tip D, with velocity  $V$ ; surface tension force along AB, no fluid-solid contact along AB; adherence along BC with relative velocity  $-V$  (Bui et al. 1991).

pressurised cracked tubes. The LBB studies of a longitudinally cracked cylinder are based on the use of crack singularity analyses provided by D. Bergez and D. Radenkovic (1972). The singularities of cracked plates or shells are characterised by 5 coefficients:  $k_1, k_2, k_3$  (the classical three modes in the middle surface of the shell) and the rates of change through the thickness  $k'_1 = \partial K_1(x_3)/\partial x_3, k'_2 = \partial K_2(x_3)/\partial x_3$  of Modes I and II stress-intensity factors.

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