Obstruction classes of crossed modules of Lie algebroids and Lie groupoids linked to existence of principal bundles

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Abstract

Let K be a Lie group and P be a K-principal bundle on a manifold M. Suppose given furthermore a central extension

$$1 \to Z \to \hat{K} \to K \to 1$$

of K. It is a classical question whether there exists a $\hat{K}$-principal bundle $\hat{P}$ on M such that $\hat{P}/Z \cong P$. In [11], Neeb defines in this context a crossed module of topological Lie algebras whose cohomology class $[\omega_{\rm top\ alg}]$ is an obstruction to the existence of $\hat{P}$. In the present paper, we show that $[\omega_{\rm top\ alg}]$ is up to torsion a full obstruction for this problem, and we clarify its relation to crossed modules of Lie algebroids and Lie groupoids, and finally to gerbes.

Introduction

It is well known that 3-cohomology in an algebraic category like the categories of (discrete) groups, Lie algebras or associative algebras is related to crossed modules of groups, Lie algebras or associative algebras respectively. A *crossed module of groups* is, roughly speaking, a homomorphism of groups $\mu : M \to N$ together with an action by automorphisms of N on M which is compatible in some sense with μ. Passing to kernel and cokernel of μ, one gets a 4-term exact sequence of groups

$$1 \to V = \ker \mu \to M \to N \to G = coker\, \mu \to 1,$$

such that V is an abelian group and a G-module. The general algebraic picture associates to such a crossed module a cohomology class in $H^3(G, V)$ which is the obstruction to come from an extension of G by M. Related notions which take into account topology exist for Lie groupoids [7], [1] and for topological Lie algebras [11], [14].

In [11], Karl-Hermann Neeb defines for a crossed module of topological Lie algebras which is split as a sequence of topological vector spaces a cohomology class $[\omega_{\rm top\ alg}]$. He shows that $[\omega_{\rm top\ alg}]$ has a specific meaning in the following context: let K be some Lie group and P be a K-principal bundle on some manifold M. Suppose given furthermore a central extension

$$1 \to Z \to \hat{K} \to K \to 1$$

of K. The question is now whether there exists a $\hat{K}$-principal bundle $\hat{P}$ on M such that $\hat{P}/Z \cong P$. Neeb uses the ingredients of the problem to associate a crossed module of topological Lie algebras to it such that its obstruction class $[\omega_{\rm top\ alg}]$ is a 3-de Rham cohomology class on M which is an obstruction to the existence of $\hat{P}$.

The origin of this paper is the question of Neeb at the end of section IV of [11] whether $[\omega_{\rm top\ alg}]$ is the full obstruction to the existence of $\hat{P}$, which 3-de Rham forms arise as obstructions, and on the relation of $[\omega_{\rm top\ alg}]$ to gerbes.

We answer his questions in the framework of crossed modules of Lie algebroids and groupoids and show that $[\omega_{\rm top\ alg}]$ can be identified with the obstruction class of a certain crossed module of Lie algebroids associated to the above problem (theorem 1), and, up to torsion, even to the obstruction class of a certain crossed module of Lie groupoids associated to the above problem (theorem 2, main theorem of this paper), which is known [7] to be the full obstruction to the existence of $\hat{P}$.

In section 5, we show that it follows from Serre's identification of the Brauer group $Br(M)$ of M (cf [5]) that Neeb's obstruction class is zero for finite dimensional structure group. In section 7, we deduce from the observation that gerbes and crossed modules of Lie groupoids are classified by the same kind of cohomology classes a direct relation between these two kinds of objects.

In the appendix on Deligne cohomology, we present the necessary material for the proof of theorem 2.

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1 The Atiyah sequence

In this section, we recall the Atiyah sequence associated to a K-principal bundle on a fixed base manifold M, and we explain the main object of this article.

Let M be a (finite dimensional, connected, second countable, smooth) manifold with finite dimensional de Rham cohomology, and let K be a (not necessarily finite dimensional) Lie group with Lie algebra $\mathfrak{k}$. We usually take infinite dimensional Lie groups to be modeled on locally convex spaces. Furthermore, let $\pi : P \to M$ be a K-principal bundle on M. As for finite dimensional structure groups, P can be represented by a smooth Čech 1-cocycle, cf [10]. Connections on P can be constructed by patching on the finite dimensional manifold M. K acts on P from the right, and this action induces an action on

TP. The induced map between tangent bundles $T\pi : TP \to TM$ factors to a map $\pi_* : TP/K \to TM$. The kernel of π_* can be identified (cf [8] p. 92 or [11] IV) with $(P \times \mathfrak{k})/K$, where K acts on the product by the diagonal action, using the adjoint action on the second factor. We denote this bundle by $\mathrm{Ad}(P)$. Therefore, we get the *Atiyah sequence* of vector bundles

$$0 \to \mathrm{Ad}(P) \to TP/K \to TM \to 0.$$

The main question we address in this paper is the following: given a central extension

$$1 \to Z \to \hat{K} \to K \to 1$$

of K by Z, is there a $\hat{K}$-principal bundle $\hat{P}$ on M such that $\hat{P}/Z \cong P$? More precisely, one wants to construct computable obstructions for the existence of such a $\hat{P}$. This is the point of view expressed in Neeb's paper [11] section VI.

2 Crossed modules of topological Lie algebras

In [11], Neeb associates to a given principal bundle P on M and a central extension

$$1 \to Z \to \hat{K} \to K \to 1$$

of its structure group K, a differential 3-form $\omega_{\text{top alg}}$ defining a class in $H^3_{\text{dR}}(M, \mathfrak{z}) = H^3_{\text{dR}}(M) \otimes \mathfrak{z}$ which is an obstruction to the existence of a principal $\hat{K}$-bundle $\hat{P}$ such that $\hat{P}/Z \cong P$. Let us recall its construction:

Definition 1 *A crossed module of Lie algebras is a homomorphism of Lie algebras $\mu : \mathfrak{m} \to \mathfrak{n}$ together with an action η of $\mathfrak{n}$ on $\mathfrak{m}$ by derivations such that*

(a) $\mu(\eta(n) \cdot m) = [n, \mu(m)]$ for all $n \in \mathfrak{n}$ and all $m \in \mathfrak{m}$,

(b) $\eta(\mu(m)) \cdot m' = [m, m']$ for all $m, m' \in \mathfrak{m}$.

In the framework of *topological Lie algebras*, one requires all maps to be continuous and topologically split, cf [11].

To an Atiyah sequence

$$0 \to \mathrm{Ad}(P) \to TP/K \to TM \to 0,$$

and a central extension of the structure group

$$1 \to Z \to \hat{K} \to K \to 1,$$

one associates a crossed module of topological Lie algebras. For this, denoting $\mathfrak{z}$, $\mathfrak{k}$ and $\hat{\mathfrak{k}}$ the Lie algebras of Z, K and $\hat{K}$ respectively, we first associate to the extension the sequence of vector bundles

$$0 \to M \times \mathfrak{z} \to \widehat{\mathrm{Ad}}(P) \to \mathrm{Ad}(P) \to 0, \tag{1}$$

where $\widehat{\mathrm{Ad}}(P)$ is the bundle $(P \times \hat{\mathfrak{k}})/K$. Observe that the adjoint action of $\hat{K}$ on $\hat{\mathfrak{k}}$ factors to an action of K on $\hat{\mathfrak{k}}$. The bundle $\widehat{\mathrm{Ad}}(P)$, which is isomorphic to $\mathrm{Ad}(\hat{P})$ in case $\hat{P}$ exists, is constructed from the ingredients of the problem and thus exists even if $\hat{P}$ does not exist.

We now pass to the spaces of global sections of the above vector bundles. For the Atiyah sequence, we get $\mathfrak{aut}(P) = \Gamma(TP/K) = \mathcal{V}(P)^K$ the Lie algebra of K-invariant vector fields on P, $\mathcal{V}(M) = \Gamma(TM)$ the Lie algebra of vector fields on M, and $\mathfrak{gau}(P) = \mathfrak{n} = \Gamma(\mathrm{Ad}(P))$ the *gauge Lie algebra*, i.e. the Lie algebra of vertical K-invariant vector fields. For the sequence (1), we get Lie algebras $\mathfrak{c} = \Gamma(M \times \mathfrak{z})$, $\mathfrak{n} = \Gamma(\mathrm{Ad}(P))$ and $\hat{\mathfrak{n}} = \Gamma(\widehat{\mathrm{Ad}}(P))$. All these are given the topology of uniform convergence on compact sets of the function and all of its derivatives, and become in this way locally convex topological Lie algebras.

One gets a crossed module of topological Lie algebras $\mu : \hat{\mathfrak{n}} \to \mathfrak{aut}(P)$ by projecting onto $\mathfrak{n}$ and including then $\mathfrak{n}$ into $\mathfrak{aut}(P)$, or, passing to the kernel and the cokernel of μ, we get a four term exact sequence

$$0 \to \mathfrak{c} \stackrel{i}{\to} \hat{\mathfrak{n}} \stackrel{\mu}{\to} \mathfrak{aut}(P) \stackrel{\pi}{\to} \mathcal{V}(M) \to 0.$$

The action η of $\mathfrak{aut}(P)$ on $\hat{\mathfrak{n}}$ is induced by the derivation action of $\mathcal{V}(P)$ on $\hat{\mathfrak{n}} \subset \mathcal{C}^\infty(P, \hat{\mathfrak{k}})$.

The differential 3-form $\omega_{\text{top alg}}$ in $\Omega^3(M, \mathfrak{z})$ is constructed as a cocycle associated to this crossed module (cf lemma VI.2 in [11]). Namely, take a principal connection 1-form $\theta \in \Omega^1(P, \mathfrak{k})^K$. It serves two purposes: first, it defines a section σ of π, second, it gives a projection from $\mathfrak{aut}(P)$ to $\mathfrak{n}$. The curvature of θ is the 2-form

$$R_\theta = d\theta + \frac{1}{2}[\theta, \theta] \in \Omega^2(P, \mathfrak{k})^K.$$

Now, regarding R_θ as an $\mathrm{Ad}(P)$-valued 2-form, it can be lifted to an $\widehat{\mathrm{Ad}}(P)$-valued 2-form Ω. The section σ, and therefore the connection θ, permit to define an *outer action*

$$S = \eta \circ \sigma : \mathcal{V}(M) \to \mathfrak{der}(\hat{\mathfrak{n}}),$$

meaning that the linear map S is a homomorphism when projected to $\mathfrak{out}(\hat{\mathfrak{n}})$ and thus gives, in particular, an action of $\mathcal{V}(M)$ on $\mathfrak{c}$. The class $\omega_{\text{top alg}}$ is then just $d_S\Omega$, but interpreted as an element of $\Omega^3(M, \mathfrak{z})$. Here d_S means that one takes the Chevalley-Eilenberg coboundary with values in $\hat{\mathfrak{n}}$ as if S were an action:

$$\begin{aligned} d_S\Omega(x_1, x_2, x_3) &= S(x_1) \cdot \Omega(\sigma x_2, \sigma x_3) + S(x_2) \cdot \Omega(\sigma x_3, \sigma x_1) + S(x_3) \cdot \Omega(\sigma x_1, \sigma x_2) \\ &- \Omega(\sigma[x_1, x_2], \sigma x_3) - \Omega(\sigma[x_2, x_3], \sigma x_1) - \Omega(\sigma[x_3, x_1], \sigma x_2) \end{aligned}$$

for $x_1, x_2, x_3 \in \mathcal{V}(M)$. Geometrically speaking, $d_S\Omega$ is the horizontal component of $d\Omega$, cf [11] section VI.

Remark 1 *We consider that we have fixed a decomposition as vector spaces $\hat{\mathfrak{k}} = \mathfrak{k} \oplus \mathfrak{z}$, and that with respect to this decomposition, the above lift sends $x \in \mathfrak{k}$*

to $(x, 0) \in \mathfrak{k} \oplus \mathfrak{z}$. *Observe that as* $d_S\Omega(x_1, x_2, x_3)$ *has values in* $\mathfrak{z}$*,* R_θ *has values in* $\mathfrak{k}$ *and each term* $S(x_i) \cdot \Omega(\sigma x_j, \sigma x_k)$ *has a component in* $\mathfrak{z}$ *and one in* $\mathfrak{k}$*, the sum in the second line of the equation for* $d_S\Omega(x_1, x_2, x_3)$ *together with the components with values in* $\mathfrak{k}$ *from the first line must give zero.* $d_S\Omega$ *is thus just the component with values in* $\mathfrak{z}$ *of the cyclic sum of the terms of the form* $S(x_i) \cdot \Omega(\sigma x_j, \sigma x_k)$.

3 Crossed modules of Lie algebroids

We recall in this section the main definitions on Lie algebroids, their cohomology and crossed modules, cf [1], [8].

Definition 2 *Let* M *be a fixed manifold. A Lie algebroid* A *over* M *is a vector bundle* $q : A \to M$ *together with a morphism of vector bundles* $a : A \to TM$ *from* M *to the tangent bundle* TM *of* A*, called the anchor, and an* $\mathbb{R}$*-linear skewsymmetric bracket* $[,] : \Gamma A \times \Gamma A \to \Gamma A$ *on the space of smooth sections* ΓA *of* A *which satisfies the Jacobi identity, and such that*

(1) $[X, fY] = f[X, Y] + a(X)(f)Y$

(2) $a([X, Y]) = [a(X), a(Y)]$ *(this second point being in fact a consequence of the first one).*

A Lie algebroid A *is called transitive if* a *is fibrewise surjective. In this case,* a *is of constant rank and* TM *is a quotient bundle of* A*. Given two Lie algebroids* A *and* A' *over the same manifold* M*, a morphism* $\phi : A \to A'$ *of vector bundles such that, first,* $q' \circ \phi = q$*, second* ϕ *induces a Lie algebra homomorphism on the spaces of sections, and such that* $a' \circ \phi = a$ *is said to be a Lie algebroid homomorphism over the identity of* M*.*

We may say simply *Lie algebroid morphism*, when it is obvious that it is over the identity of M.

Note that if the anchor map is trivial at each point of M, a Lie algebroid is precisely what is called a *Lie algebra bundle*, i.e. a vector bundle, endowed fiberwise with a Lie algebra structure, which is required to satisfy the following assumption of regularity: for two smooths sections of the vector bundle, taking the bracket pointwise yields an other smooth section.

The main example of a Lie algebroid and Lie algebroid morphisms which we have in mind is the Atiyah algebroid and its associated sequence of a principal K-bundle P on a manifold M, cf [8] p. 97. It is a transitive Lie algebroid.

Remark 2 *Regarding a Lie algebra as a Lie algebroid over the point defines a fully faithful functor from the category of Lie algebras to the category of Lie algebroids. Sending a Lie algebroid to its space of global sections defines a functor* Γ *from the category of Lie algebroids to the category of (topological) Lie algebras which is an inverse to the previous one when restricted to Lie algebroids over the point.*

Definition 3 *Let A be a Lie algebroid on M, and E be a vector bundle on M. A representation of A on E is a morphism of Lie algebroids $\rho : A \to \mathcal{D}(E)$ where $\mathcal{D}(E)$ is the Lie algebroid of first order differential operators $D : \Gamma E \to \Gamma E$ such that there is a vector fields $X = a_{\mathcal{D}}(D)$ with $D(fs) = fD(s) + a_{\mathcal{D}}(D)(f)s$.*

We shall then say that E is an A-module. A morphism of representations $\phi : E \to E'$ where E, E' are A-modules via ρ and ρ', is a morphism of vector bundles satisfying $\phi(\rho(X)(s)) = \rho'(X)(\phi(s))$ for all $s \in \Gamma E$, $X \in \Gamma A$.

For instance, the trivial bundle $M \times \mathbb{R}^k \to M$ is an A-module, when the Lie algebroid morphism ρ of the definition is choosen to be the anchor map (and vector fields are considered as derivations of the space of k-tuples of smooth functions).

Let us now recall the complex which computes the cohomology of a Lie algebroid with values in some representation:

Definition 4 *Let A be a Lie algebroid and E be an A-module, both being vector bundles on M. The standard complex $C^*(A, E)$ is the complex of vector bundles $C^p(A, E) = \mathrm{Alt}^p(A, E)$ together with the usual differential $d : \Gamma C^p(A, E) \to \Gamma C^{p+1}(A, E)$. Cocycles, coboundaries and cohomology spaces are defined in the usual way.*

The tangent bundle $TM \to M$ is a Lie algebroid: the anchor map is the identity and the bracket is the bracket of vector fields. With respect to the trivial representation $M \times \mathbb{R}^k \to M$, the space of cochains $C^p(A, E)$ is precisely the space of k-tuples of p-forms. It is a direct verification that the algebroid differential becomes, under this identification, the de Rham differential. In conclusion,

Lemma 1 *The algebroid cohomology of the tangent bundle $TM \to M$ taking values in the trivial module $M \times \mathbb{R}^k \to M$ is the de Rham cohomology tensored by $\mathbb{R}^k$, in short*

$$H^*(A) = \bigoplus_{i \in \mathbb{N}} H^i_{\mathrm{dR}}(M) \otimes \mathbb{R}^k.$$

This lemma stays true for values in a vector bundle with infinite dimensional fiber, seen as a trivial $TM \to M$-module.

The following definition is taken from [1]:

Definition 5 *A crossed module of Lie algebroids over the manifold M is a quadruple (L, μ, A, ρ) where $L \to M$ is a Lie algebra bundle, $A \to M$ is a transitive Lie algebroid, $\mu : L \to A$ is a morphism of Lie algebroids and $\rho : A \to \mathcal{D}(L)$ is a representation of A in L such that*

(a) $\rho(X)([U, V]) = [\rho(X)(U), V] + [U, \rho(X)(V)]$ *for all $X \in \Gamma A$ and for all $U, V \in \Gamma L$ and*

(b) $\rho(\mu(V))(W) = [V, W]$ *for all $V, W \in \Gamma L$ and*

(c) $\mu(\rho(X)(V)) = [X, \mu(V)]$ *for all* $X \in \Gamma A$ *and* $V \in \Gamma L$.

Observe that this notion is actually the notion of a crossed module of *transitive* Lie algebroids.

According to [1], Section 2, the map μ is of constant rank.

Let us recall [8] p. 272-273 how to associate a 3-cocycle

$$\omega_{\mathrm{alg}} \in C^3(coker(\mu), \ker(\mu))$$

to a crossed module of Lie algebroids. Every crossed module of Lie algebroids (L, μ, A, ρ) induces (using ρ) a *coupling*, i.e. a morphism of Lie algebroids $coker(\mu) \to \mathrm{Out}\mathcal{D}(L)$. A lift to $\nabla : coker(\mu) \to \mathcal{D}(L)$ is called a *Lie derivation law* covering the coupling. ∇ can be taken formally to be an action of $coker\mu$ on L, although it is not an action in general. The curvature of ∇ takes values in the inner derivations of L, and can thus be lifted to an alternating vector bundle map $\Lambda : (coker\ \mu) \oplus (coker\ \mu) \to L$. The cocycle is now obtained as

$$\gamma(X, Y, Z) := \sum_{\mathrm{cycl.}} \left(\nabla_X(\Lambda(Y, Z)) - \Lambda([X, Y], Z) \right),$$

i.e. as the formal Chevalley-Eilenberg coboundary operator applied to the cochain Λ using the "action" of $coker\ \mu$ on L. It is clear that ∇ can be given by a section of the quotient map $A \to coker\ \mu$ and the action ρ.

Now let us look at the special case which is of interest in this paper: let P be a K-principal bundle on a manifold M, and

$$1 \to Z \to \hat{K} \to K \to 1$$

be a central extension of the structure group. To these data, we attach the crossed module of Lie algebroids $\mu : \widehat{\mathrm{Ad}}(P) \to TP/K$. Here, $\widehat{\mathrm{Ad}}(P)$ denotes the bundle of Lie algebras $(P \times \hat{\mathfrak{k}})/K$, TP/K is the transitive Atiyah Lie algebroid, and a representation of TP/K on $\widehat{\mathrm{Ad}}(P)$ by first order differential operators is given by the action of K-invariant vector fields on P as derivations on functions from P to $\hat{\mathfrak{k}}$. This action makes μ equivariant and restricts to the action of $\mathrm{Ad}(P)$ on $\widehat{\mathrm{Ad}}(P)$ given by the central extension. In this case, $coker(\mu)$ is the tangent bundle $TM \to M$ and $\ker(\mu)$ is the trivial bundle $M \times \mathfrak{z} \to M$, which amounts to the fact that ω_{alg} is an element of $\Omega^3(M, \mathfrak{z})$, and, by Lemma 1, the class $[\omega_{\mathrm{alg}}]$ belongs to $H^3_{\mathrm{dR}}(M, \mathfrak{z}) = H^3_{\mathrm{dR}}(M) \otimes \mathfrak{z}$.

Theorem 1 *Given a* K*-principal bundle* P *on a manifold* M *and a central extension*

$$1 \to Z \to \hat{K} \to K \to 1$$

of the structure group, the 3*-cohomology class* $[\omega_{\mathrm{top\ alg}}]$ *of the crossed module of topological Lie algebras*

$$0 \to \mathfrak{c} \xrightarrow{i} \hat{\mathfrak{n}} \xrightarrow{\mu} \mathfrak{aut}(P) \xrightarrow{\pi} \mathcal{V}(M) \to 0$$

coincides with the class $[\omega_{\mathrm{alg}}]$ *associated to the crossed module of Lie algebroids*

$$0 \to M \times \mathfrak{z} \to \widehat{\mathrm{Ad}}(P) \to TP/K \to TM \to 0.$$

Proof.

Choosing a principal connection 1-form θ gives a horizontal lifting for the Lie algebroids, as for the Lie algebras of global sections. θ gives rise on the one hand to a coupling and a ∇ which corresponds on the other hand to the outer action $S = \eta \circ \sigma$ where σ is also defined by θ. The curvature R_θ of θ can be regarded as an $\mathrm{Ad}(P)$-valued 2-form. It is then lifted to an $\widehat{\mathrm{Ad}}(P)$-valued 2-form Ω on M which corresponds to Λ in the discussion preceeding the theorem. The next step is to compute the above mentioned formal Chevalley-Eilenberg coboundary of Ω with values in $\widehat{\mathrm{Ad}}(P)$ denoted by d_S in the previous section. Up to identification, we get thus that for this special crossed module of Lie algebroids, ω_{alg} has the same expression as $\omega_{\mathrm{top\ alg}}$.

□

By the above proposition, we have at our disposal all results on Lie algebroids in part 2, ch. 7 of [8]; applying the global section functor, they give results on the class of $\omega_{\mathrm{top\ alg}}$. For example, Neeb shows in [11] prop. VI.3 that if an extension $\hat{P}$, i.e. a principal $\hat{K}$-bundle $\hat{P}$ such that $\hat{P}/Z \cong P$, exists, the class of $\omega_{\mathrm{top\ alg}}$ is trivial. But on the other hand, cor. 7.3.9 on p. 281 in [8] shows that in case $[\omega_{\mathrm{alg}}] = 0$, there exists an extension of Lie algebroids

$$0 \to \widehat{\mathrm{Ad}}(P) \to R \to TM \to 0.$$

This is in some sense the converse of Neeb's prop. VI.3. However, we do not know how R is related to the existence of a $\hat{K}$-principal bundle $\hat{P}$, but we will see that the question of the existence of $\hat{P}$ can be solved in terms of Lie groupoids associated to the Lie algebroids studied here.

4 Crossed modules of Lie groupoids

Definition 6 *A Lie groupoid consists of two manifolds Γ and M, together with two surjective submersions $s, t : \Gamma \to M$, called the source and the target map, and a smooth object inclusion $M \to \Gamma$ such that for composable elements (i.e. $g, h \in \Gamma$ such that $t(g) = s(h)$), there is a smooth group law having the images of the object inclusion as its unit elements.*

A morphism of Lie groupoids $\mathcal{F} : \Gamma \to \Xi$ where Γ is on M and Ξ is on N, is a pair of smooth maps $(\mathcal{F} : \Gamma \to \Xi, f : M \to N)$ intertwining sources and targets such that $\mathcal{F}$ is a homomorphism of groups for composable elements.

A Lie groupoid where the source and target maps are equal is what is called a *Lie group bundle*, i.e. a bundle Γ over M where all the fibers admit a Lie group structure which is smooth in the sense that taking the pointwise product of two smooth local sections, or the pointwise inverse of a local smooth section, yields a smooth local section.

We do not limit ourself to the case of manifolds of finite dimension, but M and $H^*_{\mathrm{dR}}(M)$ will always be supposed finite dimensional; as remarked before, Lemma 1 can be easily adapted to (this version of) the infinite dimensional framework. If Γ is infinite dimensional, one can use the notion of submersion with the help of an implicit function theorem with parameter like in [3], as it is shown explicitly in [12].

Definition 7 *Let Γ be a Lie groupoid on M, and let $\pi : F \to M$ be a bundle of Lie groups on M. A representation of Γ on F is a smooth map $\rho : \Gamma \times_M F \to F$ (where the fiber product is taken with respect to the source map s), such that*

- $\pi \circ \rho = t \circ \mathrm{proj}_1$, *with the target map t,*
- ρ *is an action for composable elements, and*
- $\rho(x) : F_{s(x)} \to F_{t(x)}$ *is a Lie group isomorphism for all $x \in \Gamma$.*

Definition 8 *A crossed module of Lie groupoids $\mathcal{F} : F \to \Gamma$ is a Lie groupoid Γ on M, a Lie group bundle F on M, a morphism of Lie groupoids $\mathcal{F} : F \to \Gamma$ (with f the identity), and a representation ρ of Γ on F such that*

- $\mathcal{F}(\rho(x, y)) = x\mathcal{F}(y)x^{-1}$ *for all $(x, y) \in \Gamma \times_M F$,*
- $\rho(\mathcal{F}(x), y) = xyx^{-1}$ *for all $x, y \in F$ avec $\pi(x) = \pi(y)$, and*
- *the image of $\mathcal{F}$ is a closed regularly embedded submanifold of Γ.*

We will only work here with crossed modules of Lie groupoids $\mathcal{F} : F \to \Gamma$ such that *coker* $\mathcal{F} = M \times M$ (the pair groupoid) and $\ker(\mathcal{F}) \simeq M \times Z$ for some Abelian Lie group Z. As a crossed module of transitive Lie algebroids is in a sense a crossed module of TM by a trivial $\mathfrak{z}$-bundle, the crossed modules of Lie groupoids we discuss here are crossed modules of $M \times M$ by a trivial Lie group bundle $Z \times M$. In other words, we restrict ourself to the case of crossed modules over *transitive* Lie groupoids.

An important point is that, in our case, Γ and F may be infinite dimensional, while *coker* $\mathcal{F}$ is finite dimensional.

Suppose given a K-principal bundle P and a central extension $\hat{K}$ of the structure group K. Note that K acts by conjugation, not only on K itself, but also on $\hat{K}$. Denote by $P_K(\hat{K}) = \mathcal{C}^\infty(P, \hat{K})^K$ the space of K-equivariant smooth maps from P_m to $\hat{K}$, where P_m is the fiber over an arbitrary $m \in M$. $P_K(\hat{K})$ is naturally a group bundle over M, and there is a natural groupoid homomorphism $\mathcal{F} : P_K(\hat{K}) \to (P \times P)/K$ mapping $\phi \in \mathcal{C}^\infty(P_m, \hat{K})^K$ to $\overline{(p \cdot \nu(\phi(p)), p)}$ where p is an arbitrary element of P_m, ν is the map from $\hat{K}$ to P, and the bar means the class in $(P \times P)\,/\,K$. It is easy to check that the groupoid homomorphism $\mathcal{F}$ is a crossed module. Conversely, we have:

Proposition 1 *Let $\mathcal{F} : F \to \Gamma$ be a crossed modules of Lie groupoids such that* coker $\mathcal{F} = M \times M$ *and such that the kernel of $\mathcal{F}$ is trivial:* $\ker \mathcal{F} = Z \times M$.

Fix a point $m \in M$, and define the Lie group K to be the quotient of $\hat{K} := F_m$ by Z. Then there exists a K-principal bundle P such that $\mathcal{F} : F \to \Gamma$ is isomorphic to the crossed module

$$0 \to M \times Z \to P_K(\hat{K}) \to (P \times P)/K \to M \times M \to 0.$$

Proof. We fix a point $m \in M$, and define the Lie group K to be the quotient of $\hat{K} := F_m$ by Z. Denote by s and t the source and target maps of Γ. The space $t^{-1}(m)$ is a submanifold of Γ (because the target map is a submersion) on which K acts freely by right multiplication. Now, the source map $s : t^{-1}(m) \to M$ is a surjective submersion onto M as $coker(\mathcal{F}) = M \times M$, and the fibers are precisely given the right K-action. Hence $t^{-1}(m) \to M$ is a principal K-bundle that we denote by P.

It is easy to check that the groupoid Γ is isomorphic to the Atiyah groupoid $(P \times P)/K$. The isomorphism Ψ is as follows. Any element of Γ can be written in the form $\gamma = \gamma_1 \cdot \gamma_2$ with $\gamma_1, \gamma_2^{-1} \in t^{-1}(m)$. Define $\Psi(\gamma)$ to be $\overline{(\gamma_1, \gamma_2^{-1})} \in (P \times P)/K$.

Let us also define a map $\Phi : F \to P_K(\hat{K}) = \mathcal{C}^\infty(P, \hat{K})^K$. To an $f \in F$, we associate the map $(x \mapsto \rho(x, f))$, where ρ is the action of Γ on F, given in the data of the crossed module, and let us recall that the map $\mathcal{G} : P_K(\hat{K}) \to (P \times P) \,/\, K$ of the Atiyah crossed module corresponding to the central extension of the structural group K of P to $\hat{K}$, is given in our context by

$$\phi \mapsto \mathcal{G}(\phi) := \overline{(p \cdot_\Gamma \mathcal{F}(\phi(p)), p)},$$

where $p \in P$ is an arbitrary point, $\cdot_\Gamma$ is the multiplication in Γ.

Let us show that the square

$$\begin{array}{ccc} P_K(\hat{K}) & \xrightarrow[\mathcal{G}]{} & (P \times P) \,/\, K \\ \uparrow{\scriptstyle \Phi} & & \uparrow{\scriptstyle \Psi} \\ F & \xrightarrow[\mathcal{F}]{} & \Gamma \end{array}$$

is commutative. Indeed, by the axioms of a crossed module, we have a commutative square

$$\begin{array}{ccc} F_n & \xrightarrow[\mathcal{F}_n]{} & \Gamma_n^n \\ {\scriptstyle \rho(p,1)}\downarrow & & \uparrow{\scriptstyle \mathrm{Conj}(p^{-1})} \\ F_m & \xrightarrow[\mathcal{F}_m]{} & \Gamma_m^m \end{array}$$

where $n = t'(p)$ and $m = s'(p)$, t' and s' being the source and target maps of P (i.e. the restrictions of those of Γ to $P \subset \Gamma$). In n, we have $\mathcal{F}(f) = \gamma_1 \gamma_2^{-1}$ with $\gamma_1, \gamma_2 \in \Gamma_n^m$ and $f \in F_n$. Now compute

$$\Psi(\mathcal{F}(f)) = \overline{(\gamma_1, \gamma_2)} = \overline{(\mathcal{F}(f)p, p)} = \overline{(pp^{-1}\mathcal{F}(f)p, p)} = \overline{(p\mathcal{F}(\rho(p, f)), p)}.$$

□

There is a standard way to associate to a crossed module of Lie groupoids a characteristic class ω_{grp} (cf [7] p. 197 or [1] p. 13): let us choose a covering (trivializing the principal bundle P described above) $\mathcal{U} = \{U_i\}$ on the manifold M, and denote as usual $U_{ij} = U_i \cap U_j$, $U_{ijk} = U_i \cap U_j \cap U_k$. The principal bundle P is given by transition functions $g_{ij} : U_{ij} \to K$, which we may as well regard as a Čech 1-cocycle. Lift the functions g_{ij} to functions $\hat{g}_{ij}$ with values in $\hat{K}$, and denote by $\hat{g}_{ij}\hat{g}_{jk}\hat{g}_{ki} = h_{ijk}$ their default for forming a Čech 1-cocycle. h_{ijk} has values in the sheaf $\underline{Z}$ of differentiable Z-valued functions, as the g_{ij} do form a cocycle. The Čech 2-cocycle h_{ijk} with values in $\underline{Z}$ is by definition $\omega_{\rm grp}$. Its class will also be regarded as a class in $H^3(M, \pi_1(Z))$, provided that Z is a connected regular abelian Lie group, according to the isomorphism

$$H^2(M, \underline{Z}) \cong H^3(M, \pi_1(Z)),$$

which stems from the fact that for a connected regular abelian Lie group

$$1 \to \pi_1(Z) \to \mathfrak{z} \to Z \to 1$$

is an exact sequence of groups, where the abelian group $\mathfrak{z}$ is the Lie algebra of Z, and that the sheaf $\underline{\mathfrak{z}}$ is fine.

The last step is to consider $H^3(M, \pi_1(Z))$ as a subspace of the de Rham cohomology groups $H^3_{\rm dR}(M) \otimes \mathfrak{z}$ obtained by composing the map $H^3(M, \pi_1(Z)) \to H^3(M, \mathbb{R}) \otimes \mathfrak{z}$ coming from the inclusion $\pi_1(Z)$ into $\mathfrak{z}$ with the Čech-de Rham isomorphism $H^3(M, \mathbb{R}) \simeq H^3_{\rm dR}(M)$.

The following theorem is the main result of our paper:

Theorem 2 *Given a K-principal bundle P on a manifold M and a central extension*

$$1 \to Z \to \hat{K} \to K \to 1$$

of the structure group, the 3-cohomology class $[\omega_{\rm top\ alg}]$ of the crossed module of topological Lie algebras

$$0 \to \mathfrak{c} \xrightarrow{i} \hat{\mathfrak{n}} \xrightarrow{\mu} \mathfrak{aut}(P) \xrightarrow{\pi} \mathcal{V}(M) \to 0$$

defines the same de Rham cohomology class as the 3-cocycle $\omega_{\rm grp}$ associated to the crossed module of Lie groupoids

$$0 \to M \times Z \to P_K(\hat{K}) \to (P \times P)/K \to M \times M \to 0.$$

Proof. We use the previously established notations. Let us choose a connection on the principal bundle $P \to M$ given, in local trivializing coordinates $\{U_i\}_{i \in I}$, by a family θ_i of $\mathfrak{k}$-valued 1-forms on U_i. As usual, the relation $\theta_j = {\rm Ad}_{g_{ij}} \theta_i + g_{ij}^{-1} dg_{ij}$ expresses how to pass from θ_i to θ_j by gauge transformation. Lifting to $\hat{\mathfrak{k}}$, we get $\theta_i - {\rm Ad}_{\hat{g}_{ij}} \theta_j = \hat{g}_{ij}^{-1} d\hat{g}_{ij} - \alpha_{ij}$ with $\alpha_{ij} \in \mathfrak{z}$. In the same way, we get for the curvature $R_i = d\theta_i - \frac{1}{2}[\theta_i, \theta_i]_{\hat{\mathfrak{k}}}$, and further $dR_i = [\theta_i, R_i]_{\hat{\mathfrak{k}}}$.

The Čech 2-cocycle h_{ijk} represents by definition the class $\omega_{\rm grp}$. In order to establish the link with the class $\omega_{\rm top\ alg}$, we translate Čech cocycles into differential forms via the Čech-de Rham bicomplex: a straight forward computation, using that h_{ijk} and α_{ij} have values in Z, gives

$$\alpha_{ij} + \alpha_{jk} + \alpha_{ki} = h_{ijk}^{-1} dh_{ijk}. \qquad (2)$$

Then we get $R_i - \mathrm{Ad}_{\hat{g}_{ij}} R_j = d\alpha_{ij}$ using once again that α_{ij} is central. We conclude, denoting $p_{\mathfrak{z}}$ some continuous linear projection onto $\mathfrak{z} \subset \hat{\mathfrak{k}}$, that $p_{\mathfrak{z}}(R_i) - p_{\mathfrak{z}}(R_j) = d\alpha_{ij}$, as the adjoint action acts trivially on the center. Now the β_i defined by $d\alpha_{ij} = \beta_i - \beta_j$ is a 2-form with values in $\mathfrak{z}$.

The relation (2) means that (h, α) forms a Čech 2-cocycle with values in the complex of sheaves

$$d\log : \Omega^0_M(Z) \to \Omega^1_M(\mathfrak{z}).$$

With $\beta_i = p_{\mathfrak{z}}(R_i)$, we get

$$d\beta_i = p_{\mathfrak{z}}(dR_i) = p_{\mathfrak{z}}([R_i, \theta_i]_{\hat{\mathfrak{k}}}).$$

Observe that this is exactly the expression of a representative of $\omega_{\text{top alg}}$ by remark 1.

The general procedure means here that (h, α, β) forms a Čech 2-cocycle with values in the complex of sheaves

$$\Omega^0_M(Z) \stackrel{d\log}{\to} \Omega^1_M(\mathfrak{z}) \stackrel{d}{\to} \Omega^1_M(\mathfrak{z}). \tag{3}$$

The rest of the proof of the theorem is inspired by the proof of proposition 4.2.7 in [2] p. 174: By corollary 2 in the appendix, (h, α, β) defines a cohomology class in the smooth Deligne cohomology group $H^3(M, \pi_1(Z)(3)^\infty_D)$. Furthermore, the class is sent to the Čech-connecting homomorphism of g_{ij}, i.e. h_{ijk}, under the map in cohomology from $H^3(M, \pi_1(Z)(3)^\infty_D)$ to $H^3(M, \pi_1(Z)) \cong H^2(M, \underline{Z})$, induced by the canonical projection from the complex of sheaves $\pi_1(Z)(3)^\infty_D$ to its first term (which is the constant sheaf $\pi_1(Z)$), see proposition 2 in the appendix. On the other hand, the Deligne cohomology class maps to $[d\beta]$, and thus to $[\omega_{\text{top alg}}]$, under the map of complexes of sheaves given by prolongation of the complex one step further, see proposition 3 of the appendix. It is therefore clear that the image of $[h_{ijk}]$ or $[\omega_{\text{grp}}]$ under the map

$$H^2(M, \underline{Z}) \cong H^3(M, \pi_1(Z)) \to H^3(M, \mathfrak{z})$$

is the class $[d\beta]$ or $[\omega_{\text{top alg}}]$. □

Remark 3 *In conclusion, ω_{grp} and ω_{alg} coincide up to torsion, in the above context.*

Corollary 1 *Let $\mathcal{F} : F \to \Gamma$ be a croosed module of Lie groupoids such that* coker $\mathcal{F} = M \times M$ *and such that the Lie group bundle* ker $\mathcal{F} =: Z \times M$ *is trivial. Set as in proposition 1 $K := F_x/Z$, and P for the principal K-bundle $t^{-1}(m) \to M$.*

Then the 3*-cohomology class $[\omega_{\text{top alg}}]$ of the crossed module of topological Lie algebras*

$$0 \to \mathfrak{c} \stackrel{i}{\to} \hat{\mathfrak{n}} \stackrel{\mu}{\to} \mathfrak{aut}(P) \stackrel{\pi}{\to} \mathcal{V}(M) \to 0$$

defines the same de Rham cohomology class as the 3*-cocycle ω_{grp} associated to the crossed module of Lie groupoids*

$$0 \to M \times Z \to P_K(\hat{K}) \to (P \times P)/K \to M \times M \to 0.$$

Proof. This follows immediately from theorem 2 and proposition 1. □

5 Finite dimensional structure group

In this section, we look at the special case of a finite dimensional structure group K. Let us start with $K = PU_n$, the projective unitary group of an n-dimensional complex vector space. K-principal bundles on M which are non-trivial in the sense that they cannot be lifted to a principal U_n-bundle define elements of the *Brauer group* $Br(M)$, cf [5]. A theorem of Serre determines $Br(M)$:

Theorem 3 (Serre) *On the manifold M, $Br(M)$ can be identified with the torsion subgroup of the sheaf cohomology group $H^2(M, \underline{\mathbb{C}^*}) \cong H^3(M, \mathbb{Z})$.*

The identification is given by the obstruction class $[\omega_{\rm grp}]$ which measures the obstruction for a given PU_n-principal bundle P to be lifted to a U_n-principal bundle. By Serre's theorem together with theorem 2, we arrive thus at the following conclusion:

Corollary 2 *Neeb's class $[\omega_{\rm top\ alg}]$ associated to the problem of lifting a given PU_n-principal bundle P to a U_n-bundle is always zero.*

By functoriality of the class $[\omega_{\rm top\ alg}]$, this remains true for all structure group extension problems which embed into

$$1 \to \mathbb{C}^* \to U_n \to PU_n \to 1$$

or powers of it. This is why Neeb's class does not seem to be interesting in the finite dimensional setting.

On the other hand, the significance of the (torsion) class $[\omega_{\rm grp}]$ is well known; see examples in [7] pp. 206-207. To cite just one example, for $K = SO(n)$ and a principal K-bundle P on M, $n \geq 3$, the obstruction class in $H^2(M, \mathbb{Z}_2)$ for the existence of a $Spin(n)$-bundle lifting P is the second Stiefel-Whitney class and gives a 2-torsion element in $H^3(M, \mathbb{Z})$ under the Bockstein map.

6 Existence of principal bundles

We will now use the previously established results to answer the question concerning the existence of a principal $\hat{K}$-bundle $\hat{P}$ such that $\hat{P} \,/\, Z \cong P$. We use for this the general theory of fiber spaces set up by Grothendieck [4], and we will get back the classification of principal bundles via cohomology classes of Lie groupoids due to Mackenzie [7].

Indeed, the short exact sequence of groups

$$1 \to Z \to \hat{K} \to K \to 1$$

induces (regardless of the finiteness of the dimension of K) an exact sequence

$$H^1(M, \underline{\hat{K}}) \to H^1(M, \underline{K}) \to H^2(M, \underline{Z}).$$

The obstruction class $[\omega_{\mathrm{grp}}]$ is nothing else than the image of the class of a given K-principal bundle P under the connecting homomorphism, i.e. the map on the right, and the exactness of the sequence means that there exists a $\hat{K}$-principal bundle $\hat{P}$ extending P if and only if $[\omega_{\mathrm{grp}}] = 0$:

$$\begin{array}{ccccc} H^1(M, \underline{\hat{K}}) & \longrightarrow & H^1(M, \underline{K}) & \longrightarrow & H^2(M, \underline{Z}) \\ & & & & \\ [\hat{g}_{ij}] & \vdash - - - - - \dashrightarrow & [g_{ij}] & \longmapsto & [h_{ijk} = \omega_{\mathrm{grp}}] \end{array}$$

This reasoning gives back theorem 3.4′ in [7] which shows the existence of a principal bundle $\hat{P}$ if and only if $[\omega_{\mathrm{grp}}] = 0$. By Grothendieck's theory, we get of course a Čech cocycle $\hat{g}_{ij}$ of transition functions which is only continuous, but by [10], we may choose a smooth representative of the same equivalence class.

Up to torsion, we get thus an equivalence between the existence of a principal $\hat{K}$-bundle $\hat{P}$ such that $\hat{P}/Z \cong P$ and the vanishing of Neeb's class $[\omega_{\mathrm{top\ alg}}]$.

Remark 4 *Actually, in the main problem of our paper, we prescribe not only the outer action of the Lie groupoid (or principal bundle) whose existence we study, but also the Lie algebroid (i.e. $P_K(\hat{\mathfrak{k}})$). This is the reason why the Čech cohomology class ω_{grp} takes values in a sheaf of germs of constant maps, cf [7] p.* 203.

7 Gerbes

In this section, we reinterprete our results in terms of gerbes and note that crossed modules of Lie groupoids and gerbes are related to similar cohomological problems. We then explore a more direct link between these two objects.

The theory of gerbes and stacks is rather heavy from the point of view of definitions, so we content ourselves with intuitive explanations for the objects of study in this section, in order not to blow up this paper.

Differentiable stacks are more general objects than differentiable manifolds; their notion grew out of Grothendieck's attempt to characterize the functor of points of an algebraic variety by abstract conditions. A stack is in particular a sheaf of categories on a site, but here, our stacks will always be sheaves on the manifold M. The guiding example for us here is the stack given by the local liftings of the structure group of a K-principal bundle P from K to a Z central extension $\hat{K}$ of K, cf [2] pp. 171-172. On an open set U of M, the associated category C_U consists of $\hat{K}$-principal bundles $\hat{P}$ on U together with an isomorphism of K-principal bundles $f : \hat{P}/Z \cong P$. Morphisms in C_U are bundle morphisms which commute with the isomorphisms f. Note that the category C_U is never empty (for $U \neq \emptyset$), and that up to refinement of U, any two objects of C_U are isomorphic. These are the basic properties of a sheaf of groupoids which is a *gerbe*.

The functor of points associates to each Lie groupoid a (pre)stack. Up to Morita equivalence, this gives a fully faithful functor from Lie groupoids to differentiable stacks. In this sense a Morita equivalence class of Lie groupoids defines a differentiable stack.

The manifold M, seen as a Lie groupoid, is Morita equivalent to the groupoid associated to an open covering $\coprod_{i\in I} U_i$ of M. Now given a class $\omega \in H^3(M,\mathbb{Z})$, there is a central extension of Lie groupoids (see [13] p. 863)

$$R_\omega \longrightarrow \coprod_{i,j\in I} U_{ij} \rightrightarrows \coprod_{i\in I} U_i$$

On the other hand, the central extension $1 \to \mathbb{C}^* \to U(\mathcal{H}) \to PU(\mathcal{H}) \to 1$ of the projective unitary group $PU(\mathcal{H})$ of an infinite dimensional separable Hilbert space $\mathcal{H}$ gives rise to an extension of Lie groupoids

$$U(\mathcal{H}) \longrightarrow PU(\mathcal{H}) \rightrightarrows \mathrm{pt}$$

over the point pt. Now it is shown in *loc. cit.* that there exists a morphism of stacks from the first extension of Lie groupoids to the second, and that this implies the existence of a $PU(\mathcal{H})$-principal bundle P on M whose obstruction class is ω.

All this can be summarized by the morphism Φ

$$\Phi : H^1(M,\underline{K}) \times H^2(K,\underline{S^1}) \to H^2(M,\underline{S^1})$$

to be found in [13] p. 860. Choosing a particular S^1-central extension of K, gives an isomorphism

$$\Phi' : H^1(M,\underline{K}) \to H^2(M,\underline{S^1}),$$

see [13] p. 861. We see thus that K-principal bundles (LHS) correspond in a one-to-one manner to integral 3-cohomology of M (RHS) via the obstruction mechanism. Note that the proofs in this section of [13] do certainly not extend to an infinite dimensional center Z, and that it is not even clear how they extend to a non-compact group Z.

In the framework of [13], the space $H^2(M,\underline{S^1})$ is interpreted as the space of equivalence classes of gerbes; the natural gerbe associated to our initial problem of lifting the structure group of a principal bundle to a central extension is the one described in the beginning. The above mentioned central extension of Lie groupoids also gives rise to a gerbe. The *band* of a gerbe is the sheaf of groups given by the outer action (see [9] p.15 def. 3.5). In the above central extension, the band happens to be trivial.

On the other hand, we started our study from equivalence classes of crossed modules of Lie groupoids. We are therefore led to search a direct link, i.e. one which does not pass by cohomology classes, between gerbes and crossed modules of Lie groupoids.

Theorem 4 *There is a one-to-one correspondence between gerbes with abelian, trivialized band on a connected manifold M (also called bundle gerbes), and crossed modules of Lie groupoids with trivial kernel and cokernel $M \times M$, which induces the above one-to-one correspondence between cohomology classes when passing to equivalence classes.*

Proof. We will first describe a map ϕ associating to a crossed module (F, δ, Γ) of Lie groupoids

$$1 \to M \times Z \to F \to \Gamma \to M \times M \to 1$$

a gerbe. ϕ will respect cohomology classes in $H^2(M, \underline{Z})$. Recall that M is connected. Call H the fiber F_{x_0} of the Lie group bundle F over $x_0 \in M$. As described in [1] p. 13, one may associate to (F, δ, Γ) a cocycle of transition functions $s_{ij} : U_{ij} \to \delta(H)$. Denote by $\hat{s}_{ij} : U_{ij} \to H$ a lift of s_{ij} with values in H. We now define a gerbe as a sheaf of categories on M by taking for C_U (for an open set U of M) the category of H-principal bundles, given for example by a Čech cocycle $\hat{s}_{ij} : U_{ij} \to H$, together with an isomorphism of $\delta(H)$-principal bundles $(\delta \circ \hat{s}_{ij}) \cong (s_{ij})$. By the general theory, it is clear that this defines a gerbe $C_{(F,\delta,\Gamma)}$ on M. We define thus ϕ by

$$\phi(F, \delta, \Gamma) \;=\; C_{(F,\delta,\Gamma)}.$$

It is also clear from [2] p. 172 that ϕ respects cohomology classes.

Let us now define a map ψ in the reverse direction: a gerbe $\mathfrak{G}$ (with trivialized abelian band) on M comes together with an identification of the sheaf of (abelian) groups of automorphisms of the objects of the local categories C_U; let this sheaf be $\underline{Z}$ and the abelian group be Z. Choosing a local section $s : U \to \mathfrak{G}$ (whose existence is due to the local existence of objects in C_U - this is one of the axioms of a gerbe), this identification implies that $U \times_{\mathfrak{G}} U \to U$ is a locally trivial principal bundle $\hat{P}_U$, cf [6] rem. 5.2. The local isomorphy of any two objects in C_U (which is the other axiom of a gerbe) implies that the fiber of $U \times_{\mathfrak{G}} U \to U$ is a group. Denote it by $\hat{K}$. Now the locally defined $\hat{P}_U/Z =: P_U$ form a globally defined K-principal bundle P on M, because the $\mathrm{Aut}(C_U) = Z$-valued Čech cochain defined by the local sections becomes a cocycle for P. This defines as in the beginning of section 4 a crossed module of Lie groupoids (F, δ, Γ) of $M \times M$ by $\underline{Z}$. We set

$$\psi(\mathfrak{G}) \;=\; (F, \delta, \Gamma).$$

ϕ and ψ are mutually inverse bijections, descending to bijections of equivalence classes. In this sense, gerbes and crossed modules of Lie groupoids are the same objects. □

Remark 5 *Note that this construction does not involve the construction of a principal bundle P for a given integral cohomology class in [13]. Indeed, passing by their construction, one could show a relation between gerbes and crossed modules only for $K = S^1$.*

A Smooth Deligne cohomology

In this appendix, we present the definitions and results from smooth Deligne cohomology which we need in section 4. Our main reference is [2] Ch. 1.5, while our definition differs slightly from his.

Deligne cohomology in our sense is the hypercohomology of truncated complexes of sheaves. The complex we consider in section 4, denoted by $\pi_1(Z)(3)_D^\infty$, is

$$\pi_1(Z) \stackrel{\text{incl}}{\to} \Omega^0_M(\mathfrak{z}) \stackrel{d}{\to} \Omega^1_M(\mathfrak{z}) \stackrel{d}{\to} \Omega^2_M(\mathfrak{z}).$$

The notation $\pi_1(Z)(3)_D^\infty$ indicates that the 0th term is the simple (or constant) sheaf $\pi_1(Z)$, and that the complex is truncated at the 3rd place. It will be related to the complex

$$\Omega^0_M(Z) \stackrel{d\log}{\to} \Omega^1_M(\mathfrak{z}) \stackrel{d}{\to} \Omega^2_M(\mathfrak{z}).$$

Let us explain the ingredients: Z is here some (possibly infinite dimensional) Lie group, $\mathfrak{z}$ is its Lie algebra. The only property from the setting of infinite dimensional Lie groups we might choose is that the composition of smooth maps must be smooth. The main geometric property of Z and $\mathfrak{z}$ we use is that $\mathfrak{z}$ is a universal covering space for Z, in particular

$$1 \to \pi_1(Z) \to \mathfrak{z} \stackrel{\pi}{\to} Z \to 1$$

is an exact sequence of groups. In the above complex, $\Omega^0_M(Z)$ denotes the sheaf of smooth maps on M with values in Z, and $\Omega^i_M(\mathfrak{z})$ for $i = 1, 2$ have a similar meaning. $d\log$ is the *logarithmic derivative*, i.e. $d\log f = f^{-1}df$ for $f \in \Omega^0_M(Z)$. $d\log f$ for $x \in M$ is thus the composition

$$L_{f^{-1}(x)} \circ d_x f : T_x M \to T_{f(x)} Z \to T_e Z = \mathfrak{z}.$$

Proposition 2 *The following square is commutative:*

$$\begin{array}{ccc} \Omega^0_M(\mathfrak{z}) & \stackrel{d}{\longrightarrow} & \Omega^1_M(\mathfrak{z}) \\ \downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle \text{id}} \\ \Omega^0_M(Z) & \stackrel{d\log}{\longrightarrow} & \Omega^1_M(\mathfrak{z}) \end{array}$$

Proof. It is enough to have $d(\pi \circ f) = (\pi \circ f) \cdot df$ for each $f \in \Omega^0_M(\mathfrak{z})$ (where $\cdot$ means once again the left translation of df in Z).

Observe that for $a \in \mathfrak{z}$, the composition

$$L_{\pi(a)^{-1}} \circ d\pi(a) : \mathfrak{z} \to T_{\pi(a)} Z \to T_e Z = \mathfrak{z}$$

is the identity, as $\pi : \mathfrak{z} \to Z$ is the universal covering. This gives $d\pi(a) = L_{\pi(a)}$ for all $a \in \mathfrak{z}$.

Therefore, $d(\pi \circ f) = (d\pi \circ f) \cdot df = (\pi \circ f) \cdot df$ as claimed. $\square$

Corollary 3 *We have a quasi-isomorphism of complexes of sheaves:*

$$\begin{array}{ccccccc}
\pi_1(Z) & \xrightarrow{\text{incl}} & \Omega^0_M(\mathfrak{z}) & \xrightarrow{d} & \Omega^1_M(\mathfrak{z}) & \xrightarrow{d} & \Omega^2_M(\mathfrak{z}) \\
\downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} \\
0 & \xrightarrow{0} & \Omega^0_M(Z) & \xrightarrow{d\log} & \Omega^1_M(\mathfrak{z}) & \xrightarrow{d} & \Omega^2_M(\mathfrak{z})
\end{array}$$

Proposition 3 *Let M be a smooth paracompact manifold of dimension n, and let*

$$0 \to \mathcal{F}^0 \to \mathcal{F}^1 \to \ldots \to \mathcal{F}^n \to 0$$

be a complex of sheaves. Denote by $H^q(M, \mathcal{F}^0(p)^\infty_D)$ the q-th Deligne cohomology group, i.e. the qth hypercohomology group of the truncated sheaf complex $\{\mathcal{F}^i\}_{0 \leq i \leq p-1}$. Then the projection of $\{\mathcal{F}^i\}_{0 \leq i \leq p-1}$ onto $\mathcal{F}^0$ induces an epimorphism

$$H^q(M, \mathcal{F}^0(p)^\infty_D) \;\to\; H^q(M, \underline{\mathcal{F}}^0)$$

for each $q \geq p$.

Proof. This proposition is analoguous to a part of theorem 1.5.3 in [2] p. 48 and is shown in the same way. $\square$

Proposition 4 *In the same setting as the previous proposition, the one-step-extension of the truncated complex $\{\mathcal{F}^i\}_{0 \leq i \leq p-1}$*

$$\begin{array}{ccccccc}
\mathcal{F}^0 & \longrightarrow & \mathcal{F}^1 & \longrightarrow & \cdots & \longrightarrow & \mathcal{F}^{p-1} \\
\downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \downarrow \\
0 & \xrightarrow{0} & 0 & \xrightarrow{0} & \cdots & \longrightarrow & \mathcal{F}^p
\end{array}$$

induces a map

$$H^p(M, \mathcal{F}^0(p)^\infty_D) \;\to\; \mathcal{F}^p(M).$$

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