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# STABILITY OF A QUASI-STATIC EVOLUTION OF A VISCO-ELASTIC, VISCO-PLASTIC OR ELASTIC-PLASTIC SOLID

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The problem of stability of a quasi-static evolution of a solid is discussed in the framework of standard plasticity and visco-plasticity. General results concerning the criterion of second variation of energy are presented. Two different approaches are considered to discuss this problem for visco-elastic, visco-plastic or elastic-plastic solids. The first approach is based upon the linearization method and gives the asymptotic stability of the evolution of a visco-elastic solid when the associated dissipation potential is quadratic and positive-definite. The second approach introduces a direct analysis of the evolution in a nonsmooth but convex framework of plasticity and visco-plasticity.

**Keywords:** stability, evolution, quasi-static, visco-elasticity, visco-plasticity, plasticity, energy, criterion, second variation,

## 1. INTRODUCTION

In solid mechanics, the problem of stability of an equilibrium has been intensively discussed for both time-dependent and time-independent be-

haviour. For example, Liapunov's theorem plays a key role in stability analysis of an equilibrium in the context of visco-elasticity and Hill's criteria of stability and non-bifurcation have been widely applied in incremental plasticity, e.g. Hill, 1958.

In this paper, the stronger problem of stability of an evolution is considered. The stability of an evolution expresses the continuity of the solution of an evolution with respect to the perturbation of initial conditions. This important problem has been principally developed for the study of orbital stability of periodic evolutions. For a mechanical system, this evolution may be dynamic as the case of an elastic solid in a dynamic transformation. But it may also be quasi-static as the case of a visco-elastic or visco-plastic or elastic-plastic solid submitted to a slow loading path. The discussion is here limited to the stability of a quasi-static evolution of solids. Few discussions of the literature have been devoted to this problem because of mathematical difficulties relating to non-autonomous differential equations. Such a discussion is however necessary to address the stability problem of an visco-elastic structure in quasi-static evolution under applied loads i.e. the stability of time-dependent systems. It is well known that the buckling analysis of a visco-elastic structure is a very difficult problem and that no general and operational criterion has been derived, in contrast with the case of elastic or elastic-plastic solids. But this discussion also concerns with the problem of stability of an elastic-plastic response under perturbation of loads or of initial or current positions which remains an open problem in plastic buckling.

This paper presents a contribution to the problem of stability of an evolution in the simple framework of standard plasticity and viscoplasticity. General results concerning the criterion of second variation of energy are here presented. Two different approaches are followed to discuss this criterion for visco-elastic or visco-plastic solids (time-dependent behaviour) or for elastic-plastic solids (time-independent behaviour) obeying to the generalized standard model. The first approach is based upon the linearization method and gives the possibility to discuss this result for visco-elastic solids when the associated dissipation potential is quadratic and positive-definite. The second approach introduces a direct analysis of the evolution in a nonsmooth but convex framework. The continuity of the solution with respect to initial conditions is discussed and the criterion of second variation of energy is established for certain class of visco-plastic or elastic-plastic generalized standard models under a certain assumption on the loading path. Our discussion is here restricted to discrete or discretized equations in order to avoid the well known difficulties of functional analysis in three-

dimensional solids, e.g. Ciarlet, 1988 or Panagiotopoulos, 1985. A more detailed presentation of the results can also be found in Abed Meraim, 1999a, Abed Meraim, 1999b, Nguyen, 2000.

## 2. STABILITY OF AN EVOLUTION AND LINEARIZATION METHOD

A general non-autonomous differential equation in  $\mathbf{R}^n$  is first considered:

$$\dot{y} = F(y, t), \quad y(0) = y_o^o \quad (1.1)$$

for  $t \in [0, \infty[$ . It is well known that if  $F(y, t)$  is sufficiently smooth, Cauchy-Lipschitz's theorem can be applied and leads to the existence of a solution  $y_o(t)$  associated to an initial value  $y_o^o$ . For example, the fact that  $F(y, t)$  is a  $C^1$ -function in  $y$  and  $t$  ensures the existence of one unique solution on any bounded interval  $[0, T]$ . It is assumed here that the existence of an unique solution  $y_o(t)$  on time interval  $[0, +\infty[$  is ensured. For a physical system characterized by parameters  $y$ , this solution describes the evolution  $y_o$  of the considered system associated to initial value  $y_o^o$ . By definition,  $y_o$  is a stable evolution if a small perturbation of the initial value leads to a small variation of this evolution i.e. for all  $t > 0$

$$\| y_p^o - y_o^o \| \text{ small} \Rightarrow \| y_o(t) - y_p(t) \| \text{ small} \quad \forall t \geq 0. \quad (1.2)$$

Thus this definition is simply a straightforward extension of the notion of stability of an equilibrium. In the same spirit, asymptotic stability of the evolution is obtained if

$$\lim_{t \rightarrow \infty} \| y_o(t) - y_p(t) \| = 0.$$

Linearized equations can again be considered. The fact that the distance  $y_p(t) - y_o(t)$  must remain small leads to the introduction of the linearized differential equation associated with the evolution  $y_o$  from the first order of the development of  $F(y(t), t)$ . Linearized equations are

$$\dot{y}^* = \Psi(t) \cdot y^* \quad \text{with} \quad \Psi(t) = F_{,y}(y_o(t), t). \quad (1.3)$$

Without additional assumptions, a few general results are known on non-autonomous differential equations. For instance, the following classical results have been obtained, cf. Hahn, 1967 or Roseau, 1966:

**Proposition 1** *If for all  $t > 0$ , there exists a symmetric and positive-definite matrix  $G(t)$  such that the matrix  $\dot{G} + \Psi^T G + G \Psi$  is negative-definite, the evolution  $y_o$  is asymptotically stable.*

In particular, this proposition leads to

**Proposition 2** *The evolution  $y_o$  is asymptotically stable:*

- *if, for all  $t \geq 0$ , the symmetric part  $\Psi^s$  of matrix  $\Psi$  admits only strictly negative eigenvalues.*

- *or if matrix  $\Psi$  can be decomposed as  $\Psi = R^{-1}S$ , where symmetric matrices  $R, S$  are such that  $\dot{R}(t) + 2S(t)$  is a negative-definite matrix for all  $t \geq 0$ .*

Indeed, the previous proposition can be applied with  $G = I$  in the first case and  $G = R$  in the second case. These propositions furnish sufficient conditions to ensure the asymptotic stability of the considered evolution. This is an extension of the linearization method and of Liapunov's theorem. It is clear that:

- Time interval  $[0, \infty[$  must be considered since the solution is in general continuous with respect to initial value on any bounded interval  $[0, T]$  for regular differential equations (1.1).

- A stable evolution on interval  $[0, \infty[$  is also stable on any interval  $[T, \infty[$  for  $T > 0$ . Evolution stability means thus stability with respect to perturbation of the position at any time.

- In the particular case of an equilibrium  $y_o(t) = y_e \quad \forall t$ , the first proposition does not recover Liapunov's theorem with the choice  $G = I$ . This proposition is thus not optimal. But without additional assumptions, no stronger statement can be derived. In particular, elementary counter-examples can be given to prove that the stability of the considered evolution may not be ensured when the real parts of the eigenvalues of matrix  $\Psi(t)$  are all negative.

- Linearized equation is a linear differential equation in  $y^*$  but time-dependent when  $y_o$  is a real evolution. No explicit expression of the linearized solution is available, in contrast with the particular case of an equilibrium.

- In the same spirit, implicit differential equations  $\Phi(y, \dot{y}, t) = 0$  can be considered. For example, visco-elastic standard systems are governed by Biot's equation

$$\mathbf{D}_{,q} + \mathbf{E}_{,q} = 0 \tag{1.4}$$

in which  $\mathbf{D} = \mathbf{D}(\dot{q}, q)$  and  $\mathbf{E} = \mathbf{W}(q, t)$  denotes respectively the dissipation and the energy potentials,  $q$  the parameters of the system. Biot's equation is an implicit differential equation.

- Nonsmooth differential equation can also be considered. If the dissipation potential is a differentiable function for visco-elastic solids, it isn't for visco-plastic or for elastic-plastic solids, dissipation potential is only a pseudo-potential in the sense that it is convex but not differentiable at the origin. For such a system, Biot's equation must be understood in

the sense of sub-gradient. This differential equation cannot be linearized and a special discussion must be given in the study of the stability of an evolution.

### 3. STABILITY OF A VISCO-ELASTIC EVOLUTION

The case of a visco-elastic solid in finite transformation is now considered. It is assumed that the material admits a generalized standard constitutive laws cf. Halphen and Nguyen, 1975, Nguyen, 1994 defined by state variables  $(\nabla u, \alpha)$  in Lagrange description,  $\alpha$  denotes internal parameters. From the expression of the free energy  $\mathcal{W}(\nabla u, \alpha)$  and of the dissipation potential  $D(\nabla \dot{u}, \dot{\alpha})$ , constitutive equations are defined by state equations

$$b = b^R + b^{IR}, \quad b^R = \mathcal{W}_{,\nabla u}, \quad A = -\mathcal{W}_{,\alpha} \quad (1.5)$$

and by force-flux relations

$$b^{IR} = D_{,\nabla \dot{u}}, \quad A = D_{,\dot{\alpha}}, \quad (1.6)$$

where tensor  $b$  denotes the Piola-Lagrange's stress,  $b^R$  is the reversible stress and  $b^{IR}$  the irreversible stress.

It is assumed that the solid is submitted to conservative loads, for example to implied forces  $r^d(\lambda)$  and implied displacements  $u^d(\lambda)$ ,  $\lambda$  is a time-dependent control parameter. Function  $\lambda(t)$  is given in  $[0, +\infty[$ . The static equilibrium of the solid is expressed by virtual work equation

$$\int_V b : \nabla \delta u \, dV = \int_{S_r} r^d(\lambda) \cdot \delta u \, dS \quad (1.7)$$

which ensures force equilibrium while moment equilibrium is implicitly ensured by the symmetry of the Cauchy stress  $\sigma$  which implies the condition  $bF^T = Fb^T$  to be fulfilled by the constitutive equations.

For example, combined visco-elastic model consists to take as  $\alpha$  the viscous strain  $\epsilon^v$  and as energy

$$W = \frac{1}{2}(\epsilon - \epsilon^v) : L^e : (\epsilon - \epsilon^v) + \frac{1}{2}\epsilon^v : h : \epsilon^v$$

in which  $\epsilon = 1/2(\nabla u + \nabla u^T + \nabla u^T \nabla u)$  is Green deformation,  $L^e$  is the tensor of elastic modulus and  $h$  is the hardening parameter.

From a given initial state  $u(0) = u_o^o$ ,  $\alpha(0) = \alpha_o^o$ , can be associated an evolution of the solid defined by  $u_o(t)$ ,  $\alpha_o(t)$ . With the compact notation  $q = (u, \alpha)$ , the energy potential and the dissipation potential of the solid

can be introduced

$$\mathbf{E}(q, \lambda) = \int_V \mathcal{W}(\nabla u, \alpha) dV - \int_{S_r} r^d(\lambda) \cdot u dS, \quad (1.8)$$

$$\mathbf{D}(\dot{q}) = \int_V D(\nabla \dot{u}, \dot{\alpha}) dV \quad (1.9)$$

to write the evolution under the form of Biot's equation

$$(\mathbf{D}_{,\dot{q}} + \mathbf{E}_{,q}) \cdot \delta q = 0. \quad (1.10)$$

Linearized equation associated to a particular evolution  $q_o$  can be then written as

$$\mathbf{E}_{qq}^o[\delta q, q^*] + \mathbf{D}_{\dot{q}\dot{q}}^o[\delta q, \dot{q}^*] = 0 \quad \forall \delta q, \quad \forall t > 0. \quad (1.11)$$

It is assumed first that dissipation potential is quadratic, state independent and strictly convex. The associated bilinear form  $\mathbf{D}_{\dot{q}\dot{q}}[\delta q, \dot{q}^*]$  is then time-independent and positive-definite

$$\mathbf{D}_{\dot{q}\dot{q}}[\delta q, \delta q] > c \|\delta q\|^2.$$

In these conditions, proposition (2) can be applied with  $R = \mathbf{D}_{,\dot{q}\dot{q}}, S = \mathbf{E}_{,qq}^o$ . The positive-definiteness of the second derivative of energy

$$\mathbf{E}_{qq}^o[\delta q, \delta q] \geq \delta \|\delta q\|^2 \quad (1.12)$$

ensures then the asymptotic stability of evolution  $q_o$ :

**Proposition 3** *If dissipation potential is quadratic, state-independent and strictly convex, the positive-definiteness of the second variation of energy of the solid with respect to variables  $q = (u, \alpha)$  at current points of an evolution ensures the asymptotic stability of this evolution.*

If the deformation is not in itself a dissipative mechanism, dissipation potential depends only on the rate of internal parameters  $\dot{\alpha}$ . If quadratic form  $\mathbf{D}_{,\dot{\alpha}\dot{\alpha}}[\alpha^*, \alpha^*]$  is positive-definite, the result still holds under the same condition. Indeed, the same argument shows already that the distance to the considered evolution  $\|\epsilon_p^v(t) - \epsilon_o^v(t)\|$  remains small for all  $t$  and tends to 0 for  $t \rightarrow \infty$ . If the second variation of energy is positive-definite, equilibrium equation  $\mathbf{E}(u, \epsilon_p^v)_{,u} \cdot \delta u = 0$  gives  $u = u(\epsilon_p^v)$  in a unique manner in the vicinity of  $\epsilon_o^v$  after implicit function theorem since  $\mathbf{E}_{,uu}^o$  is then regular. Displacement  $u$  can be then expressed in function of the loading and of the viscous strain. Thus the perturbed displacement  $u_p(t)$  remains close to the solution  $u_o(t)$  and the difference tends to 0 when  $t \rightarrow \infty$ .

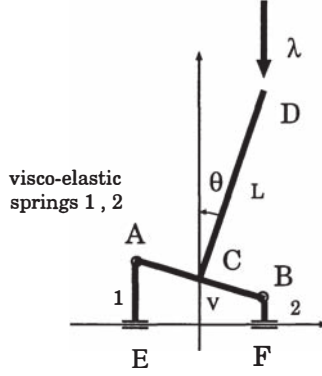


Figure 1.1 Discrete visco-elastic Shanley's model

In the general case, dissipation potential is not quadratic, the property of asymptotic stability in the spirit of Propositions 1 and 2 is not obtained.

As example, the Shanley's column of Figure 1 is here considered again by assuming that the springs are elastic-visco-elastic following the Maxwell's rheological model of kinematic hardening. If the applied vertical load is a given function of time  $\lambda = \lambda(t)$ , evolution equations are

$$\begin{aligned} \sigma_1 + \sigma_2 + \lambda(t) &= 0, & \sigma_2 - \sigma_1 &= \frac{\lambda L}{\ell} \tan \theta. \\ v + \ell \sin \theta &= \frac{\sigma_1}{E} + \epsilon_1^v, & v - \ell \sin \theta &= \frac{\sigma_2}{E} + \epsilon_2^v \\ \eta \dot{\epsilon}_1^v &= \sigma_1 - h \epsilon_1^v, & \eta \dot{\epsilon}_2^v &= \sigma_2 - h \epsilon_2^v. \end{aligned}$$

Let  $u = (v, \theta)$  be displacement parameters,  $\alpha = (\epsilon_1^v, \epsilon_2^v)$  the internal parameters and  $q = (u, \alpha)$ . Energy and dissipation potentials are

$$\begin{aligned} \mathbf{E}(\mathbf{q}, \lambda) &= \frac{1}{2} E (v + \ell \sin \theta - \epsilon_1^v)^2 + \frac{1}{2} E (v - \ell \sin \theta - \epsilon_2^v)^2 + \frac{1}{2} h (\epsilon_1^v)^2 \\ &\quad + \frac{1}{2} h (\epsilon_2^v)^2 + \lambda (v + L \cos \theta), & \mathbf{D}(\dot{\alpha}) &= \frac{1}{2} \eta (\dot{\epsilon}_1^v)^2 + \frac{1}{2} \eta (\dot{\epsilon}_2^v)^2. \end{aligned}$$

Evolution equations can also be written under the form of Biot's equation (1.4). The symmetric evolution associated with symmetric initial condition  $\epsilon_1^v(0) = \epsilon_2^v(0) = 0$  is considered

$$\theta_o(t) = 0, \quad v = v_o(t), \quad \epsilon_{o1}^v = \epsilon_{o2}^v = \epsilon_o^v(t).$$

Linearization leads to

$$\begin{cases} \eta \dot{\epsilon}_1^{v*} = -\frac{\lambda E L}{2(2\ell^2 E - \lambda L)} (\epsilon_2^{v*} - \epsilon_1^{v*}) - h \epsilon_1^{v*} \\ \eta \dot{\epsilon}_2^{v*} = -\frac{\lambda E L}{2(2\ell^2 E - \lambda L)} (\epsilon_1^{v*} - \epsilon_2^{v*}) - h \epsilon_2^{v*}. \end{cases}$$



Thus the following expression holds

$$\Psi(t) = -\frac{1}{\eta} \begin{bmatrix} h - z & z \\ z & h - z \end{bmatrix} \quad \text{with} \quad z = \frac{\lambda EL}{2(2\ell^2 E - \lambda L)}.$$

Eigenvalues are  $\mu_1 = -\frac{h}{\eta}$  and  $\mu_2 = -\frac{h-2z}{\eta}$  or with the notations  $\lambda_E = \frac{2\ell^2}{L}E$ ,  $\lambda_T = \frac{2\ell^2}{L}\frac{Eh}{E+h} = \frac{2\ell^2}{L}E_T$

$$\mu_1 = -\frac{h}{\eta}, \quad \mu_2 = -\frac{h}{\eta} \frac{\lambda_E}{\lambda_T} \frac{\lambda_T - \lambda}{\lambda_E - \lambda}.$$

After Proposition 2, it is then concluded that the symmetric solution is stable if  $\lambda(t) < \lambda_T$  for all  $t$ .

#### 4. STABILITY OF A VISCO-PLASTIC EVOLUTION

Since the method of linearization cannot be applied in this case, a direct method is here considered. It is available for both time-dependent and time-independent behaviour, in particular for plastic or visco-plastic generalized standard models.

Visco-plastic laws can also be considered as a particular case of non-linear visco-elasticity with an elastic domain. The introduction of the dual dissipation potential, classically denoted in this framework as visco-plastic potential

$$D^*(A) = \max_{\dot{\alpha}} A \cdot \dot{\alpha} - D(\dot{\alpha}), \quad (1.13)$$

by Legendre-Fenchel transform leads to the evolution law of internal parameters

$$\dot{\alpha} = D^*_{,A}(A). \quad (1.14)$$

For example, Perzyna's model consists to take as  $\alpha$  the visco-plastic strain,  $A$  is then the Kirchhoff stress and  $D^*(A) = \frac{1}{2\eta} < \sqrt{A : A} - k >^2$ . Norton-Hoff's model consists to take as visco-plastic potential  $D^*(A) = \frac{1}{\eta} < f(A) >^m$ ,  $m > 1$ , where  $f$  is a convex function,  $f(A) < 0$  defines the elastic domain  $\mathbf{C}$ .

In the same spirit as in plasticity, the dissipation per unit volume  $\mathcal{P}(\dot{\alpha}) = A \cdot \dot{\alpha}$  must be non-negative. Thus the elastic domain must contain the origin of force 0. Moreover, if the origin of force is strictly contained inside in the sense that a sphere of radius  $r$  and centered at the origin is also included in  $\mathbf{C}$ , then

$$\mathcal{P}(\dot{\alpha}) \geq r \|\dot{\alpha}\|. \quad (1.15)$$

The quasi-static equation of evolution for a generalized standard visco-plastic solid is

$$\dot{\alpha} = \mathbf{D}^*_{,A}, \quad A = -\mathbf{E}_{,\alpha}, \quad \mathbf{E}_{,u} \cdot \delta u = 0, \quad \alpha(0) = \alpha_o. \quad (1.16)$$

where  $q \in \mathbf{R}^n$  or  $q \in H$ , an abstract Hilbert space.

Let  $q_o = (u_o, \alpha_o)$  be a bounded solution  $\|q_o(t)\| \leq M_o \quad \forall t \geq 0$  of this equation. The following assumptions on energy are admitted  $\forall \|q - q_o(t)\| \leq M$  and  $\forall t$

$$\mathbf{E}_{,q}(q, \lambda) \cdot \delta q \leq k_1 \|\delta q\|, \quad |\mathbf{E}_{,\lambda}(q, \lambda)| \leq k_2, \quad (1.17)$$

$$\mathbf{E}_{,qq}(q, \lambda)[p, r] \leq K_{11} \|p\| \|r\|, \quad \mathbf{E}_{,q\lambda}(q, \lambda) \cdot p \leq K_{12} \|p\| \quad (1.18)$$

$$\mathbf{E}_{,qq}(q, \lambda)[\delta q, \delta q] \geq k_{11} \|\delta q\|^2, \quad (1.19)$$

$$|\{\mathbf{E}_{,qq}(q, \lambda) - \mathbf{E}_{,qq}(p, \mu)\}[\delta q, \delta q]| \leq L (\|q - p\| + |\lambda - \mu|) \|\delta q\|^2 \quad (1.20)$$

$$|\{\mathbf{E}_{,q\lambda}(q, \lambda) - \mathbf{E}_{,q\lambda}(p, \mu)\} \cdot \delta q| \leq L (\|q - p\| + |\lambda - \mu|) \|\delta q\| \quad (1.21)$$

where coefficients  $M, k_1, k_2, k_{11}, K_{11}, K_{12}, L$  are positive constants.

Let  $q(t)$  be a different solution associated with a different initial condition  $q(0) = p_o \neq q_o$ . As in the proof of Liapunov's theorem, the method consists to assume first a working hypothesis which states that  $\|q(t) - q_o(t)\| \leq M$  for all  $t$  in order to take the advantage of the introduced assumptions. A better estimate of this distance is then given to show that this distance is in fact small if the initial distance is sufficiently small. This result justifies the working hypothesis and leads to the stability of the evolution  $q_o$ .

It is also assumed that the loading path, defined by function  $\lambda(t)$  on the interval  $[0, \infty[$ , satisfies

$$\int_0^{+\infty} |\dot{\lambda}(t)| dt < +\infty \quad \text{i.e.} \quad \dot{\lambda} \in L^1(0, \infty). \quad (1.22)$$

The solutions  $q_o(t)$  et  $q(t)$  satisfy some property. Equilibrium equation  $\mathbf{E}_{,u} \cdot \delta u = 0$  leads to the energy balance  $\mathbf{E}_{,u} \cdot \dot{u} = 0$  which can be also be written as

$$\dot{\mathbf{E}} + A \cdot \dot{\alpha} = \mathbf{E}_{,\lambda} \dot{\lambda}.$$

From the assumption on the elastic domain  $\mathbf{C}$ , the following inequality follows

$$r \int_0^t \|\dot{\alpha}(s)\| ds \leq \mathbf{E}_0 - \mathbf{E}_t + K \int_0^t |\dot{\lambda}| ds.$$

Thus, if  $\mathbf{E}_t$  remains bounded for all  $t$ , then

$$\int_0^{+\infty} \|\dot{\alpha}\| dt < +\infty. \quad (1.23)$$

Effectively, since  $q_o(t)$  is bounded for all  $t$ ,  $\mathbf{E}_{ot}$  is also bounded. The working hypothesis  $\|q(t) - q_o(t)\| \leq M$  for all  $t$  implies that  $q(t)$  remains bounded, i.e.  $\mathbf{E}_t$  remains bounded for all  $t$ .

The same conclusion concerning  $\dot{u}$  is also available. Indeed, equilibrium equation  $\mathbf{E}_{,u} \cdot \delta u = 0$  gives after time differentiation

$$\mathbf{E}_{,uu} [\dot{u}, \delta u] + \delta u \cdot \mathbf{E}_{,u\alpha} \cdot \dot{\alpha} + \dot{\lambda} \mathbf{E}_{,u\lambda} \cdot \delta u = 0$$

and leads to the desired result after the introduced assumptions since

$$k_{11} \|\dot{u}\| \leq K_{12} \|\dot{\alpha}\| + K_{11} |\dot{\lambda}|.$$

The rates  $\dot{q}_o, \dot{q}$  thus belong to  $L^1(0, +\infty, H)$ .

On the other hand, evolution equation, written for solutions  $q_o$  and  $q$ , gives after multiplication by  $Q(t) - Q_o(t)$  and after combination of the obtained results

$$(\mathbf{E}_{,q} - \mathbf{E}^{\circ}_{,q}) \cdot (\dot{q} - \dot{q}_o) \leq 0 \quad \forall t$$

taking account of the fact that  $(\mathbf{D}^*_{,Q}(Q) - \mathbf{D}^*_{,Q}(Q_o)) \cdot (Q - Q_o) \geq 0$ . Since

$$(\mathbf{E}_{,q} - \mathbf{E}^{\circ}_{,q}) \cdot \Delta \dot{q} = \mathbf{E}^{\circ}_{,qq} [\Delta q, \Delta \dot{q}] + r_1, \quad |r_1| \leq L \|\Delta q\|^2 \|\Delta \dot{q}\|$$

$$\mathbf{E}^{\circ}_{,qq} [\Delta q, \Delta \dot{q}] = \frac{d}{dt} \left( \frac{1}{2} \mathbf{E}^{\circ}_{,qq} [\Delta q, \Delta q] \right) - r_2,$$

$$|r_2| \leq L \|\Delta q\|^2 (\|\dot{q}_o\| + |\dot{\lambda}|)$$

it follows finally that

$$\dot{h} \leq m(t) h(t), \quad m(t) = \frac{L}{k_{11}} (2 \|\dot{q}_o\| + \|\dot{q}\| + |\dot{\lambda}(t)|) \quad (1.24)$$

if  $\|q(s) - q_o(s)\| \leq M \quad \forall 0 \leq s \leq t$ . Thus, the estimate

$$h(t) \leq h(0) \exp \mathcal{M}(t), \quad \mathcal{M}(t) = \int_0^t m(s) ds \leq B \quad (1.25)$$

holds, constant  $B$  exists from the fact that  $\dot{q} \in L^1(0, +\infty, H)$ ,  $\dot{q}_o \in L^1(0, +\infty, H)$  and  $\dot{\lambda} \in L^1(0, +\infty)$ . Thus

$$\|\Delta q(t)\|^2 \leq \frac{K_{11}}{k_{11}} \|\Delta q(0)\|^2 \exp \mathcal{M}(t). \quad (1.26)$$

## 5. STABILITY OF AN ELASTIC-PLASTIC EVOLUTION

For an elastic-plastic solid in quasi-static evolution, the stability analysis of an evolution  $q_o$  associated to a loading path  $\lambda(t)$  can be done in the same spirit. The governing equations of quasi-static evolutions are

$$\dot{\alpha} = \mathbf{N}_C(A), \quad A = -\mathbf{E}_{,\alpha}, \quad \mathbf{E}_{,u} \cdot \delta u = 0, \quad \alpha(0) = \alpha_o. \quad (1.27)$$

The same proof can be applied without modification since

$$A \cdot \dot{\alpha} \geq r \|\dot{\alpha}\|$$

as in visco-plasticity. A direct proof can also be given. From the assumption of state-independent elastic domain, it has been established that flux and force rates are necessarily orthogonal

$$\dot{\alpha} \cdot \dot{A} = 0.$$

Moreover, it follows from the equilibrium condition

$$\frac{d}{dt}(\mathbf{E}_{,u} \cdot \delta u) = 0.$$

These equations lead to

$$\mathbf{E}_{,qq} [\dot{q}, \dot{q}] = -\mathbf{E}_{,q\lambda} \cdot \dot{q} \dot{\lambda}$$

and to the estimate

$$k_{11} \|\dot{q}\| \leq K_{12} |\dot{\lambda}|. \quad (1.28)$$

The proof can be continued as it has been done previously and leads to the following inequality

$$\dot{h} \leq m(t) h(t), \quad m(t) = \frac{L}{k_{11}} \left( 3 \frac{K_{12}}{k_{11}} + 1 \right) |\dot{\lambda}(t)| = C |\dot{\lambda}(t)|.$$

This inequality leads as before to the desired conclusion.

## 6. CRITERION OF SECOND VARIATION OF FREE ENERGY

In the previous stability analysis, it has been shown that the essential assumption concerns with the positive-definiteness of the second variation of energy

$$\delta^2 \mathbf{E}_t = \mathbf{E}_{,qq}(q(t), \lambda(t))[\delta q, \delta q] \geq k_{11} \|\delta q\|^2, \quad \forall t \in [0, +\infty]. \quad (1.29)$$

with

$$\delta^2 \mathbf{E}_t = \int_V (\nabla \delta u : \mathcal{W}_{,\nabla u \nabla u} : \nabla \delta u + 2 \nabla \delta u : \mathcal{W}_{,\nabla u \alpha} \cdot \delta \alpha + \delta \alpha \cdot \mathcal{W}_{,\alpha \alpha} \cdot \delta \alpha) dV.$$

A simpler criterion can also be introduced if the energy density  $\mathcal{W}$  is strictly convex with respect to  $\alpha$  for fixed  $\nabla u$ . Since  $\delta \alpha \cdot \mathcal{W}_{,\alpha \alpha} \cdot \delta \alpha$  is positive-definite, the internal parameter can be eliminated and the criterion of second variation of energy of the solid can be ensured by the positive-definiteness of the quadratic form

$$\delta^2 \mathcal{E}_t = \int_V \nabla \delta u : \mathcal{L} : \nabla \delta u dV > 0, \quad (1.30)$$

$$\mathcal{L} = \mathcal{W}_{,\nabla u \nabla u} - \mathcal{W}_{,\nabla u \alpha} \mathcal{W}_{,\alpha \alpha}^{-1} \mathcal{W}_{,\alpha \nabla u}.$$

By definition,  $\mathcal{L}$  is the limit modulus. In an experiment of traction at constant strain rates, the limit modulus represents exactly the tangent modulus of the associated elastic-plastic curve obtained at vanishing strain rate. In an experiment of viscous flow at constant load, the limit modulus can also be interpreted as the long term tangent modulus for a visco-elastic material. The following proposition holds

**Proposition 4** *The criterion of limit modulus ensures the stability of an evolution.*

In particular, for a visco-elastic material obeying to Kelvin model, the associated second variation is exactly the second variation of the potential energy of the system. Thus, elastic stability of the solid at current positions will ensure the stability of the evolution in this case.

For a material obeying to kinematic hardening model, in the case of isotropic elasticity, the limit modulus is given by

$$\dot{\kappa} = K \operatorname{tr}(\dot{\epsilon}) I + \frac{Gh}{G+h} \dot{\epsilon}'.$$

The criterion of limit modulus is

$$\delta^2 \mathcal{E}_t = \int_V \left\{ K (\operatorname{tr}(\delta e))^2 + \frac{Gh}{G+h} \delta e' : \delta e' + \nabla \delta u \kappa \nabla \delta u^T \right\} dV > 0$$

if  $\nabla u_o(t)$  is neglected.

It may be interesting to compare the limit modulus to the tangent modulus associated with the kinematic hardening model. In particular, in the example of Shanley's model, the critical load  $\lambda_T$  has been recovered. More generally, the following proposition holds

**Proposition 5** *The limit modulus is exactly the tangent modulus if the normal cone to the elastic domain generates the whole internal parameter space.*

Indeed, in the definition of the tangent modulus, the fact that the rate of internal parameter must be an external normal to the elastic domain by normal law is essential. If any arbitrary direction can be generated by normal vectors, then the limit load is exactly the tangent load. For an elastic-plastic beam, the tangent modulus is exactly the limit modulus, the following proposition holds:

**Proposition 6** *A quasi-static evolution of a system of elastic-plastic beams is stable if the criterion of the elastic comparison solid is satisfied at current points of this evolution.*

If the space generated by the normal cone is not the whole internal space, the tangent modulus is stiffer than the limit modulus. The criterion of limit modulus is for that reason very conservative compared to the Hill's criterion of stability and cannot be operational. The problem of stability of an evolution is for this reason remains largely open. In particular, for stability analysis of visco-plastic solids, the obtention of efficient methods and estimates have been the scope of several recent discussions, cf. for example Tvergaard, 1979, Fressengeas and Molinari, 1987, Massin et al., 1999.

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