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► **To cite this version:**

Quoc Son Nguyen. Standard dissipative systems and stability analysis. Gérard Maugin; Raymonde Drouot; François Sidoroff. Continuum thermodynamics: the art and science of modeling matter's behavior, Springer, pp.343-354, 2000, 9789048155019. 10.1007/0-306-46946-4\_26 . hal-00112276

**HAL Id: hal-00112276**

**<https://hal.science/hal-00112276>**

Submitted on 15 Apr 2024

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# Standard dissipative systems and stability analysis

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**Abstract :** Stability and bifurcation analyses are discussed here for time-independent standard dissipative systems of materials and structures. This discussion is illustrated by some simple applications in plasticity and fracture mechanics.

**Keywords :** generalized standard model, standard dissipative system, static or dynamic stability, bifurcation, stability criterion.

## 1. INTRODUCTION

It is well known that the concept of energy and dissipation potentials, cf. for example [Germain, 1973], [Halphen et al., 1975], [Lemaitre et al., 1985], [Maugin, 1992], leads to a general framework in the study of dissipative effects in materials and structures. This framework is considered again in order to derive an operational and general formulation of stability and bifurcation criteria in the stability analysis of equilibrium of a time-independent standard dissipative system.

It is recalled that in finite and isothermal deformation, a generalized standard material admits as state variables  $\mathbf{q} = (\nabla \mathbf{u}, \boldsymbol{\alpha})$  and as energy density per unit volume  $\mathcal{W}(\nabla \mathbf{u}, \boldsymbol{\alpha})$ . The dissipation is a product of force and flux:  $d_{in} = (\mathbf{b} - \mathbf{b}_R) : \nabla \dot{\mathbf{u}} + \mathbf{A} \cdot \dot{\boldsymbol{\alpha}}$ , where  $\mathbf{b}_R$  and  $\mathbf{A}$  denote the associated forces  $\mathbf{b}_R = \mathcal{W}_{,\nabla \mathbf{u}}$ ,  $\mathbf{A} = -\mathcal{W}_{,\boldsymbol{\alpha}}$ . Complementary laws must be introduced to relate force and flux. These laws are written in terms of a dissipation potential as  $\mathbf{A} = \mathbf{D}_{,\dot{\boldsymbol{\alpha}}}(\dot{\boldsymbol{\alpha}}, \boldsymbol{\alpha})$ . It has been assumed that dissipation potential  $D = D(\dot{\boldsymbol{\alpha}}, \boldsymbol{\alpha})$  is convex with respect to flux  $\dot{\boldsymbol{\alpha}}$  and may depend on the present state through the present value of  $\boldsymbol{\alpha}$ . The dual function of  $D$ , obtained by Legendre-Fenchel's transform as  $D^*(\mathbf{A}, \boldsymbol{\alpha}) = \max_{\dot{\boldsymbol{\alpha}}} \mathbf{A} \cdot \dot{\boldsymbol{\alpha}} - D(\dot{\boldsymbol{\alpha}}, \boldsymbol{\alpha})$ , permits an equivalent expression of complementary laws  $\dot{\boldsymbol{\alpha}} = D^*_{,\mathbf{A}}(\mathbf{A}, \boldsymbol{\alpha})$ .

When the strain path  $\nabla \mathbf{u}(\mathbf{t})$  is given, the associated internal parameter can be obtained from its initial value by solving the system of equations

$$\mathbf{A} = -\mathcal{W}_{,\boldsymbol{\alpha}}, \quad \dot{\boldsymbol{\alpha}} = D^*_{,\mathbf{A}}(\mathbf{A}, \boldsymbol{\alpha}), \quad \boldsymbol{\alpha}(0) = \boldsymbol{\alpha}^0 \quad (1)$$

which can also be written as a differential equation, called Biot's equation cf. [Biot, 1965]

$$\mathcal{W}_{,\boldsymbol{\alpha}} + D_{,\dot{\boldsymbol{\alpha}}} = \mathbf{0}, \quad \boldsymbol{\alpha}(0) = \boldsymbol{\alpha}^0. \quad (2)$$

The assumption of dissipation potential can be expressed with any pair of associated force and flux. Indeed, the dissipation is

$$\mathbf{d} = \mathbf{A} \cdot \dot{\boldsymbol{\alpha}} = \mathbf{G} \cdot \boldsymbol{\gamma}, \quad \boldsymbol{\gamma} = \mathbf{S}(\dot{\boldsymbol{\alpha}}), \quad \mathbf{A} = \mathbf{S}^T(\mathbf{G})$$

where  $\mathbf{S}$  denotes any state-dependent linear operator,  $\boldsymbol{\gamma}$  is the transported flux resulting from this operator and  $\mathbf{G}$  is the associated force. If there exists a dissipation potential  $\mathbf{D}(\boldsymbol{\gamma})$ , depending on the present state such that  $\mathbf{G} = \mathbf{D}_{,\boldsymbol{\gamma}}$  then the function  $\mathbf{D}(\dot{\boldsymbol{\alpha}}) = \mathbf{D}(\boldsymbol{\gamma}) = \mathbf{D}(\mathbf{S}(\dot{\boldsymbol{\alpha}}))$  is a dissipation potential in the sense that  $\mathbf{A} = \mathbf{D}_{,\dot{\boldsymbol{\alpha}}}$ . The convexity of  $\mathbf{D}(\boldsymbol{\gamma})$  is also equivalent to the convexity of  $\mathbf{D}(\dot{\boldsymbol{\alpha}})$  since convexity is conserved in a linear transformation. The dissipation potential is a *priori* state-dependent, the dependence on the present state  $\boldsymbol{\alpha}$  has been here omitted for the sake of clarity.

In particular, the notion of generalized standard materials is stable with respect to a change of variables. Indeed, a change of variables  $\boldsymbol{\beta} = \boldsymbol{\beta}(\boldsymbol{\alpha})$  leads to new force  $\mathbf{B} = -\mathcal{W}_{,\boldsymbol{\beta}} = \mathbf{A} \cdot \boldsymbol{\alpha}_{,\boldsymbol{\beta}}$  while  $\dot{\boldsymbol{\beta}} = \boldsymbol{\beta}_{,\boldsymbol{\alpha}} \cdot \dot{\boldsymbol{\alpha}}$ . It is clear that  $\mathbf{d} = \mathbf{A} \cdot \dot{\boldsymbol{\alpha}} = \mathbf{B} \cdot \dot{\boldsymbol{\beta}}$  and all the ingredients of the model (energy potential, force, dissipation potential, convexity) remain valid.

For example, the model of plasticity with relaxed configuration, discussed by Lee and by Mandel, cf. [Mandel, 1971], is a generalized standard model defined by state variables  $(\mathbf{F}, \boldsymbol{\alpha}, \mathbf{T})$  with  $\boldsymbol{\alpha} = \mathbf{P}$ . The energy density is  $\mathcal{W}(\mathbf{F}, \mathbf{P})$  in isothermal transformation. In this case, the rate  $\dot{\boldsymbol{\alpha}} = \dot{\mathbf{P}}$  is associated with force  $\mathbf{A} = -\mathcal{W}_{,\mathbf{P}}$  while the rate  $\boldsymbol{\gamma} = \dot{\mathbf{P}}\mathbf{P}^{-1}$  is associated with the stress  $\mathbf{G} = \boldsymbol{\Psi}$ . The existence of a convex function (dual dissipation potential)  $\mathbf{D}^*(\boldsymbol{\Psi})$  such that  $\dot{\mathbf{P}}\mathbf{P}^{-1} = \mathbf{D}^*_{,\boldsymbol{\Psi}}$ , is strictly equivalent to the existence of a (state-dependent) convex function  $\mathbf{D}^*(\mathbf{A})$  such that  $\dot{\mathbf{P}} = \mathbf{D}^*_{,\mathbf{A}}$ . The introduction of suitable expressions of force and flux is only a matter of choice, principally motivated by physical considerations.

The generalized standard model can be extended to a mechanical system of solids if Biot's equation is the governing equation of the system in a quasi-static transformation

$$\mathbf{E}_{,\boldsymbol{q}} + \mathbf{D}_{,\boldsymbol{q}} = 0, \quad \boldsymbol{q}(0) = \boldsymbol{q}^0. \quad (3)$$

If  $\boldsymbol{q} = (\mathbf{u}, \boldsymbol{\alpha})$  where  $\mathbf{u}$  denotes displacement components and  $\boldsymbol{\alpha}$  the internal parameters, dynamic transformation of the system can be introduced with governing equation

$$\mathbf{J} + \mathbf{E}_{,\mathbf{u}} + \mathbf{D}_{,\dot{\mathbf{u}}} = 0, \quad \mathbf{E}_{,\boldsymbol{\alpha}} + \mathbf{D}_{,\dot{\boldsymbol{\alpha}}} = 0, \quad \boldsymbol{q}(0) = \boldsymbol{q}^0, \quad \dot{\mathbf{u}} = \mathbf{v}^0 \quad (4)$$

where the generalized inertia force  $\mathbf{J}$  depends linearly on  $\dot{\mathbf{u}}$ .

By definition, such a system is denoted as a **dissipative standard system**. Dissipative standard systems are governed in quasi-static transformation by Biot's differential equation (3), and in dynamic transformation by the second order differential equation (4).

If  $\mathbf{u}$  is a reversible variable, then  $\mathbf{D} = \mathbf{D}(\dot{\boldsymbol{\alpha}}, \boldsymbol{q})$ . In this case, the governing equations in quasi-static transformation can also be written as

$$\mathbf{E}_{,\mathbf{u}} = 0, \quad \mathbf{E}_{,\boldsymbol{\alpha}} + \mathbf{D}_{,\dot{\boldsymbol{\alpha}}} = 0, \quad \boldsymbol{\alpha}(0) = \boldsymbol{\alpha}^0, \quad (5)$$

or, in an equivalent way as the system of equations

$$\mathbf{E}_{,\mathbf{u}} = 0, \quad \mathbf{A} = -\mathbf{E}_{,\boldsymbol{\alpha}}, \quad \dot{\boldsymbol{\alpha}} = \mathbf{D}^*_{,\mathbf{A}}, \quad \boldsymbol{\alpha}(0) = \boldsymbol{\alpha}^0. \quad (6)$$

Equation  $\mathbf{E}_{,u}(\mathbf{u}, \boldsymbol{\alpha}, \lambda) = \mathbf{0}$  gives an implicit representation  $\mathbf{u} = \mathbf{u}(\boldsymbol{\alpha}, \lambda)$  when the quadratic form  $\mathbf{E}_{,uu}(\mathbf{u}, \boldsymbol{\alpha}, \lambda)[\delta\mathbf{u}, \delta\mathbf{u}]$  is positive-definite. In this case, let  $\mathbf{E}^r$  be the reduced energy potential

$$\mathbf{E}^r(\boldsymbol{\alpha}, \lambda) = \mathbf{E}(\mathbf{u}(\boldsymbol{\alpha}, \lambda), \boldsymbol{\alpha}, \lambda). \quad (7)$$

The quasi-static evolution of the system is also described by the reduced equations of evolution

$$\mathbf{A} = -\mathbf{E}^r_{,\boldsymbol{\alpha}}, \quad \dot{\boldsymbol{\alpha}} = \mathbf{D}^*_{,\mathbf{A}}, \quad \boldsymbol{\alpha}(0) = \boldsymbol{\alpha}^0. \quad (8)$$

The abstract equations (3) or (4) can be broadly understood by following the nature of variables  $u$  and  $\boldsymbol{\alpha}$ . For discrete systems, these variables are vectors. But for continuous systems, they may be vector functions defined on a curve, a surface or a domain. In each case, it is sufficient to define the meaning of the differentiation operations and the associated duality of force and flux.

If energy and dissipation potentials  $\mathbf{E}$ ,  $\mathbf{D}$  are regular functions, a system of first-order differential equations is obtained for quasi-static transformation. For example, the study of the quasi-static evolution of a visco-elastic structure obeying a generalized standard model of visco-elasticity and subjected to implied forces and displacements can be given in this framework. Elastic visco-plastic or elastic-plastic materials however are associated with non-smooth dissipation potentials. In this case, the concept of sub-differential of a convex function can be introduced, cf. [Moreau, 1971], to generalize the operation of differentiation and to write the governing equations of materials and structures in the same framework.

## 2. TIME-INDEPENDENT STANDARD DISSIPATIVE SYSTEMS

### 2.1 Evolution equation

A non-viscous or time-independent behaviour arises when the dissipation potential is positively homogeneous of degree 1 with respect to the flux:

$$\mathbf{D}(m \dot{\boldsymbol{\alpha}}, \boldsymbol{\alpha}) = m \mathbf{D}(\dot{\boldsymbol{\alpha}}, \boldsymbol{\alpha}) \quad \forall m > 0. \quad (9)$$

Such a function is not differentiable at point  $\dot{\boldsymbol{\alpha}} = \mathbf{0}$ , but is sub-differentiable. The set  $\mathbf{C}$  of sub-gradients at this point

$$\mathbf{C} = \partial_{\dot{\boldsymbol{\alpha}}} \mathbf{D}(\mathbf{0}, \boldsymbol{\alpha}) = \{ \mathbf{A}^* \mid \mathbf{A}^* \cdot \delta\boldsymbol{\alpha} \leq \mathbf{D}(\delta\boldsymbol{\alpha}, \boldsymbol{\alpha}) \quad \forall \delta\boldsymbol{\alpha} \} \quad (10)$$

is a convex domain of admissible forces. The dual dissipation potential is in this case the indicator function of the convex domain of admissible forces. Force-flux relation  $\dot{\boldsymbol{\alpha}} = \mathbf{D}^*_{,\mathbf{A}}$  can be written under the form of the normality law

$$\dot{\boldsymbol{\alpha}} = \mathbf{N}_{\mathbf{C}}(\mathbf{A}) \quad (11)$$

which states that rate  $\dot{\boldsymbol{\alpha}}$  must be an external normal to the admissible domain at the present state of force  $\mathbf{A}$ . It is well known that this evolution law can also be equivalent to the maximum dissipation principle which is classical in plasticity under the name of the principle of maximum plastic work

$$\mathbf{D}(\dot{\boldsymbol{\alpha}}, \boldsymbol{\alpha}) = \max_{\mathbf{A}^* \in \mathbf{C}} \mathbf{A}^* \cdot \dot{\boldsymbol{\alpha}}. \quad (12)$$

Governing equations (1) can be written as

$$\mathbf{E}_{,\mathbf{u}} \cdot \mathbf{0}, \quad \mathbf{A} = -\mathbf{E}_{,\alpha}, \quad \dot{\alpha} = \mathbf{N}_{,\mathbf{C}}(\mathbf{A}), \quad \alpha(0) = \alpha^0 \quad (13)$$

or in an equivalent form

$$\mathbf{E}_{,\mathbf{u}} \cdot (\delta \mathbf{u} - \dot{\mathbf{u}}) + \mathbf{E}_{,\alpha} \cdot (\delta \alpha - \dot{\alpha}) + \mathbf{D}(\delta \alpha, \alpha) - \mathbf{D}(\dot{\alpha}, \alpha) \geq 0 \quad \forall (\delta \mathbf{u}, \delta \alpha). \quad (14)$$

## 2.2 Rate problem

Let  $\boldsymbol{\gamma}(\mathbf{t})$  and  $\mathbf{G}(\mathbf{t})$  be two histories of flux and force associated by the normality law. Then, flux  $\boldsymbol{\gamma}(\mathbf{t})$  and force rate  $\dot{\mathbf{G}}(\mathbf{t})$  are related by the following proposition

**Proposition 1** *Let  $\boldsymbol{\gamma}(\mathbf{t})$  and  $\mathbf{G}(\mathbf{t})$  denote a flux and force associated by the normality law with a convex  $\mathbf{C}$  of non void interior, depending on a given function  $\mathbf{y}(\mathbf{t})$  for  $\mathbf{t} \in [0, \mathbf{T}]$ . If  $\dot{\mathbf{y}}(\mathbf{t}), \boldsymbol{\gamma}(\mathbf{t})$  and  $\dot{\mathbf{G}}(\mathbf{t})$  are piecewise continuous, then the following expressions hold for right-hand-side (r.h.s.) derivatives*

$$-\dot{\mathbf{G}} \cdot \boldsymbol{\gamma} + \dot{\mathbf{y}} \cdot \mathbf{D}_{,\mathbf{y}}(\boldsymbol{\gamma}, \mathbf{y}) = 0, \quad (15)$$

$$-\dot{\mathbf{G}} \cdot \boldsymbol{\gamma}^* + \dot{\mathbf{y}} \cdot \mathbf{D}_{,\mathbf{y}}(\boldsymbol{\gamma}^*, \mathbf{y}) \geq 0 \quad \forall \boldsymbol{\gamma}^* \text{ admissible}. \quad (16)$$

By definition, a rate  $\boldsymbol{\gamma}^*$  is admissible if  $\boldsymbol{\gamma}^* \in \mathbf{N}_{\mathbf{C}}(\mathbf{y})$ .

The proof of this proposition follows simply from the maximum dissipation principle (12) and can be found in [Nguyen, 2000]. As a consequence of the proposition, it should be noted that, if the assumption of state-independence is satisfied, i.e. if the dissipation potential does not depend on the present state, then the r.h.s. rates satisfy  $\dot{\mathbf{G}} \cdot \boldsymbol{\gamma} = 0$ . This orthogonality property is classical in perfect plasticity and gives  $\dot{\boldsymbol{\sigma}} : \dot{\boldsymbol{\epsilon}}^p = 0$ .

Relations (15), (16), written for  $\mathbf{y}(\mathbf{t}) = \mathbf{q}(\mathbf{t})$ , together with equations  $\mathbf{A} = -\mathbf{E}_{,\alpha}$ ,  $\mathbf{E}_{,\mathbf{u}} = 0$ , lead to the following description of the rate problem which consists in obtaining the rate response of the system  $\dot{\mathbf{q}}$  as a function of rate data  $\dot{\boldsymbol{\lambda}}$  when the present state is assumed to be known:

**Proposition 2** *The rate response  $\dot{\mathbf{q}}$  is a solution of the variational inequality*

$$(\mathbf{E}_{,\mathbf{q}\mathbf{q}} \cdot \dot{\mathbf{q}} + \mathbf{E}_{,\mathbf{q}\boldsymbol{\lambda}} \cdot \dot{\boldsymbol{\lambda}}) \cdot (\delta \mathbf{q} - \dot{\mathbf{q}}) + \dot{\mathbf{q}} \cdot (\mathbf{D}_{,\mathbf{q}}(\delta \mathbf{q}, \mathbf{q}) - \mathbf{D}_{,\mathbf{q}}(\dot{\mathbf{q}}, \mathbf{q})) \geq 0 \quad (17)$$

$$\forall \delta \mathbf{q} = (\delta \mathbf{u}, \delta \alpha) \text{ admissible}.$$

If  $\mathbf{D}$  depends on the present state, this variational inequality is symmetric if

$$\dot{\mathbf{q}} \cdot \mathbf{D}_{,\mathbf{q}}(\mathbf{q}^*, \mathbf{q}) = \mathbf{q}^* \cdot \mathbf{D}_{,\mathbf{q}}(\dot{\mathbf{q}}, \mathbf{q}) \quad \forall \text{ admissible rates } \dot{\mathbf{q}}, \mathbf{q}^*. \quad (18)$$

In this case, the obtained variational inequality is also quadratic. The uniqueness of the rate response  $\dot{\mathbf{q}}$  is ensured if the following positivity is ensured for all admissible rates  $\delta \mathbf{q}_1 \neq \delta \mathbf{q}_2$

$$(\delta \mathbf{q}_1 - \delta \mathbf{q}_2) \cdot \mathbf{E}_{,\mathbf{q}\mathbf{q}} \cdot (\delta \mathbf{q}_1 - \delta \mathbf{q}_2) + (\delta \mathbf{q}_1 - \delta \mathbf{q}_2) \cdot (\mathbf{D}_{,\mathbf{q}}(\delta \mathbf{q}_1, \mathbf{q}) - \mathbf{D}_{,\mathbf{q}}(\delta \mathbf{q}_2, \mathbf{q})) > 0. \quad (19)$$

The following proposition holds

**Proposition 3** Under the assumptions of symmetry (18) and of positivity (19), the rate solution  $\dot{q}$  minimizes among all admissible rates the rate functional

$$\mathbf{H}(q^*) = \frac{1}{2} ( q^* \cdot \mathbf{E}_{,qq} \cdot q^* + q^* \cdot \mathbf{D}_{,q} (q^*, q) ) + \dot{\lambda} \mathbf{E}_{,q\lambda} \cdot q^*. \quad (20)$$

Indeed, for all admissible rate  $q^*$

$$\begin{aligned} \mathbf{H}(q^*) - \mathbf{H}(\dot{q}) &= \frac{1}{2} \{ (q^* - \dot{q}) \cdot \mathbf{E}_{,qq} \cdot (q^* - \dot{q}) + (q^* - \dot{q}) \cdot (\mathbf{D}_{,q} (q^*, q) - \mathbf{D}_{,q} (\dot{q}, q)) \} + \\ & (\mathbf{E}_{,qq} \cdot \dot{q} + \mathbf{E}_{,q\lambda} \cdot \dot{\lambda}) \cdot (q^* - \dot{q}) + \dot{q} \cdot (\mathbf{D}_{,q} (q^*, q) - \mathbf{D}_{,q} (\dot{q}, q)) \geq 0 \end{aligned}$$

after (17) and (19). This functional is an extension of Hill's functional to standard dissipative systems. It is difficult however to eliminate the internal parameters as a function of displacement components in order to obtain a rate functional of the displacement rates. The existence of a solution  $\dot{q}$  is ensured for all  $\dot{\lambda}$  under the condition

$$\mathbf{E}_{,qq} [q^*, q^*] + q^* \cdot \mathbf{D}_{,q} (q^*, q) > 0 \quad \forall q^* \text{ admissible } \neq 0. \quad (21)$$

### 3. STABILITY AND BIFURCATION ANALYSIS

#### 3.1 Stability criterion

Condition (21) can be interpreted as a criterion of stability of the current equilibrium in a certain sense, cf. [Petryk, 1985], in relation to the notion of static or directional or dynamic stability. Indeed, in a perturbation of the system out of equilibrium, the energy injected by the external world in a time interval  $[0, \tau]$  is

$$\mathbf{W}_{per}(\tau) = \mathbf{W}^s(\tau) + C_\tau, \quad \mathbf{W}^s(\tau) = [\mathbf{E}(q, \lambda)]_0^\tau + \int_0^\tau \mathbf{D}(\dot{q}, q) dt$$

where  $C_t \geq 0$  denotes the kinetic energy of the system. Thus if  $\tau$  is sufficiently small, the expansion

$$\mathbf{W}^s = W_2^s \frac{\tau^2}{2} + o(\tau^2), \quad W_2^s = q_1 \cdot \mathbf{E}_{,qq} \cdot q_1 + q_1 \cdot \mathbf{D}_{,q} (q_1, q)$$

can be associated with the expansion  $q_\tau = q + q_1 \tau + q_2 \frac{\tau^2}{2} + \dots$ . It follows from condition (19) that it is necessary to inject energy initially into the system in order to remove it from the considered equilibrium, i.e. static stability as well as directional stability, are obtained. It is expected that the symmetry condition (18) is necessary to interpret this criterion as a sufficient condition of dynamic stability of the considered equilibrium.

In some particular cases, the dissipated energy may be a function of the present state if there exists a function  $\mathbf{W}^d(q)$  such that

$$\int_0^t \mathbf{D}(\dot{q}, q) dt = \mathbf{W}^d(q(t)) - \mathbf{W}^d(q(0)). \quad (22)$$

These systems, denoted in [Ehrlacher,1985] as *simple dissipative systems*, are frequently found in fracture and damage mechanics. Such a system is almost a conservative system in the sense that a total potential energy  $\mathbf{E}^a$  exists:

$$\mathbf{E}^a(q, \lambda) = \mathbf{E}(q, \lambda) + \mathbf{W}^d(q),$$

but the rate  $\dot{q}$  must be admissible. For simple dissipative systems, condition (21) can be written as

$$\delta^2 \mathbf{E}^a = \mathbf{E}^a_{,qq} [\delta q, \delta q] > 0 \quad \forall \delta q \text{ admissible.} \quad (23)$$

The following proposition holds, cf. [Nguyen, 2000]

**Proposition 4** *Condition (21) is a static or directional criterion of stability. This criterion ensures also the dynamic stability of the considered equilibrium if the assumption of state-independent potential is satisfied or if the system is a simple dissipative system.*

### 3.2 Non-bifurcation criterion

The non-uniqueness of the rate response indicates a critical point and eventually a bifurcation point. This idea leads to Hill's criterion of non-bifurcation. This criterion is available in the study of angular bifurcation as well as of tangent bifurcation.

**Proposition 5** *Condition (19) is a non-bifurcation criterion in the sense of Hill.*

From the definition of admissible rates, the linear space  $V(A)$  generated by the external normals to the convex of admissible forces  $\mathbf{C}$  at the present value  $A$  can be introduced. Since  $\delta\alpha_1 - \delta\alpha_2 \in V(A)$  if the rates  $\delta\alpha_1, \delta\alpha_2$  are admissible, the non-bifurcation condition can also be written as

$$\mathbf{E}_{,qq} [\delta q, \delta q] + \delta q \cdot \mathbf{D}_{,q} (\delta q, q) > 0 \quad \forall \delta q = (\delta u, \delta \alpha), \quad \delta \alpha \in V(A). \quad (24)$$

## 4. ILLUSTRATION IN PLASTICITY

The case of a generalized standard elastic-plastic material admitting as elastic domain a non-smooth convex, defined by several inequalities:

$$f^i(A, \alpha) \leq 0, \quad i = 1, N, \quad (25)$$

is considered in order to illustrate the symmetry condition (18). Such a model is necessary in the study of monocrystals for example, where plastic strains arise from different gliding mechanisms obeying Schmid's law. Functions  $f^i$  are classically denoted as plastic potentials. The normality law states that

$$\dot{\alpha} = \mu^i f^i_{,A} \quad \text{with} \quad \mu^i \geq 0, \quad \mu^i f^i = 0, \quad f^i \leq 0. \quad (26)$$

The rate equations can be written after the computation of the plastic multipliers  $\mu_i$  in terms of  $\nabla \dot{u}$ . For this, it can be noted that

$$\mu^i \geq 0, \quad (m^i - \mu^i) f^i \geq 0 \quad \forall m^i \geq 0.$$

It follows that  $\mu$  must satisfy

$$\begin{aligned} (m^i - \mu^i) (C_{ij}\mu^j + B_i : \nabla \dot{\mathbf{u}}) &\geq 0, \quad \forall m \in M, \quad \text{with} & (27) \\ M &= \{ m = (m^1, \dots, m^N) \mid m^i \geq 0, \quad m^i f^i = 0 \text{ (no sum)} \} \\ C_{ij} &= f^i_{,A} \cdot \mathcal{W}_{,\alpha\alpha} \cdot f^j_{,A} - f^i_{,A} f^j_{,\alpha}, \quad B_j = f^j_{,A} \cdot \mathcal{W}_{,\alpha} \nabla \mathbf{u}. & (28) \end{aligned}$$

This variational inequality gives  $\mu^i$  in terms of  $\nabla \dot{\mathbf{u}}$  in a unique manner if the matrix  $C$  is positive-definite. However the explicit expressions of these relations cannot be derived when  $N > 1$ . Matrix  $C$  is not necessarily symmetric; its symmetry is ensured only if the interaction matrix  $\mathcal{H}$  defined by

$$\mathcal{H}_{ij} = f^i_{,A} f^j_{,\alpha} \quad (29)$$

is symmetric. This matrix satisfies the relation

$$\delta \alpha \cdot D_{,\alpha}(\dot{\alpha}, \alpha) = -m^j \mathcal{H}_{ji} \mu^i \quad \forall \quad \delta \alpha = m^j f^j_{,A}, \quad \dot{\alpha} = \mu^j f^j_{,A}. \quad (30)$$

Indeed, the expression of the dissipation potential

$$D(\dot{\alpha}, \alpha) = A \cdot \dot{\alpha} - \mu^i f^i(A, \alpha),$$

gives after differentiation in the direction  $\delta \alpha = m^j f^j_{,A}$

$$\delta \alpha \cdot D_{,\alpha}(\dot{\alpha}, \alpha) = \delta A \cdot (\dot{\alpha} - \mu^i f^i_{,A}) - \mu^i f^i_{,\alpha} \cdot \delta \alpha - \delta \mu^i f^i.$$

Thus

$$\delta \alpha \cdot D_{,\alpha}(\dot{\alpha}, \alpha) = -\mu^i f^i_{,\alpha} \cdot \delta \alpha = -m^j H_{ji} \mu^i.$$

The symmetry of the interaction matrix is exactly the symmetry (18). If  $I$  denotes the set of index of active mechanisms, i.e.  $f^i = 0$ ,  $f^i_{,\alpha} = 0$  for  $i \in I$ , the plastic modulus  $\mathcal{L}_I^p$  associated with these active mechanisms can be computed. Indeed since

$$C_{ij}\mu^j + B_i : \nabla \dot{\mathbf{u}} = 0 \quad i, j \in I$$

it follows that  $\dot{\mathbf{b}} = \mathcal{L}^p : \nabla \dot{\mathbf{u}}$  with

$$L_I^p = \mathcal{W}_{,\nabla \mathbf{u} \nabla \mathbf{u}} - B_I^T C_I^{-1} B_I$$

where  $C_I$  and  $B_I$  are sub-matrices of  $C, B$  and related to active index  $I$ . It is concluded that the plastic modulus admits the major symmetry if and only if the interaction matrix  $\mathcal{H}$  is symmetric.

## 5. CRACK PROPAGATION, STABILITY AND BIFURCATION

The analyses of crack nucleation, crack propagation and crack stability are the objective of fracture mechanics. In brittle fracture, the stability of a Griffith crack has been considered in many discussions. Its generalization to study the propagation of a system of interacting



linear cracks or of a plane crack of arbitrary shape in an elastic solid is relatively straightforward as a particular example of standard dissipative systems.

### 5.1 System of interacting linear cracks

We consider the equilibrium problem of a solid in two-dimensional deformation, i.e. plane strain or plane stress, admitting in its volume  $V$  a system of linear cracks of lengths  $\ell_i, i = 1, n$ , undergoing small transformation under the action of implied forces and implied displacements defined by a load parameter  $\lambda$ . To simplify, it is assumed that surface forces  $\mathbf{r}^d(\lambda)$  are applied on the portion  $S_r$  of the boundary and on the complementary part  $S_u$ , displacements  $\mathbf{u}^d(\lambda)$  are implied. If the solid is elastic, the response of the system is reversible when there is no crack propagation and irreversible when the crack lengths change. Variables  $\ell_i$  thus describe the irreversible behaviour of the system and represent the internal variables  $\alpha$ . The set of admissible displacements of the solid depends on the present state of cracks, and can be written as

$$U(\ell, \lambda) = \{ \mathbf{u} \mid \mathbf{u} = \mathbf{u}^d \text{ on } S_u, [u_n] \geq 0 \text{ on } \Gamma_i \} \quad (31)$$

where  $\Gamma_i$  denotes the crack surfaces. If  $W(\epsilon)$  is the elastic energy density, the total potential energy of the system is

$$\mathbf{E}(\mathbf{u}, \ell, \lambda) = \int_{V_t} W(\epsilon(\mathbf{u})) dV - \int_{S_r} \mathbf{r}^d \cdot \mathbf{u} dS. \quad (32)$$

The displacement at equilibrium must satisfy the virtual work equation which can be written in the form of a variational inequality in order to take into account the possibility of unilateral contact, assumed to be frictionless, on crack surfaces:

$$\mathbf{E}_{,u}(\mathbf{u}, \ell, \lambda) \cdot \delta \mathbf{u} \geq 0 \quad \forall \delta \mathbf{u} = \mathbf{v} - \mathbf{u}, \mathbf{v} \in U(\ell, \lambda).$$

If the energy is strictly convex, the equilibrium displacement must minimise the total potential energy of the system, and permits the introduction of the energy at equilibrium  $\mathbf{E}^r(\ell, \lambda)$ :

$$\mathbf{E}^r(\ell, \lambda) = \mathbf{E}(\mathbf{u}(\ell, \lambda), \ell, \lambda) = \min_{\mathbf{u}^* \in U(\ell, \lambda)} \mathbf{E}(\mathbf{u}^*, \ell, \lambda). \quad (33)$$

The associated generalized force  $\mathbf{G}_i = -\mathbf{E}^r_{,\ell_i}$  is by definition the energy release rate associated with crack length  $\ell_i$ . Griffith's law states that

$$\begin{cases} \text{If } G_i < G_c \text{ then } \dot{\ell}_i = 0 \text{ (no propagation)} \\ \text{If } G_i = G_c \text{ then } \dot{\ell}_i \geq 0 \text{ (possible propagation).} \end{cases}$$

The critical surface energy  $G_c$  is often considered as a constant of the material. In order to interpret the resistance effects due to the presence of plastic strains, it has been also assumed in certain applications that the value  $G_c$  can depend on the effective length of propagation  $\Delta \ell_i = \ell_i - \ell_i^0$ ; function  $G_c = G_c(\Delta \ell_i)$  describes the resistance curve. Such a criterion introduces a domain of admissible forces:

$$\mathbf{C} = \{ G \mid G_i - G_c(\Delta \ell_i) \leq 0 \quad i = 1, n \}$$

and a dissipation potential  $\mathbf{D}(\dot{\ell}, \ell) = G_c(\Delta \ell_i) \dot{\ell}_i$  when  $\dot{\ell}_i \geq 0$ . The energy dissipated by crack propagation  $\mathbf{W}^d(\ell) = \sum_i \int_{\ell_i^0}^{\ell_i} G_c(\Delta \ell_i) d\ell_i$  depends only on the present state of cracks. A system of simple dissipation is thus obtained with total energy  $\mathbf{E}^a(\ell, \lambda) = \mathbf{E}^r(\ell, \lambda) + \mathbf{W}^d(\ell)$ .

In order to obtain the description of the rate problem at a given state, let  $I$  be the set of indices  $i$  such that the propagation limit is reached:  $G_i = G_c(\Delta \ell_i)$ . Admissible rate  $\delta \ell$  must satisfy  $\delta \ell_i \geq 0$  for all  $i \in I$ . The previous discussion leads to the following statements:

- Propagation rate  $\dot{\ell}$  is a solution of the quadratic and symmetric variational inequality

$$(\mathcal{A}_{ij} \cdot \dot{\ell}_j + \mathbf{E}^r_{, \lambda i} \cdot \dot{\lambda}) (\delta \ell_i - \dot{\ell}_i) \geq 0 \quad \forall \delta \ell \text{ admissible.}$$

where  $\mathcal{A}$  denotes the symmetric matrix

$$\mathcal{A}_{ij} = \mathbf{E}^r_{, ij} + \frac{dG_c}{d\ell}(\ell_i - \ell_i^0) \delta_{ij} \quad \text{with} \quad \mathbf{E}^r_{, ij} = \frac{\partial^2 \mathbf{E}^r}{\partial \ell_i \partial \ell_j}, \quad i, j \in I. \quad (34)$$

This problem can also be written as a linear complementarity problem (cf. [Cottle et al., 1992]) which consists of finding  $X$  such that

$$Y = \mathcal{A}X - F, \quad Y_i \geq 0, \quad X_i \geq 0, \quad X_i Y_i = 0 \quad (\text{no sum}) \quad (35)$$

where  $X, F$  are respectively vectors of components  $\dot{\ell}_i$  and  $\dot{\lambda} \mathbf{E}^r_{, \lambda i}$ .

- The present equilibrium state is stable with respect to crack propagation and to displacement in the dynamic sense if matrix  $\mathcal{A}$  satisfies the co-positivity condition:

$$\delta \ell \cdot \mathcal{A} \cdot \delta \ell > 0 \quad \forall \delta \ell \neq 0, \quad \delta \ell \geq 0. \quad (36)$$

This condition can also be written as

$$(\delta G_c - \delta G) \cdot \delta \ell > 0 \quad \forall \delta \ell \neq 0, \quad \delta \ell \geq 0. \quad (37)$$

- The present equilibrium state is not a bifurcation state if the matrix  $\mathcal{A}$  is positive-definite. The stability criterion is less restrictive than the non-bifurcation criterion since positive-definiteness is more restrictive than co-positivity.

The computation of matrix  $\mathcal{A}$  or of the second derivatives of energy has been discussed in [Nguyen et al., 1990], [Suo et al., 1992]. The difference between co-positivity and positive-definiteness has been illustrated in many simple analytical examples, cf. [Nguyen, 2000].

## 5.2 Stability and configurational stability of plane cracks

This problem is of interest in various applications, for example in the debonding of interface cracks, of surface coating by thin films, cf. [Berest, 1989], [Hutchinson et al., 1992] [Jensen, 1995], in the delamination of multi-layer composites, cf. [Cochelin, 1994], [Destuynder, 1987], [Pradeilles-Duval, 1992], [Storakers, 1988]. Moreover, the mechanical modelling of brittle damage also leads to the study of the propagation of damage zones in

an elastic solid, a problem of the same mathematical nature, cf. for example [Dems et al., 1985], (Stolz, 1987).

For a plane crack, the crack surface is a plane domain  $\Omega$  of contour  $\Lambda$ ; this domain represents the irreversible variable  $\alpha$ . The displacement at equilibrium  $\mathbf{u}(\Omega, \lambda)$  leads again to the energy of the system at equilibrium  $\mathbf{E}^r(\Omega, \lambda)$ . The generalized force associated with the irreversible variable is defined from the partial derivative of energy at equilibrium with respect to  $\Omega$ . In order to compute this partial derivative, a rate of variation of the boundary of domain  $\Omega$  can be described by the rate of normal extension. This normal rate, which is a scalar function defined on the present contour  $\Lambda$ , is denoted as  $\dot{\Omega}(s)$ . Since the crack surface can only increase, it follows that

$$\dot{\Omega} \text{ admissible} \Leftrightarrow \dot{\Omega}(s) \geq 0 \quad \forall s \in \Lambda. \quad (38)$$

Let  $\mathbf{E}^r_{,\Omega} \cdot \partial\Omega$  be the directional derivative of energy with respect to domain  $\Omega$  in the direction  $\partial\Omega$ . This directional derivative is a linear form for plane cracks, and can be expressed as

$$\mathbf{E}^r_{,\Omega} \cdot \partial\Omega = -\mathbf{G} \cdot \partial\Omega = - \int_{\Lambda} G(s) \partial\Omega(s) ds \quad (39)$$

where  $G(s)$  is a function defined on the present contour  $\Lambda$ . By definition, the local value  $G(s)$  is denoted as the local energy release rate, and the associated generalized force to the crack extension or the crack driving force  $\mathbf{G} = -\mathbf{E}^r_{,\Omega}$  is function  $\mathbf{G}$ .

The computation of the crack driving force  $\mathbf{G}$  has been discussed for different particular cases. For example, for a plane crack in a three-dimensional solid,  $G(s)$  is still given by the limit value of the  $J$ -integral. The dissipation of the whole system due to crack extension is

$$D_{in} = \mathbf{G} \cdot \dot{\Omega} = \int_{\Lambda} G(s) \dot{\Omega}(s) ds. \quad (40)$$

In particular, if  $\mathbf{G}_c$  is a constant, the dissipated energy by crack propagation is proportional to the cracked surface. The system of solid with crack is then an irreversible system of simple dissipation, of total energy  $\mathbf{E}^{ra}(\Omega, \lambda) = \mathbf{E}^r(\Omega, \lambda) + \mathbf{W}^d(\Omega)$ , and leads to the variational inequality

$$\int_{\Lambda_c} (\dot{G} - \dot{G}_c) (\partial\Omega - \dot{\Omega}) ds \leq 0 \quad \forall \partial\Omega \geq 0 \text{ on } \Lambda_c, \quad (41)$$

where  $\Lambda_c$  is the portion at yield of the contour. The rate problem is thus described by the variational inequality

$$(\partial\Omega - \dot{\Omega}) \cdot (\mathbf{E}^r_{,\Omega\Omega} \cdot \dot{\Omega} + \mathbf{E}^r_{,\Omega\lambda} \cdot \dot{\lambda}) + \dot{\Omega} \cdot (\mathbf{D}_{,\Omega}(\partial\Omega, \Omega) - \mathbf{D}_{,\Omega}(\dot{\Omega}, \Omega)) \geq 0 \quad (42)$$

for all admissible  $\partial\Omega$ , i.e. satisfying  $\partial\Omega \geq 0$  on  $\Lambda_c$ . The general form of the rate problem is then recovered. In particular, the stability criterion follows:

$$\int_{\Lambda_c} (\partial G_c - \partial G) \partial\Omega ds > 0 \quad \forall \partial\Omega \neq 0 \text{ admissible.} \quad (43)$$

In particular, if  $\mathbf{G}_c$  is constant, the stability criterion is reduced to the co-positivity condition

$$\partial\Omega \cdot \mathbf{E}^{ra}_{,\Omega\Omega} \cdot \partial\Omega > 0 \quad \forall \partial\Omega \neq 0 \text{ admissible,} \quad (44)$$

while the non-bifurcation criterion requires the positive-definiteness of the same quadratic form.

The principal difficulty stems from the calculation of the rate  $\delta G$  following the crack motion. Several analytical discussions have been given recently in the literature for the circular crack problem, cf. [Gao et al., 1987], [Berest, 1987] or of the tunnel crack problem cf. [Leblond et al., 1996], [Jensen et al., 1995]. For example, a circular crack may be stable in displacement control, but stable bifurcation in a star-shaped mode can be observed. The fact that the circular form may be lost is also known as a configurational instability in the literature. For the tunnel crack, a bifurcated mode to a wavy form has been also computed.

## 6. CONCLUDING REMARKS

For standard systems satisfying the symmetry condition, stability and bifurcation analysis leads to a general expression of stability and non-bifurcation criteria in terms of energy and dissipation potentials. This is an extension of the classical second variation criterion to dissipative systems.

The fact that stability and bifurcation analysis can be discussed in a satisfactory manner in fracture mechanics and in plasticity is due principally to the symmetry of the rate problem which permits a static analysis. For unsymmetric systems, for example in the context of unilateral contact with friction, it is clear that a static analysis presents less interest since stability must be considered in dynamics, in the same spirit as for non-conservative systems, cf. for example [Nguyen, 2000] for a review on the subject.

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