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# The influence of domain geometry in the boundary behavior of large solutions of degenerate elliptic problems.

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## Abstract

In this paper we study the asymptotic boundary behavior of large solutions of the equation  $\Delta u = d^\alpha u^p$  in a regular bounded domain  $\Omega$  in  $\mathbb{R}^N$ ,  $N \geq 2$ , where  $d(x)$  denotes the distance from  $x$  to  $\partial\Omega$ ,  $p > 1$  and  $\alpha > 0$ . We precise the expansion which depends on the mean curvature of the boundary.

## 1 Introduction : notations and main results

Let  $\Omega$  be a regular bounded domain in  $\mathbb{R}^N$ ,  $N \geq 2$ ,  $p > 1$  and  $\alpha > 0$ . We denote by  $d(x)$  the distance from  $x$  to  $\partial\Omega$ , the boundary of  $\Omega$ . In this paper we consider the semilinear degenerate equation

$$\Delta u = d^\alpha u^p \quad \text{in } \Omega \quad (1)$$

and we are interesting in the large solutions of (1), that is solutions of (1) which blow up at the boundary :

$$u(x) \rightarrow +\infty \quad \text{as } d(x) \rightarrow 0. \quad (2)$$

Note already that the maximum principle implies that the solutions  $u \in C^2(\Omega)$  of (1)-(2) are positive in  $\Omega$ .

Equation (1) registers in problems of the form

$$\Delta u = p(x)f(u) \quad \text{in } \Omega. \quad (3)$$

Those problems were first studied by Bieberbach [4] for the case  $p(x) = 1$ ,  $f(u) = e^u$  and  $N = 2$ , in the context of Riemannian surfaces of constant negative curvature, and the theory of automorphic functions. The case  $p(x) > 0$  for all  $x \in \overline{\Omega}$  has been, largely dealt with in the literature ( see [7], [12], [8], [5] for example).

Existence of solutions of (1)-(2) was established by Lair and Wood [9]. The question of the uniqueness of solutions of (1)-(2) is more delicate. When  $\alpha = 0$  and  $p > 1$ , it is well know that problem (1)-(2) has a unique solution which satisfies

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$$\lim_{d(x) \rightarrow 0} u(x) d(x)^{\frac{2}{p-1}} = \left( \frac{2(p+1)}{(p-1)^2} \right)^{\frac{1}{p-1}}. \quad (4)$$

This was first established by Loewner and Nirenberg [10] for the case  $p = (N+2)/(N-2)$ . Later we can find many extensions, see for example [1], [2] and [14] and the references cited there. The case  $\alpha < 0$  and  $p > 0$  is studied in [6]. In the general case  $\alpha \geq 0$ , Marcus and Véron proved the uniqueness of the solutions of (1)-(2) under the condition  $1 < p < (N+1+\alpha)/(N-1)$ . Our first theorem completes this result and gives the rate of the blow-up.

**Theorem 1.1** *Let  $u \in C^2(\Omega)$  be a solution of (1) – (2) . Then it satisfies*

$$\lim_{d(x) \rightarrow 0} d(x)^{\frac{\alpha+2}{p-1}} u(x) = l \quad (5)$$

where  $l$  is given by

$$l = \left[ \frac{(\alpha+2)(\alpha+p+1)}{(p-1)^2} \right]^{\frac{1}{p-1}}. \quad (6)$$

This theorem allows us to establish the uniqueness of solutions of (1)-(2) with no conditions on  $p$  and  $\alpha$ .

**Theorem 1.2** *Problem (1) possesses a unique large solution.*

In the second time we are interested in the influence of the geometry of  $\Omega$  in the boundary behavior. When  $\alpha = 0$ , this problem was first studied by Bandle and Marcus [3] for the radially symmetric solutions of (1)-(2) in a ball. Later their result was extended by del Pino and Letelier [13] for general solutions. They proved that a lower-order term, still explosive, appears in the expansion of  $u$  wich depends linearly of the mean curvature of the boundary of  $\Omega$ . More precisely, if  $1 < p < 3$  and  $\alpha = 0$ , then on a sufficiently small neighborhood of  $\partial\Omega$  we have the expansion

$$u(x) = \left( \frac{2(p+1)}{(p-1)^2} \right)^{\frac{1}{p-1}} d(x)^{-\frac{2}{p-1}} \left\{ 1 + \frac{N+1}{p+3} H(\bar{x}) d(x) + o(d(x)) \right\}. \quad (7)$$

Here, for all  $x$  in a neighborhood of  $\partial\Omega$ ,  $\bar{x}$  denotes the unique point of the boundary such that  $d(x) = |x - \bar{x}|$  and  $H(\bar{x})$  the mean curvature of the boundary at that point. Estimate (7) generalizes to our case  $\alpha \geq 0$  in the following way.

**Theorem 1.3** *Let  $u \in C^2(\Omega)$  a large solution of (1). Then, on a sufficiently small neighborhood of  $\partial\Omega$  :*

$$u(x) = l d(x)^{-\frac{2+\alpha}{p-1}} \left\{ 1 + \frac{N-1}{\alpha+p+3} H(\bar{x}) d(x) + o(d(x)) \right\}. \quad (8)$$

This theorem implies that on a sufficiently small neighborhood of  $\partial\Omega$  :

$$u(x) - ld(x)^{-\frac{2+\alpha}{p-1}} = \frac{N-1}{\alpha+p+3} H(\bar{x}) d(x)^{-\frac{\alpha+3-p}{p-1}} + o\left(d(x)^{-\frac{\alpha+3-p}{p-1}}\right). \quad (9)$$

Therefore, we obtain that

- if  $p > \alpha + 3$ , then the first member of (9) tends to 0 at the boundary,
- if  $p = \alpha + 3$ , then  $u(x) - ld(x)^{-\frac{2+\alpha}{p-1}} = \frac{N-1}{\alpha+p+3} H(\bar{x}) + o(1)$ ,
- if  $p < \alpha + 3$ , then the first member of (9) is not bounded and the blow-up depends on the mean curvature. Roughly, the "more curved" or "sharper" towards the exterior of  $\Omega$  is around a given point of  $\partial\Omega$ , the higher the explosion rate at that point is.

That is a generalization of the results of Bandle and Marcus [3] for the radially symmetric solutions of (1)-(2) in a ball  $\Omega = B(0, R)$  :

- if  $p > 3$ , then  $u(r) - l(R-r)^{-\frac{2}{p-1}} \rightarrow 0$  when  $r \rightarrow R$ ,
- if  $p = 3$ , then  $u(r) - l(R-r)^{-\frac{2}{p-1}} \rightarrow \frac{C(N)}{R}$  when  $r \rightarrow R$ , which represents the mean curvature of the ball,
- if  $p < 3$ , then  $u(r) - l(R-r)^{-\frac{2}{p-1}} \rightarrow \infty$  when  $r \rightarrow R$ .

Our paper is organized as follows :

1. Introduction
2. Asymptotic behavior and uniqueness
3. Boundary influence in the explosion rate.

## 2 Asymptotic behavior and uniqueness.

We begin this section by proving a classical estimate for all solution  $u$  of (1). ( see [12]).

**Proposition 2.1** : (*Osserman estimate*): *There exist two positive constants  $a = a(\partial\Omega)$  and  $C = C(\Omega, \alpha, p)$  such that for all solution  $u \in C^2(\Omega)$  of equation (1), we have :*

$$u(x) \leq Cd(x)^{-\frac{2+\alpha}{p-1}} \quad (10)$$

for all  $x \in \Omega$  such that  $d(x) < a$ .

**Proof** : Since  $\Omega$  is regular there exist  $\tilde{a} = \tilde{a}(\Omega) > 0$  and  $M = M(\Omega) > 0$  such that

$$|\Delta d(x)| \leq M, \quad |\nabla d(x)| = 1 \quad (11)$$

for all  $x \in \Omega$  such that  $d(x) < \tilde{a}$ . Set  $a = \min(1, \frac{\tilde{a}}{2})$ . Let  $x_0 \in \Omega$  such that  $d(x_0) < a$  and  $r_0 = d(x_0)/2$ . We denote by  $B_0$  the ball centered at  $x_0$  of radius  $r_0$  and we define the function  $w$  in  $B_0$  as follows :

$$w(x) = \lambda d(x)^{-\frac{\alpha}{p-1}} (r_0^2 - |x - x_0|^2)^{-\frac{2}{p-1}} \quad (12)$$

with  $\lambda > 0$  to determine such that

$$-\Delta w + d^\alpha w^p \geq 0 \quad \text{in } B_0. \quad (13)$$

A straightforward computation gives :

$$\begin{aligned}
-\Delta w + d^\alpha w^p &= \lambda d^{-\frac{\alpha}{p-1}} (r_0^2 - |x - x_0|^2)^{-\frac{2p}{p-1}} \times \\
&\left[ -\frac{\alpha(\alpha + p - 1)}{(p - 1)^2} (r_0^2 - |x - x_0|^2)^2 d^{-2} + \frac{\alpha}{p - 1} (r_0^2 - |x - x_0|^2)^2 d^{-1} \Delta d \right. \\
&\quad + \frac{8\alpha}{(p - 1)^2} (r_0^2 - |x - x_0|^2) d^{-1} \nabla d \cdot (x - x_0) - \frac{8(p + 1)}{(p - 1)^2} |x - x_0|^2 \\
&\quad \left. - \frac{4N}{p - 1} (r_0^2 - |x - x_0|^2) + \lambda^{p-1} \right].
\end{aligned}$$

Since  $|x - x_0| < d(x_0) \leq 1$ ,  $d(x) \geq d(x_0)/2$  and  $r_0^3 < r_0^2$ , there exists a constant  $L = L(\alpha, p, M) > 0$  such that

$$-\Delta w + d^\alpha w^p \geq \lambda d^{-\frac{\alpha}{p-1}} (r_0^2 - |x - x_0|^2)^{-\frac{2p}{p-1}} (-Lr_0^2 + \lambda^{p-1})$$

in  $B_0$ . Therefore, we choose  $\lambda = L^{\frac{1}{p-1}} r_0^{\frac{2}{p-1}}$  and we obtain (13). Note that  $w(x) = +\infty$  if  $x \in \partial B_0$  because  $-2/(p - 1) < 0$ . The comparison principle implies  $u \leq w$  in  $B_0$  and in particular

$$u(x_0) \leq w(x_0) = L^{\frac{1}{p-1}} \left( \frac{d(x_0)}{2} \right)^{\frac{2}{p-1}} (d(x_0))^{-\frac{\alpha}{p-1}} \left( \frac{d(x_0)}{2} \right)^{-\frac{4}{p-1}}$$

which gives inequality (10).

We now establish an estimate from below for the solutions of (1)-(2). The results of [1] and [2] can't be used because the distance function  $d$  is not positive in  $\bar{\Omega}$ . Nevertheless we can adapt them as follows.

**Proposition 2.2** *Let  $u \in C^2(\Omega)$  be a solution of (1) – (2). Then*

$$\liminf_{d(x) \rightarrow 0} d(x)^{\frac{\alpha+2}{p-1}} u(x) \geq l \tag{14}$$

where  $l$  is defined in (6).

**Proof :** Let  $\varepsilon > 0$ ,  $\tilde{a}$  be as the proof of proposition 2.1 and  $\beta \in (0, 1)$ . We define

$$\underline{u}(x) = \beta l ((d(x) + \varepsilon)^{-\frac{\alpha+2}{p-1}} - (\bar{a} + \varepsilon)^{-\frac{\alpha+2}{p-1}})$$

where  $\bar{a}$  will be determined such that  $\bar{a} < \tilde{a}$ . We have  $\underline{u} > 0$  on  $\partial\Omega$  and  $\underline{u}(x) = 0$  for all  $x$  such that  $d(x) = \bar{a}$ . Moreover a straightforward computation yields

$$-\Delta \underline{u} + d^\alpha \underline{u}^p = \beta \left[ \Delta d \left( \frac{\alpha + 2}{p - 1} \right) l (d + \varepsilon)^{-\frac{\alpha+p+1}{p-1}} - l^p (d + \varepsilon)^{-\frac{\alpha+2p}{p-1}} \right]$$

$$+d^\alpha \beta^{p-1} l^p \left[ (d + \varepsilon)^{-\frac{\alpha+2}{p-1}} - (\bar{a} + \varepsilon)^{-\frac{\alpha+2}{p-1}} \right]^p$$

in  $0 < d(x) < \bar{a}$ . Using inequality (11), we obtain

$$\begin{aligned} -\Delta \underline{u} + d^\alpha \underline{u}^p &\leq \beta l^p (d + \varepsilon)^{-\frac{\alpha+2p}{p-1}} \left[ M \left( \frac{\alpha+2}{p-1} \right) l^{1-p} (d + \varepsilon) \right. \\ &\quad \left. - 1 + \beta^{p-1} \left( \frac{d}{d + \varepsilon} \right)^\alpha \right] \end{aligned}$$

which implies

$$-\Delta \underline{u} + d^\alpha \underline{u}^p \leq \beta l^p (d + \varepsilon)^{-\frac{\alpha+2p}{p-1}} [\bar{M}(d + \varepsilon) - (1 - \beta^{p-1})]$$

with  $\bar{M} = M \left( \frac{\alpha+2}{p-1} \right) l^{1-p}$ . We now choose  $\bar{a} = \frac{1}{2} \min(\tilde{a}, \frac{1-\beta^{p-1}}{M})$  and impose  $\varepsilon < \frac{1}{2} \left( \frac{1-\beta^{p-1}}{M} \right)$ .

Then  $\bar{u}$  is a subsolution of (1) in  $0 < d(x) < \bar{a}$ . By the maximum principle we derive  $\underline{u} \leq u$  in  $0 < d(x) < \bar{a}$ . Letting  $\varepsilon$  tend to 0, this implies for all  $\beta \in (0, 1)$  and  $x$  such that  $d(x) < \bar{a}$ :

$$\beta l \left[ 1 - \left( \frac{d(x)}{\bar{a}} \right)^{\frac{\alpha+2}{p-1}} \right] \leq d(x)^{\frac{\alpha+2}{p-1}} u(x).$$

Therefore for all  $\beta \in (0, 1)$ :

$$\beta l \leq \liminf_{d(x) \rightarrow 0} d(x)^{\frac{\alpha+2}{p-1}} u(x)$$

which ends the proof.

Because of proposition 2.2, we can describe the asymptotic behavior of radially symmetric solutions of (1)-(2).

**Proposition 2.3** *Let  $R > 0$  and  $v \in C^2(0, R)$  a solution of*

$$-v'' - \frac{N-1}{r} v' + (R-r)^\alpha v^p = 0 \tag{15}$$

*in  $(0, R)$  such that*

$$\lim_{r \rightarrow R} v(r) = +\infty.$$

*Then*

$$\lim_{r \rightarrow R} (R-r)^{\frac{\alpha+2}{p-1}} v(r) = l \tag{16}$$

*where  $l$  is defined in (6).*

We omit the proof of this proposition because it follows the idea of [14]: the function  $w(t) = (R-r)^{\frac{\alpha+2}{p-1}}v(r)$  with  $R-r = e^{-t}$  is bounded and satisfies a second order differential equation in a neighborhood of infinity and the  $\omega$ -limit set of a trajectory of that equation is  $\{0\}$  or  $\{l\}$ . Therefore proposition 2.2 implies proposition 2.3. Those results allows us to prove theorem 1.1.

**Proof** of theorem 1.1 : In view of (14) we must only prove that

$$\limsup_{d(x) \rightarrow 0} d(x)^{\frac{\alpha+2}{p-1}} u(x) \leq l. \quad (17)$$

Still the results of [1], [2] or [14] don't apply directly but we can adapt them. Let  $y \in \partial\Omega$ . Since  $\partial\Omega$  is smooth, there exists a ball  $B_y$  centered at a point  $Y$  of radius  $R_y$  such that  $B_y \subset \Omega$  and  $\overline{B_y} \cap \partial\Omega = \{y\}$ . We introduce the function  $V$  defined by  $V(x) = v(|x|)$  for all  $x \in B_{R_y}$  where  $v$  is a function as in proposition 2.3 with  $R = R_y$ . The function  $v$  exists because it is the radial solution of (1)-(2) for  $\Omega = B$  (see [9]). Let  $k > 1$ . Finally we introduce the function  $V_k$  defined by  $V_k(x) = k^{\frac{2}{p-1}}V(k(x-Y))$  for all  $x \in B(Y, \frac{R_y}{k})$ . Note that  $B(Y, \frac{R_y}{k}) \subset B_y$  and  $V_k$  is solution of

$$-\Delta V_k + (R_y - k|x - Y|)^{\alpha} V_k^p = 0$$

in  $B(y, \frac{R_y}{k})$  and satisfies

$$\lim_{|x-Y| \rightarrow \frac{R_y}{k}} V_k(x) = +\infty.$$

Since  $x \in B(Y, \frac{R_y}{k})$  implies  $d(x) \geq R_y - |x - Y| \geq R_y - k|x - Y|$ , the comparison principle involves  $u \leq V_k$  in  $B(Y, \frac{R_y}{k})$ . Letting  $k$  tend to 1, we obtain

$$u(x) \leq v(|x - Y|) \quad \text{in } B_y. \quad (18)$$

Because of proposition 2.3, for all  $\varepsilon > 0$  there exists  $\eta > 0$  such that

$$|s^{\frac{\alpha+2}{p-1}}v(R_y - s) - l| < \varepsilon \quad \forall s \in (0, \eta). \quad (19)$$

Let  $\tilde{\eta} > 0$  be sufficiently small so that for all  $x \in \Omega$  with  $d(x) < \tilde{\eta}$  there exists a unique  $y \in \partial\Omega$  such that  $|x - y| = d(x)$ . Then for all  $x \in \Omega$  such that  $d(x) < \min(\eta, \tilde{\eta})$ , both inequalities (18) and (19) imply

$$d(x)^{\frac{\alpha+2}{p-1}} u(x) \leq d(x)^{\frac{\alpha+2}{p-1}} v(R_y - d(x)) < l + \varepsilon$$

and inequality (17) holds.

**Proof** of theorem 1.2 : Large solutions of (1) satisfy (5). Then two large solutions  $u_1$  and  $u_2$  of (1) are such that

$$\lim_{d(x) \rightarrow 0} \frac{u_1(x)}{u_2(x)} = 1.$$

and the result follows as in [1] or [11].

### 3 Boundary influence in the explosion rate.

In this section we prove theorem 1.3. As in [13] we construct suitable sub- and supersolutions of (1) in a neighborhood of  $\partial\Omega$  which are inspired of the radial study that we omit here.

Since  $\Omega$  is regular there exists  $\bar{b} > 0$  such that  $d$  is a function of class  $C^2$  in  $\{x \in \Omega / d(x) < \bar{b}\}$ ,  $|\nabla d(x)| = 1$  and

$$\Delta d(x) = -(N-1)H(\bar{x}) + o(1) \quad \text{as } d(x) \rightarrow 0. \quad (20)$$

Let  $b_0 \in (0, \bar{b})$ ,  $b \in (0, b_0)$  and  $\varepsilon > 0$ . We introduce the function  $\Psi$  defined in  $E_{b,b_0} = \{x \in \Omega / b < d(x) < b_0\}$  by

$$\Psi(x) = l(d(x) - b)^{-\frac{\alpha+2}{p-1}} + \frac{l}{\alpha + p + 3} ((N-1)H(\bar{x}) + \varepsilon)(d(x) - b)^{-\frac{\alpha+3-p}{p-1}}.$$

We claim that if  $b_0$  is chosen sufficiently small, independently of  $\varepsilon$  and  $b$ , then  $\Psi$  is a supersolution in  $E_{b,b_0}$ . Indeed, a straightforward computation using (20) gives :

$$\begin{aligned} \Delta \Psi &= l^p(d(x) - b)^{-\frac{\alpha+2p}{p-1}} + l(d(x) - b)^{-\frac{\alpha+p+1}{p-1}} \left[ \frac{\alpha+2}{p-1}(N-1)H(\bar{x}) \right. \\ &\quad \left. + \frac{(\alpha+3-p)(\alpha+2)}{(\alpha+p+3)(p-1)^2} ((N-1)H(\bar{x}) + \varepsilon) + o(1) \right. \\ &\quad \left. + \frac{\alpha+3-p}{(p-1)(\alpha+p+3)} ((N-1)H(\bar{x}) + \varepsilon)((N-1)H(\bar{x}) + o(1))(d(x) - b) \right]. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} d(x)^\alpha \Psi^p &\geq (d(x) - b)^\alpha \Psi^p \\ &\geq l^p(d(x) - b)^{-\frac{\alpha+2p}{p-1}} \left[ 1 + \frac{p}{\alpha+p+3} ((N-1)H(\bar{x}) + \varepsilon)(d(x) - b) + o(d(x) - b) \right]. \end{aligned}$$

Then

$$\begin{aligned} -\Delta \Psi + d^\alpha \Psi^p &\geq l(d(x) - b)^{-\frac{\alpha+p+1}{p-1}} \times \\ &\quad \left[ -\frac{\alpha+2}{p-1}(N-1)H(\bar{x}) - \frac{(\alpha+3-p)(\alpha+2)}{(\alpha+p+3)(p-1)^2} ((N-1)H(\bar{x}) + \varepsilon) \right. \\ &\quad \left. - \frac{\alpha+3-p}{(p-1)(\alpha+p+3)} ((N-1)H(\bar{x}) + \varepsilon)((N-1)H(\bar{x}) + o(1))(d(x) - b) \right. \\ &\quad \left. + \frac{l^{p-1}p}{\alpha+p+3} ((N-1)H(\bar{x}) + \varepsilon) + o(1) \right] \end{aligned}$$

Since

$$-\frac{\alpha+2}{p-1} - \frac{(\alpha+3-p)(\alpha+2)}{(\alpha+p+3)(p-1)^2} + \frac{l^{p-1}p}{\alpha+p+3} = 0,$$

and since the coefficient of  $\varepsilon$  is  $(\alpha+2)/(p-1)$ , it implies that there exists  $b_0 = b_0(\varepsilon) \in (0, \bar{b})$  such that for all  $0 < b < b_0$  :

$$-\Delta\Psi + d^\alpha\Psi^p \geq 0 \quad \text{in} \quad E_{b,b_0}.$$

Consider the solution  $u$  of (1)-(2). We claim that there exists a positive number  $K$  independent of  $b \in (0, b_0)$  such that :

$$\Psi(x) + K \geq u(x) \tag{21}$$

for all  $x \in \Omega$  with  $d(x) = b_0$ . In fact, if we define

$$M_0 = \max_{d(x)=b_0} u(x),$$

we can compute for all  $x$  such that  $d(x) = b_0$  :

$$\Psi(x) = l(b_0 - b)^{-\frac{\alpha+2}{p-1}} + \frac{l}{\alpha+p+3}((N-1)H(\bar{x}) + \varepsilon)(b_0 - b)^{-\frac{\alpha+3-p}{p-1}}.$$

Since  $\partial\Omega$  is regular, there exists a real  $b_1 \in (0, b_0)$  such that

$$\left| \frac{1}{\alpha+p+3}((N-1)H(\bar{x}) + \varepsilon)(b_0 - b) \right| \leq \frac{1}{2}$$

for all  $b \in (b_1, b_0)$ , where  $\bar{x}$  is such that  $d(x) = |x - \bar{x}|$ . Therefore

$$1 + \frac{1}{\alpha+p+3}((N-1)H(\bar{x}) + \varepsilon)(b_0 - b) \geq \frac{1}{2}$$

and then

$$\Psi(x) \geq \frac{l}{2}(b_0 - b)^{-\frac{\alpha+2}{p-1}} \geq \frac{l}{2}(b_0 - b_1)^{-\frac{\alpha+2}{p-1}}$$

for all  $b \in (b_1, b_0)$ , where  $\bar{x}$  is such that  $d(x) = |x - \bar{x}|$ . On the other hand, for all  $b \in (0, b_1]$  and  $d(x) = b_0$  :

$$\begin{aligned} \Psi(x) &= l(b_0 - b)^{-\frac{\alpha+2}{p-1}} + \frac{l}{\alpha+p+3}((N-1)H(\bar{x}) + \varepsilon)(b_0 - b)^{-\frac{\alpha+3-p}{p-1}} \\ &\geq lb_0^{-\frac{\alpha+2}{p-1}} - C(b_0 - b_1)^{-\frac{\alpha+3-p}{p-1}} \end{aligned}$$

with  $C > 0$  and because the assumption if we assume  $\alpha + 3 - p > 0$  (we omit the proof in the case  $\alpha + 3 - p \leq 0$  which is simpler). Finally we obtain for all  $b \in (0, b_0)$  :

$$\Psi(x) \geq L = \min \left( \frac{l}{2}(b_0 - b_1)^{-\frac{\alpha+2}{p-1}}, lb_0^{-\frac{\alpha+2}{p-1}} - C(b_0 - b_1)^{-\frac{\alpha+3-p}{p-1}} \right),$$

then, for all  $x$  such that  $d(x) = b_0$ ,  $u \leq M_0 \leq \max(1, M_0 - L) + L \leq \max(1, M_0 - L) + \psi$  which implies (21).

On the other hand the function  $\Psi + K$  is itself a supersolution of equation (1) in  $E_{b,b_0}$ . Therefore the comparison principle implies (21) in  $E_{b,b_0}$ . Letting  $b$  tend to 0, we obtain

$$u(x) \leq ld(x)^{-\frac{\alpha+2}{p-1}} + \frac{l}{\alpha+p+3}((N-1)H(\bar{x}) + \varepsilon)d(x)^{-\frac{\alpha+3-p}{p-1}} + K$$

for all  $x \in \Omega$  such that  $0 < d(x) < b_0$ . In the same way, by considering subsolutions in the form

$$\phi(x) = l(d(x) + b)^{-\frac{2+\alpha}{p-1}} + \frac{l}{\alpha+p+3}((N-1)H(\bar{x}) - \varepsilon)(d(x) + b)^{-\frac{\alpha+3-p}{p-1}} - \bar{K}$$

we obtain expansion (8).

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## References

- [1] C. Bandle and M. Marcus, Large solutions of semilinear elliptic equations : existence, uniqueness and asymptotic behavior, J. Anal. Math. 58 (1992) 9-24
- [2] C. Bandle and M. Marcus, Asymptotic behavior of solutions and their derivatives, for semilinear elliptic problems with blow-up on the boundary, Annal. Inst. Henri Poincaré-Anal. non lin. 12 (1995) 155-171.
- [3] C. Bandle and M. Marcus, On second-order effects in the boundary behavior of large solutions of semilinear elliptic problems, Diff. Integ. Eq. 11 (1998) 23-34.
- [4] L. Bieberbach,  $\Delta u = e^u$  und die automorphen funktionen, Math. Ann. 77 (1916), 173-212.
- [5] K. S. Cheng and W. M. Ni, On the structure of the conformal scalar curvature equation on  $\mathbb{R}^N$ , Indiana Univ. Math. J. 41 (1992) 261-278.
- [6] M. Chuaqui, C. Cortázar, M. Elgueta, C. Flores, R. Letelier and J. Garcí-Melián, On an elliptic problem with boundary blow-up and a singular weight : the radial case, Proceed. of the Roy. Soc. of Edinburgh 133A (2003) 1283-1297.
- [7] J. B. Keller, On solution of  $\Delta u = f(u)$ , Comm. Pure Appl. Math. 10 (1957) 503-510.
- [8] A. V. Lair, A necessary and sufficient condition for existence of large solutions to semilinear elliptic equations, J. Math. Anal. and Appl. 240 (1999) 205-218.
- [9] A. V. Lair and A. W. Wood, Large solutions of semi-linear elliptic problems, Nonlin. Anal. 37 (1999) 805-812.

- [10] C. Loewner and L. Nirenberg, Partial differential equations invariant under conformal projective transformations, Contributions to Analysis, ed. L. Ahlfors, Academic Press, New York (1974) 245-272.
- [11] M. Marcus and L. Véron, Uniqueness and asymptotic behavior of solutions with boundary blow-up for a class of nonlinear elliptic equations, Annal. Inst. Henri Poincaré. Anal. non lin. 14 (1997) 237-274.
- [12] R. Osserman, On the inequality  $\Delta u \geq f(u)$ , Pacific J. Math. 7 (1957) 1641-1647.
- [13] M. del Pino and R. Letelier, The influence of domain geometry in boundary blow-up elliptic problems, Nonlin. Anal. 48 (2002) 897-904.
- [14] L. Véron, Semilinear elliptic equations with uniform blow-up on the boundary, J. Anal. Math. 59 (1992) 231-250.