



**HAL**  
open science

# Crack propagation from a pre-existing flaw at a notch root. I. Introduction and general form of the stress intensity factors at the initial crack tip

Jean-Baptiste Leblond, Pierre Mouro

## ► To cite this version:

Jean-Baptiste Leblond, Pierre Mouro. Crack propagation from a pre-existing flaw at a notch root. I. Introduction and general form of the stress intensity factors at the initial crack tip. *International Journal of Fracture*, 1999, 327, pp.581-587. 10.1023/A:1007684632479 . hal-00111632

**HAL Id: hal-00111632**

**<https://hal.science/hal-00111632v1>**

Submitted on 28 Feb 2019

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Crack propagation from a pre-existing flaw at a notch root. I. Introduction and general form of the stress intensity factors at the initial crack tip

JEAN-BAPTISTE LEBLOND<sup>1</sup> and PIERRE MOURO<sup>2</sup>

<sup>1</sup>*Laboratoire de Modélisation en Mécanique, Université Pierre et Marie Curie, Tour 66, 4 place Jussieu, 75005 Paris, France*

<sup>2</sup>*Laboratoire de Mécanique des Solides, Ecole Polytechnique, Route de Saclay, 91128 Palaiseau, France*

**Abstract.** This paper and its companion are devoted to the study of crack kinking from some *small* pre-existing crack originating from a notch root (the notch root radius being zero). Both the notch boundaries and the initial crack are allowed to be curved; also, the geometry of the body and the loading are totally arbitrary. The ingredients required are knowledge of the stress intensity factors at the initial crack tip and use of a suitable mixed mode propagation criterion. This paper is devoted to the first point, and more specifically to establishing the *general* (that is, not yet fully explicit) form of the formulae giving these stress intensity factors. The method used is based on changes of scale (homogeneity properties of the equations of elasticity) on the one hand, and on continuity of the displacement and stresses at a given, fixed point with respect to the crack length on the other hand. The formulae derived for the stress intensity factors at the tip of the small crack are of *universal* value: they apply to any situation, whatever the geometry of the body, the notch and the crack and whatever the loading, the stress intensity factors depending always only upon the ‘stress intensity factor of the notch’ (the multiplicative coefficient of the singular stress field near the notch root in the absence of the crack), the length of the crack, the aperture angle of the notch and the angle between its bisecting line and the direction of the crack.

**Key words:** Notch, pre-existing crack, mixed mode, stress intensity factors, homogeneity and continuity properties, universality properties.

## 1. Introduction

Predicting crack growth from some notch root is a problem of obvious practical importance. If one adopts the classical theory of LEFM, however, one immediately encounters a problem : indeed crack propagation cannot occur directly from a notch root because the relevant energy-release-rate is zero. One is therefore forced to postulate the presence of some pre-existing small flaw originating from the notch root, and study crack propagation from this initial crack.

We therefore consider the general problem depicted schematically in Fig. 1. An arbitrarily shaped isotropic elastic body  $\Omega$  under plane strain conditions is subjected to arbitrary prescribed displacements  $\mathbf{u}^p$  on the portion  $\partial\Omega_u$  of its boundary  $\partial\Omega$  and to arbitrary prescribed tractions  $\mathbf{T}^p$  on the complementary portion  $\partial\Omega_T$ . This body contains a traction-free notch of apex  $O$ , with a zero root radius and an aperture angle  $2(\pi - \psi)$  (so that  $\psi$  represents the angle between the bisecting line  $Ox$  of the notch, oriented towards the material, and the notch boundaries), with  $\pi/2 \leq \psi \leq \pi$ . The notch boundaries are allowed to be curved; we denote by  $C_1$  the curvature of the upper boundary at the point  $O$  and by  $C_2$  the curvature of the lower boundary at the same point. An edge traction-free crack (in full line in Fig. 1) of *small* length

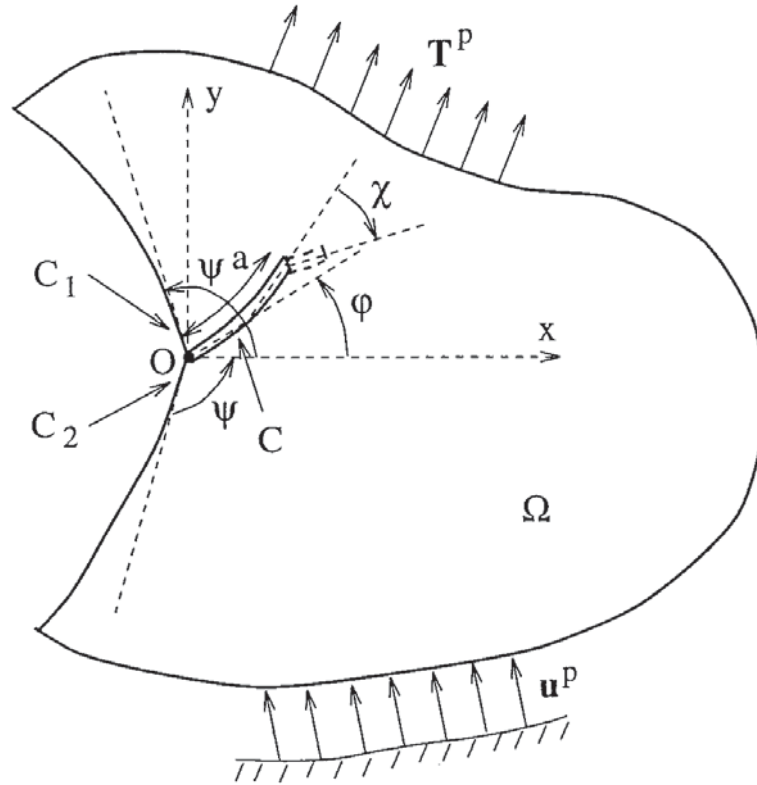


Figure 1. The general problem considered.

$a$  originates from the notch apex  $O$ . It makes an initial angle  $\varphi$  ( $-\psi < \varphi < +\psi$ ) with the bisecting line  $Ox$  of the notch. It is also allowed to be curved; we denote by  $C$  its curvature at the point  $O$ <sup>1</sup>. Since in general the crack is loaded in mixed mode I+II, subsequent propagation will occur along a deviated branch (in dotted line in Fig. 1). *The purpose of this paper and its companion is to predict the beginning of the propagation path, and more specifically the kink angle  $\chi$ , that is, the angle between the old and new tangents to the crack at its initial tip (see Fig. 1)*<sup>2</sup>.

This goal will be achieved through use of some appropriate mixed mode propagation criterion; we shall choose the well-known and widely accepted *principle of local symmetry* of Goldstein and Salganik (1974) in Part II. However, prediction of the kink angle  $\chi$  through this criterion of course requires knowledge of the stress intensity factors (SIFs)  $K_p(a)$  ( $p = I, II$ ) at the initial crack tip. Calculation of these SIFs, in the limit case  $a \rightarrow 0$ , therefore appears as an indispensable prerequisite.

This problem has never been envisaged previously with this degree of generality. However, partial results are available for various particular cases which all involve a notch with straight boundaries and a straight crack ( $C_1 = C_2 = C = 0$ ), and angles  $\psi$  and  $\varphi$  taking special values. For instance, it is a classical result that for  $\psi = \pi/2$  and  $\varphi = 0$  (surface crack originating perpendicularly from a regular, corner-free free surface),  $K_I(a) \simeq 1.12\sigma_{yy}^\infty\sqrt{\pi a}$  and  $K_{II}(a) = 0$ , where  $\sigma_{yy}^\infty$  denotes the remote tensile stress. The case where  $\psi = \pi$  but  $\varphi$  is arbitrary (semi-infinite crack with an arbitrarily kinked extension) has been extensively

<sup>1</sup>Rigorously speaking, one should introduce parameters characterizing the geometry of the notch boundaries and the crack with a higher degree of accuracy than that envisaged here, that is, consider successive derivatives of the curvatures at the point  $O$ . It can easily be verified at the end, however, that this would not modify the results derived in any way.

<sup>2</sup> $\chi$  is represented as negative in Fig. 1 because  $\varphi$  is taken as positive; we shall see in Part II that these angles are necessarily of opposite signs.

studied, for instance by Bilby and Cardew (1975), Wu (1978), Amestoy *et al.* (1979), Amestoy and Leblond (1992); accurate numerical results are available for that situation. The case where  $\psi = \pi/2$  and  $\varphi$  is arbitrary (oblique edge crack emanating from a regular free surface) has been considered by several authors (Wilson, 1969; Sha and Yang, 1985; Aliabadi *et al.*, 1987), whose (numerical) results have recently been compiled and compared by Fett and Munz (1997). Finally, these authors have also considered the case where  $\varphi = 0$  but  $\psi$  is arbitrary (crack lying along the bisecting line of an arbitrary notch), using previous works of Gross (1970), Hasebe and Iida (1978) and Isida (1979); they have derived an approximate but accurate formula for  $K_I(a)$  ( $K_{II}(a)$  being zero) in that case.

Treating the problem in its full generality (arbitrary geometry of the body, the notch and the small crack, arbitrary loading) obviously requires a radically new approach. That which will be used here, which is basically similar to that employed by Leblond (1989) and Amestoy and Leblond (1992) to study crack kinking in an arbitrarily shaped body subjected to an arbitrary loading, consists of two steps. The present Part I is devoted to Step 1. It consists of establishing the *general form* (that is, not yet fully explicit because it involves some unknown functions) of the asymptotic expression of the SIFs  $K_p(a)$  for a small crack length  $a$ . The method is based on two ingredients : first, changes of scale and homogeneity properties of the equations of elasticity (invariance of these equations upon homothetical transformations), and second, continuity properties of the mechanical fields (displacement, stresses) at a given, fixed point of the body with respect to the crack length  $a$  for  $a = 0$ . The SIFs at the tip of the small crack are shown to depend on the ‘stress intensity factor’  $\kappa$  of the notch (that is, the multiplicative factor of the singular stress field near the notch root, in the absence of the crack), the length  $a$  of the crack, the aperture angle  $2(\pi - \psi)$  of the notch and the angle  $\varphi$  between its bisecting line  $Ox$  and the initial direction of the crack. The formulae derived are of *universal* value, that is, they apply to any situation, whatever the geometry of the body, the notch and the crack and whatever the loading, the SIFs at the tip of the small crack depending always solely upon the mechanical and geometrical parameters just mentioned.

The formulae obtained involve some (universal but) unknown functions of the angles  $\psi$  and  $\varphi$ . Although the approach used does not provide the values of these functions, it does yield a method for their determination through solution of a plane strain elasticity problem corresponding to some special case. Not surprisingly, this special case involves an *infinite* body containing a notch with arbitrary aperture angle  $2(\pi - \psi)$  but *straight* ( $C_1 = C_2 = 0$ ) boundaries and a *straight* ( $C = 0$ ) crack of *unit* ( $a = 1$ ) length making an arbitrary angle  $\varphi$  with the bisecting line  $Ox$  of the notch.

Step 2 will be envisaged in Part 2. It will consist of explicitly solving the elasticity problem just mentioned, using Muskhelishvili’s (1952) complex potentials method. The determination of the kink angle  $\chi$  of the future extension of the small crack, as a function of the angles  $\psi$  and  $\varphi$ , will be envisaged in conclusion.

## 2. The stress field near the notch root in the absence of a crack

The problem of finding the asymptotic form of the stress field near the apex of a notch with aperture angle  $2(\pi - \psi)$  ( $\pi/2 \leq \psi \leq \pi$ ) is a very classical one, which was solved for the first time by Williams (1952). He showed that for  $\psi < \pi$ , the stress field is of the form, in polar coordinates  $r, \theta$ :

$$\begin{cases} \sigma_{rr}(r, \theta) = -\kappa \frac{\alpha}{2} (2\pi r)^{\alpha-1} \left[ \frac{(3-\alpha) \sin((1+\alpha)\psi)}{(1-\alpha) \sin((1-\alpha)\psi)} \cos((1-\alpha)\theta) + \cos((1+\alpha)\theta) \right], \\ \sigma_{\theta\theta}(r, \theta) = \kappa \frac{\alpha}{2} (2\pi r)^{\alpha-1} \left[ -\frac{(1+\alpha) \sin((1+\alpha)\psi)}{(1-\alpha) \sin((1-\alpha)\psi)} \cos((1-\alpha)\theta) + \cos((1+\alpha)\theta) \right], \\ \sigma_{r\theta}(r, \theta) = \kappa \frac{\alpha}{2} (2\pi r)^{\alpha-1} \left[ -\frac{\sin((1+\alpha)\psi)}{\sin((1-\alpha)\psi)} \sin((1-\alpha)\theta) + \sin((1+\alpha)\theta) \right], \end{cases} \quad (1)$$

where  $\alpha$  is the smallest positive solution (varying from 1 for  $\psi = \pi/2$  to  $1/2$  for  $\psi = \pi$ ) of the equation

$$\sin(2\alpha\psi) + \alpha \sin(2\psi) = 0 \quad (2)$$

and  $\kappa$  a constant (depending on the far geometry and loading) which we shall call the ‘mode I stress intensity factor’, or more briefly the ‘stress intensity factor’, ‘of the notch’. The reason why  $\kappa$  is said to correspond to ‘mode I’ is that the associated stress field is symmetric with respect to the bisecting line  $\theta = 0$  of the notch ( $\sigma_{rr}$  and  $\sigma_{\theta\theta}$  are even, and  $\sigma_{r\theta}$  odd, functions of  $\theta$ ). A ‘mode II stress intensity factor of the notch’ does also exist but it corresponds to some stress field which is less singular (that is, proportional to some greater power of  $r$ ) than that associated to  $\kappa$  (except for  $\psi = \pi$ : limit case of a crack) and therefore negligible in the limit  $r \rightarrow 0$ .

In fact, formulae (1) involve some degree of arbitrariness in that one can always multiply  $\kappa$  by some factor while dividing the rest of the expression of the stresses by the same factor. Choosing this factor (here as unity) means choosing a certain definition for the ‘stress intensity factor of the notch’. The definition adopted here is the ‘best’ possible one, in the sense that it possesses the nice properties that  $\kappa$  reduces to the ordinary mode I SIF  $K_I$  for  $\psi = \pi$  (case of a crack) and to the uniform tensile stress  $\sigma_{yy}$  for  $\psi = \pi/2$  (case of a regular, corner-free boundary). This is established in detail in the Appendix.

### 3. Continuity of the mechanical fields at a given point with respect to the crack length

We now wish to establish that *the displacement and stresses at some given, fixed point of the body are right-hand continuous functions of the crack length  $a$  for  $a = 0^+$* .

Let us first consider the initial situation where the prescribed displacements  $\mathbf{u}^p$  are applied on  $\partial\Omega_u$  and the prescribed tractions  $\mathbf{T}^p$  on  $\partial\Omega_T$ , in the absence of any crack emanating from the notch root  $O$ . The displacement and stresses at the generic point  $M$  of the body are then denoted  $\mathbf{u}(M)$  and  $\boldsymbol{\sigma}(M)$ . One can always suppose that a crack of length  $a$  in fact originates from the point  $O$  provided that one exerts on the lips of this crack some suitable closing tractions  $\mathbf{T}^+(s) \equiv \boldsymbol{\sigma}(s) \cdot \mathbf{n}^+(s)$ ,  $\mathbf{T}^-(s) \equiv \boldsymbol{\sigma}(s) \cdot \mathbf{n}^-(s)$ , where  $s$  denotes the curvilinear length along the crack from the point  $O$  and  $\mathbf{n}^+(s)$ ,  $\mathbf{n}^-(s)$  the unit normal vectors to the upper (+) and lower (−) lips at the point  $s$ , oriented towards the opposite lip. These tractions are  $O(s^{\alpha-1})$  like  $\boldsymbol{\sigma}(s)$ .

Let us now consider the final situation where  $\mathbf{u}^p$  is still imposed on  $\partial\Omega_u$  and  $\mathbf{T}^p$  on  $\partial\Omega_T$ , but a traction-free crack of length  $a$  emanates from the point  $O$ ; equivalently, the tractions  $\mathbf{T}^+(s)$ ,  $\mathbf{T}^-(s)$  applied on the crack lips in the initial situation are released. The displacement and stresses at the point  $M$  are then denoted  $\mathbf{u}(M, a)$  and  $\boldsymbol{\sigma}(M, a)$ . If we take the difference between the final and initial situations, taking advantage of the linearity of the equations of

elasticity, we obtain some *Problem A* where a zero displacement is imposed on  $\partial\Omega_u$  and a zero traction on  $\partial\Omega_T$  while tractions  $-\mathbf{T}^+(s)$ ,  $-\mathbf{T}^-(s)$  are exerted on the crack lips. The displacement and stresses at the point  $M$  are then  $\mathbf{u}(M, a) - \mathbf{u}(M)$  and  $\boldsymbol{\sigma}(M, a) - \boldsymbol{\sigma}(M)$ .

We further define some *Problem B* in the following way. Let us consider the same geometry as before, including a crack of length  $a$  originating from the point  $O$ . Let  $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$  denote an arbitrary orthonormal basis. We again impose a zero displacement on  $\partial\Omega_u$  and a zero traction on  $\partial\Omega_T$  and the crack lips, but now exert a unit point force parallel to  $\mathbf{e}_i$  on the point  $M$ . The resulting displacements on the upper and lower crack lips at the point  $s$  are denoted  $\mathbf{v}^{(i)+}(M, a; s)$ ,  $\mathbf{v}^{(i)-}(M, a; s)$ .

Application of Betti's reciprocity theorem to *Problems A and B* yields

$$u_i(M, a) - u_i(M) = - \int_0^a [\mathbf{T}^+(s) \cdot \mathbf{v}^{(i)+}(M, a; s) + \mathbf{T}^-(s) \cdot \mathbf{v}^{(i)-}(M, a; s)] ds . \quad (3)$$

Also, differentiating this equation with respect to the coordinates  $x_j$  of the point  $M$ , we get

$$\frac{\partial u_i}{\partial x_j}(M, a) - \frac{\partial u_i}{\partial x_j}(M) = - \int_0^a \left[ \mathbf{T}^+(s) \cdot \frac{\partial \mathbf{v}^{(i)+}}{\partial x_j}(M, a; s) + \mathbf{T}^-(s) \cdot \frac{\partial \mathbf{v}^{(i)-}}{\partial x_j}(M, a; s) \right] ds . \quad (4)$$

The quantities  $(\partial \mathbf{v}^{(i)+} / \partial x_j)(M, a; s)$ ,  $(\partial \mathbf{v}^{(i)-} / \partial x_j)(M, a; s)$  in this equation must not be interpreted as strains, since the coordinates with respect to which differentiation is performed are those of the point of application of the force, not those of the point of observation of the displacement. Rather, they should be interpreted as displacements on the upper and lower crack lips at the point  $s$  resulting from the application of a unit 'dipole' at the point  $M$  (that is, two opposite infinite forces parallel to  $\mathbf{e}_i$  applied on points separated by an infinitesimal vector parallel to  $\mathbf{e}_j$ , the product of the intensity of the forces and the distance between the points being equal to unity).

The *displacements*  $\mathbf{v}^{(i)+}(M, a; s)$ ,  $\mathbf{v}^{(i)-}(M, a; s)$  are obviously bounded by some positive constant  $A$ . Similarly,  $(\partial \mathbf{v}^{(i)+} / \partial x_j)(M, a; s)$  and  $(\partial \mathbf{v}^{(i)-} / \partial x_j)(M, a; s)$  being also displacements, as just noted, they are also bounded by some positive constant  $B$ . It then follows from Equations (3) and (4) that

$$|u_i(M, a) - u_i(M)| \leq A \int_0^a (\|\mathbf{T}^+(s)\| + \|\mathbf{T}^-(s)\|) ds ; \quad (5)$$

$$\left| \frac{\partial u_i}{\partial x_j}(M, a) - \frac{\partial u_i}{\partial x_j}(M) \right| \leq B \int_0^a (\|\mathbf{T}^+(s)\| + \|\mathbf{T}^-(s)\|) ds . \quad (6)$$

Since the tractions  $\mathbf{T}^+(s)$ ,  $\mathbf{T}^-(s)$  are  $O(s^{\alpha-1})$ , it follows from inequalities (5) and (6) that  $|u_i(M, a) - u_i(M)|$  and  $|(\partial u_i / \partial x_j)(M, a) - (\partial u_i / \partial x_j)(M)|$  are  $O(a^\alpha)$ . Since  $\alpha$  is positive (see Section II above), this implies that the displacement and its gradient (and hence the stresses) at any given, fixed point  $M$  are continuous functions of  $a$  for  $a = 0^+$ , q.e.d.

#### 4. Asymptotic form of the stress intensity factors at the tip of a small crack

We shall now examine the asymptotic form of the SIFs  $K_p(a)$  at the tip of the crack originating from the notch root  $O$  for  $a \rightarrow 0$ .

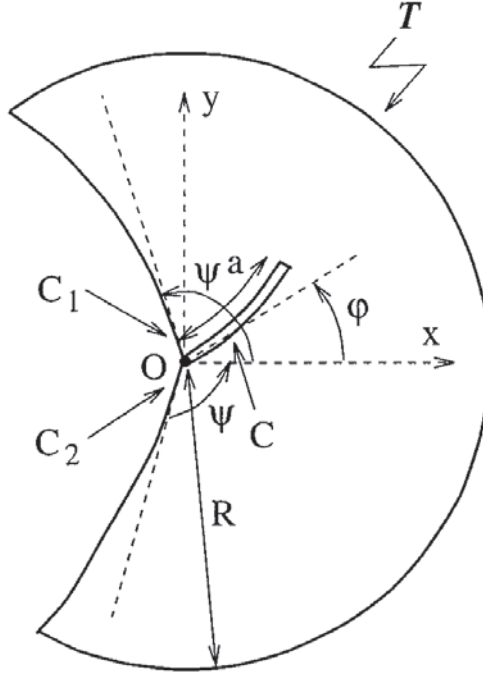


Figure 2. The special case of a circular disk.

We first consider the special case where the body considered is a circular disk of center  $O$ , radius  $R$ , endowed with a notch with apex at  $O$  and a crack originating from the same point, and subjected to some boundary traction field  $\mathcal{T}$  (see Fig. 2). The geometric parameters of the problem are the radius  $R$ , the aperture angle  $2(\pi - \psi)$  of the notch, the angle  $\varphi$  between its bisecting line  $Ox$  and the initial tangent to the crack, the curvatures  $C_1, C_2$  of the upper and lower boundaries of the notch, the curvature  $C$  of the crack and its length  $a$ . Thus the vector  $\mathbf{K}(a) \equiv (K_I(a), K_{II}(a))$  of SIFs at the crack tip may be written symbolically as

$$\mathbf{K}(a) \equiv \mathbb{L}(R, \psi, \varphi, C_1, C_2, C, a) \cdot \mathcal{T} , \quad (7)$$

where  $\mathbb{L}$  is a (vectorial) linear functional, depending on the geometric parameters  $R, \psi, \varphi, C_1, C_2, C, a$ .

It is a classical property of the equations of elasticity that one can obtain a new solution from an old one by multiplying both the distances and displacements by some arbitrary positive factor  $\lambda$  while keeping the strains and stresses (and therefore the surface tractions) unchanged. In this operation, the parameters  $R, \psi, \varphi, C_1, C_2, C, a$  become  $\lambda R, \psi, \varphi, C_1/\lambda, C_2/\lambda, C/\lambda, \lambda a$ . Furthermore, since the SIFs are limits of certain stress components times the square root of some vanishingly small distance, they are multiplied by  $\sqrt{\lambda}$  in the process. Thus the functional  $\mathbb{L}$  satisfies the following ‘positive homogeneity’ property :

$$\begin{aligned} \mathbb{L}(\lambda R, \psi, \varphi, C_1/\lambda, C_2/\lambda, C/\lambda, \lambda a) \cdot \mathcal{T} &= \sqrt{\lambda} \mathbb{L}(R, \psi, \varphi, C_1, C_2, C, a) \cdot \mathcal{T} \Rightarrow \\ \mathbb{L}(\lambda R, \psi, \varphi, C_1/\lambda, C_2/\lambda, C/\lambda, \lambda a) &= \sqrt{\lambda} \mathbb{L}(R, \psi, \varphi, C_1, C_2, C, a) \quad (\forall \lambda > 0) . \end{aligned} \quad (8)$$

We now introduce the hypothesis that  $\mathbf{K}(a)$  behaves like a certain power of the crack length  $a$  for  $a \rightarrow 0$ ; this implies that

$$\mathbb{L}(R, \psi, \varphi, C_1, C_2, C, a) \equiv \mathbb{L}^{(\beta)}(R, \psi, \varphi, C_1, C_2, C) a^\beta + o(a^\beta) , \quad (9)$$



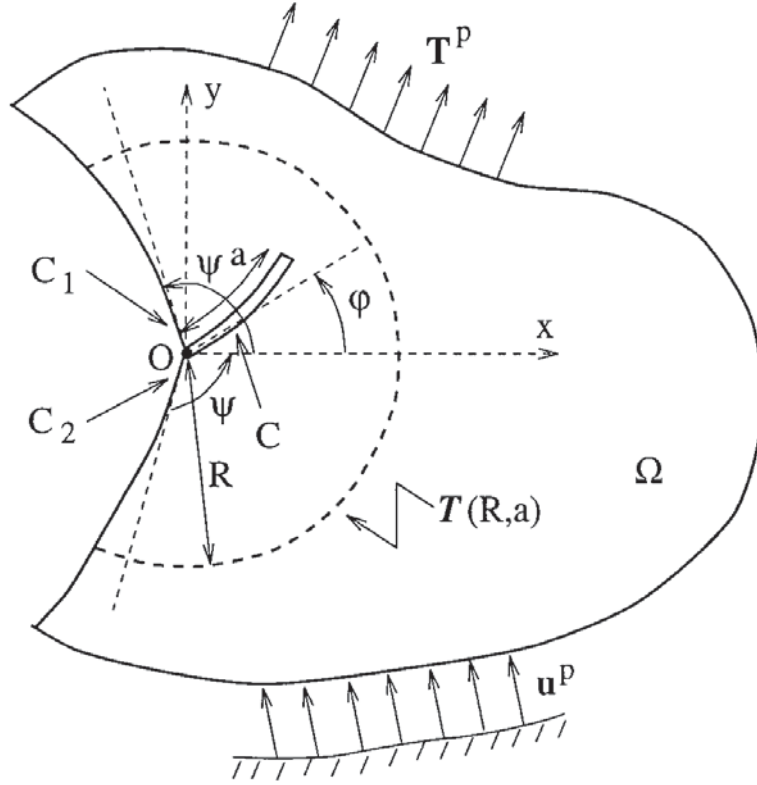


Figure 3. Circular disk within an arbitrary body.

where  $\beta$  is an unknown exponent to be determined and  $\mathbb{L}^{(\beta)}(R, \psi, \varphi, C_1, C_2, C)$  some new linear functional<sup>1</sup>. Inserting Equation (9) into Equation (8), one gets

$$\begin{aligned} \mathbb{L}^{(\beta)}(\lambda R, \psi, \varphi, C_1/\lambda, C_2/\lambda, C/\lambda)(\lambda a)^\beta + o(a^\beta) &= \\ &= \sqrt{\lambda} [\mathbb{L}^{(\beta)}(R, \psi, \varphi, C_1, C_2, C)a^\beta + o(a^\beta)] ; \end{aligned}$$

identifying terms proportional to  $a^\beta$  in this equation, one gets the following ‘positive homogeneity’ property for the functional  $\mathbb{L}^{(\beta)}$ :

$$\mathbb{L}^{(\beta)}(\lambda R, \psi, \varphi, C_1/\lambda, C_2/\lambda, C/\lambda) = \lambda^{\frac{1}{2}-\beta} \mathbb{L}^{(\beta)}(R, \psi, \varphi, C_1, C_2, C) \quad (\forall \lambda > 0). \quad (10)$$

We now come back to the general case where the shape of the body  $\Omega$  is arbitrary. We consider, within  $\Omega$ , circular disks of center  $O$  and radius  $R$  (intended to be shrunk to zero at the end, but fixed for the moment), containing the entire crack ( $R > a$ ): see Fig. 3. We denote by  $\mathcal{T}(R, a)$  the traction field exerted on the boundary of the disk of radius  $R$ , when the crack length is  $a$ , as a result of the application of the prescribed loading ( $\mathbf{u}^p, \mathbf{T}^p$ ) on  $\partial\Omega_u$  and  $\partial\Omega_T$ . Clearly, the SIFs  $K_p(a)$  are unchanged if one eliminates the exterior of the disk of radius  $R$  while preserving the traction field  $\mathcal{T}(R, a)$  exerted on its boundary; thus Equation (7) holds for that loading :

$$\mathbf{K}(a) \equiv \mathbb{L}(R, \psi, \varphi, C_1, C_2, C, a) \cdot \mathcal{T}(R, a). \quad (11)$$

Now we take the limit  $a \rightarrow 0$  in the previous equation. Then,  $R$  being fixed,  $\mathcal{T}(R, a)$  tends towards the traction field  $\mathcal{T}(R)$  exerted on the boundary of the disk *prior to initiation of the crack*, because of the property of continuity of the stresses at a fixed point with respect to  $a$  for  $a = 0^+$  established in the previous section. Thus, by Equation (9),

<sup>1</sup>The notation  $o(x^\gamma)$  is recalled to stand for a function of the form  $x^\gamma f(x)$ , with  $\lim_{x \rightarrow 0} f(x) = 0$ .



$$\mathbf{K}(a) \equiv \mathbf{K}^{(\beta)} a^\beta + o(a^\beta), \quad (12)$$

where

$$\mathbf{K}^{(\beta)} \equiv \mathbb{L}^{(\beta)}(R, \psi, \varphi, C_1, C_2, C) \cdot \mathcal{T}(R). \quad (13)$$

This equation shows that  $\mathbf{K}^{(\beta)}$  possesses the remarkable property of *depending on the mechanical fields only through their value prior to initiation of the crack*.

We shall finally show that they depend in fact only on the *asymptotic* value of the initial mechanical fields. In order to do so, we shall now let  $R \rightarrow 0$ . Prior to doing that, however, we rewrite Equation (13), using Equation (10) with  $\lambda = 1/R$ , as

$$\mathbf{K}^{(\beta)} \equiv R^{\frac{1}{2}-\beta} \mathbb{L}^{(\beta)}(1, \psi, \varphi, RC_1, RC_2, RC) \cdot \mathcal{T}(R). \quad (14)$$

Furthermore, we note that in view of Equations (1), the stresses prior to initiation of the crack are of the form

$$\boldsymbol{\sigma}(r, \theta) \equiv \kappa r^{\alpha-1} \mathbf{f}(\psi; \theta) + o(r^{\alpha-1})$$

where  $r$  and  $\theta$  denote polar coordinates,  $\kappa$  the ‘stress intensity factor of the notch’ in the absence of the crack and  $\alpha$  the ‘exponent of the displacements’ as in Section 2, and  $\mathbf{f}(\psi; \theta)$  some universal (depending only on  $\psi$  and  $\theta$ ) symmetric second-rank tensor. It follows that the traction field  $\mathcal{T}(R)$  exerted on the boundary of the disk of radius  $R$  prior to crack initiation is of the form

$$\mathcal{T}(R) \equiv \kappa R^{\alpha-1} \{\mathbf{f}(\psi; \theta) \cdot \mathbf{e}_r(\theta)\} + o(R^{\alpha-1}) \quad (15)$$

where  $\mathbf{e}_r(\theta)$  denotes the unit radial vector and  $\{\mathbf{T}(\theta)\}$  the surface traction field defined by the traction  $\mathbf{T}(\theta)$ . Inserting Equation (15) into Equation (14) and using linearity, we get

$$\mathbf{K}^{(\beta)} \equiv \kappa R^{\alpha-\frac{1}{2}-\beta} \mathbb{L}^{(\beta)}(1, \psi, \varphi, 0, 0, 0) \cdot \{\mathbf{f}(\psi; \theta) \cdot \mathbf{e}_r(\theta)\} + o(R^{\alpha-\frac{1}{2}-\beta}). \quad (16)$$

Now the left-hand side in this formula is in fact independent of  $R$  by definition (it is connected to the SIFs at the crack tip, which are governed only by the geometry of the body and the loading applied, not by the radius of the fictitious disk envisaged); it follows that the exponent  $\alpha - \frac{1}{2} - \beta$  in the right-hand side must necessarily be zero, which implies that

$$\beta = \alpha - \frac{1}{2}. \quad (17)$$

Since  $\alpha > 1/2$  except for  $\psi = \pi$  (case where the notch is in fact a crack),  $\beta > 0$  except in that case, which means, by Equation (12), that *the SIFs  $K_p(a)$  tend to zero with  $a$  except for  $a = \pi$* . Equation (16) may be rewritten, using Equation (17), as

$$\mathbf{K}^{(\beta)} = \kappa \mathbb{L}^{(\beta)}(1, \psi, \varphi, 0, 0, 0) \cdot \{\mathbf{f}(\psi; \theta) \cdot \mathbf{e}_r(\theta)\} + o(R^0).$$

The left-hand side and the first term of the right-hand side here are independent of  $R$ . Hence the second term of the right-hand side must also be independent of  $R$ . Since it tends to zero with this parameter, it must be zero. Hence we finally get

$$\mathbf{K}^{(\beta)} = \kappa \mathbb{L}^{(\beta)}(1, \psi, \varphi, 0, 0, 0) \cdot \{\mathbf{f}(\psi; \theta) \cdot \mathbf{e}_r(\theta)\}, \quad (18)$$

that is,

$$\mathbf{K}^{(\beta)} \equiv \mathbf{L}(\psi, \varphi)\kappa, \quad (19)$$

where  $\mathbf{L}(\psi, \varphi)$  is a *universal* vectorial function of the angles  $\psi$  and  $\varphi$ . The term *universal* means that Equation (19) holds for all envisageable geometries and loadings,  $\mathbf{K}^{(\beta)}$  being always given by the product of  $\mathbf{L}(\psi, \varphi)$  and  $\kappa$  where the function  $\mathbf{L}$  depends solely on the arguments  $\psi$  and  $\varphi$  (and is for instance completely independent of the curvatures  $C_1, C_2, C$  of the notch boundaries and the crack).

## 5. Practical calculation of the universal functions $L_p(\psi, \varphi)$

The reasoning expounded in the preceding section has evidenced the *existence* of the universal vectorial function  $\mathbf{L}(\psi, \varphi)$  but has not provided the *values of its components*  $L_p(\psi, \varphi)$ . We shall now see how this reasoning can be pursued to yield a practical method for effective determination of these components.

We start from Equations (18) and (19), which imply that

$$\mathbf{L}(\psi, \varphi) = \mathbb{L}^{(\beta)}(1, \psi, \varphi, 0, 0, 0). \{\mathbf{f}(\psi; \theta). \mathbf{e}_r(\theta)\}.$$

Now, by the very definition (9) of the functional  $\mathbb{L}^{(\beta)}$ , the preceding expression becomes

$$\begin{aligned} \mathbf{L}(\psi, \varphi) &= \lim_{a \rightarrow 0} a^{-\beta} \mathbb{L}(1, \psi, \varphi, 0, 0, 0, a). \{\mathbf{f}(\psi; \theta). \mathbf{e}_r(\theta)\} = \\ &= \lim_{a \rightarrow 0} a^{\frac{1}{2}-\beta} \mathbb{L}\left(\frac{1}{a}, \psi, \varphi, 0, 0, 0, 1\right). \{\mathbf{f}(\psi; \theta). \mathbf{e}_r(\theta)\} \end{aligned}$$

(by Equation (8) with  $\lambda = a$ )

$$= \lim_{\mathcal{R} \rightarrow +\infty} \mathcal{R}^{\alpha-1} \mathbb{L}(\mathcal{R}, \psi, \varphi, 0, 0, 0, 1). \{\mathbf{f}(\psi; \theta). \mathbf{e}_r(\theta)\}$$

(with  $\mathcal{R} \equiv 1/a$ , and account being taken of Equation (17)). Using linearity, one therefore finally gets, for  $p = I, II$  :

$$L_p(\psi, \varphi) = \lim_{\mathcal{R} \rightarrow +\infty} \mathbb{L}_p(\mathcal{R}, \psi, \varphi, C_1 = 0, C_2 = 0, C = 0, a = 1). \{\mathcal{R}^{\alpha-1} \mathbf{f}(\psi; \theta). \mathbf{e}_r(\theta)\}. \quad (20)$$

What Equation (20) says is the following. Consider a circular disk of center  $O$ , radius  $\mathcal{R}$ , endowed with a notch with apex at  $O$ , *straight* ( $C_1 = C_2 = 0$ ) boundaries and aperture angle  $2(\pi - \psi)$ , and a *straight* ( $C = 0$ ) crack originating from  $O$  of *unit* ( $a = 1$ ) length, making an angle  $\varphi$  with the bisecting line  $Ox$  of the notch (Fig. 4). Enforce the ‘fundamental’ traction field  $\{\mathcal{R}^{\alpha-1} \mathbf{f}(\psi; \theta). \mathbf{e}_r(\theta)\}$  on the boundary of that disk, the boundaries of the notch and the crack lips remaining traction-free. Then  $L_p(\psi, \varphi)$  is *identical to the  $p$ -th SIF at the tip of the crack, in the limit  $\mathcal{R} \rightarrow +\infty$* . It is equivalent to saying that one must consider an infinite body with the following conditions on the notch boundaries and the crack lips and at infinity :

$$\left\{ \begin{array}{l} \sigma(r, \theta = \pm\psi). \mathbf{e}_\theta(\theta = \pm\psi) = \mathbf{0}, \forall r > 0 \text{ (conditions on the notch boundaries),} \\ \sigma(r, \theta = \varphi^\pm). \mathbf{e}_\theta(\theta = \varphi) = \mathbf{0}, \forall r, 0 < r < 1 \text{ (conditions on the crack lips),} \\ \sigma(r, \theta) = \Sigma(r, \theta) + o(r^{\alpha-1}) \text{ for } r \rightarrow +\infty, \forall \theta, -\psi < \theta < +\psi \text{ (condition at infinity),} \end{array} \right. \quad (21)$$

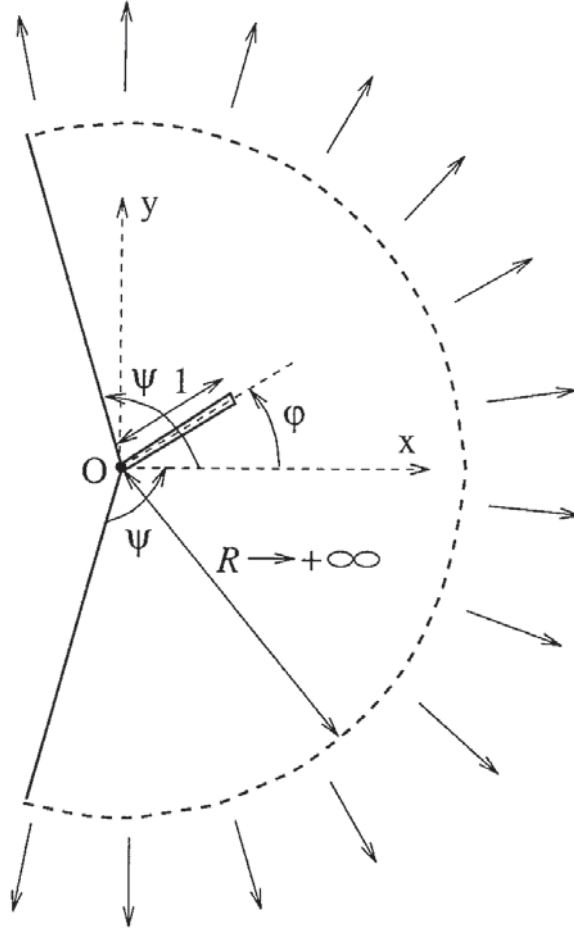


Figure 4. The geometry to be considered to determine the functions  $L_p(\psi, \varphi)$ .

where  $\mathbf{e}_\theta(\theta)$  denotes the unit orthoradial vector and  $\{\Sigma(r, \theta)\} \equiv \{r^{\alpha-1}\mathbf{f}(\psi; \theta)\}$  the ‘fundamental’ stress field defined by Equations (1) except for the omission of the factor  $\kappa$ .

Using a classical LEFM trick, one can remove the imposed stresses from infinity to the crack lips. The conditions on the notch boundaries and the crack lips and at infinity then become, instead of (21) :

$$\left\{ \begin{array}{l}
 \sigma(r, \theta = \pm\psi) \cdot \mathbf{e}_\theta(\theta = \pm\psi) = \mathbf{0}, \forall r > 0 \text{ (conditions on the notch boundaries),} \\
 \sigma(r, \theta = \varphi^+) \cdot (-\mathbf{e}_\theta(\theta = \varphi)) = -\Sigma(r, \theta = \varphi) \cdot (-\mathbf{e}_\theta(\theta = \varphi)) = \Sigma(r, \theta = \varphi) \cdot \mathbf{e}_\theta(\theta = \varphi) \\
 \forall r, 0 < r < 1 \text{ (condition on the upper lip of the crack),} \\
 \sigma(r, \theta = \varphi^-) \cdot \mathbf{e}_\theta(\theta = \varphi) = -\Sigma(r, \theta = \varphi) \cdot \mathbf{e}_\theta(\theta = \varphi) \\
 \forall r, 0 < r < 1 \text{ (condition on the lower lip of the crack),} \\
 \sigma(r, \theta) = o(r^{\alpha-1}) \text{ for } r \rightarrow +\infty, \forall \theta, -\psi < \theta < +\psi \text{ (condition at infinity) .}
 \end{array} \right. \quad (22)$$

One will in fact need a more precise information on the behavior of the stresses at infinity than Equation (22)<sub>4</sub>. For  $r \rightarrow +\infty$ , the influence of the crack vanishes so that the stresses behave as if it were absent. Hence  $\sigma(r, \theta)$  must behave as  $r^{\alpha'-1}$  where  $\alpha'$  is an exponent ‘adapted’ to

the notched, crack-free geometry, and therefore satisfying one of the equations

$$\sin(2\alpha'\psi) + \alpha' \sin(2\psi) = 0, \quad \sin(2\alpha'\psi) - \alpha' \sin(2\psi) = 0$$

(see the Appendix); since  $\sigma(r, \theta)$  is  $o(r^{\alpha-1})$  for  $r \rightarrow +\infty$ ,  $\alpha'$  must in fact be the greatest solution of one of these equations smaller than  $\alpha$ . Since  $\alpha$  is the smallest positive solution of one of these equations,  $\alpha'$  cannot be positive. Hence it is the greatest negative<sup>1</sup> solution. It follows that it must be equal to  $-\alpha$ . Thus Equation (22)<sub>4</sub> may be refined in the following way :

$$\sigma(r, \theta) = O(r^{-\alpha-1}) \text{ for } r \rightarrow +\infty, \forall \theta, -\psi < \theta < +\psi. \quad (23)$$

Part II will be devoted to the effective calculation of the functions  $L_p(\psi, \varphi)$  through solution of the plane elasticity problem defined by Equations (22), (23).

### Appendix: Definition of the stress intensity factor of a notch

The aim of this Appendix is to provide some justification for the definition of the ‘stress intensity factor of a notch’  $\kappa$  implicitly implied in Equations (1) of the text.

It is first necessary to briefly recall some elements of Williams’ (1952) classical treatment of the problem, based on the search of the form of the Airy stress function  $\phi(r, \theta)$ . This function is looked for in the form  $\phi(r, \theta) \equiv r^{\alpha+1}\chi(\theta)$  where  $\alpha$  is the a priori unknown ‘exponent of the displacements’, that is, the exponent of  $r$  in their asymptotic expression for  $r \rightarrow 0$ . Beltrami’s equation  $\Delta\Delta\phi(r, \theta) = 0$  yields the differential equation  $\chi^{iv}(\theta) + ((\alpha - 1)^2 + (\alpha + 1)^2)\chi''(\theta) + (\alpha - 1)^2(\alpha + 1)^2\chi(\theta) = 0$ , the general solution of which is

$$\chi(\theta) = A \cos((\alpha - 1)\theta) + B \sin((\alpha - 1)\theta) + C \cos((\alpha + 1)\theta) + D \sin((\alpha + 1)\theta)$$

where  $A, B, C, D$  are arbitrary constants. The boundary conditions  $\sigma_{\theta\theta}(r, \pm\psi) = \phi_{,rr}(r, \pm\psi) = 0$ ,  $\sigma_{r\theta}(r, \pm\psi) = -(\phi_{,\theta}/r)_{,r}(r, \pm\psi) = 0$  then yield the two systems of equations

$$\begin{cases} A \cos((\alpha - 1)\psi) + C \cos((\alpha + 1)\psi) = 0, \\ (\alpha - 1)A \sin((\alpha - 1)\psi) + (\alpha + 1)C \sin((\alpha + 1)\psi) = 0 \end{cases}$$

and

$$\begin{cases} B \sin((\alpha - 1)\psi) + D \sin((\alpha + 1)\psi) = 0, \\ (\alpha - 1)B \cos((\alpha - 1)\psi) + (\alpha + 1)D \cos((\alpha + 1)\psi) = 0. \end{cases}$$

The condition for the first system to possess a non-trivial (non-zero) solution ( $A, C$ ), that is, for its determinant to be zero, reads

$$\sin(2\alpha\psi) + \alpha \sin(2\psi) = 0$$

while the analogous condition for the second system is

$$\sin(2\alpha\psi) - \alpha \sin(2\psi) = 0.$$

<sup>1</sup>The solution  $\alpha' = 0$  can be ruled out because it only represents some rigid-body (translatory) motion.

It is easily verified that for  $\psi < \pi$ , the *smallest* positive<sup>1</sup> value of  $\alpha$  satisfying one of these equations, corresponding to the most singular stress field, is in fact a solution of the *first* equation. Provided that  $\alpha$  takes this particular value, the coefficient  $A$  is arbitrary but  $C$  is tied to it through the relation

$$C = -\frac{(1 - \alpha) \sin((1 - \alpha)\psi)}{(1 + \alpha) \sin((1 + \alpha)\psi)} A$$

while the coefficients  $B$  and  $D$  are necessarily zero. It follows that the Airy stress function is of the form

$$\phi(r, \theta) = Ar^{\alpha+1} \left[ \cos((1 - \alpha)\theta) - \frac{(1 - \alpha) \sin((1 - \alpha)\psi)}{(1 + \alpha) \sin((1 + \alpha)\psi)} \cos((1 + \alpha)\theta) \right].$$

Now define

$$\kappa \equiv -2(2\pi)^{1-\alpha} \frac{(1 - \alpha) \sin((1 - \alpha)\psi)}{\sin((1 + \alpha)\psi)} A;$$

the expression of the stress function becomes

$$\phi(r, \theta) = \frac{\kappa}{2} (2\pi)^{\alpha-1} r^{\alpha+1} \left[ -\frac{\sin((1 + \alpha)\psi)}{(1 - \alpha) \sin((1 - \alpha)\psi)} \cos((1 - \alpha)\theta) + \frac{1}{1 + \alpha} \cos((1 + \alpha)\theta) \right]$$

and it follows that the stresses are given by

$$\begin{aligned} \sigma_{rr}(r, \theta) &= \left( \frac{\phi_{,\theta\theta}}{r^2} + \frac{\phi_{,r}}{r} \right) (r, \theta) = \\ &= -\kappa \frac{\alpha}{2} (2\pi r)^{\alpha-1} \left[ \frac{(3 - \alpha) \sin((1 + \alpha)\psi)}{(1 - \alpha) \sin((1 - \alpha)\psi)} \cos((1 - \alpha)\theta) + \cos((1 + \alpha)\theta) \right]; \end{aligned}$$

$$\begin{aligned} \sigma_{\theta\theta}(r, \theta) &= \phi_{,rr}(r, \theta) = \\ &= \kappa \frac{\alpha}{2} (2\pi r)^{\alpha-1} \left[ -\frac{(1 + \alpha) \sin((1 + \alpha)\psi)}{(1 - \alpha) \sin((1 - \alpha)\psi)} \cos((1 - \alpha)\theta) + \cos((1 + \alpha)\theta) \right]; \end{aligned}$$

$$\begin{aligned} \sigma_{r\theta}(r, \theta) &= -\left( \frac{\phi_{,\theta}}{r} \right)_{,r} (r, \theta) = \\ &= \kappa \frac{\alpha}{2} (2\pi r)^{\alpha-1} \left[ -\frac{\sin((1 + \alpha)\psi)}{\sin((1 - \alpha)\psi)} \sin((1 - \alpha)\theta) + \sin((1 + \alpha)\theta) \right], \end{aligned}$$

that is, Equations (1) of the text.

To check that our definition of the ‘stress intensity factor of the notch’  $\kappa$  is the best possible one, let us now consider the special situations where  $\psi = \pi$  (case of a crack) and  $\psi = \pi/2$  (case of a corner-free boundary).

If  $\psi = \pi$ ,  $\alpha = 1/2$  so that  $\sin((1 + \alpha)\psi) = -1$ ,  $\sin((1 - \alpha)\psi) = 1$ , and the expressions of the stresses become

<sup>1</sup>The ‘displacement exponent’  $\alpha$  must be positive because the ‘stress exponent’  $\alpha - 1$  must be greater than  $-1$  in order for the total elastic energy to be finite.

$$\begin{cases} \sigma_{rr}(r, \theta) = \frac{\kappa}{4\sqrt{2\pi r}} \left( 5 \cos \frac{\theta}{2} - \cos \frac{3\theta}{2} \right) \\ \sigma_{\theta\theta}(r, \theta) = \frac{\kappa}{4\sqrt{2\pi r}} \left( 3 \cos \frac{\theta}{2} + \cos \frac{3\theta}{2} \right) \\ \sigma_{r\theta}(r, \theta) = \frac{\kappa}{4\sqrt{2\pi r}} \left( \sin \frac{\theta}{2} + \sin \frac{3\theta}{2} \right) ; \end{cases}$$

they are identical to the classical formulae for a crack loaded in mode I, with  $\kappa \equiv K_I$ .

If  $\psi = \pi/2$ ,  $\alpha = 1$  so that  $\sin((1 - \alpha)\psi) = \sin((1 + \alpha)\psi) = 0$ , and we get the indeterminate form  $\frac{\sin((1 + \alpha)\psi)}{(1 - \alpha) \sin((1 - \alpha)\psi)}$  in the expression of the stresses. To find the value of this indeterminate form, it suffices to note that since the value of  $\alpha$  is such that the determinant of the system in  $A$  and  $C$  be zero,

$$(1 + \alpha) \cos((1 - \alpha)\psi) \sin((1 + \alpha)\psi) - (1 - \alpha) \sin((1 - \alpha)\psi) \cos((1 + \alpha)\psi) = 0$$

so that

$$\frac{\sin((1 + \alpha)\psi)}{(1 - \alpha) \sin((1 - \alpha)\psi)} = \frac{\cos((1 + \alpha)\psi)}{(1 + \alpha) \cos((1 - \alpha)\psi)} \rightarrow \frac{\cos \pi}{2 \cos 0} = -\frac{1}{2}$$

for  $\psi \rightarrow \pi/2$ . From there, one immediately derives the expressions of the stresses :

$$\begin{cases} \sigma_{rr}(r, \theta) = \frac{\kappa}{2} (1 - \cos(2\theta)) = \kappa \sin^2 \theta \\ \sigma_{\theta\theta}(r, \theta) = \frac{\kappa}{2} (1 + \cos(2\theta)) = \kappa \cos^2 \theta \\ \sigma_{r\theta}(r, \theta) = \frac{\kappa}{2} \sin(2\theta) = \kappa \sin \theta \cos \theta, \end{cases}$$

which correspond to some uniform stress field  $\sigma_{yy} \equiv \kappa$ .

We have thus shown that with our definition of  $\kappa$ , this parameter possesses the nice properties of reducing to  $K_I$  for  $\psi = \pi$  (case of a crack) and to  $\sigma_{yy}$  for  $\psi = \pi/2$  (case of a regular boundary), which justifies our choice.

## References

- Aliabadi, M.H., Rooke, D.P. and Cartwright, D.J. (1987). Mixed-mode Bueckner weight functions using boundary element analysis. *International Journal of Fracture* **34**, 131–147.
- Amestoy, M., Bui, H.D. and Dang Van, K. (1979). Déviation infinitésimale d'une fissure dans une direction arbitraire. *Comptes-Rendus de l'Académie des Sciences de Paris* **289B**, 99–102.
- Amestoy, M. and Leblond, J.-B. (1992). Crack paths in plane situations - II. Detailed form of the expansion of the stress intensity factors. *International Journal of Solids and Structures* **29**, 465–501.
- Bilby, B.A. and Cardew, G.E. (1975). The crack with a kinked tip. *International Journal of Fracture* **11**, 708–712.
- Fett, T. and Munz, D. (1997). *Stress Intensity Factors and Weight Functions*, Computational Mechanics Publications, Southampton, UK.
- Goldstein, R.V. and Salganik, R.L. (1974). Brittle fracture of solids with arbitrary cracks. *International Journal of Fracture* **10**, 507–523.
- Gross, B. (1970). *Some Plane Problem Elastostatic Solutions for Plates Having a V-Notch*, Ph.D. Thesis, Case Western Reserve University, USA.

- Hasebe, N. and Iida, J. (1978). A crack originating from a triangular notch on rim of a semi-infinite plate. *Engineering Fracture Mechanics* **10**, 773–782.
- Isida, M. (1979). Tension of a half-plane containing array cracks, branched cracks and cracks emanating from sharp notches. *Transactions of the Japan Society of Mechanical Engineers* **45**, 306–317.
- Leblond, J.-B. (1989). Crack paths in two-dimensional situations - I. General form of the expansion of the stress intensity factors. *International Journal of Solids and Structures* **25**, 1311–1325.
- Muskhelishvili, N.I. (1952). *Some Basic Problems of the Mathematical Theory of Elasticity*, Noordhoff, Leiden, the Netherlands.
- Sha, G.T and Yang, C.T. (1985). Weight function calculations for mixed-mode fracture problems with the virtual crack extension technique. *Engineering Fracture Mechanics* **21**, 1119–1149.
- Williams, M.L. (1952). Stress singularities resulting from various boundary conditions in angular corners of plates in extension. *Transactions of the American Society of Mechanical Engineers. Journal of Applied Mechanics* **19**, 526–528.
- Wilson, W.K. (1969). *On Combined Mode Fracture Mechanics*, Research Report 69-1E7 FMECH-R1, Westinghouse Research Laboratories, Pittsburgh, USA.
- Wu, C.H. (1978). Elasticity problems of a slender Z-crack. *Journal of Elasticity* **8**, 183–205.