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# Reciprocity principle and crack identification

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**Abstract.** In this paper we are concerned with the planar crack identification problem defined by a unique complete elastostatic overdetermined boundary datum. Based on the reciprocity gap principle, we give a direct process for locating the host plane and we establish a new constructive identifiability result for 3D planar cracks.

## 1. Introduction

Let  $\Omega$  be a smooth open domain of  $\mathbb{R}^2$  or  $\mathbb{R}^3$ , and  $\Sigma$  a curve of  $\mathbb{R}^2$  or a surface of  $\mathbb{R}^3$  smooth enough but not necessarily simply connected (cracks). We consider the following elastostatic problem (P):

$$(P) \quad \begin{cases} \operatorname{div} \sigma = 0 & \text{in } \Omega \setminus \Sigma \\ \sigma \cdot n = F_m & \text{on } \partial\Omega \\ \sigma \cdot N = 0 & \text{on } \Sigma \end{cases} \quad (1)$$

with:

- the linear elastic stress–strain law

$$\sigma = A\varepsilon(u), \quad (2)$$

- the relation between displacement and linearized strain

$$\varepsilon(u) = (\nabla u)^s, \quad (3)$$

where  $A$  is the stiffness tensor,  $\nabla$  is the gradient operator and  $(\cdot)^s$  denotes the symmetric part of the tensor  $(\cdot)$ . Equations (4) are added to ensure the existence of a solution. Since the problem involves only the ‘Neumann’ boundary conditions, the solution is known up to a rigid displacement. The conditions (5) are added to select a solution:

$$\int_{\partial\Omega} F_m = 0 \quad \text{and} \quad \int_{\partial\Omega} OM \wedge F_m = 0, \quad (4)$$

$$\int_{\partial\Omega} u = 0 \quad \text{and} \quad \int_{\partial\Omega} OM \wedge u = 0, \quad (5)$$

where  $O$  is an arbitrarily fixed point of the space. We denote by  $u$  the unique solution of the considered problem. The aim of this paper is to give results on the identifiability of

cracks  $\Sigma$  lying inside the domain  $\Omega$ , when the overspecified boundary data of the problem (P) are available on the whole exterior boundary  $\partial\Omega$  of the domain (prescribed surface force  $F_m$  and displacement response on the boundary, denoted by  $u_m$  which is supposed to be known). To our knowledge, theoretical results on the identifiability of cracks are sparse and are concerned mainly with the thermal conductivity equation in two dimensions [8, 1, 6]. For the scalar conductivity equation and in the particular case of plane cracks (or straight cracks in 2D), with given overspecified data on the whole boundary, Andrieux and Ben Abda [3] have proved a theorem on crack identifiability based on the concept of reciprocity gap. Andrieux [2] has extended the result to other types of operators (non-symmetrical). The proof of the theorem given in [3] is direct: after showing the identity between the support of the field discontinuity and the domain occupied by the crack, one identifies the temperature discontinuity via its Fourier series. This result has been numerically checked by Bannour *et al* [4] in two and three spatial dimensionals. Here we propose to study the elastostatic case. Consider a three-dimensional domain  $\Omega$ , containing one or many cracks lying on the same plane  $\Pi$  of  $\mathbb{R}^3$ ,  $\Sigma \subset \Pi$ . The space is referred to the direct orthonormal frame of reference  $(O, e_1, e_2, e_3)$ , with  $x_1, x_2, x_3$  denoting the associated coordinates. The plane  $\Pi$  is defined by the equation:  $N \cdot x + c = 0$ , where  $N$  is the unit normal  $\Pi$ . In section 2, we shall give explicit formulae for localizing the plane  $\Pi$ . Section 3 is devoted to the complete identification of the crack  $\Sigma$ .

## 2. Localization of the crack plane

Let  $V$  be the set of regular fields, such that the associated stress fields, via the elastic law, are in equilibrium inside  $\Omega$ :

$$V = \{v \in [H^1(\Omega)]^3, \quad \text{div } A\varepsilon(v) = 0 \quad \text{in } \Omega\}. \quad (6)$$

The reciprocity functional gap for considered problem is defined on the space  $V$  by:

$$RG(v) = \int_{\partial\Omega} \{F_m \cdot v - u_m \cdot A\varepsilon(v) \cdot n\}.$$

Remark that whenever the domain is uncracked ( $\Sigma = \emptyset$ ) the functional  $RG$  is identically zero on  $V$ , which is nothing but the reciprocity theorem of Maxwell–Betti in elastostatics. This remark justifies the name ‘reciprocity gap functional’. The following lemma expresses the value of  $RG$ , defined with the boundary data by an integral over the crack surfaces.

**Lemma 1.** *For any  $v$  of  $V$ ,*

$$RG(v) = \int_{\Sigma} [[u]] \cdot A\varepsilon(v) \cdot N, \quad (7)$$

where  $[[u]]$  is the jump of  $u$  across  $\Sigma$ .

**Proof.** The result is obtained by applying Green’s formula. □

**Remarks.** Denoting by  $[[u]]$  the extension of  $u$  to the whole plane  $\Pi$ , by letting it equal zero outside  $\Sigma$ , we obtain immediately:

$$RG(v) = \int_{\Pi \cap \Omega} [[u]] \cdot A\varepsilon(v) \cdot \vec{N}.$$

Let us consider the particular case of isotropic elasticity, with  $E$  and  $\nu$  being the Young modulus and the Poisson ratio respectively. We introduce the following family of displacement field  $(v^{ij})$ , with the parameters (or indexes)  $(ij)$ , the  $(k)$  component of which is defined by:

$$v_k^{ij} = A_{kl\ mn}^{-1} E_{mn}^{ij} x_l, \quad (8)$$

with

- $(E_{ij})$  canonical basis of second-order symmetrical tensors of  $\mathbb{R}^3$ :

$$E_{mn}^{ij} = \frac{1}{2}(\delta_m^i \delta_n^j + \delta_n^i \delta_m^j).$$

- $A^{-1}$  compliance tensor of the medium (inverse of  $A$ ).

The indices take values in the set  $\{1, 2, 3\}$  and  $\delta$  is the Kronecker delta. We use the convention of summation for repeated indexes. The stress fields  $\sigma^{ij}$  associated to  $v^{ij}$  are constant ones (zero divergence and hence belonging to the space  $V$ ):

$$\sigma^{ij} = A\varepsilon(v^{ij}) = E^{ij}. \quad (9)$$

Using lemma 1, it is easy to show that:

$$RG(v^{ij}) = \left( N \otimes \int_{\Sigma} [[u]] \right)^s ij \equiv R_{ij}. \quad (10)$$

Denoting the tensor  $R$  with components  $R_{ij}$  by:

$$R = \left( N \otimes \int_{\Sigma} [[u]] \right)^s \quad (11)$$

one can establish the following two propositions.

**Proposition 1.** *The norm of the mean value of the displacement discontinuity, across the crack, is given explicitly by:*

$$\left| \int_{\Sigma} [[u]] \right| = \sqrt{2R^2 - (\text{Tr } R)^2}, \quad (12)$$

moreover, denoting the tangential discontinuity by:  $[[u_t]] = [[u]] - [[u]] \cdot N N$ , one gets

$$\begin{cases} \left| \int_{\Sigma} [[u_t]] \right| = \sqrt{2R^2 - (\text{Tr } R)^2} \\ \int_{\Sigma} [[u]] \cdot N = \text{Tr } R. \end{cases} \quad (13)$$

We assume now that the applied loadings result in a tangential displacement discontinuity  $[[u_t]]$ , with a non-zero mean value. The selection of such a loading is still an open problem. We only observe that formula (12) allows us to check easily if the hypothesis is satisfied by a given loading system. Define the unit vector

$$U = \frac{\int_{\Sigma} [[u]]}{\left| \int_{\Sigma} [[u]] \right|}.$$

Equation (10) shows that the identification of the normal to the crack plane, by the  $RG$  functional, consists of the identification of two unit vectors  $N$  and  $U$  using the data  $\bar{R}$ :

$$\bar{R} = \frac{R}{\sqrt{2R^2 - (\text{Tr } R)^2}}.$$

**Proposition 2.** *The two non-vanishing principal values of the tensor  $\bar{R}$  are:*

$$\lambda_1 = \frac{1 + \text{Tr } \bar{R}}{2} \quad \text{and} \quad \lambda_2 = \frac{1 - \text{Tr } \bar{R}}{2}, \quad \text{Tr } \bar{R} \leq 1.$$

All possible vectors  $U$  and  $N$  can be chosen by the permutation and the change of signs in the following pairs of vectors, in the basis of eigenvectors  $(\Phi_1, \Phi_2, \Phi_3)$  of  $\bar{R}$ :

$$\left\{ \left[ \sqrt{\lambda_1}, \sqrt{-\lambda_2}, 0 \right], \left[ \sqrt{\lambda_1}, -\sqrt{-\lambda_2}, 0 \right] \right\}. \quad (14)$$

Since the normal  $N$  can be restricted without loss of generality to pointing into an *a priori* chosen hemisphere, two systems of applied loads are necessary for the selection of the normal. If two systems of loadings 1 and 2 lead to the tensors  $\bar{R}_1$  and  $\bar{R}_2$ , one then obtains:

$$N = \frac{\Phi_3^1 \wedge \Phi_3^2}{|\Phi_3^1 \wedge \Phi_3^2|}, \quad (15)$$

with the condition that these systems of loadings must be discriminative for the crack detection (in the sense that the vectors  $\Phi_3^1$  and  $\Phi_3^2$  are not parallel). Having determined the normal  $N$  by the previous proposition, let  $(T, V, N)$  be an orthonormal direct basis. With  $X_k, k = 1, 2, 3$  as the new coordinates in the basis  $(O, T, V, N)$ , the equation of the crack plane  $\Pi$  is given by  $X_3 + C = 0$ . Under the condition  $U_t \neq 0$ , we have the following proposition.

**Proposition 3 (identification of the crack plane).** *The constant  $C$  is given by:*

$$|C| = \frac{1}{\sqrt{2R^2 - 2Tr(R)^2}} \sqrt{\sum_{\alpha=T,V} RG^2(v_\alpha)}, \quad (16)$$

where the fields  $v_\alpha$  are defined by:

$$\begin{aligned} v_T^1 &= -X_1^2/2E - \nu X_2^2/2E + (2 + \nu)X_3^2/2E, \\ v_T^2 &= \nu X_1 X_2/E, \quad v_T^3 = \nu X_1 X_3/E, \end{aligned} \quad (17a)$$

$$\begin{aligned} v_V^2 &= -X_2^2/2E - \nu X_1^2/2E + (2 + \nu)X_3^2/2E, \\ v_V^1 &= \nu X_1 X_2/E, \quad v_V^3 = \nu X_1 X_3/E. \end{aligned} \quad (17b)$$

**Remark.** Results from the lemma and the above propositions are valid for edge cracks, when  $\Sigma$  intersects  $\partial\Omega$ . It must be emphasized that unilateral contact problems, with or without friction on the crack surfaces, fall without restriction in the framework of the above results. The only additional difficulty lies in the *a priori* choice of external loadings giving rise to non-negative normal crack displacement discontinuity and to tangential displacement blocked by friction.

### 3. Complete identification

In this section, we give a constructive method for proving the complete identification of the crack, once the crack plane has been determined. Again, the right tool is the linear form  $RG$ . Based on lemma 1, the identification of  $\Sigma$  is obtained by the interpretation of  $[[u]]$  as a continuous linear form of  $L^2(\Pi)$  (Riesz's representation theorem [5]). The proof consists of two steps. In the first step, we show that the support of the displacement discontinuity coincides with the domain occupied by the crack (in other words, the displacement discontinuity cannot vanish in a null open set of  $\Sigma$ ). In the second step, we construct two new families of fields  $v$  which enables us to identify the Fourier transform of the displacement discontinuity  $[[u]]$ .

**Lemma 2.** *The support of  $[[u]]$  coincides with the domain occupied by the crack:*

$$\text{supp}[[u]] = \Sigma.$$

**Proof.** Clearly one has  $\text{supp}[[u]] \subset \Sigma$ . Suppose that the inclusion is strict. The proof is based on the statement that the discontinuity must be identically equal to zero, so that there

is a contradiction with what has been assumed, that is that the loading leads to a jump of the tangential displacement with a non-vanishing mean value.

Suppose that  $\text{supp}[[u]] \subset \Sigma$ . Then there is an open subset  $I$  in  $\Sigma$  where:

$$\begin{cases} [[u]] = 0 \\ \sigma \cdot N = 0. \end{cases}$$

*Step 1.* Denote by  $S_1$  the intersection of  $\Pi$  and  $\Omega$ , and consider  $u_0$ , the trace of the displacement  $u$  on  $S_1$ :

$$u_0 = u_0^S + u_0^A$$

where  $u_0^A$  is the normal component of  $u$ ,  $u_0^S$  its tangential component and  $S$  splits  $\Omega$  into two subsets denoted by  $\Omega^+$  and  $\Omega^-$ .

Consider, now, the following elastostatic problem defined on  $\Omega^+$ :

$$\begin{cases} \text{div } \sigma_1^+ = 0 & \text{in } \Omega^+ \\ \sigma_1^+ \cdot n = 0 & \text{on } \partial\Omega^+ \cap \partial\Omega \\ u_1^+ = u_0^S & \text{on } S_1 \\ \sigma_1^+ = A\varepsilon(u_1^+) \end{cases}$$

and

$$\begin{cases} \text{div } \sigma_2^+ = 0 & \text{in } \Omega^+ \\ \sigma_2^+ \cdot n = F_m & \text{on } \partial\Omega^+ \cap \partial\Omega \\ u_2^+ = u_0^A & \text{on } S_1 \\ \sigma_2^+ = A\varepsilon(u_2^+). \end{cases}$$

By the superposition principle, one has:

$$\sigma^+ = \sigma|_{\Omega^+} = \sigma_1^+ + \sigma_2^+$$

and

$$u^+ = u|_{\Omega^+} = u_1^+ + u_2^+.$$

Let  $S$  be the symmetry with respect to the plane  $\Pi$ . One extends  $\sigma_1^+$  and  $\sigma_2^+$  to  $S(\Omega^+)$  by:

$$\begin{aligned} u_1^-(SX) &= Su_1^+(X), \\ u_2^-(SX) &= -Su_2^+(X). \end{aligned}$$

Denote by  $(\tilde{\sigma}_i, \tilde{u}_i)$  the fields:

$$\tilde{\sigma}_i = \begin{cases} \sigma_i^+ & \text{in } \Omega^+ \\ \sigma_i^- & \text{in } S(\Omega^+), \end{cases}$$

where  $\sigma_i^- = Au_i^-$  and

$$\tilde{u}_i = \begin{cases} u_i^+ & \text{in } \Omega^+ \\ u_i^- & \text{in } S(\Omega^+). \end{cases}$$

Therefore  $(\tilde{\sigma}_i, \tilde{u}_i)$  satisfies the equilibrium equations in  $\Omega^+ \cup S(\Omega^+) \cup \Sigma$ .

*Step 2.* Let  $(\tilde{\sigma}, \tilde{u})$  be equal to  $(\tilde{\sigma}_1 + \tilde{\sigma}_2, \tilde{u}_1 + \tilde{u}_2)$  by the Almansi lemma (i.e. unique continuation property for the elastic system),  $(\tilde{\sigma}, \tilde{u})$  coincides with  $(\sigma, u)$  on  $(\Omega_1^+ \cup \Omega_1^- \cup \Sigma) \cap \Omega$ , therefore  $[u] = 0$  on  $\Sigma$ .  $\square$

**Theorem.** *If  $\Omega$  contains a plane crack  $\Sigma$  not necessarily connected and if the loads  $F_m$  are chosen in such a manner that  $\int_{\Sigma} [[u]] \neq 0$ , then  $\Sigma$  is determined uniquely by the boundary data  $(F_m, u_m)$ .*

**Proof.** After lemma 2, identifying  $\Sigma$  is equivalent to identifying the support of  $[[u]]$ . As in [3] the reconstruction of  $[[u]]$  is based on the linear form  $RG(v)$  (with appropriate fields  $v$  of  $V$ ). The elastic case is more complicated and one has to identify the Fourier transform of  $[[u]]$  using two families of fields of  $V$ . After a new change of frame of reference  $(O', T, V, N)$  with  $O'$  in the plane  $\Pi$ , one introduces the index  $\xi' := (\xi_1, \xi_2, 0)$  running over  $R^2$  and the family of vector fields  $Z_{\xi'}$  and  $Z_{\xi'}^*$  of the complex space  $R^3 + iR^3$ , with the family parameter  $\xi'$ :

$$Z_{\xi'} = (\xi' + i|\xi'|N), \quad Z_{\xi'}^* = (\xi' - i|\xi'|N). \quad (18)$$

Following a method introduced by Calderon [7], we construct the families of fields:

$$w^+(X, \xi') = \nabla \exp(-iZ_{\xi'} \cdot X) + \nabla \exp(-iZ_{\xi'}^* \cdot X), \quad (19a)$$

$$w^-(X, \xi') = \nabla \exp(-iZ_{\xi'} \cdot X) - \nabla \exp(-iZ_{\xi'}^* \cdot X). \quad (19b)$$

These fields belong to the space  $V$  since the divergences of the associated stress fields  $\sigma^+(X, \xi')$  and  $\sigma^-(X, \xi')$  vanish. Applying lemma 1, we get:

$$RG(w^+(\xi')) = \frac{2E}{(1+\nu)} |\xi'|^2 [[\hat{u}]]_N(\xi'),$$

$$RG(w^-(\xi')) = -\frac{2E}{(1+\nu)} i|\xi'| \xi' \cdot [[\hat{u}]]_t(\xi'),$$

where  $[[\hat{u}]]_N$  and  $[[\hat{u}]]_t$  denote respectively the Fourier transforms of the normal and the tangential components of the crack displacement discontinuity  $[[u]]$ , with respect to the plane  $\Pi$ . Hence, we are led to the following equations:

$$[[\hat{u}]]_N = \frac{(1+\nu)}{2E|\xi'|^2} RG(w^+(\xi')), \quad (20a)$$

$$|\xi'| \xi' \cdot [[\hat{u}]]_t(\xi') = \frac{(1+\nu)}{2iE} RG(w^-(\xi')), \quad (20b)$$

which show that the Fourier transform  $[[\hat{u}]]$  can be obtained uniquely by the boundary data  $(F_m, u_m)$  since the parameter (index)  $\xi'$  runs over  $R^2$ . For example, if we take  $\xi' = (\xi_1, 0, 0)$  (respectively  $\xi' = (0, \xi_2, 0)$ ) equation (20b) gives us the partial Fourier transform of  $[[\hat{u}]]_t$  with respect to  $\xi_1$  (respectively  $\xi_2$ ). The support of the discontinuity and hence the crack itself, can be identified by the boundary data.  $\square$

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