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# Double Time Scale Analysis of the Consolidation of a Two-Strata Poroelastic Layer with High Permeability Contrast

E. BOURGEOIS, A. CORFDIR and L. DORMIEUX

*Laboratoire de Mécanique des Solides (URA 317 CNRS), ENPC-CERCSO, 6–8 avenue Blaise Pascal, Cité Descartes, F-77455 Marne La Vallée Cedex 2, France*

**Abstract.** The consolidation of a saturated porous medium submitted to a vertical load is a classical problem, and has been given a complete analytical solution within the framework of linear poroelasticity. For a layer constituted by several homogeneous strata, the problem can be dealt with by means of analytical as well as numerical methods, provided that the contrast between the properties of the strata remain moderate. In this paper, we consider a layer constituted by two strata and discuss the situation in which the contrast between the properties of the strata becomes very large and fluid conduction exhibits very different characteristic times in each stratum. The method used consists in introducing a double time scale, and gives an analytical solution, which makes it possible to discuss the influence of the ratio between strata thicknesses on the overall consolidation time of the layer.

**Key words:** double time scale, asymptotic method, permeability contrast, layered soil, consolidation, poroelasticity.

## 1. Introduction

The consolidation of a saturated porous layer submitted to a vertical load is a classical problem, first studied by Terzaghi (1925, 1943) and Biot (1941) among others, and has been given a complete analytical solution for a homogeneous layer within the framework of linear poroelasticity. This solution is recalled in the first section: all quantities (notably the strain in the layer and the global vertical settlement) can be deduced from the excess pore pressure within the layer, which takes the form of a series of functions decreasing with time. The consolidation of a layer constituted by several homogeneous compressible poroelastic strata can be dealt with by means of analytical as well as numerical methods, provided that the contrast between the characteristic times of consolidation of the strata remain moderate. We consider a layer constituted by two strata and discuss the situation in which the contrast between the properties of the strata becomes very large and fluid conduction exhibits very different characteristic times in each stratum. We propose an analytical approach, based on the introduction of a double time scale: the excess pore pressure within each stratum is expanded into a power series of the ratio of the characteristic times of both strata.

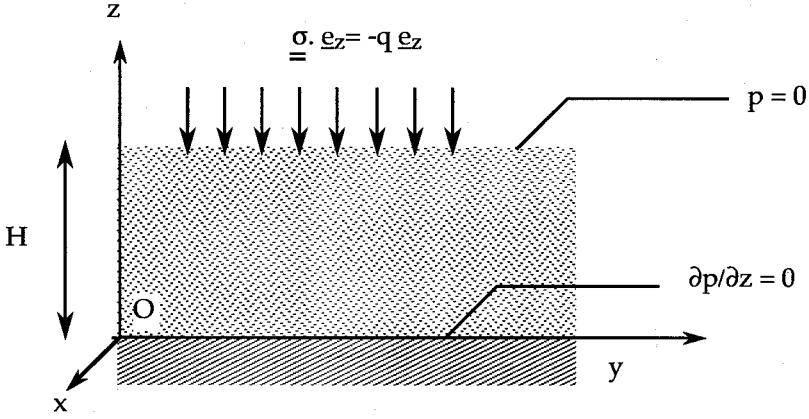


Figure 1. Layer subjected to a uniform overload.

## 2. Consolidation of a Homogeneous Compressible Porous Layer

Consider a horizontal layer of a compressible poroelastic medium, of infinite extent in the Oxy plane and of thickness  $H$  (Figure 1). On its lower boundary (located in the plane  $z = 0$ ), the layer lies on a horizontal rigid and impervious substratum. The upper boundary of the layer is perfectly drained, so that the fluid pressure in the plane  $z = H$  remains equal to zero. Initially ( $t < 0$ ) the layer is at rest, and the pore pressure field  $P^o(z)$  is hydrostatic.

At time  $t = 0$ , the layer is instantaneously subjected to a uniform density of vertical overload:

$$\forall t > 0 \quad \underline{\sigma} \cdot \underline{e}_z = -q \underline{e}_z \quad \text{at any point of the plane } z = H, \quad (1)$$

where  $\underline{\sigma}$  denotes the Cauchy stress tensor and  $\underline{e}_z$  the unit vector of the upward vertical direction. Given the symmetry of both the geometry and the loading, all quantities are assumed to be independent of the horizontal coordinates  $x$  and  $y$ .

When the overload is applied, the fluid pressure is instantaneously uniformly increased in the layer, then decreases progressively as the fluid leaves the layer. The problem consists in finding out the time evolution of the pore pressure  $P(z, t)$ . We denote by  $p(z, t) = P(z, t) - P^o(z)$  the difference between the current and initial pore pressure fields, and the initial fluid mass density is denoted by  $\rho_o^f$ .

It is recalled that the relative motion of the fluid with respect to the skeleton of the porous medium is represented by the Eulerian vector  $\mathbf{w}$  of relative fluid mass flow, which is related to the pressure field through the fluid mass conduction law (or Darcy's law):

$$\mathbf{w} / \rho_o^f = -[\mathbf{k} / \mu] \text{grad } p, \quad (2)$$

where  $\mathbf{k}$  is the (intrinsic) permeability tensor and  $\mu$  the viscosity of the fluid. In the case of a homogeneous and isotropic layer, the permeability tensor takes the form  $\mathbf{k} = k\mathbf{I}$ . Projecting (2) upon the vertical direction, one gets

$$w_z/\rho_o^f = -k/\mu \frac{\partial p}{\partial z}. \quad (3)$$

The behavior is characterized by the undrained Lamé coefficients  $\lambda$  and  $G$  and by the Biot modulus  $M$  and Biot coefficient  $b$  (see, for instance, Coussy (1995)). Denoting by  $c_m$  the diffusion coefficient ( $c_m = (k/\mu)M(\lambda - Mb^2 + 2G)/(\lambda + 2G)$ ), a classical reasoning shows that the problem consists in solving the well-known equation of consolidation

$$\frac{\partial p}{\partial t} = c_m \frac{\partial^2 p}{\partial z^2} \quad (4)$$

with the conditions of impermeability of the substratum on the lower boundary and of nullity of the pore pressure on the upper boundary

$$w_z(z = 0, t) = 0 \Rightarrow \frac{\partial p}{\partial z}(z = 0, t) = 0 \quad (5)$$

$$p(z = H, t) = 0 \quad (6)$$

together with the initial condition

$$p(z, t = 0^+) = Rq \quad \text{with} \quad R = Mb/(\lambda + 2G). \quad (7)$$

The solution is looked for under the form of a linear combination of the elementary solutions of (4) given by

$$u_n(z) \exp\left(-\frac{\pi^2}{4}(2n+1)^2 t/t_c\right) \quad (8)$$

where the functions  $u_n(z)$  are the eigenvectors of the differential operator  $d^2/dz^2$  on the interval  $[0, H]$  that satisfy (5) and (6):

$$u_n(z) = \cos\left((2n+1)\frac{\pi}{2}z/H\right) \quad (9)$$

and  $t_c$  is the characteristic time of consolidation of the layer

$$t_c = \frac{H^2}{c_m}. \quad (10)$$

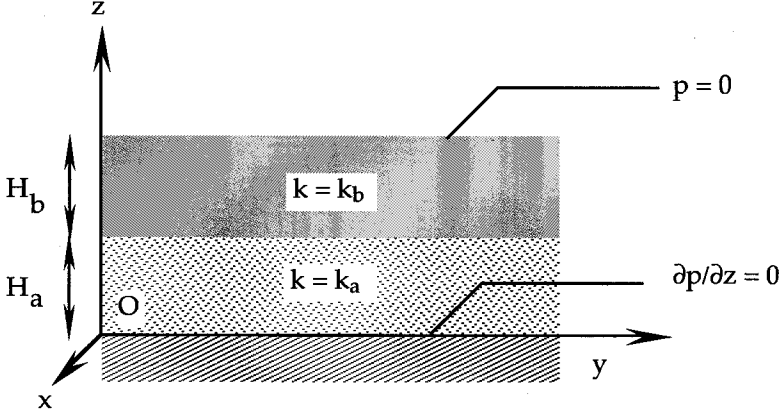


Figure 2. Layer constituted by two strata of different permeabilities.

The unknown function  $p(z, t)$  is then obtained as a series whose coefficients are the Fourier coefficients of the initial condition (7) with respect to the functions  $u_n(z)$ , which yields the classical result (see, for instance, Taylor (1948)):

$$\begin{aligned}
 P(z, t) - P^o(z) &= Rq \sum_{n=0}^{\infty} \frac{4(-1)^n}{\pi(2n+1)} \cos((2n+1)z/H) \exp\left(-\frac{\pi^2}{4}(2n+1)^2 t/t_c\right). \quad (11)
 \end{aligned}$$

This series is convergent for any  $z$  if  $t > 0$ ; for  $t = 0$ , it is convergent at any point of the interval  $[0, H[$ , and the limit is the value prescribed by (7).

### 3. Layer Constituted by Two Strata

Consider now the case of a layer constituted by two strata (Figure 2), both homogeneous and isotropic but in which the phenomenon of conduction of the fluid has very different characteristic times. This situation occurs, for instance, if both layers have very different permeabilities but mechanical properties roughly of the same order; permeability in clays may be lower by up to 9 orders of magnitude than typical permeabilities of sands, whereas their bulk and shear moduli are of similar order. We denote with a subscript  $a$  (resp.  $b$ ) quantities relative to the lower (resp. upper) stratum. Thus, the permeability and the elastic constants of the lower stratum ( $0 \leq z \leq H_a$ ) are denoted by  $k_a, \lambda_a, M_a, b_a$ , and that of the upper stratum ( $H_a \leq z \leq H_a + H_b$ ) by  $k_b, \lambda_b, M_b, b_b$ . The characteristic times are given by

$$t_a = H_a^2 \frac{\lambda_a + 2G_a}{k_a M_a (\lambda_a - M_a b_a^2 + 2G_a)}; \quad t_b = H_b^2 \frac{\lambda_b + 2G_b}{k_b M_b (\lambda_b - M_b b_b^2 + 2G_b)}. \quad (12)$$

In what follows, we discuss the case of very contrasted characteristic times (i.e.  $t_a \gg t_b$  or  $t_b \gg t_a$ ).

- If  $t_b \ll t_a$ , the upper stratum reacts much faster than the lower one. In a first phase, whose duration is of the order of a few times  $t_b$ , the excess pore pressure in the upper stratum vanishes, whereas the lower stratum behaves as an impervious substratum. Then, at a larger time scale, the excess pore pressure progressively disappears in the lower stratum as if it were perfectly drained on its upper boundary. In spite of its simplicity, this reasoning enlightens the existence of two different time scales related to the conduction of fluid mass in each stratum, and provides sufficient information to understand the overall evolution of the pore pressure in the layer and to predict the evolution of the layer thickness.

- If  $t_a \ll t_b$ , such a simple analysis is no longer effective. The fact that fluid mass conduction is faster in the lower stratum ensures that the difference between the pore pressure and its initial hydrostatic value rapidly becomes uniform in the stratum, but its value depends on the quantity of fluid leaving the lower stratum, which is determined by the evolution of the upper stratum. Thus, the evolution of the layer on a large time scale remains to be discussed.

### 3.1. ASYMPTOTIC METHOD. CHOICE OF A SMALL PARAMETER

We first define a reduced coordinate for each stratum

$$Z_a = z/H_a; \quad Z_b = (z - H_a)/H_b \quad (13)$$

and we denote by  $p_a(Z_a, t)$  and  $p_b(Z_b, t)$  the difference between the current value of the pore pressure and the initial hydrostatic value. The PDEs governing the conduction of the fluid in each stratum and the conditions on the boundaries become:

$$\frac{\partial p_a}{\partial t} = \frac{1}{t_a} \frac{\partial^2 p_a}{\partial Z_a^2}, \quad \frac{\partial p_b}{\partial t} = \frac{1}{t_b} \frac{\partial^2 p_b}{\partial Z_b^2}, \quad (14)$$

$$\frac{\partial p_a}{\partial Z_a}(Z_a = 0, t) = 0, \quad (15)$$

$$p_b(Z_b = 1, t) = 0. \quad (16)$$

Besides, the solution is bound to satisfy the following conditions at the interface  $z = H$ :

$$p_a(Z_a = 1, t) = p_b(Z_b = 0, t) \quad (\text{continuity of the pore pressure}) \quad (17)$$

$$\frac{k_a}{H_a} \frac{\partial p_a}{\partial Z_a}(Z_a = 1, t) = \frac{k_b}{H_b} \frac{\partial p_b}{\partial Z_b}(Z_b = 0, t) \quad (\text{continuity of the fluid mass flow } w_z). \quad (18)$$

The characteristic times of the strata being very different, we suggest to use here an asymptotic method: such methods are used in a wide range of problems involving more than one scale, usually in space rather than in time (see for instance (Bensoussan *et al.*, 1978)). They consist in introducing a small parameter to take into account the existence of two different scales and in assuming that physical quantities can be expanded into power series of the small parameter.

In the problem under consideration, given the existence of two time scales, the dependency with respect to time of any quantity  $A(t)$  is taken into account by means of two independent variables

$$A(t) = A(\tau_a, \tau_b) \quad \text{with} \quad \tau_a = t/t_a; \tau_b = t/t_b \quad (19)$$

Denoting by  $\eta$  the little parameter  $\eta = t_a/t_b \ll 1$ , Equations (14) take the form

$$\frac{\partial p_a}{\partial \tau_a} + \eta \frac{\partial p_a}{\partial \tau_b} = \frac{\partial^2 p_a}{\partial Z_a^2}, \quad \frac{1}{\eta} \frac{\partial p_b}{\partial \tau_a} + \frac{\partial p_b}{\partial \tau_b} = \frac{\partial^2 p_b}{\partial Z_b^2}. \quad (20)$$

Physical quantities are expanded into power series of  $\eta$ :

$$A(t) = A^{(0)}(\tau_a, \tau_b) + \eta A^{(1)}(\tau_a, \tau_b) + \dots, \quad (21)$$

where the  $A^{(i)}$  are of same order of magnitude. Note that a perturbation technique has been used by Fallou *et al.* (1992) to deal with the subsidence of a layered soil due to pumping, in which the different layers exhibit very different permeabilities and mechanical properties.

### 3.2. RESOLUTION

Limiting the developments to terms of order lower than 1, Equations (20) yield

$$\frac{\partial p_a^{(0)}}{\partial \tau_a} = \frac{\partial^2 p_a^{(0)}}{\partial Z_a^2}, \quad (22)$$

$$\frac{\partial p_a^{(1)}}{\partial \tau_a} + \frac{\partial p_a^{(0)}}{\partial \tau_b} = \frac{\partial^2 p_a^{(1)}}{\partial Z_a^2}, \quad (23)$$

$$\frac{\partial p_b^{(0)}}{\partial \tau_a} = 0, \quad (24)$$

$$\frac{\partial p_b^{(0)}}{\partial \tau_b} + \frac{\partial p_b^{(1)}}{\partial \tau_a} = \frac{\partial^2 p_b^{(0)}}{\partial Z_b^2}. \quad (25)$$

The boundary conditions (15) and (16) and the interface conditions (17) and (18) can be rewritten as

$$\frac{\partial p_a^{(0)}}{\partial Z_a}(Z_a = 0, \tau_a, \tau_b) = \frac{\partial p_a^{(1)}}{\partial Z_a}(Z_a = 0, \tau_a, \tau_b) = 0, \quad (26)$$

$$p_b^{(0)}(Z_b = 1, \tau_a, \tau_b) = p_b^{(1)}(Z_b = 1, \tau_a, \tau_b) = 0, \quad (27)$$

$$p_a^{(0)}(Z_a = 1, \tau_a, \tau_b) = p_b^{(0)}(Z_b = 0, \tau_a, \tau_b), \quad (28)$$

$$p_a^{(1)}(Z_a = 1, \tau_a, \tau_b) = p_b^{(1)}(Z_b = 0, \tau_a, \tau_b), \quad (29)$$

$$\frac{\partial p_a^{(0)}}{\partial Z_a}(Z_a = 1, \tau_a, \tau_b) = 0, \quad (30)$$

$$\alpha \frac{\partial p_a^{(1)}}{\partial Z_a}(Z_a = 1, \tau_a, \tau_b) = \frac{\partial p_b^{(0)}}{\partial Z_b}(Z_b = 0, \tau_a, \tau_b), \quad (31)$$

where

$$\alpha = \frac{H_a(\lambda_a + 2G_a)}{M_a(\lambda_a + M_a b_a^2 - 2G_a)} / \frac{H_b(\lambda_b + 2G_b)}{M_b(\lambda_b + M_b b_b^2 - 2G_b)}$$

is a nondimensional parameter that depends on the ratio between the thicknesses of the strata and on the contrast between their elastic constants. Notice that (24) shows that  $p_b^{(0)}$  does not depend on the rapid time variable  $\tau_a$ . In a general way, it is clear that the evolution of the upper stratum only involves the slow time variable  $\tau_b$ , and it can be shown that all functions  $p_b^{(i)}$  are independent of  $\tau_a$ . It then follows from (25) that the function  $p_b^{(0)}(Z_b, \tau_b)$  satisfies the familiar PDE

$$\frac{\partial p_b^{(0)}}{\partial \tau_b} = \frac{\partial^2 p_b^{(0)}}{\partial Z_b^2} \quad (32)$$

with the conditions (27) and (31). The latter has an unusual form, and involves the function  $p_a^{(1)}$  through the value taken by its derivative with respect to  $Z_a$  at the interface.

Given (22), and the boundary conditions (30) and (28), it is readily seen that the function  $p_a^{(0)}$  does not depend on  $\tau_a$ :

$$p_a^{(0)}(Z_a, \tau_a, \tau_b) = p_b^{(0)}(Z_b = 0, \tau_b) \quad (33)$$

In the particular case of the loading discussed here, the overall evolution of the layer is only controlled by that of the upper stratum: no rapid evolution of the lower



stratum is expected after the load is applied, since the initial excess pore pressure is uniform according to (7). Thus, given the form of the initial condition, it is assumed that the functions  $p_a^{(i)}$  do not depend on  $\tau_a$ . It then follows from (23) and (33) that:

$$\frac{\partial^2 p_a^{(1)}}{\partial Z_a^2} = \frac{\partial}{\partial \tau_b} [p_b^{(0)}(Z_b = 0, \tau_b)] \quad (34)$$

Taking into account (26), the integration of (34) with respect to  $Z_a$  leads to

$$\frac{\partial p_a^{(1)}}{\partial Z_a} = Z_a \frac{\partial}{\partial \tau_b} [p_b^{(0)}(Z_b = 0, \tau_b)]. \quad (35)$$

The function  $p_b^{(0)}(Z_b, \tau_b)$  is solution to (32), with the boundary conditions (27) and (31), or, using (35):

$$\begin{aligned} \frac{\partial p_b^{(0)}}{\partial \tau_b} &= \frac{\partial^2 p_b^{(0)}}{\partial Z_b^2}, & p_b^{(0)}(Z_b = 1, \tau_b) &= 0, \\ \frac{\partial p_b^{(0)}}{\partial Z_b}(Z_b = 0, \tau_b) &= \alpha \frac{\partial}{\partial \tau_b} [p_b^{(0)}(Z_b = 0, \tau_b)]. \end{aligned} \quad (36)$$

The originality of the system lies in the form of the condition written in  $Z_b = 0$ , which accounts for the coupling between strata. Denoting by  $Y$  the Heaviside function, the initial condition takes the form

$$p_b^{(0)}(Z_b, \tau_b = 0^+) = Rq Y(1 - Z_b). \quad (37)$$

Solutions to (36) are looked for under the form  $f(Z_b)h(\tau_b)$ . We denote by  $f'$  (resp.  $f''$ ) the derivative (resp. the second derivative) of  $f$ , and by  $\dot{h}$  the derivative of  $h$ . One first classically obtains

$$\dot{h}(\tau_b)/h(\tau_b) = -\omega^2 \Rightarrow h(\tau_b) = h(0) \exp(-\omega^2 \tau_b), \quad (38)$$

while  $f$  is solution to the differential equation

$$f'' = -\omega^2 f, \quad (39)$$

with the conditions

$$f(Z_b = 1) = 0, \quad (40)$$

$$\begin{aligned} f'(Z_b = 0) &= \alpha \dot{h}(\tau_b)/h(\tau_b) f(Z_b = 0) \\ &= -\alpha \omega^2 f(Z_b = 0) = \alpha f''(Z_b = 0). \end{aligned} \quad (41)$$

Solutions are given by

$$f_n(Z_b) = \sin(\omega_n(1 - Z_b)) \quad (n \geq 0), \quad (42)$$

where  $\omega_n$  is the only solution in the interval  $](n - 1/2)\pi, (n + 1/2)\pi[$  to the equation

$$\omega \tan \omega = 1/\alpha. \quad (43)$$

We now look for a linear combination of the elementary solutions  $f_n(Z_b) \times \exp(-\omega_n^2 \tau_b)$  that comply with the initial condition (37). Let us introduce the bilinear form  $\langle , \rangle$  defined by:

$$\langle f, g \rangle = \int_0^1 (\alpha f' - f)(\alpha g' - g) dZ_b. \quad (44)$$

It is readily seen that this bilinear form is symmetric and takes a strictly positive value for any differentiable function that comply with (40). Besides, for any  $n \neq m$ , the functions  $f_n$  and  $f_m$  are orthogonal. To prove this, we evaluate the quantity  $B_{nm}$  defined as

$$B_{nm} = \int_0^1 d/dZ_b [(\alpha f_n'' - f_n')(\alpha f_m' - f_m) - (\alpha f_m'' - f_m')(\alpha f_n' - f_n)] dZ_b. \quad (45)$$

Developing the integrand, and remembering the functions  $f_n(Z_b)$  are eigenvectors of  $d^2/dZ^2$ , one gets

$$B_{nm} = (\omega_m^2 - \omega_n^2) \langle f_n, f_m \rangle \quad (46)$$

Besides, it is readily seen that  $B_{nm} = F_{nm}(1) - F_{nm}(0)$ , with

$$\begin{aligned} F_{nm}(x) &= (\alpha f_n'' - f_n')(\alpha f_m' - f_m) - (\alpha f_m'' - f_m')(\alpha f_n' - f_n) \\ &= -\omega_n^2 f_n(\alpha f_m' - f_m) + \omega_m^2 f_m(\alpha f_n' - f_n) + \\ &\quad + (f_n' f_m - f_m' f_n) \end{aligned} \quad (47)$$

$F_{nm}(1)$  is zero because all functions  $f_n(Z_b)$  take the value zero in  $Z_b = 1$ , and condition (41) yields

$$f_n'(Z_b = 0) = f_n''(Z_b = 0) = -\alpha \omega_n^2 f_n(Z_b = 0) \quad (48)$$

which implies that  $F_{nm}(0)$  is also equal to zero. It follows that  $B_{nm} = 0$ , and with (46), that  $f_n$  and  $f_m$  are orthogonal.

Eventually the solution of the problem is given by

$$p_b^{(0)}(Z_b, \tau_b) = Rq \sum_{n=0}^{\infty} A_n \exp(-\omega_n^2 \tau_b) \sin(\omega_n(1 - Z_b)) \quad (49)$$

where the  $A_n$  are the Fourier coefficients of the initial condition with respect to the functions  $f_n(Z_b)$ :

$$A_n = Rq \langle Y(1 - Z_b), f_n \rangle / \langle f_n, f_n \rangle. \quad (50)$$

Note that the bilinear form used here is more complicated than that used in the usual problem of consolidation; its definition (44) involves the derivatives of  $f$  and  $g$ : it is therefore necessary to write the initial condition under the form (37) (instead of  $p_b^{(0)}(Z_b, \tau_b = 0^+) = Rq$ ) to get the proper values of the  $A_n$ . For any  $Z_b$ , this series is convergent at  $\tau_b = 0$  and absolutely convergent for  $\tau_b > 0$ .

One thus gets an approximation of the evolution of the pore pressure in both strata at the lowest order, i.e. in the case of an infinite contrast of permeability. Terms of higher orders could be obtained by solving successively problems of the same kind. However, (35) gives the gradient of the function  $p_a^{(1)}$ , which is responsible for the motion of fluid in the lower stratum. Note that when the contrast becomes very large ( $\eta \rightarrow 0$ ), the lower stratum is progressively drained in spite of the fact that the excess pore pressure remains uniform at any time.

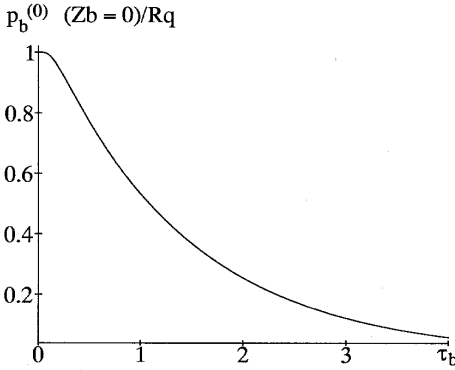


Figure 3.  $p_b^{(0)}(Z_b)$  for different times  $\tau_b$ : (a)  $\tau_b = 5 \times 10^{-4}$ ; (b)  $\tau_b = 10^{-2}$ ; (c)  $\tau_b = 10^{-1}$ ; (d)  $\tau_b = 1$ .

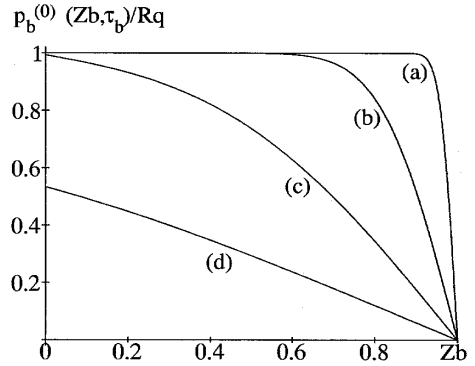


Figure 4. Excess pore pressure at the interface between strata  $p_b^{(0)}(Z_b = 0)$  as a function of  $\tau_b$ .

#### 4. Application

Figures 3 and 4 sum up the results obtained in the particular case where  $\alpha = 1$ : they give the time evolution of the excess pore pressure profile across the upper stratum and the evolution of the value of the excess pore pressure at the interface between both strata.

Assuming the materials constituting both strata have the same elastic constants, and given the thickness  $H_b$  of the upper stratum, it is interesting to discuss the influence of the thickness  $H_a$  of the lower stratum on the time necessary to the

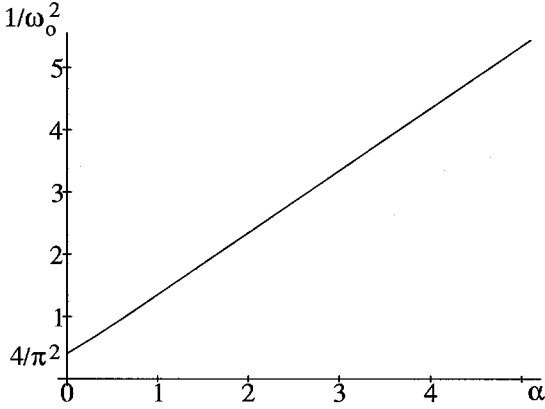


Figure 5. Influence of the thickness of the lower stratum on the overall consolidation time of the layer.

consolidation of the whole layer. This is achieved on Figure 5 by plotting the non-dimensional time appearing in the first term of the series (which rapidly becomes preponderant as time increases), namely  $1/\omega_o^2$ , against  $\alpha$ . For small values of  $\alpha$ , one gets the value  $4/\pi^2$ , which means that the upper stratum reacts as though it were lying on an impervious substratum (see the usual result (11)). Besides, asymptotic developments of the value of  $\omega_o$  provide the slope of the curve:

- for small values of  $\alpha$ , it follows from (43) that  $\omega_o$  tends to  $\pi/2$ . Let:

$$\omega_o = \pi/2 - \varepsilon \quad (51)$$

It is readily seen from (43) that  $\varepsilon \cong \alpha\pi/2$ . One then derives from (51) the approximation  $1/\omega_o^2 \cong 4/\pi^2(1 + 2\alpha)$  for small values of  $\alpha$ , and the slope tends to  $8/\pi^2$ ;

- for great values of  $\alpha$ , it is readily seen (cf. (43)) that  $\omega_o$  tends to 0. It then results from (43) that

$$\omega_o^2 \cong \omega_o \tan \omega_o = 1/\alpha \quad (52)$$

and the slope tends to 1.

## 5. Conclusion

The introduction of a double time scale has proved to be a successful method to get analytically the exact solution of the problem under discussion. The main difficulty of the resolution is due to the nonusual form of the boundary condition at the interface between the strata, and results in the introduction of an unusual bilinear form to obtain the coefficients  $A_n$  of the series (49). The small parameter we choose is the ratio between the characteristic times of the strata, which has a clear physical meaning and makes the interpretation of the results simple.

The results obtained make it possible to discuss the influence of the ratio of strata thicknesses on the overall consolidation time of the layer. When the thickness of the lower stratum tends to zero, the consolidation time is the same as if the upper stratum would lie on an impervious substratum. We provide an estimation of the increase of the consolidation time with the thickness of the lower stratum.

Eventually, the analytical solution obtained herein may be useful to test the validity of existing numerical softwares simulating vertical compaction when high contrasts between characteristic times occur.

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