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► **To cite this version:**

Eric Vasseur, Pirouz Navidi, Saïd Hariri. Direct determination of limiting cycle during cyclic loading. Computational Materials Science, 1996, 7 (1-2), pp.11-15. 10.1016/S0927-0256(96)00053-5 . hal-00111528

**HAL Id: hal-00111528**

**<https://hal.science/hal-00111528>**

Submitted on 21 Aug 2020

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# Direct determination of limiting cycle during cyclic loading

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## Abstract

A general framework for the modelling of mechanical behavior of structures has been recently proposed by J. Zarka et al. Its essential feature consists of the introduction of a group of internal parameters which characterize the local inelastic mechanisms of the structure, and a group of transformed internal parameters which are linearly linked to the previous ones through a symmetrical non-negative matrix. With this approach, the treatment of the local plastic yield conditions can be made easily from simple elastic analysis. The problem of the determination of the limit state in the case of cyclic loading is then reduced to the determination of the asymptotic values of the transformed internal parameters. In most cases, this is done by simple geometrical constructions. As an example, we will present a simple structure made of two connected elastoplastic workhardening cylinders of different radius, subjected to cyclic imposed displacement at one end, in order to show the efficiency of the method.

**Keywords:** Simplified inelastic analysis; Cyclic loading; Limit state

Fatigue analysis of structures submitted to cyclic loading is now assumed as a part of the design process for mechanical parts. In order to be able to include such analysis in an optimization cycle, it is necessary to compute quickly the limiting cycle (in the case when a limiting cycle exists).

It is obvious that the loading path should be maintained in the elastic range, but it is not always possible. In such cases, we have also to take into account the non-linear part of the response of the structure.

A general framework, which enables the designers to effectively modelize the main non-linear effects observed during experiments, has been proposed [1].

In this paper, the principle of the method will be briefly exposed and its tractability will be shown on a simple analytical example.

To describe mathematically the behavior of materials, a micro-macro type procedure can be used: A representative fundamental volume element is complex and composed of various elements. Its global behavior is linked to some local elementary inelastic mechanisms and their distribution in the volume. Some qualitative and quantitative data relating to those inelastic mechanisms can be determined, by means of monotonic or cyclic, one dimensional or multidimensional, very slow or fast loading on a test specimen. We shall have:

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1. mechanisms with dry friction (mechanisms  $\alpha$ ) which will replicate instantaneous inelastic strains with a threshold; they will include all dry slip mechanisms, dislocations ...;
2. mechanisms with time effects (mechanisms  $\beta$ ) which will replicate delayed inelastic strains without a threshold; they will include all diffusion processes;
3. mechanisms with dry friction and viscosity (mechanisms  $\gamma$ ) which will replicate delayed inelastic strains with a threshold; they will include all thermally activated processes.

Those laboratory tests allow to determine the simplest model which is able to represent the observed behavior.

All those inelastic mechanisms are linked by an elastic media; their inelastic strains,  $\mathbf{X}$ , will be considered as our internal parameters (they are tensorial objects).  $\mathbf{X} = [\alpha \ \beta \ \gamma]^T$

Let us denote by  $\underline{\Sigma}$  the applied global stress tensor on the volume element. The local stresses at the level of each inelastic mechanism can be written in the form:  $\sigma = [\sigma_\alpha \ \sigma_\beta \ \sigma_\gamma]^T = \mathbf{A} \cdot \underline{\Sigma} - \mathbf{b} \cdot \mathbf{X} = \mathbf{A} \cdot \underline{\Sigma} - \mathbf{y}$ , with:  $\mathbf{y} = [y_\alpha \ y_\beta \ y_\gamma]^T = \mathbf{b} \cdot \mathbf{X}$

- $(\sigma_\alpha, \sigma_\beta, \sigma_\gamma)$  are the local stresses which are associated with the local inelastic strains;
- $\mathbf{A}$  is the elastic localization matrix of the inelastic mechanisms;
- $\mathbf{b}$  is the inelastic mechanisms interaction matrix (symmetrical and non-negative);
- $(y_\alpha, y_\beta, y_\gamma)$  are the transformed internal parameters. Two cases have to be considered:
  - if  $\mathbf{b}$  is regular, there is a one to one relation between  $\mathbf{X}$  and  $\mathbf{y}$ , and we may follow either  $\mathbf{X}$  or  $\mathbf{y}$ ;
  - if  $\mathbf{b}$  is singular, at any  $\mathbf{X}$ , we can find a  $\mathbf{y}$  which will belong to a compatibility subspace  $\mathcal{L}$ . We shall have to follow  $\mathbf{X}$  and  $\mathbf{y}$  simultaneously.

Let us denote by  $E^P$  the plastic strain of the volume element.  $E^P$  is defined by:  $E^P = \mathbf{A}^T \cdot \mathbf{X}$  We define an evolution law for each of the inelastic mechanisms:

1.  $\alpha$  mechanisms: for those mechanisms there is a convex set,  $C_0$ , centered on the origin and which is such as  $\sigma_\alpha \in C_0$  and  $\dot{\alpha} \in \partial \mathcal{V}_{C_0}(\sigma_\alpha)$  i.e.  $\dot{\alpha}$  is an external normal to  $C_0$  in  $\sigma_\alpha$ .

2.  $\beta$  mechanisms: for those mechanisms, in the simplest form, we have  $\dot{\beta} = \sigma_\beta / \eta$  with  $\dot{\beta} = 0$  if  $\sigma_\beta = 0$ , where  $\eta$  is the damping factor of the  $\beta$  mechanism.
3.  $\gamma$  mechanisms: for those mechanisms there is a convex set,  $D_0$ , centered on the origin and which is such as  $\dot{\gamma} = \partial \omega / \partial \sigma_\gamma$ , where  $\omega$  is the viscoplastic potential of the  $\gamma$  mechanism.

We shall now consider a structure made of such materials and subjected to an arbitrary loading. The materials have been characterized: we have identified the nature of the internal parameters  $\mathbf{X}$ , the various matrices  $\mathbf{A}$  and  $\mathbf{b}$ , the transformed internal parameters and the local evolution laws. We shall limit ourselves to the case of small deformations and quasi-static loading. Moreover, we shall assume that the decomposition of the boundary of the structure (surface forces–displacement), remain constant during the loading path.

Let us denote by  $\underline{U}_{el}(t)$ ,  $\underline{E}_{el}(t)$  and  $\underline{\Sigma}_{el}(t)$ , the displacement, strain and stress response of the structure remaining virtually elastic; and by  $\underline{U}(t)$ ,  $\underline{E}(t)$  and  $\underline{\Sigma}(t)$ , the real response of the structure. We have:

- $\underline{E}^{el}(t) = \underline{M} \cdot \underline{\Sigma}^{el}(t) + \underline{E}^l(t)$ , where  $\underline{E}^l(t)$  is the initial strain field and  $\underline{M}$  the elastic matrix;
- $\underline{E}(t) = \underline{E}^l(t) + \underline{E}^e(t) + \underline{E}^p(t)$ , which is the strain decomposition in initial, elastic and plastic parts;
- $\underline{\Sigma}(t) = \underline{M}^{-1} \cdot \underline{E}^e(t)$ ;

The real response of the structure can be decomposed into an elastic and an inelastic part, the inelastic one being auto-equilibrated

$$\underline{U}(t) = \underline{U}^{el}(t) + \underline{U}^{inc}(t)$$

$$\underline{E}(t) = \underline{E}^{el}(t) + \underline{E}^{inc}(t)$$

$$\text{with: } \underline{E}^{inc}(t) = \underline{M} \cdot \underline{R}(t) + \underline{E}^p(t)$$

$$\underline{\Sigma}(t) = \underline{\Sigma}^{el}(t) + \underline{R}(t)$$

This last relation means that we have a linear elastic problem which implies symbolically:  $\underline{R}(t) = \underline{Z}_0 \cdot \underline{E}^p(t)$ , where  $\underline{Z}_0$  is a linear operator which is symmetrical, non-positive and generally singular.

Now we shall use a mathematical trick in order to

come down to the resolution of a simple problem. We may write:

$$\begin{aligned}\underline{\sigma}(t) &= \underline{\mathbf{A}} \cdot \underline{\Sigma}(t) - \underline{\mathbf{y}}(t) \\ &= \underline{\mathbf{A}} \cdot \underline{\Sigma}^{\text{el}}(t) - (\underline{\mathbf{y}}(t) - \underline{\mathbf{A}} \cdot \underline{\mathbf{R}}(t)) \\ &= \underline{\mathbf{A}} \cdot \underline{\Sigma}^{\text{el}}(t) - \underline{\mathbf{Y}}(t),\end{aligned}$$

$$\begin{aligned}\text{with: } \underline{\mathbf{Y}}(t) &= \underline{\mathbf{y}}(t) - \underline{\mathbf{A}} \cdot \underline{\mathbf{R}}(t) \\ &= \underline{\mathbf{b}} \cdot \underline{\mathbf{X}}(t) - \underline{\mathbf{A}} \cdot \underline{\mathbf{Z}}_0 \cdot \underline{\mathbf{A}}^{\text{T}} \cdot \underline{\mathbf{X}}(t) \\ &= \underline{\mathbf{B}} \cdot \underline{\mathbf{X}}(t)\end{aligned}$$

$\underline{\mathbf{Y}}(t)$  is a new field of transformed parameters for the structure and  $\underline{\mathbf{B}}$  is symmetrical and non-negative. If  $\underline{\mathbf{B}}$  is regular, the knowledge of  $\underline{\Sigma}^{\text{el}}(t)$  and  $\underline{\mathbf{Y}}(t)$  is then sufficient to define the global evolution of the structure: by inverting  $\underline{\mathbf{B}}$ , we deduce  $\underline{\mathbf{X}}(t) = \underline{\mathbf{B}}^{-1} \cdot \underline{\mathbf{Y}}(t)$ , then  $\underline{\mathbf{E}}^{\text{p}}(t) = \underline{\mathbf{A}}^{\text{T}} \cdot \underline{\mathbf{X}}(t)$ ,  $\underline{\mathbf{R}}(t) = \underline{\mathbf{Z}}_0 \cdot \underline{\mathbf{E}}^{\text{p}}(t)$  and  $\underline{\Sigma}(t) = \underline{\Sigma}^{\text{el}}(t) + \underline{\mathbf{R}}(t)$ .

When all the local  $\underline{\mathbf{b}}$  matrices are regular, we may prove that  $\underline{\mathbf{B}}$  is regular too, which implies that to obtain the global evolution of the structure, we just have to determine  $\underline{\mathbf{Y}}(t)$ , the evaluation of  $\underline{\Sigma}^{\text{el}}(t)$  being straightforward. In such a case, we have:

$$\begin{aligned}\left. \begin{aligned}\underline{\mathbf{E}}^{\text{p}}(t) &= \underline{\mathbf{A}}^{\text{T}} \cdot \underline{\mathbf{b}}^{-1} \cdot (\underline{\mathbf{Y}}(t) + \underline{\mathbf{A}} \cdot \underline{\mathbf{R}}(t)) \\ \underline{\mathbf{E}}^{\text{ine}}(t) &= \underline{\mathbf{M}} \cdot \underline{\mathbf{R}}(t) + \underline{\mathbf{E}}^{\text{p}}(t)\end{aligned}\right\} \\ \Rightarrow \underline{\mathbf{E}}^{\text{ine}}(t) &= (\underline{\mathbf{M}} + \underline{\mathbf{A}}^{\text{T}} \cdot \underline{\mathbf{b}}^{-1} \cdot \underline{\mathbf{A}}) \cdot \underline{\mathbf{R}}(t) \\ &\quad + \underline{\mathbf{A}}^{\text{T}} \cdot \underline{\mathbf{b}}^{-1} \cdot \underline{\mathbf{Y}}(t)\end{aligned}$$

We shall now look at a particular application of this framework: structures containing only  $\alpha$ -mechanisms and subjected to cyclic loading. In the case of cyclic loading, we may prove that when all the local matrices  $\underline{\mathbf{b}}$  are regular, only elastic or plastic shakedown can occur; elastic and plastic shakedown being obtained when the stabilized asymptotic cycle is respectively elastic ( $\underline{\mathbf{Y}}(t)$  is constant) and periodic ( $\underline{\mathbf{Y}}(t)$  is periodic).

We shall limit ourselves to the case where all the local matrices  $\underline{\mathbf{b}}$  are regular.

We have  $\underline{\sigma}(t) \in C_0$  and  $\underline{\sigma}(t) = \underline{\mathbf{A}} \cdot \underline{\Sigma}^{\text{el}}(t) - \underline{\mathbf{Y}}(t)$ , which implies  $\underline{\mathbf{Y}}(t) \in \overline{C(\underline{\mathbf{A}} \cdot \underline{\Sigma}^{\text{el}}(t))} = C_0 + \underline{\mathbf{A}} \cdot \underline{\Sigma}^{\text{el}}(t)$ . We shall notice that  $\overline{C(\underline{\mathbf{A}} \cdot \underline{\Sigma}^{\text{el}}(t))}$  is known once the elastic response of the structure has been computed, i.e. obtaining  $\underline{\Sigma}^{\text{el}}(t)$  for one cycle  $[0, T]$ , and that  $C(\underline{\mathbf{A}} \cdot \underline{\Sigma}^{\text{el}}(t))$  is locally constructed, i.e. we

treat each inelastic mechanism independently of the other ones.

Moreover,  $\underline{\dot{\alpha}}(t) \in \partial \Psi_{C_0}(\underline{\sigma}(t))$ , which implies that  $\underline{\dot{\alpha}}(t)$  belongs to an *internal* normal to  $C(\underline{\mathbf{A}} \cdot \underline{\Sigma}^{\text{el}}(t))$  in  $\underline{\mathbf{Y}}(t)$  (i.e.  $\underline{\dot{\alpha}}(t) \in -\partial \Psi_{C(\underline{\mathbf{A}} \cdot \underline{\Sigma}^{\text{el}}(t))}(\underline{\mathbf{Y}}(t))$ ), and we have  $\underline{\dot{\mathbf{Y}}}(t) = \underline{\mathbf{B}} \cdot \underline{\dot{\alpha}}(t)$ .

Thus, to obtain the nature of the limit state, for each mechanism, we have to check if all the plastic yield surfaces defined by  $C(\underline{\mathbf{A}} \cdot \underline{\Sigma}^{\text{el}}(t))$ , for each time  $t$  in  $[0, T]$ , have a common part. In such a case the elastic shakedown will occur (for each mechanism,  $\underline{\mathbf{Y}}(t)$  is constant and inside the common part); otherwise, plastic shakedown occurs.

Finally, to compute the stabilized response of the structure, according to the nature of the limit state, the asymptotic values of  $\underline{\mathbf{Y}}(t)$  have to be locally determined (which is done by simple geometrical constructions).

We shall now consider a structure composed of two connected cylinders (cylinder 1: length  $L_1$ , section  $S_1$ ; and cylinder 2: length  $L_2$ , section  $S_2$  with  $S_2 > S_1$ ).

This structure is made of a kinematical hardening material (hardening modulus =  $h$ ) which is supposed to be elastically isotropic (Young modulus =  $E$ , Poisson ratio =  $\nu$ , elastic limit in tension =  $\sigma_y$ ).

We suppose that the initial state is virgin (no initial strain). This structure is embedded at the end of the cylinder 1, and subjected to a cyclic imposed axial displacement,  $U_2(t)$ , at the end of the cylinder 2;  $U_2(t)$  varying monotonically between 0 and  $\bar{U}$  (Fig. 1).

The problem is to determine analytically the limit

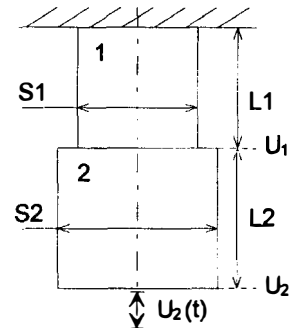


Fig. 1. Structure made of two connected elastoplastic workhardening cylinders, subjected to cyclic imposed displacement.

state of the structure, according to its geometry, the mechanical properties of the material and the imposed loading. We shall neglect the influence of the geometrical discontinuity, i.e. we shall consider that strain and stress fields are uniaxial.

For a kinematical hardening material ( $\alpha$ -mechanism), the yield criterion can be written in the form:  $f = f(\sigma) = f(\Sigma - h \cdot E^p)$ , which implies, by identification with  $\sigma = \mathbf{A} \cdot \Sigma - \mathbf{b} \cdot \mathbf{X}$ :  $\mathbf{A} = 1$  and  $\mathbf{b} = h$  ( $\mathbf{b}$  is regular).

First of all, we shall begin by establish the elastic response of the structure:

\* Compatibility equations: ( $E^{el}(t)$  has to be kinematically admissible with  $U_2(t)$ )

$$\begin{cases} E_1^{el}(t) = U_1^{el}(t)/L1 \\ E_2^{el}(t) = (U_2(t) - U_1^{el}(t))/L2 \end{cases}$$

\* Statical equilibrium equation:  $S1 \cdot \Sigma_1^{el}(t) = S2 \cdot \Sigma_2^{el}(t)$

\* Elastic behavior law: (we have no initial strain, i.e.  $E^l(t) = 0$ )

$$\begin{cases} \Sigma_1^{el}(t) = E \cdot (E_1^{el}(t) - 0) \\ \Sigma_2^{el}(t) = E \cdot (E_2^{el}(t) - 0) \end{cases}$$

The resolution of this system implies:  $U_1^{el}(t) = (S2 \cdot L1 / (S1 \cdot L2 + S2 \cdot L1)) \cdot U_2(t)$

$$E_1^{el}(t) = \frac{S2}{S1 \cdot L2 + S2 \cdot L1} \cdot U_2(t)$$

$$E_2^{el}(t) = \frac{S1}{S1 \cdot L2 + S2 \cdot L1} \cdot U_2(t)$$

$$\Sigma_1^{el}(t) = \frac{E \cdot S2}{S1 \cdot L2 + S2 \cdot L1} \cdot U_2(t)$$

$$\Sigma_2^{el}(t) = \frac{E \cdot S1}{S1 \cdot L2 + S2 \cdot L1} \cdot U_2(t)$$

Let us assume that  $U_2(t)$  is such as there is some plastic strain just in the cylinder 1. This condition implies:

$$\max[\Sigma_1^{el}(t)] > \sigma_y \text{ and } \max[\Sigma_2^{el}(t)] < \sigma_y.$$

The nature of the limit state of the structure will depend on the elastic response of the cylinder number one:

The convex set  $C(\Sigma_1^{el}(t)) = C_0 + \Sigma_1^{el}(t)$ , has a constant radius,  $\sigma_y$ , and moves linearly between  $C(\min[\Sigma_1^{el}(t)] = 0) = C_0$  and  $C(\max[\Sigma_1^{el}(t)] = C_0 + (E \cdot S2 / (S1 \cdot L2 + S2 \cdot L1)) \cdot \bar{U}$ .

Elastic shakedown will occur when those two sets have a common part and  $\max[\Sigma_1^{el}(t)] > \sigma_y$ , i.e.:  $\sigma_y < (E \cdot S2 / (S1 \cdot L2 + S2 \cdot L1)) \cdot \bar{U} < 2 \cdot \sigma_y$ . Otherwise, i.e.  $(E \cdot S2 / (S1 \cdot L2 + S2 \cdot L1)) \cdot \bar{U} > 2 \cdot \sigma_y$ , plastic shakedown occurs.

According to the nature of the limit state, we have now to determine the stabilized response of the structure.

So, we have to express the inelastic response of the structure:

\* Compatibility equations: ( $E^{ine}(t)$  has to be kinematically admissible with  $U_2(t) = 0$ )

$$E_1^{ine}(t) = U_1^{ine}(t)/L1$$

$$E_2^{ine}(t) = (0 - U_1^{ine}(t))/L2$$

\* Statical equilibrium equation:  $S1 \cdot R_1(t) = S2 \cdot R_2(t)$

\* Behavior laws: (we have:  $\mathbf{A} = 1$ ,  $\mathbf{b} = h$ ,  $E^p(t) \neq 0$  and  $E_2^p(t) = 0$ )

– Cylinder 1:  $R_1(t) = (M + 1/h)^{-1} \cdot (E_1^{ine}(t) - Y_1(t)/h) = E' \cdot (E_1^{ine}(t) - Y_1(t)/h)$  with:  $E' = (E \cdot h) / (E + h)$

– Cylinder 2:  $R_2(t) = M^{-1} \cdot (E_2^{ine}(t) - E_2^p(t)) = E \cdot (E_2^{ine}(t) - 0)$

The resolution of this system implies:  $U_1^{ine}(t) = (L1 \cdot L2 \cdot S1 \cdot E' / (E' \cdot S1 \cdot L2 + E \cdot S2 \cdot L1)) \cdot (Y_1(t)/h)$

$$E_1^{ine}(t) = \frac{L2 \cdot S1 \cdot E'}{E' \cdot S1 \cdot L2 + E \cdot S2 \cdot L1} \cdot \frac{Y_1(t)}{h}$$

$$E_2^{ine}(t) = - \frac{L1 \cdot S1 \cdot E'}{E' \cdot S1 \cdot L2 + E \cdot S2 \cdot L1} \cdot \frac{Y_1(t)}{h}$$

$$R_1(t) = - \frac{S2 \cdot L1 \cdot E \cdot E'}{E' \cdot S1 \cdot L2 + E \cdot S2 \cdot L1} \cdot \frac{Y_1(t)}{h}$$

$$R_2(t) = - \frac{S1 \cdot L1 \cdot E \cdot E'}{E' \cdot S1 \cdot L2 + E \cdot S2 \cdot L1} \cdot \frac{Y_1(t)}{h}$$

The stabilized inelastic response of the structure is

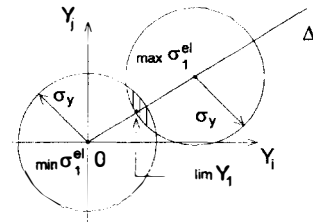


Fig. 2. Elastic shakedown evolution of the plastic yield surface in the  $Y$  space.

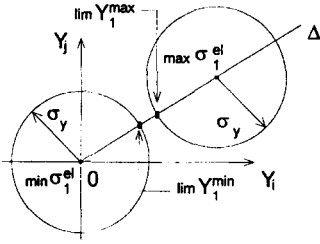


Fig. 3. Plastic shakedown evolution of the plastic yield surface in the  $Y$  space.

obtained by determining the asymptotic value of  $Y_1(t)$ . We have:

$$\begin{cases} Y_1(t) \in C(\Sigma_1^{el}(t)) \forall t & (\text{spherical convex set}) \\ \dot{E}_1^p(t) = \dot{\alpha}_1(t) \in -\partial \Psi_{C(\Sigma_1^{el}(t))}(Y_1(t)) \\ \dot{Y}_1(t) = \mathbf{B} \cdot \dot{\alpha}_1(t) & (\mathbf{B} \text{ is linear and regular}) \end{cases}$$

which implies (Figs. 2 and 3):

\* in the case of elastic shakedown:  $Y_1(t)$  tends

toward the constant value  $\lim Y_1 = (E \cdot S_2 / (S_1 \cdot L_2 + S_2 \cdot L_1)) \cdot \bar{U} - \sigma_y$ ,

\* in the case of plastic shakedown:  $Y_1(t)$  varies periodically between  $\lim Y_1^{\min}$  and  $\lim Y_1^{\max}$ .

$$\lim Y_1^{\max} = \frac{E \cdot S_2}{S_1 \cdot L_2 + S_2 \cdot L_1} \cdot \bar{U} - \sigma_y;$$

$$\lim Y_1^{\min} = \sigma_y$$

The stabilized response of the structure is then finally computed by adding the elastic response with the stabilized inelastic one. We have expressed it in terms of the geometry, the characteristics of the material and the imposed loading.

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